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HARNACK'S INEQUALITY FOR DEGENERATE SCHRÖDINGER OPERATORS

CRISTIAN E. GUTIERREZ

ABSTRACT. We prove a Harnack inequality for nonnegative weak solutions of certain Schrödinger equations of the form $Lu - Vu = 0$ where L is a second order degenerate elliptic operator in divergence form and V is a potential in certain class.

1. INTRODUCTION

The purpose of this paper is to establish a Harnack inequality for nonnegative weak solutions of certain degenerate Schrödinger equations of the form

$$Lu - Vu = \sum_{i,j=1}^{n} D_{x_i}(a_{ij}(x)D_{x_j}u) - Vu = 0, \tag{1.1}$$

$x \in R^n$. The coefficients a_{ij} are measurable real-valued functions, the coefficient matrix $a = (a_{ij})$ is symmetric and

$$\lambda^{-1}w(x)|\xi|^2 \le \langle a(x)\xi, \xi\rangle \le \lambda w(x)|\xi|^2,$$

where $\lambda > 0$, $\xi = (\xi_1, \ldots, \xi_n)$, $\langle,\rangle$ is the Euclidean inner product and w is a weight satisfying either

(i) $w \in A_2$, that is

$$\sup_B \left(\fint_B w(x)\,dx \right) \left(\fint_B w(x)^{-1}\,dx \right) = c_0 < \infty,$$

where the supremum is taken over all balls B in R^n and $\fint_B w(x)\,dx$ denotes the average of w over B. The constant c_0 is referred to as the A_2 constant of w; or

(ii) $w(x) = |f'(x)|^{1-2/n}$, where $f\colon R^n \to R^n$ is a quasiconformal mapping and $|f'(x)|$ denotes the absolute value of the Jacobian determinant of f. Quasiconformal means $f = (f_1, \ldots, f_n)$ is one-to-one, the distributional derivatives of f_i belong to $L^n_{\mathrm{loc}}(R^n)$ and there is a constant $C_0 > 0$, called the

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dilation constant of f, such that a.e.

$$\left[\sum_{i,j=1}^{n}(D_{x_j}f_i)^2\right]^{1/2} \leq C_0|f'(x)|^{1/n}.$$

We shall assume that the potential V satisfies the following condition

$$\lim_{\delta\to 0}\sup_{B_{R/4}(x_0)}\int_{|x-y|<\delta}|V(y)|\int_{|x-y|}^{R}\frac{s^2}{w(B_s(x))}\frac{ds}{s}\,dy = 0, \tag{1.2}$$

for every $x_0 \in R^n$ and $R > 0$, $B_s(x)$ denotes the ball centered at x with radius s.

Let Ω be an open, bounded and connected set in R^n. We say that the function u is a weak solution of $Lu - Vu = 0$ in Ω if $u \in H^1_{\text{loc}}(\Omega, w)$ and

$$-\int \langle a(x)\nabla u(x), \nabla\psi(x)\rangle\,dx = \int V(x)u(x)\psi(x)\,dx$$

for every $\psi \in H^1_0(\Omega, w)$ (see definitions in §2).

Given Ω an open bounded subset of R^n let B_R be the smallest ball containing Ω. If η is a nondecreasing function defined for $r > 0$ and such that $\lim_{r\to 0}\eta(r) = 0$ then we set

$$K_\eta = \left\{V\colon \sup_{x\in B_R}\int_{|x-y|\leq r}|V(y)|\int_{|x-y|}^{4R}\frac{s^2}{w(B_s(x))}\frac{ds}{s}\,dy \leq \eta(r),\ r>0\right\}.$$

By c_0 we denote either the A_2 constant of w or the dilation constant of f if w satisfies (ii). The main result is the following:

Theorem. *Let Ω be an open, bounded subset of R^n, w is a weight satisfying* (i) *or* (ii) *and $V \in K_\eta$. Then there exist positive constants r_0 and γ only depending on λ, n, c_0, Ω and η such that if u is any nonnegative solution of* (1.1) *in Ω then for any ball B_r with $B_{8r} \subset \Omega$ and $0 < r \leq r_0$ we have*

$$\sup_{B_{r/2}} u \leq \gamma \inf_{B_{r/2}} u.$$

The theorem in the nondegenerate case, i.e. $w \equiv 1$, was obtained by Chiarenza, Fabes and Garofalo in [1]. In this case (1.2) means that V belongs to the Kato-Stummel class. In the degenerate case (1.2) is suggested by the following approximate formula for the Green's function $G_L(x, y)$ for L in $B_R(x_0)$ valid when w satisfies (i) or (ii),

$$G_L(x, y) \simeq \int_{|x-y|}^{R}\frac{s^2}{w(B_s(x))}\frac{ds}{s}, \tag{1.3}$$

for $x, y \in B_{R/4}(x_0)$ (see [4] for a proof of this formula). It is easy to see that if $V/w \in L^p_w$ locally for $p > (n/2)\mu$ then V satisfies (1.2). Here μ means the doubling order of w, i.e. $w \in D_\mu$ (see §2 for definitions).

The proof of our Theorem is based on the method developed in [1] and [6] which basically consists of estimating powers of the solution u. One of the ingredients used in the proof is a weighted interpolation inequality (Lemma (3.3)) having some independent interest.

As in the nondegenerate case our result implies the continuity of solutions.

The paper is organized as follows: in §2 we state some preliminary definitions and results, in §3 we show an L^∞-estimate for solutions and in §4 we establish some properties of the Green's function for $L - V$ and the infimum estimate.

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2. Preliminaries

Ω denotes a bounded, open and connected subset of R^n. $L^p(\Omega, w)$ denotes the class of functions f such that

$$\|f\|_{p,w}^p = \int_\Omega |f(x)|^p w(x)\,dx < \infty.$$

$\mathrm{Lip}(\overline{\Omega})$ denotes the class of Lipschitz functions in $\overline{\Omega}$. We say $\psi \in \mathrm{Lip}_0(\Omega)$ if $\psi \in \mathrm{Lip}(\overline{\Omega})$ and ψ has compact support contained in Ω. For $\psi \in \mathrm{Lip}(\overline{\Omega})$ we define the norm

$$\int_\Omega |\psi(x)|^2 w(x)\,dx + \int_\Omega |\nabla\psi(x)|^2 w(x)\,dx. \tag{2.1}$$

$H^1(\Omega, w)$ denotes the closure of $\mathrm{Lip}(\overline{\Omega})$ under the norm (2.1). $H_0^1(\Omega, w)$ denotes the closure of $\mathrm{Lip}_0(\Omega)$ under the norm (2.1). $H^{-1}(\Omega, w)$ denotes the dual space of $H_0^1(\Omega, w)$. When w satisfies (i) or (ii) and $u \in H^1(\Omega, w)$ the gradient of u is uniquely defined (see [5, §2]). It can be shown (see [4, p. 154]) that

$$H^{-1}(\Omega, w) = \{f_0 - \operatorname{div}\vec{f} : \vec{f} = (f_1, \dots, f_n), f_i/w \in L^2(\Omega, w), \quad i = 0, 1, \dots, n\}.$$

We say $u \in H_{\mathrm{loc}}^1(\Omega, w)$ if $u \in H^1(\Omega', w)$ for every Ω' with closure contained in Ω. Let $u \in H^1(\Omega, w)$, $E \subset \overline{\Omega}$, then $u \geq 0$ on E in the sense of $H^1(\Omega, w)$ if there exists a sequence $u_n \in \mathrm{Lip}(\overline{\Omega})$ such that $u_n(x) \geq 0$ for $x \in E$ and $u_n \to u$ in $H^1(\Omega, w)$. If w satisfies (i) or (ii) then Poincaré's inequality holds, i.e. there exist constants C and $\tau > 1$ depending only on c_0 such that

$$\left(\fint_B |u - u_B|^{2\tau} w(x)\,dx\right)^{1/2\tau} \leq C|B|^{1/n} \left(\fint_B |\nabla u|^2 w(x)\,dx\right)^{1/2} \tag{2.2}$$

for all $u \in H^1(\Omega, w)$, $u_B = \frac{1}{W(B)} \int_B uw\,dx$. Also, if $u \in H_0^1(\Omega, w)$ we have Sobolev's inequality

$$\left(\fint_B |u|^{2\tau} w(x)\,dx\right)^{1/2\tau} \leq C|B|^{1/n} \left(\fint_B |\nabla u|^2 w\,dx\right)^{1/2}. \tag{2.3}$$

For a proof of (2.2) and (2.3) see [5]. We say that the weight w satisfies a doubling condition of order μ if there exists a constant $C > 0$ such that

$$w(B_{tr}(x_0)) \leq Ct^{n\mu} w(B_r(x_0))$$

for every $x_0 \in R^n$, $r > 0$ and $t \geq 1$. In this case we write $w \in D_\mu$. It is well known that if w satisfies (i) or (ii) in §1 then $w \in D_\mu$ for some $\mu \geq 1$.

3. The L^∞-estimate

In this section we will show the following

Theorem (3.1). *Given $p > 0$ there exist positive constants r_0 and C only depending on p, λ, n, η and Ω such that if u is any solution of $Lu - Vu = 0$ in Ω and B_r is any ball with $r \leq r_0$ and $B_{2r} \subset \Omega$ then we have*

$$\sup_{B_{r/2}} |u| \leq C \left(⨍_{B_r} |u|^p w(x)\, dx \right)^{1/p}.$$

The proof will be a consequence of the following lemmas.

Lemma (3.2). *Let u be a solution of $Lu - Vu = 0$ in Ω. Then there exists a positive constant $C = C(\lambda, n, c_0, \eta, \Omega)$ such that if $0 < s < t$ and $B_t \subset \Omega$ then we have*

$$\int_{B_s} |\nabla u(x)|^2 w(x)\, dx \leq C \frac{1}{(t-s)^2} \int_{B_t} u(x)^2 w(x)\, dx.$$

Proof. Take $\phi \in C_0^\infty(B_t)$ such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ on B_s and $\|\nabla \phi\|_\infty \leq \frac{C}{t-s}$. We have

$$\begin{aligned}\int |\nabla u(x)|^2 \phi(x)^2 w(x)\, dx &\leq \lambda \int \langle a(\nabla u), \nabla u\rangle \phi(x)^2\, dx \\ &= \lambda \int \langle a(\nabla u), \nabla(u\phi^2)\rangle\, dx - 2\lambda \int \langle a(\nabla u), \nabla\phi\rangle \phi(x) u(x)\, dx \\ &= -\lambda \int u(x)^2 \phi(x)^2 V(x)\, dx - 2\lambda \int \langle a(\nabla u), \nabla \phi\rangle \phi(x) u(x)\, dx.\end{aligned}$$

To estimate the second term we use the fact that for every $\varepsilon > 0$ we have

$$|\langle a(\phi\nabla u), u\nabla\phi\rangle| \leq \frac{\varepsilon}{2}\langle a(\phi\nabla u), \phi\nabla u\rangle + \frac{1}{2\varepsilon}\langle a(u\nabla\phi), u\nabla\phi\rangle.$$

Hence by taking $\varepsilon\lambda^2 = 1/2$ we obtain

$$\int |\nabla u|^2 \phi^2 w\, dx \leq -2\lambda \int u^2\phi^2 V\, dx + 4\lambda^4 \int u^2 |\nabla\phi|^2 w\, dx.$$

To estimate the second integral in the last inequality we use the following embedding lemma. (For a proof of this lemma in the unweighted case see [9, p. 138].)

Lemma (3.3). *Let Ω be an open, bounded and connected set in R^n and let V be a potential satisfying* (1.2). *Then given $\varepsilon > 0$ there exists a constant $C_\varepsilon = C(\varepsilon, \Omega, w, n, V, \lambda)$ such that for any $u \in H_0^1(\Omega, w)$ we have*

$$\int u^2(x)|V(x)|\,dx \le \varepsilon \int |\nabla u|^2 w(x)\,dx + C_\varepsilon \int u^2 w(x)\,dx.$$

Proof. Given D an open and bounded set in R^n let B_R be a ball such that $D \subset B_{R/4}$ and let $G(x, y)$ be the Green's function for L in B_R. We define

$$\eta_D(s) = \sup_{x \in D} \int_{|x-y|<s} |V(y)| G(x, y)\,dy.$$

By (1.3) we have that (1.2) is equivalent to

$$\lim_{s \to 0} \eta_D(s) = 0$$

for every bounded and open set $D \subset R^n$. It is enough to prove the lemma for $u \in \mathrm{Lip}_0(\Omega)$, then the desired result follows by passing to the limit. Let us suppose first that u has support contained in a ball $B_r \subset \Omega$, then we claim that for every $\varepsilon > 0$ there exists a constant C_ε independent of u and V such that

$$\int u^2 V\,dx \le \varepsilon \int |\nabla u|^2 w\,dx + C_\varepsilon \left[\eta_{B_r}(3r) \int u^2 V\,dx + \eta_{B_r}(3r)^2 \int |\nabla u|^2 w\,dx\right].$$

To prove the claim assume V has support contained in B_r, $V \ge 0$, and set

$$F(x) = \int V(y) G(x, y)\,dy$$

where G is the Green's function of a ball B such that $\frac{1}{4}B \supset \Omega$. Then F is the solution of $LF = -V$ in B and $F \in H_0^1(B, w)$. This is because since V satisfies (1.2) then

$$\int_B \int_B G(x, y) V(x) V(y)\,dx\,dy < \infty,$$

and therefore as in the proof of Theorem 4.8 of [4] we can conclude that $\chi_B V \in H^{-1}(B, w)$, ($\chi_B$ denotes the characteristic function of B). Therefore

$$\begin{aligned}\int u^2 V\,dx &= \int \langle a(\nabla F), \nabla u^2 \rangle\,dx = 2 \int \langle a(u \nabla F), \nabla u \rangle\,dx \\ &\le \varepsilon \int \langle a(\nabla u), \nabla u \rangle\,dx + \frac{1}{\varepsilon} \int \langle a(u \nabla F), u \nabla F \rangle\,dx \\ &\le \lambda \varepsilon \int |\nabla u|^2 w(x)\,dx + \frac{\lambda}{\varepsilon} \int |\nabla F|^2 u^2 w\,dx.\end{aligned}$$

Now observe that

$$\begin{aligned}\int |\nabla F|^2 u^2 w\,dx &\le \lambda \int \langle a(\nabla F), \nabla F\rangle u^2\,dx\\ &= \lambda \int \langle a(\nabla F), \nabla(Fu^2)\rangle dx - 2\lambda \int \langle a(\nabla F), \nabla u\rangle uF\,dx\\ &= \lambda \int u^2 FV\,dx - 2\lambda \int \langle a(u\nabla F), F\nabla u\rangle\,dx\\ &\le \lambda \int u^2 F(x)V(x)\,dx + \lambda\delta \int |\nabla F|^2 u^2 w\,dx + \frac{\lambda}{\delta}\int F^2 |\nabla u|^2 w\,dx\end{aligned}$$

for every $\delta > 0$. Note that for $x \in B_r$, $F(x) \le \eta_{B_r}(3r)$ and then by taking $\delta = (2\lambda)^{-1}$ we get

$$\int |\nabla F|^2 u^2 w\,dx \le 2\eta_{B_r}(3r)\lambda \int u^2 V\,dx + 4\lambda^2 \eta_{B_r}(3r)^2 \int |\nabla u|^2 w\,dx.$$

The claim follows with $C_\varepsilon = 4\lambda^4/\varepsilon$.

To complete the proof of the lemma, given $0 < \delta < 1$, let $\{\psi_j^2\}_1^N$ be a finite partition of unity of $\overline{\Omega}$ such that $\operatorname{supp}\psi_j \subseteq B_{r_j}(x_j)$ with $x_j \in \overline{\Omega}$ and $0 < r_j \le \delta$. Set $\tilde{\Omega} = \{x : d(x, \overline{\Omega}) \le 1\}$. Therefore

$$\begin{aligned}\int (u\psi_j)^2 V\,dx &\le \frac{\varepsilon}{2}|\nabla(u\psi_j)|^2 w\,dx\\ &\quad + C_\varepsilon\left[\eta_{B_{r_j}}(3r_j)\int (\psi_j u)^2 V\,dx + \eta_{B_{r_j}}(3r_j)^2 \int |\nabla(u\psi_j)|^2 w\,dx\right]\\ &\le \frac{\varepsilon}{2}\int |\nabla u|^2 \psi_j^2 w\,dx + \frac{\varepsilon}{2}\int u^2 |\nabla\psi_j|^2 w\,dx\\ &\quad + C_\varepsilon\left[\eta_{\tilde{\Omega}}(3\delta)\int (\psi_j u)^2 V\,dx\right.\\ &\qquad \left. + \eta_{\tilde{\Omega}}(3\delta)^2 \int |\nabla u|^2 \psi_j^2 w\,dx + \eta_{\tilde{\Omega}}(3\delta)^2 \int |\nabla\psi_j|^2 u^2 w\,dx\right].\end{aligned}$$

We now choose $\delta = \delta(\varepsilon) < 1$ such that

$$C_\varepsilon \eta_{\tilde{\Omega}}(3\delta) < \frac{1}{2} \quad \text{and} \quad C_\varepsilon \eta_{\tilde{\Omega}}(3\delta)^2 < \frac{\varepsilon}{2}.$$

Hence

$$\frac{1}{2}\int (u\psi_j)^2 V\,dx \le \varepsilon \int |\nabla u|^2 \psi_j^2 w\,dx + \varepsilon \int u^2 |\nabla\psi_j|^2 w\,dx.$$

By summing in j it follows that

$$\frac{1}{2}\int u^2 V\,dx \le \varepsilon \int |\nabla u|^2 w\,dx + \varepsilon \frac{N(\varepsilon)}{\delta(\varepsilon)^2}\int u^2 w\,dx.$$

Remarks. (1) The constant C_ε only depends on $\varepsilon, \Omega, \eta, n, c_0$ and λ.

(2) By Sobolev's inequality (2.3) Lemma (3.3) implies the following two-weights Sobolev inequality

$$\int u^2(x)V(x)\,dx \le C\int |\nabla u|^2 w(x)\,dx.$$

Lemma (3.4). *There exists a constant $C = C(n, \lambda, \eta, \Omega, c_0)$ such that if u is a solution of $Lu - Vu = 0$ in Ω and $B_{2r}(x_0) \subset \Omega$ then*

$$\left(\fint_{B_{r/2}(x_0)} u^2 w \, dx \right)^{1/2} \leq C \fint_{B_r(x_0)} |u| w \, dx.$$

Proof. We claim that it is enough to prove the lemma when $r = 1$ and $x_0 = 0$. In fact $u_{x_0}(x) = u(x - x_0)$ is defined in $\Omega + x_0$ and if $a_{x_0}(x) = a(x - x_0)$, $V_{x_0}(x) = V(x - x_0)$ and $w_{x_0}(x) = w(x - x_0)$ then u_{x_0} is a solution of $\operatorname{div}(a_{x_0}(x)\nabla) - V_{x_0} = 0$ in $\Omega + x_0$. Note that the constant c_0 of w_{x_0} does not change and V_{x_0} is in K_η defined with w_{x_0}. Therefore by translations we can assume $x_0 = 0$. Set $u_r(x) = u(rx)$, then u_r is defined in $\frac{1}{r}\Omega$ (in particular in B_2) and if we set $a_r(x) = a(rx)$, $V_r(x) = r^2 V(rx)$ and $w_r(x) = w(rx)$ then u_r is a solution of $\operatorname{div}(a_r(x)\nabla) - V_r = 0$ in $\frac{1}{r}\Omega$. Note again that the constant c_0 of w_r does not change. Also by changing variables is easy to see that

$$\begin{aligned} &\sup_{x \in 1/r\Omega} \int_{|x-y|<\delta} |V_r(y)| \int_{|x-y|}^{4R/r} \frac{s}{w_r(B_s(x))} \, ds \, dy \\ &= \sup_{x \in \Omega} \int_{|x-y|<\delta r} |V(y)| \int_{|x-y|}^{4R} \frac{s}{w(B_s(x))} \, ds \, dy, \end{aligned}$$

which if $r \leq 1$ implies that V_r belongs to the class K_η defined with w_r. Let us assume $\fint_{B_1} |u| w \, dx = 1$ and for $\frac{1}{2} < s < 1$ consider

$$I(s) = \left(\frac{1}{w(B_{1/2})} \int_{B_s} u^2 w \, dx \right)^{1/2}.$$

If $I(\frac{1}{2}) \leq 1$ then there is nothing to show, so suppose $I(\frac{1}{2}) > 1$. We want to show $I(\frac{1}{2}) \leq C$, C only depends on n, λ, η, Ω and c_0. Let τ be the exponent in the Poincaré inequality and choose $0 < \theta < 1$ such that $(2-\theta)/(1-\theta) = 2\tau$. By doubling and Poincaré we have

$$\begin{aligned} I(s) = \left(\frac{1}{w(B_{1/2})} \int_{B_s} |u|^{2-\theta} |u|^{\theta} w \, dx \right)^{1/2} &\leq C \left(\fint_{B_s} |u|^{(2-\theta)/(1-\theta)} w \, dx \right)^{(1-\theta)/2} \\ \leq C s^{(1-\theta)\tau} \left(\fint_{B_s} |\nabla u|^2 w \, dx \right)^{((1-\theta)/2)\tau} &+ C \left(\fint_{B_s} |u|^2 w \, dx \right)^{((1-\theta)/2)\tau}. \end{aligned}$$

If $\frac{1}{2} \leq s < t \leq 1$ then by Lemma (3.2) and doubling we obtain $I(s) \leq C[(t-s)^{-1} I(t)]^{(1-\theta)\tau}$ which implies $I(\frac{1}{2}) \leq C$. (See Lemma 1.2 of [1].)

Lemma (3.5). *Let $\Omega = B_2(0)$, $w \in D_\mu$ and $p > (n/2)\mu$. There exist constants $\delta_0 = \delta_0(\lambda, n, \eta, c_0)$ and $C = C(\lambda, n, \eta, c_0)$ such that if*

$$\sup_{B_2} \int_{B_6} |V(y)| \int_{|x-y|}^{8} \frac{s}{w(B_s(x))} \, ds \, dy < \delta_0$$

then given $f/w \in L^p_w(\Omega)$ *there exists a unique* $u \in H^1_0(\Omega, w)$ *such that* $Lu - Vu = f$ *in* Ω *and*

$$\|u\|_{L^\infty(\Omega)} \le \frac{c}{w(B_1)^{1/p}} \|f/w\|_{L^p_w(\Omega)}.$$

Proof. The bilinear form

$$\alpha(u, v) = \int_\Omega \langle a(\nabla u), \nabla v\rangle \, dx + \int_\Omega uvV \, dx$$

is continuous and coercive in $H^1_0(\Omega, w)$ provided δ_0 is small enough. This follows by the claim made in the proof of Lemma (3.3). Now if $p \ge 2$ and $f \in L^p_{w^{1-p}}(\Omega)$ implies $f \in H^{-1}(\Omega, w)$ and consequently the existence and uniqueness of u is a consequence of the Lax-Milgram theorem, (see [7]).

Let u_0 be the solution of the problem $Lu = f$ in Ω, $u/\partial\Omega = 0$ (i.e. $u \in H^1_0(\Omega, w)$), and for $j \ge 1$, let u_j be the solution of $Lu - Vu_{j-1} = f$ in Ω, $u/\partial\Omega = 0$. Then we have

$$u_0(x) = \int_\Omega G_L(x, y) f(y) \, dy,$$

where $G_L(x, y)$ is the Green's function of L in Ω. By the maximum principle

$$G_L(x, y) \le C \int_{|x-y|}^{8} \frac{s}{w(B_s(x))} \, ds.$$

It is easy to see that

$$\left(\int_\Omega G_L(x, y)^{p'} w(y) \, dy\right)^{1/p'} = \frac{C}{w(B_1(0))^{1/p}} = C_1 < \infty, \quad \text{for } p > \frac{n}{2}\mu.$$

Then

$$|u_0(x)| \le C_1 \|f/w\|_{L^p_w(\Omega)}, \qquad x \in \Omega.$$

We also have

$$u_1(x) = \int_\Omega G_L(x, y) f(y) \, dy + \int_\Omega G_L(x, y) V(y) u_0(y) \, dy$$

which implies

$$|u_1(x)| \le C_1 \|f\|_{L^p_{w^{1-p}}} + C_1 \delta \|f\|_{L^p_{w^{1-p}}},$$

provided $\sup_\Omega \int G_L(x, y)|V(y)| \, dy < \delta$. Continuing in this manner we obtain

$$u_j(x) = \int_\Omega G_L(x, y) f(y) \, dy + \int_\Omega G_L(x, y) V(y) u_{j-1}(y) \, dy \tag{3.6}$$

and

$$|u_j(x)| \le C_1 \cdot C_\delta \|f/w\|_{L^p_w(\Omega)}, \quad \text{for } j = 2, 3, \ldots.$$

We claim that u_j is a Cauchy sequence in $H^1_0(\Omega, w)$ and $u_j \to u$ in $H^1_0(\Omega, w)$. By (3.6) we have

$$u_{j+1}(x) - u_j(x) = \int_\Omega G_L(x, y) V(y) [u_j(y) - u_{j-1}(y)] \, dy$$

and therefore

$$\|u_{j+1} - u_j\|_{L^\infty(\Omega)} \le \delta \|u_j - u_{j-1}\|_{L^\infty(\Omega)}.$$

Consequently for $m > n$

$$\|u_m - u_n\|_{L^\infty(\Omega)} \le \sum_{j=n}^{m-1} \|u_{j+1} - u_j\|_{L^\infty(\Omega)}$$

$$\le \|u_0 - u_1\|_{L^\infty(\Omega)} \sum_{j=n}^{m-1} \delta^j.$$

Therefore for $\delta < 1$ $\{u_j\}$ is a Cauchy sequence in $L^\infty(\Omega)$ and therefore in $L^2(\Omega, w)$. Also

$$\int_\Omega |\nabla(u_m - u_n)|^2 w(x)\,dx \le \lambda \int_\Omega \langle a(\nabla(u_m - u_n)), \nabla(u_m - u_n)\rangle\,dx$$

$$= -\int (u_m - u_n)(u_{m-1} - u_{n-1})V(x)\,dx$$

$$\le \left(\int |u_m - u_n|^2 V(x)\,dx\right)^{1/2} \left(\int |u_{m-1} - u_{n-1}|^2 V(x)\,dx\right)^{1/2},$$

and since $V \in L^1(\Omega)$ we have $\|\nabla(u_m - u_n)\|_{L^2(\Omega, w)}$ tends to 0 as $m, n \to \infty$. Consequently $u_j \to \tilde{u}$ in $H_0^1(\Omega, w)$ and by (3.6) $\tilde{u} = u$.

Remark (3.7). Lemma (3.5) implies the existence and integrability of the Green's function $G(x, y)$ of $L - V$. In fact, if $p > (n/2)\mu$ then

$$\int_\Omega G(x, y) w(y)^{(p-1)/p} f(y) w(y)^{(1-p)/p}\,dy \le \frac{C}{w(B_1)^{1-p}} \|fw^{(1-p)/p}\|_{L^p(\Omega)},$$

which implies

$$\left(\int_\Omega G(x, y)^q w(y)\,dy\right)^{1/q} = \frac{C}{w(B_1)^{1/p}} < \infty \quad \text{for } 1 < q < \frac{n\mu}{n\mu - 2}$$

and a.e. $x \in \Omega$ $(\frac{1}{p} + \frac{1}{9} = 1)$.

Theorem 3.8. *There exist* $r_0 = r_0(\lambda, n, c_0, \eta)$ *and for each* $p > 0$ *a constant* $C = C(p, \lambda, n, c_0, \eta)$ *such that if* u *is a solution of* $Lu - Vu = 0$ *in* Ω *and* $B_{2r} \subset \Omega$ *with* $r \le r_0$ *then*

$$\sup_{B_{r/2}} |u| \le C \left(⨍_{B_r} |u(x)|^p w(x)\,dx \right)^{1/p}.$$

Proof. By translation we may assume B_r is centered at 0. As before $u_r(x) = u(rx)$ is a solution in $B_2 = B_2(0)$ of $L_r u_r - V_r u_r = 0$ where L_r and V_r are defined in Lemma (3.4). Also observe that

$$\int_{|x-y|<\delta} |V_r(y)| \int_{|x-y|}^{8} \frac{s}{w_r(B_s(x))}\,ds\,dy$$

$$= \int_{|rx-z|<r\delta} \int_{|rx-z|}^{8r} \frac{s}{w_r(B_s(x))}\,ds\,dy \le \eta(\delta r).$$

Therefore

$$\sup_{x\in B_2}\int_{|x-y|<\delta}|V_r(y)|\int_{|x-y|}^{8}\frac{s}{w_r(B_s(x))}\,ds\,dy\le\eta(\delta r).$$

Since $\eta(\delta r)\to 0$ as $\delta\to 0$ then it is enough to show that if δ_0 is the number specified in Lemma (3.5) and

$$\sup_{B_2}\int_{B_6}|V(y)|\int_{|x-y|}^{8}\frac{s}{w(B_s(x))}\,ds\,dy<\delta_0$$

then we have

$$\sup_{B_{1/2}}|u|\le C\left(\fint_{B_1}|u|^p w(y)\,dy\right)^{1/p},$$

with $C=C(\lambda,\eta,p,n,c_0)$.

Let $G(x,y)$ be the Green's function of $L-V$ in B_2. Given $\frac{1}{2}\le s<t\le 1$, let ψ be in $C_0^\infty(B_{t-(t-s)/4}(0))$ such that $0\le\psi\le 1$, $\psi\equiv 1$ on $B_{(t+s)/2}(0)$ and $|\nabla\psi|\le C/(t-s)$. We have

$$\begin{aligned}u(x)\psi(x)=&\int_{B_2}\langle a(\nabla_y G(x,y)),\nabla\psi(y)\rangle u(y)\,dy\\&-\int_{B_2}\langle a(\nabla u),\nabla\psi\rangle G(x,y)\,dy=J_1-J_2.\end{aligned}$$

$$\begin{aligned}J_1=&\int_{B_{t-(t-s)/4}\setminus B_{(t+s)/2}}\langle a(\nabla G(x,\cdot)),\nabla\psi\rangle u(y)\,dy\\ \le&\left(\int_{B_{t-(t-s)/4}\setminus B_{(t+s)/2}}|\nabla_y G(x,y)|^2 w(y)\,dy\right)^{1/2}\\&\times\left(\int_{B_{t-(t-s)/4}\setminus B_{(t+s)/2}}u^2|\nabla\psi|^2 w(y)\,dy\right)^{1/2}\\ \le&\frac{C}{t-s}\left(\int_{B_{t-(t-s)/4}\setminus B_{(t+s)/2}}|\nabla_y G(x,y)|^2 w(y)\,dy\right)^{1/2}\left(\int_{B_t}u^2 w(y)\,dy\right)^{1/2}.\end{aligned}$$

Analogously

$$\begin{aligned}J_2\le&\frac{C}{t-s}\left(\int_{B_{t-(t-s)/4}\setminus B_{(t+s)/2}}|G(x,y)|^2 w(y)\,dy\right)^{1/2}\\&\times\left(\int_{B_{t-(t-s)/4}\setminus B_{(t+s)/2}}|\nabla u|^2 w(y)\,dy\right)^{1/2}.\end{aligned}$$

Now by Lemma (3.2)

$$\begin{aligned}\int_{B_{t-(t-s)/4}\setminus B_{(t+s)/2}}|\nabla u|^2 w(y)\,dy\le&\int_{B_{t-(t-s)/4}}|\nabla u|^2 w(y)\,dy\\ \le&\frac{C}{(t-s)^2}\int_{B_t}u^2 w(y)\,dy.\end{aligned}$$

We cover the annulus $B_{t-(t-s)/4} \setminus B_{(t+s)/2}$ by a union of N balls $B_{(t-s)/4}(z_i)$, with $|z_i| = (t+s)/2 + (t-s)/8$ (observe that the annulus has width $(t-s)/4$). Therefore if $x \in B_s$ then $x \notin B_{(t-s)/2}(z_i)$ and then for $x \in B_s$ we have

$$\int_{B_{t-(t-s)/4}\setminus B_{(t+s)/2}} |\nabla_y G(x, y)|^2 w(y)\, dy \leq \sum_{i=1}^{N} \int_{B_{(t-s)/4}(z_i)} |\nabla G(x, y)|^2 w(y)\, dy$$

$$\leq \sum_{i=1}^{N} \frac{C}{(t-s)^2} \int_{B_{(t-s)/2}(z_i)} |G(x, y)|^2$$

$$= \frac{C}{(t-s)^2} \sum_{i=1}^{N} w(B_{(t-s)/2}(z_i)) ⨍_{B_{(t-s)/2}(z_i)} |G(x, y)|^2 w(y)\, dy$$

which by Lemma (3.4) is less than

$$\frac{C}{(t-s)^2} \sum_{i=1}^{N} w(B_{t-s}(z_i))^{-1} \cdot \left(\int_{B_{t-s}(z_i)} G(x, y) w(y)\, dy \right)^2 .$$

Now by doubling we have $w(B_1(0))(t-s)^{n\mu} \leq c \cdot w(B_{t-s}(z_i))$ and then by Remark (3.7) we obtain

$$\left(\int_{B_{t-(t-s)/4}\setminus B_{(t+s)/2}} |\nabla G(x, y)|^2 w(y)\, dy \right)^{1/2} \leq \frac{C}{(t-s)^{1+(n/2)\mu}} \frac{1}{w(B_1)^{1/2}}.$$

Analogously we have

$$\int_{B_{t-(t-s)/4}\setminus B_{(t+s)/2}} G(x, y)^2 w(y)\, dy$$

$$\leq C \sum_{i=1}^{N} w(B_{t-s}(z_i)) \left(⨍_{B_{(t-s)/2}(z_i)} G(x, y) w(y)\, dy \right)^2 .$$

Collecting estimates we obtain

$$\|u\|_{L^\infty(B_s)} \leq \frac{C}{(t-s)^{2+(n/2)\mu}} \left(\frac{1}{w(B_1)} \int_{B_t} u^2 w(y)\, dy \right)^{1/2} .$$

For $\frac{1}{2} \leq s \leq 1$ we set $I(s) = (1/w(B_1) \int_{B_s} u^2 w\, dx)^{1/2}$. Let $p > 0$ and assume $⨍_{B_1} u^p w\, dy = 1$, then if $p < 2$ we have

$$I(s) \leq \left(\sup_{B_s} |u| \right)^{\theta} , \qquad \theta = 1 - \frac{p}{2} ,$$

and therefore

$$I(s) \leq \frac{C}{(t-s)^{(2+(n/2)\mu)\theta}} I(t)^{\theta}.$$

By the argument in [6, p. 1004] we obtain the theorem.

4. The infimum estimate

We begin with the following

Lemma (4.1). *Let $u \geq 0$ be a solution of $Lu - Vu = 0$ in Ω. Then there exists a constant $C = C(\lambda, \eta, c_0)$ independent of u such that if $B_{4r} \subset \Omega$ then for every $\varepsilon > 0$ we have*

$$\frac{1}{w(B_r)} \int_{B_r} \left| \log(u+\varepsilon) - \fint_{B_r} \log(u+\varepsilon) w(y)\, dy \right|^2 w(y)\, dy \leq C.$$

Proof. Let $\psi \in C_0^\infty(B_{3r/2})$, $\psi \equiv 1$ on B_r and $0 \leq \psi \leq 1$, $|\nabla\psi| \leq c/r$, $B_r = B_r(x)$ and set $u_\varepsilon = u + \varepsilon$. Then

$$\begin{aligned}
\int |\nabla \log u_\varepsilon|^2 w\, dy &= \int \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} \psi^2 w\, dy \\
&\leq \lambda \int \langle a(\nabla u_\varepsilon), \nabla u_\varepsilon \rangle \frac{\psi^2}{u_\varepsilon^2}\, dy \\
&= 2\lambda \int \langle a(\nabla u), \nabla\psi \rangle \frac{\psi}{u_\varepsilon}\, dy - \lambda \int \left\langle a(\nabla u), \nabla\left(\frac{\psi^2}{u_\varepsilon}\right)\right\rangle dy \\
&= 2\lambda \int \langle a(\nabla u), \nabla\psi \rangle \frac{\psi}{u_\varepsilon} dy + \lambda \int V(y) \frac{u}{u_\varepsilon} \psi^2\, dy \\
&\leq 2\lambda \int \langle a(\nabla u), \nabla\psi \rangle \frac{\psi}{u_\varepsilon}\, dy + \lambda \int |V(y)| \psi^2\, dy.
\end{aligned}$$

We have

$$\int V(y)\psi^2(y)\, dy$$
$$\leq \int_{B_{2r}(x)} |V(y)| \left(\int_{|x-y|}^{4r} \frac{s}{w(B_s(x))}\, ds \right) \left(\int_{|x-y|}^{4r} \frac{s}{w(B_s(x))}\, ds \right)^{-1} dy.$$

If $y \in B_{2r}$ then by doubling we have

$$\int_{|x-y|}^{4} \frac{s}{w(B_s(x))}\, ds \geq C \cdot \frac{r^2}{w(B_r(x))}.$$

Consequently by the assumption on V we have

$$\int V(y)\psi^2(y)\, dy \leq C \frac{w(B_r(x))}{r^2}.$$

Also as in the proof of Lemma (3.2) we have for every $\delta > 0$

$$\int \langle a(\nabla u), \nabla\psi \rangle \frac{\psi}{u_\varepsilon}\, dy \leq \frac{\lambda}{\delta} \int |\nabla \log u_\varepsilon|^2 \psi^2 w(y)\, dy + \delta \cdot C \frac{w(B_r(x))}{r^2}.$$

Hence if δ is large we have

$$\int |\nabla \log u_\varepsilon|^2 w\, dy \leq C \frac{w(B_r(x))}{r^2}.$$

Now since $\log u_\varepsilon \in H^1(B_{2r}, w)$ then by Poincaré the lemma follows.

Remark (4.2). Note that if $w \in A_\infty$ then by Theorem 5 of [8], Lemma (4.1) implies that $\log(u+\varepsilon) \in BMO$, $\varepsilon > 0$.

Lemma (4.3). *Let $u > 0$ be a solution of $Lu - Vu = 0$ in Ω and let r_0 be the number in Theorem (3.8) then there exists a constant $C = C(\lambda, \eta, n, c_0)$ such that*

$$\int_{B_{2r}} u(x)w(x)\,dx \le C \int_{B_r} u(x)w(x)\,dx$$

for $0 < r \le r_0$ and $B_{8r} \subset \Omega$.

Proof. By Lemma (4.1) and Theorem 5 of [8] there exist $\delta > 0$ and $C > 0$ such that

$$\left(\frac{1}{w(B_r)}\int_{B_r} u_\varepsilon^\delta w\,dx\right)\left(\frac{1}{w(B_r)}\int_{B_r} u_\varepsilon^{-\delta} w\,dx\right) \le C,$$

for $B_{4r} \subset \Omega$, i.e. $u_\varepsilon^\delta \in A_2(w)$, for every $\varepsilon > 0$. Since w is doubling this implies

$$\int_{B_{2r}} u_\varepsilon^\delta w\,dx \le C \int_{B_r} u_\varepsilon^\delta w\,dx.$$

Hence by Theorem (3.8) and $\delta < 1$ we have

$$\begin{aligned}\frac{1}{w(B_{2r})}\int_{B_{2r}} uw\,dx &\le C\left(\frac{1}{w(B_{4r})}\int_{B_{4r}} u_\varepsilon^\delta w\,dx\right)^{1/\delta} \\ &\le C\left(\frac{1}{w(B_{r/2})}\int_{B_{r/2}} u_\varepsilon^\delta w\,dx\right)^{1/\delta} \le \frac{1}{w(B_r)}\int_{B_r} u_\varepsilon w\,dx,\end{aligned}$$

by letting $\varepsilon \to 0$ this implies the lemma.

Lemma (4.4). *Set $V^+ = \max(V, 0)$ and let $G(x, y)$ denote the Green's function of $L - V^+$ in Ω. Then for any $1 < q < n\mu/(n\mu - 2)$ $(w \in D_\mu)$ there exists a constant $C = C(q, \lambda, \eta, c_0)$ such that*

$$\left(\frac{1}{w(B_r)}\int_{B_r} G(x, y)^q w(y)\,dy\right)^{1/q} \le C\frac{1}{w(B_r)}\int_{B_r} G(x, y)w(y)\,dy,$$

for $B_{8r} \subset \Omega$ and $0 < r \le r_0$ (r_0 is as in Theorem (3.8)).

Proof. First observe that $G(x, \cdot)$ is a solution of $L - V^+ = 0$ for $y \ne x$, $x \in \Omega$. Suppose first that $x \notin B_{8r}$, then by Theorem (3.8) we have for every $q > 0$ that

$$\left(⨍_{B_r} G(x, y)^q w(y)\,dy\right)^{1/q} \le \sup_{y \in B_r} G(x, y) \le C ⨍_{B_{2r}} G(x, y)w(y)\,dy$$

which by Lemma (4.3) is less than

$$⨍_{B_r} G(x, y)w(y)\,dy.$$

Now assume $x \in B_{8r}$ and let $G_r(x, y) = G_{L-V^+, B_{8r}}(x, y)$. By the maximum principle $G_r(x, y) \le G(x, y)$ for $x, y \in B_{8r}$. Then

$$\fint_{B_r} G(x, y)^q w(y)\, dy \le 2^q \left\{ \fint_{B_r} [G(x, y) - G_r(x, y)]^q w(y)\, dy \right. \\ \left. + \fint_{B_r} G_r(x, y)^q w(y)\, dy \right\}.$$

Since $G(x, \cdot) - G_r(x, \cdot)$ is a nonnegative solution of $L - V^+ = 0$ in B_{8r} then arguing as before we obtain

$$\fint_{B_r} [G(x, y) - G_r(x, y)]^q w(y)\, dy < C \left\{ \fint_{B_r} [G(x, y) - G_r(x, y)] w(y)\, dy \right\}^q.$$

By translation we can assume that B_r is centered at 0. Let L_r and V_r^+ be defined as in Lemma (3.4) and let $\tilde{G}(x, y)$ be the Green's function to $L_r - V_r^+$ in B_8. Then for $x \in B_{8r}$ and $z \in B_8$ we have

$$G_r(x, rz) = r^{2-n} \tilde{G}(x, z).$$

Therefore

$$\fint_{B_r} G_r(x, y)^q w(y)\, dy = \frac{r^n}{w(B_r)} \left(\int_{|z| \le 1} \tilde{G}(x, z)^q w(rz)\, dz \right) r^{(2-n)q}$$

and

$$\fint_{B_r} G_r(x, y) w(y)\, dy = \frac{r^n}{w(B_r)} \left(\int_{B_1} \tilde{G}(x, z) w(rz)\, dz \right) r^{2-n}.$$

Then if we set

$$u(x) = \int_{B_1} \tilde{G}(x, z) w(rz)\, dz$$

then $(L_r - V_r^+)u = -\chi_{B_1} w_r$ $(w_r(z) = w(rz))$. Since $u/\partial B_8 = 0$ then we have

$$u(x) = \int_{B_1} G_{L_r, B_8}(x, y) w_r(y)\, dy \\ - \int_{B_1} G_{L_r, B_8}(x, y) V_r^+(y) u(y)\, dy,$$

where G_{L_r,B_8} is the Green's function to L_r in B_8. By the estimates (1.3) for G_{L_r,B_8} we have

$$\begin{aligned}\int_{B_1} G_{L_r,B_8}(x,y)w_r(y)\,dy &\geq C\int_{B_1}\left(\int_{|x-y|}^{8}\frac{s}{w_r(B_s(x))}\,ds\right)w_r(y)\,dy\\ &= C\int_{B_1}\int_0^8 \chi_{(|x-y|,8)}(s)\frac{s}{w_r(B_s(x))}\,ds w_r(y)\,dy\\ &= C\int_0^8 \frac{s}{w_r(B_s(x))}\int_{B_1}\chi_{(|x-y|,8)}(s)w_r(y)\,dy\,ds\\ &= C\int_0^8\frac{s}{w_r(B_s(x))}w_r(B_1\cap B_s(x))\,ds\\ &\geq C\int_4^8\frac{s}{w_r(B_s(x))}w_r(B_1\cap B_s(x))\,ds.\end{aligned}$$

If $x\in B_3$ and $s>4$ then $B_1\subset B_s(x)$ and by doubling we have

$$\inf_{x\in B_3}\int_{B_1} G_{L_r,B_8}(x,y)w_r(y)\,dy \geq C$$

C independent of r. Also by (1.2) we have

$$\int_{B_1} G_{L_r,B_8}(x,y)V^+(y)u(y)\,dy \leq \delta\|u\|_{L^\infty(B_1)}.$$

Now by Lemma (3.5) (with $f=\chi_{B_1}w_r$) we have $\|u\|_{L^\infty(B_1)}\leq C$ with $C=C(\lambda,n,\eta,c_0)$. This implies that there exist $\delta, C>0$ depending only on the parameters such that if

$$\sup_{B_4}\int_{B_4}|V_r(y)|\int_{|x-y|}^{8}\frac{s}{w(B_s(x))}\,ds\,dy\leq\delta$$

then we have

$$\inf_{B_3} u\geq C_1.$$

Consequently for $1<q<n\mu/(n\mu-2)$ we have

$$\begin{aligned}\fint_{B_r} G_r(x,y)^q w(y)\,dy &= \frac{r^n}{w(B_r)}\left(\int_{|z|\leq 1}\tilde{G}(x,z)^q w_r(z)\,dz\right)r^{(2-n)q}\frac{u(x)^q}{u(x)^q}\\ &\leq C\left[\frac{r^n}{w(B_r)}\right]^{1-q}\left(\fint_{B_r} G_r(x,y)w(y)\,dy\right)^q\\ &\quad\times\left(\int_{|z|\leq 1}\tilde{G}(x,z)^q w_r(z)\,dz\right).\end{aligned}$$

By Remark (3.7) we have

$$\int_{|z|\leq 1}\tilde{G}(x,z)^q w_r(z)\,dz\leq\frac{c}{w_r(B_1)^{q-1}},$$

and since $w_r(B_1)=r^{-n}w(B_r)$ then the lemma follows.

We are now in a position to show

Theorem (4.5). *Let u be a nonnegative solution of $Lu - Vu = 0$ in Ω. There exist positive constants $r_0 = r_0(\lambda, \eta, n, c_0)$, $p_0 = p_0(\lambda, c_0)$ and $C = C(\lambda, \eta, n, c_0)$ all independent of u such that*

$$\left(\fint_{B_r} u^{p_0} w(y)\, dy \right)^{1/p_0} \le C \inf_{B_{r/2}} u$$

for $B_{8r} \subset \Omega$ and $r \le r_0$.

Proof. By translation and dilation we can assume $\Omega = B_2(0)$ and as in the proof of Theorem (3.8) we can assume $r = 1$ and all balls are centered at 0. We show that if $u \ge 1$ on a closed set $\Gamma \subset B_1$ in the sense of $H^1(B_2)$ then we have

$$\inf_{B_{1/2}} u \ge C \left[\frac{w(\Gamma)}{w(B_1)} \right]^M , \tag{4.6}$$

where C and M only depend on λ, η and c_0. Set

$$z(x) = \int_\Gamma G_{L-V^+, B_2}(x, y) w(y)\, dy ,$$

then $(L - V^+) z = -\chi_\Gamma w$ in B_2. Also by Remark (3.7) we have

$$z(x) \le \left(\int_{B_2} G_{L-V^+, B_2}(x, y)^q w(y)\, dy \right)^{1/q} w(\Gamma)^{1/q'} \le C_1 , \quad \text{a.e. in } B_2 ,$$

here C_1 only depends on λ, n, η and c_0. Consequently $(1/C_1) z(x) \le 1$ in the H^1 sense in B_2 and therefore $(1/C_1) z(x) \le 1$ in the H^1 sense in Γ. Then $u(x) \ge (1/C_1) z(x)$ in H^1 sense in Γ. Since $z/\partial B_2 = 0$ and $u \ge 0$ in B_2 a.e. then we have $u \ge (1/C_1) z$ in the H^1 sense in ∂B_2. Also $(L - V^+)(u - (1/C_1) z) = -V^- u - (1/C_1) \chi_\Gamma w \le 0$, in $B_2 \setminus \Gamma$, then by the maximum principle we have $u(x) \ge (1/C_1) z(x)$ in H^1 sense in B_2 and consequently a.e. in B_2. Now Lemma (4.4) implies that there exists $C > 0$ and $M > 0$ such that

$$\int_\Gamma G_{L-V^+, B_2}(x, y) w(y)\, dy \ge C \left[\frac{w(\Gamma)}{w(B_1)} \right]^M \int_{B_{1/4}} G_{L-V^+, B_2}(x, y) w(y)\, dy.$$

By the argument used to prove Lemma (4.4) and (1.2) we obtain

$$\inf_{x \in B_{1/2}} \int_{B_{1/4}} G_{L-V^+, B_2}(x, y) w(y)\, dy \ge C ,$$

where $C = C(\lambda, n, \eta, c_0)$. This implies (4.6). We claim that

$$\inf_{B_{1/4}} u \ge C \left[\frac{w\{x \in B_1 : u(x) \ge 2 \text{ a.e.}\}}{w(B_1)} \right]^M .$$

In fact, since $u \in H^1(B_{3/2})$ there is a sequence $u_n \in \mathrm{Lip}(B_{3/2})$ such that $u_n \to u$ in the $H^1(B_{3/2})$ sense and a.e. Let $F = \{x \in B_1 : u(x) \ge 2 \text{ a.e.}\}$, then by Egorov's theorem given $\varepsilon > 0$ there exist a closed set $F_\varepsilon \subset F$ such that $w(F - F_\varepsilon) < \varepsilon$ and $u_n \to u$ uniformly in F_ε. Therefore $u \geq 1$ in the $H^1(B_{3/2})$ sense in F_ε and then

$$\inf_{B_{1/2}} u \geq C\left[\frac{w(\Gamma_\varepsilon)}{w(B_1)}\right]^M \geq C\left[\frac{w(F) - \varepsilon}{w(B_1)}\right]^M .$$

Now letting $\varepsilon \to 0$ the claim follows. By taking u/t we obtain that

$$\inf_{B_{1/2}} u \geq Ct\left[\frac{w\{x \in B_1 : u(x) \geq 2t \text{ a.e.}\}}{w(B_1)}\right]^M ,$$

and consequently the theorem follows for $0 < p_0 < 1/M$.

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Department of Mathematics, Temple University, Philadelphia, Pennsylvania 19122