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DIGITAL REPRESENTATIONS USING THE GREATEST INTEGER FUNCTION

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ABSTRACT. Let $S_d(\alpha)$ denote the set of all integers which can be expressed in the form $\sum \varepsilon_i[\alpha^i]$, with $\varepsilon_i \in \{0, \dots, d-1\}$, where $d \geq 2$ is an integer and $\alpha \geq 1$ is real, and let I_d denote the set of α so that $S_d(\alpha) = \mathbf{Z}^+$. We show that $I_d = [1, r_d) \cup \{d\}$, where $r_2 = 13^{1/4}$, $r_3 = 22^{1/3}$ and $r_4 = (d^2 - d - 2)^{1/2}$ for $d \geq 4$. If $\alpha \notin I_d$, we show that $T_d(\alpha)$, the complement of $S_d(\alpha)$, is infinite, and discuss the density of $T_d(\alpha)$ when $\alpha < d$. For $d \geq 4$ and a particular quadratic irrational $\beta = \beta(d) < d$, we describe $T_d(\beta)$ explicitly and show that $|T_d(\beta) \cap [0, n]|$ is of order $n^{e(d)}$, where $e(d) < 1$.

1. INTRODUCTION

In the usual base d digital representation, every nonnegative integer m is written (uniquely) in the form $m = \sum \varepsilon_i d^i$, where $\varepsilon_i \in [[0, d-1]]$. (Here and throughout the paper, we shall denote $[a, b] \cap \mathbf{Z}$ by $[[a, b]]$.) This remarkable fact has inspired many generalizations, especially when $d = 2$. (The survey papers [4] and [8] together contain sixty references.) In this paper, we replace d^i by $[\alpha^i]$, where $[x]$ denotes the greatest integer $\leq x$ and $\alpha \geq 1$ is a fixed real number, and allow the value of d for the range of ε_i to become a second parameter.

The fine behavior of the sequence $[\alpha^i]$ has also attracted much interest, especially in the equivalent form $\{\alpha^i\} = \alpha^i - [\alpha^i]$. For example, Mahler has asked whether there exists λ so that $\{\lambda(3/2)^n\}$ is equidistributed on $[0, 1)$. The only nonintegral α's for which $\{\alpha^i\}$ is well understood appear to be the PV numbers (algebraic integers whose conjugates lie in $|z| < 1$). A representation $m = \sum \varepsilon_i[\alpha^i]$ can also be viewed as a partition of m into the set $([\alpha^0], [\alpha^1], [\alpha^2], \dots)$ with at most $d - 1$ repetitions; this question has previously been studied when $\alpha < d$ is an integer.

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We now introduce our most important definitions. Let

$$(1.1)\qquad S_d(\alpha) = \left\{ \sum_{i=0}^{T} \varepsilon_i[\alpha^i] : \varepsilon_i \in [[0, d-1]],\ T \geq 0 \right\},$$

$$(1.2)\qquad T_d(\alpha) = \mathbf{Z}^+ \setminus S_d(\alpha),$$

$$(1.3)\qquad I_d = \{\alpha : S_d(\alpha) = \mathbf{Z}^+\}.$$

In this paper we compute I_d and discuss the cardinality and density of $T_d(\alpha)$ when $\alpha \notin I_d$. We give an explicit description of $T_d(\beta)$ for a family of pairs (d, β) in which $\beta = \beta(d)$ is a certain PV number.

If $\alpha > d$, then an easy counting argument (see §3) shows that $S_d(\alpha)$ has density 0, so $\alpha \notin I_d$. If $\alpha \leq d$ is an integer, then the base α digital representations are included in (1.1), so $\alpha \in I_d$. One might expect this argument to generalize to nonintegral $\alpha < d$ by some version of the pigeonhole principle, so that our problem is trivial. This is not true. Suppose $d \geq 3$ and $\alpha \in [(d^2-1)^{1/2}, d)$, so $[\alpha] = d-1$ and $[\alpha^2] = d^2 - 1$, and suppose $d^2 - d + 1 \leq m \leq d^2 - 2$. If $m = \sum \varepsilon_i[\alpha^i]$, then $\varepsilon_i = 0$ for $i \geq 2$ as $m < [\alpha^2]$. But then $m \leq (d-1)([\alpha^0] + [\alpha^1]) = d^2 - d$, a contradiction. A similar argument works for $d = 2$, with $m = 6$ and $\alpha \in [7^{1/3}, 2)$, by considering ε_0, ε_1 and ε_2.

This paper contains five theorems (and one conjecture) about $S_d(\alpha)$, $T_d(\alpha)$ and I_d.

Theorem A.

(i) $I_2 = [1, 13^{1/4}) \cup \{2\}$,

(ii) $I_3 = [1, 22^{1/3}) \cup \{3\}$,

(iii) $I_d = [1, (d^2 - d + 2)^{1/2}) \cup \{d\}$, $d \geq 4$.

The description of I_2 may be deduced from [7, Theorem 5]. Contrary to appearances, it is not true that $m \in S_d(\alpha)$ implies $m \in S_d(\gamma)$ for $\gamma \in [1, \alpha)$. We must prove Theorem A by an appeal to ten separate cases.

Theorem B. *If $\alpha \notin I_d$, then $|T_d(\alpha)| = \infty$.*

When $d = 2$, this may be deduced from [7, Theorem 3]. Theorem B is proved by the inductive construction of an infinite sequence in $T_d(\alpha)$. The construction follows [7], where the idea is attributed to J. Folkman.

Theorem C. *If $\alpha < d$ and $\alpha \notin I_d$, then there exists a positive integer $t = t(\alpha, d)$ so that every integer $m \geq 0$ can be written as $\sum \varepsilon_i[\alpha^i]$, with $\varepsilon_i \in [[0, d]]$ for $i \leq t$ and $\varepsilon_i \in [[0, d-1]]$ for $i > t$.*

Theorem D. *If $\alpha < d$ and $\alpha \notin I_d$, then $S_d(\alpha)$ has positive density.*

Theorems C and D are immediate consequences of the analysis of the "greedy" representations of m as $\sum \varepsilon_i[\alpha^i]$. In Theorem D, one would expect $S_d(\alpha)$ to have density one; Theorem E proves this in one special case.

Theorem E. *Suppose* $d \geq 4$ *and* $\beta = \beta(d) = \frac{1}{2}(d - 1 + (d^2 + 2d - 3)^{1/2})$. *Then* $\beta \notin I_d$ *and* $S_d(\beta)$ *has density* 1.

The proof of Theorem E hinges on the fact that β is a PV number, and so the behavior of $[\beta^i]$ is intimately related to the behavior of the sequence $\beta^r + \overline{\beta}^r$, which satisfies a second-order linear recurrence by Newton's identity. We can describe the workings of the greedy representation for (d, β) in exact detail and are able to give an explicit description of the set $T_d(\beta)$. We show that $N_r = \sharp\{m \in T_d(\beta): m \leq [\beta^{r+1}] - 1\}$ satisfies a fourth-order linear recurrence, and $N_r \approx c_d \gamma^r$ asymptotically, where $\gamma = \gamma(d) < \beta(d)$ is another quadratic irrational. Thus $S_d(\beta)$ has density 1.

Conjecture F. *If* $\alpha < d$ *and* $\alpha \notin I_d$, *then* $S_d(\alpha)$ *has density* 1.

It seems unlikely that these arguments of Theorem E can be generalized sufficiently to prove Conjecture F, especially for transcendental α. A counterintuitive bit of negative evidence is this: if $d \geq 2$ and $r \geq 2$, then there exists an open interval $I_{r,d} \subset [1, d]\backslash I_d$ so that $d^{r-1} - d^{r-2}$ integers $\leq d^r$ lie in $T_d(\alpha)$ for $\alpha \in I_{r,d}$. This result does not violate Conjecture F, since any fixed $\alpha \in [1, d]\backslash I_d$ only belongs to finitely many $I_{r,d}$. Nevertheless, it seems unlikely that a uniformity argument can be used to prove the conjecture.

The paper is organized as follows. §2 presents the literature on a related question: complete and entirely complete sequences, and gives Brown's criterion, which is essential to the sequel. In §3–§6, we prove Theorems A through E, and in §7, we discuss Conjecture F and the related partition problems.

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2. Preliminaries

Let $A = (a_0, a_1, \ldots)$ be a nondecreasing sequence of positive integers and, for $d \geq 2$, define the sets

$$S_d^n(A) = \left\{ \sum_{i=0}^{n} \varepsilon_i a_i : \varepsilon_i \in [[0, d-1]] \right\}, \tag{2.1}$$

so that, (cf. (1.1)) $S_d(A) = \bigcup_{n=0}^{\infty} S_d^n(A)$. The sequence A is called *d-complete* if $S_d(A)$ contains all sufficiently large integers and *entirely d-complete* if $S_d(A) = \mathbf{Z}^+$. In this terminology, $\alpha \in I_d$ precisely when the sequence $([\alpha^i])$ is entirely d-complete. When $d = 2$, one speaks of *complete* and *entirely complete* sequences; this was the first case discussed in the literature. Hoggatt and King ([10, 11]) posed the following problem about the Fibonacci sequence $F = (1, 1, 2, 3, 5, \ldots)$: Let F' denote F with any one element deleted and F'' denote F with any two elements deleted. Show that every F' (and no F'') is entirely complete. Brown [2] in 1960 generalized the solution to this problem, and gave a necessary and sufficient condition (Lemma 2.3 below) for a sequence to be entirely complete. Alder [1] in 1962 and Fridy [5] in 1965 independently gave the definitions for $d \geq 3$ and generalized Brown's criterion.

Given a sequence A, let

$$A^d = (a_0, \dots, a_0, a_1, \dots, a_1, a_2, \dots), \tag{2.2}$$

where each a_i from A occurs $d-1$ times in A^d. It is easy to see that the d-(entire)-completeness of A is equivalent to the (entire)-completeness of A^d. In 1964, Graham [7] considered the two-parameter family of sequences $A_t(\alpha) = ([t\alpha], [t\alpha^2], [t\alpha^3], \dots)$ with $0 < t$ and $1 < \alpha < 2$. Erdös had conjectured that $A_t(\alpha)$ is always complete, even if not entirely complete. Graham proved that, when $t < 1$, completeness implies entire completeness and determined the set of (t, α) for which $A_t(\alpha)$ is complete. Specializations of Theorems 5 and 3 in [7] to $t = \alpha^{-1}$ give the relevant portions $(d = 2)$ of Theorems A and B.

We now give Brown's criterion and its proof, which is short and illuminating.

Lemma 2.3 (Brown). *Let $A = (a_0, a_1, \dots)$ be a nondecreasing sequence of positive integers, and let $s_k = \sum_{i=0}^k a_i$. Then A is entirely complete if and only if $a_0 = 1$ and*

$$a_r \leq s_{r-1} + 1, r \geq 1. \tag{2.4}$$

If (2.4) is not satisfied and $m \in [[s_{r-1} + 1, a_r - 1]]$, then $m \notin S_2(A)$.

Proof. The smallest integer in $S_2(A)$ is a_0, so $a_0 = 1$ is both necessary and sufficient for 1 to belong to $S_2(A)$. If A is entirely complete, then for all $r \geq 1$, $a_r - 1 \in S_2(A)$; that is, $a_r - 1 = \sum \varepsilon_i a_i, \varepsilon_i \in \{0, 1\}$. As $a_i \geq a_r$ for $i \geq r$, it follows that $\varepsilon_i = 0$ for $i \geq r$. Thus,

$$a_r - 1 = \sum_{i=0}^{r-1} \varepsilon_i a_i \leq \sum_{i=0}^{r-1} a_i = s_{r-1} \tag{2.5}$$

which is (2.4). This argument also shows that, if (2.4) fails and m is in the indicated range, then $m \notin S_2(A)$. We shall use this reasoning repeatedly.

Now suppose that $a_0 = 1$ and (2.4) holds. We prove by induction that $S_2^n(A) = [[0, s_n]]$. This is clear for $n = 0$; suppose it is true for $n = r - 1$. Then by distinguishing the cases $\varepsilon_r = 0$ and 1 in (2.1), we see that

$$S_2^r(A) = S_2^{r-1}(A) \cup (S_2^{r-1}(A) + a_r) = [[0, s_{r-1}]] \cup [[a_r, s_r]]. \tag{2.6}$$

By (2.4), the two sets of integers merge without gap, completing the inductive step. Finally, $a_j \geq a_0 \geq 1$, so $m \in S_2^{m-1}(A) \subseteq S_2(A)$. □

Lemma 2.7 (Alder). *A nondecreasing sequence of positive integers, $B = (b_0, b_1, \dots)$, is entirely d-complete if and only if $b_0 = 1$ and, for all $s \geq 1$,*

$$b_s \leq 1 + (d-1)\sum_{i=0}^{s-1} b_i. \tag{2.8}$$

If (2.8) *fails and* $1 + (d-1)\sum_{i=0}^{s-1} b_i \le m \le b_s - 1$, *then* $m \notin S_d(B)$.

Proof. Apply Lemma 2.3 to $A = B^d$. If $a_r = a_{r-1}$, then (2.4) is automatic. If $a_r > a_{r-1}$, then $r = (d-1)s$, $a_{(d-1)s} = b_s$ and $a_{(d-1)s-1} = b_{s-1}$, so (2.4) reduces to (2.8). □

Henceforth, we shall abbreviate $[\alpha^i]$ and $[\beta^i]$ by α_i and β_i respectively, and write $\underline{\alpha} = (\alpha_0, \alpha_1, \ldots)$ and $\underline{\beta} = (\beta_0, \beta_1, \ldots)$. We now apply Lemma 2.7 to our situation. Let

$$\varphi(\alpha, d, k) = 1 + (d-1)\sum_{i=0}^{k-1} \alpha_i - \alpha_k = \sum_{i=0}^{k-1}(d\alpha_i - \alpha_{i+1}). \tag{2.9}$$

Since $\alpha_0 = 1$ for all α, Lemma 2.7 states that $\alpha \in I_d$ if and only if $\varphi(\alpha, d, k) \ge 0$ for all $k \ge 1$. We say that (α, d, k) is *bad* if $\varphi(\alpha, d, k) \le -1$ and $d\alpha_{k-1} - \alpha_k \le -1$. By looking at the smallest k for which $\varphi(\alpha, d, k) < 0$, we see that Lemma 2.7 gives the following criterion for I_d:

Lemma 2.10. $\alpha \notin I_d$ *if and only if* (α, d, k) *is bad for some* k.

Since $\alpha^i - 1 < \alpha_i \le \alpha^i$, α_i is "close" to $\alpha \cdot \alpha_{i-1}$; when $d > \alpha$, this leads to a useful inequality which bounds the value of k in a bad triple.

Lemma 2.11. *If* $1 < \alpha < d$ *and* $d\alpha_{k-1} - \alpha_k \le -1$, *then*

$$\alpha_{k-1} < (\alpha - 1)/(d - \alpha). \tag{2.12}$$

In particular, if $k \ge 2$, *then*

$$(\alpha^{k-2} + \cdots + \alpha + 1) < 1/(d-\alpha), \tag{2.13}$$

$$\alpha^{k-1} < (d-1)/(d-\alpha). \tag{2.14}$$

Proof. By hypothesis,

$$d\alpha_{k-1} \le \alpha_k - 1 \le \alpha^k - 1 < \alpha(\alpha_{k-1} + 1) - 1, \tag{2.15}$$

whence (2.12); (2.13) and (2.14) follow from (2.12) and $\alpha_{k-1} > \alpha^{k-1} - 1$. □

3. The proof of Theorem A

We prove Theorem A by dividing it into ten cases (this notation differs harmlessly from that in the introduction). We shall retain this case notation in the proof of Theorem B.

Theorem A.

(a) *If* $\alpha > d$, *then* $\alpha \notin I_d$.
(b) *If* $\alpha \le d$ *and* $\alpha \in \mathbf{Z}$, *then* $\alpha \in I_d$.
(c) *If* $d \ge 3$ *and* $1 \le \alpha \le d - 1$, *then* $\alpha \in I_d$.
(d) *If* $1 \le \alpha < 13^{1/4}$, *then* $\alpha \in I_2$.
(e) *If* $13^{1/4} \le \alpha < 7^{1/3}$, *then* $\alpha \notin I_2$ *and* $12 \in T_2(\alpha)$.
(f) *If* $7^{1/3} \le \alpha < 2$, *then* $\alpha \notin I_2$ *and* $6 \in T_2(\alpha)$.

(g) *If* $2 \le \alpha < 22^{1/3}$, *then* $\alpha \in I_3$.
(h) *If* $22^{1/3} \le \alpha < 8^{1/2}$, *then* $\alpha \notin I_3$ *and* $21 \in T_3(\alpha)$.
(j) *If* $d-1 \le \alpha < (d^2-d+2)^{1/2}$ *and* $d \ge 4$, *then* $\alpha \in I_d$.
(k) *If* $(d^2-d+2)^{1/2} \le \alpha < d$ *and* $d \ge 3$, *then* $\alpha \notin I_d$ *and* $d^2-d+1 \in T_d(\alpha)$.

The numerical values of the surds are: $13^{1/4} \approx 1.89883$, $7^{1/3} \approx 1.91293$, $22^{1/3} \approx 2.80204$ and $8^{1/2} \approx 2.82843$. The proofs of the first two cases were discussed in the introduction.

Proof of Theorem A(a). As in the proof of Lemma 2.3, if $m \in S_d(\alpha)$ and $m \le \alpha_n - 1$, then $m = \sum_{i=0}^{n-1} \varepsilon_i \alpha_i \in S_d^{n-1}(\alpha)$. There are only d^n formally distinct sums in $S_d^{n-1}(\alpha)$, so at most d^n of the α_n integers in $[[0, \dots, \alpha_n - 1]]$ lie in $S_d(\alpha)$. Since $\alpha > d$, $\alpha_n > \alpha^n - 1 > d^n$ for n sufficiently large. As $d^n/(\alpha^n - 1) \to 0$, the first part of Theorem B is also proved. □

Proof of Theorem A(b). Write m in base α; as $\alpha_i = \alpha^i$, $m = \sum \varepsilon_i \alpha_i$, with $\varepsilon_i \le \alpha - 1 \le d - 1$. □

In the remaining eight cases, $\alpha < d$ is not an integer. The strategy of our proof is this: we shall either present a bad triple (α, d, k), showing that $\alpha \notin I_d$, or use Lemma 2.11 to show that no bad triple exists.

Lemma 3.1. *Suppose* $1 \le \alpha < d$. *Then* $\varphi(\alpha, d, 1) \ge 0$, *and* $\varphi(\alpha, d, 2) \ge 0$ *if and only if* $1 \le \alpha < (d^2-d+2)^{1/2}$. *Thus, if* $\alpha < (d^2-d+2)^{1/2}$ *and* (α, d, k) *is bad, then* $k \ge 3$ *and* $(d-\alpha)\alpha^{k-1} < d-1$.

Proof. Write $\alpha_1 = s$ and $\alpha_2 = s^2 + u$, so $1 \le s \le d-1$ and $0 \le u \le 2s$. Then $\varphi(\alpha, d, 1) = d - s \ge 0$ and

$$\varphi(\alpha, d, 2) = d + (d-1)s - (s^2+u) = (d-s)(1+s) - u. \tag{3.2}$$

If $s \le d-2$, then $\varphi(\alpha, d, 2) \ge 2(1+s) - 2s \ge 0$. If $s = d-1$, then $\varphi(\alpha, d, 2) = d - u \ge 0$ if and only if $u \le d$, or $\alpha^2 < (d-1)^2 + (d+1) = d^2 - d + 2$. The final assertion is (2.14). □

Proof of Theorem A(c). Suppose $\alpha \le d - 1 < (d^2-d+2)^{1/2}$, $\alpha \notin I_d$ and (α, d, k) is the bad triple specified by Lemma 2.10. Then $k \ge 3$ by Lemma 3.1, but by (2.12),

$$\alpha^{k-1} - 1 < \alpha_{k-1} \le (\alpha - 1)/(d - \alpha) \le \alpha - 1 \tag{3.3}$$

since $d - \alpha \ge 1$, which contradicts $\alpha \ge 1$. □

In the remaining seven cases, $\alpha \in [d-1, d)$; the proof of A(k) was started in the introduction.

Proof of Theorem A(k). By Lemma 3.1, $\varphi(\alpha, d, 2) < 0$ if $(d^2-d+2)^{1/2} \le \alpha < d$; $(\alpha, d, 2)$ is bad, so by Lemmas 2.7 and 2.10, $\alpha \notin I_d$ and we have $[[d^2-d+1, \alpha_2 - 1]] \subseteq T_d(\alpha)$. □

Proofs of Theorems A(e), A(f) *and* A(h). These proofs are identical in form. We give enough of the vector $\underline{\alpha}$ to show that $\varphi(\alpha, d, k) \le -1$ for appropriate k. If $13^{1/4} \le \alpha < 7^{1/3}$, then $\underline{\alpha} = (1, 1, 3, 6, 13, \dots)$ and $\varphi(\alpha, 2, 4) = 1+1+1+3+6-13 = -1$, so $\alpha_4 - 1 = 12 \in T_2(\alpha)$. If $7^{1/3} \le \alpha < 2$, then $\underline{\alpha} = (1, 1, 3, 7, \dots)$ and $\varphi(\alpha, 2, 3) = 1+1+1+3-7 = -1$, so $\alpha_3 - 1 = 6 \in T_2(\alpha)$. If $22^{1/3} \le \alpha < 8^{1/2}$, then $\underline{\alpha} = (1, 2, 7, 22, \dots)$ and $\varphi(\alpha, 3, 3) = 1 + 2(1+2+7) - 22 = -1$, so $\alpha_3 - 1 = 21 \in T_3(\alpha)$. □

In the final three cases, $d - 1 \le \alpha < (d^2 - d + 2)^{1/2}$; we show that $\alpha \in I_d$ by deriving a contradiction from Lemmas 2.10 and 3.1.

Proof of Theorem A(j). Suppose (α, d, k) is bad, then $k \ge 3$ by Lemma 3.1. By (2.13),

$$1 > (d - \alpha)(1 + \alpha + \cdots + \alpha^{k-2}) \ge (d - \alpha)(1 + \alpha). \tag{3.4}$$

Thus, $H(\alpha) > 0$, where

$$H(x) = 1 - (d - x)(1 + x) = x^2 - (d - 1)x - (d - 1). \tag{3.5}$$

Since H is increasing for $x \ge (d-1)/2$ and $\alpha < (d^2 - d + 2)^{1/2}$,

$$0 < H((d^2 - d + 2)^{1/2}) = d^2 - 2d + 3 - (d - 1)(d^2 - d + 2)^{1/2}. \tag{3.6}$$

A routine calculation shows that (3.6) is equivalent to the cubic inequality

$$d^3 - 5d^2 + 7d - 7 < 0, \tag{3.7}$$

which is false for $d = 4$ by calculation, and for $d \ge 5$ by inspection. □

The positive root of H is $\beta = \beta(d)$, as defined in the statement of Theorem E. We shall need the observation that $\beta < (d^2 - d + 2)^{1/2}$ for $d \ge 4$.

Proof of Theorem A(g). We proceed as in the last case, except that we need to "bootstrap". Suppose $d = 3$, $2 < \alpha < 22^{1/3}$, $\alpha \notin I_3$ and $(\alpha, 3, k)$ is bad. Again $k \ge 3$ by Lemma 3.1, and by (2.13),

$$1 > (3 - \alpha)(1 + \alpha + \cdots + \alpha^{k-2}) \ge (3 - \alpha)(1 + \alpha). \tag{3.8}$$

This gives a contradiction if $1 \le \alpha \le \mu = 1 + 3^{1/2} \approx 2.73205$. If $\mu \le \alpha < 22^{1/3}$, then $\underline{\alpha} = (1, 2, 7, 20 \text{ or } 21, \dots)$, so $\varphi(\alpha, 3, 3) \ge 1 + 2(1 + 2 + 7) - 21 = 0$ and $(\alpha, 3, 3)$ is not bad. Thus $k \ge 4$; as in (3.8),

$$1 > (3 - \alpha)(1 + \alpha + \alpha^2) > (3 - 22^{1/3})(1 + \mu + \mu^2) \approx 2.21640, \tag{3.9}$$

which is a contradiction. □

The final case is A(d); its proof, like those of A(e) and A(f), may be deduced from [7]. We have included these proofs for completeness.

Proof of Theorem A(d). Suppose $\alpha < 13^{1/4}$, $\alpha \notin I_2$ and $(\alpha, 2, k)$ is bad. By Lemma 3.1, $k \ge 3$; as in (3.8), (2.13) implies that

$$(2 - \alpha)(1 + \alpha) < 1, \tag{3.10}$$

so $\alpha > \frac{1}{2}(1+5^{1/2}) = \Phi \approx 1.61803$. We break the interval $[\Phi, 13^{1/4})$ at $3^{1/2} \approx 1.73205$. If $\alpha \in [\Phi, 3^{1/2})$, then $\underline{\alpha} = (1, 1, 2, 4 \text{ or } 5, \dots)$; if $\alpha \in [3^{1/2}, 13^{1/4})$, then $\underline{\alpha} = (1, 1, 3, 5 \text{ or } 6, \dots)$. In either case, $\varphi(\alpha, 2, 3) \geq 0$ so $k \geq 4$. But now,

$$(2-\alpha)(1+\alpha+\alpha^2) < 1. \tag{3.11}$$

Solving the cubic, we find that $\alpha \geq 1.83929$. Thus $\underline{\alpha} = (1, 1, 3, 6, 11 \text{ or } 12, \dots)$ and $\varphi(\alpha, 2, 4) \geq 0$. Hence $k \geq 5$, and a final application of (2.13) gives

$$(2-\alpha)(1+\alpha+\alpha^2+\alpha^3) < 1, \tag{3.12}$$

which implies $\alpha \geq 1.92746\ldots \geq 13^{1/4}$, a contradiction at last. □

This final argument does not contradict $\gamma = 13^{1/4} \notin I_2$. Even though $\varphi(\gamma, 2, 5) = 1$, we have seen in the proof of A(e) that $\varphi(\gamma, 2, 4) = -1$.

4. The proof of Theorem B

In the notation of Theorem A, there are five cases in which $\alpha \notin I_d$: (a), (e), (f), (h) and (k). Theorem B states that $|T_d(\alpha)| = \infty$ in each of these cases. If $\alpha > d$, then $|T_d(\alpha)| \geq \alpha^n - d^n - 1$ for all n by the proof of Theorem A(a); this proves Theorem B(a). The remaining four cases are proved using a generic construction based on ideas from [7]. Given $m \in T_d(\alpha)$ and $m < \alpha_n$, let $m' = m + (d-1)\alpha_n$. If we are fortunate, the equation $m' = \sum \varepsilon_i \alpha_i$, $\varepsilon_i \in [[0, d-1]]$ implies $\varepsilon_n = d-1$, a contradiction, so that $m' \in T_d(\alpha)$ as well. If we are especially fortunate, we may use m' as the new m, and pick a new n' to repeat the argument. This gives an infinite increasing sequence in $T_d(\alpha)$. We need two lemmas to implement this construction.

Lemma 4.1. *If $m \in T_d(\alpha)$ and m satisfies the inequalities*

$$\alpha_{n+1} - (d-1)\alpha_n > m > (d-1)\sum_{i=0}^{n-1} \alpha_i - \alpha_n, \tag{4.2}$$

then $m' = m + (d-1)\alpha_n \in T_d(\alpha)$.

Proof. Following the argument outlined above, suppose, to the contrary, that $m' = \sum_{i=0}^{\infty} \varepsilon_i \alpha_i \in S_d(\alpha)$. Since $\alpha_{n+1} > m'$ by (4.2), $\varepsilon_i = 0$ for $i \geq n+1$. If $\varepsilon_n < d-1$, then

$$m' \leq (d-2)\alpha_n + (d-1)\sum_{i=0}^{n-1} \alpha_i, \tag{4.3}$$

which violates the other inequality of (4.2). Thus $\varepsilon_n = d-1$, so that $m = m' - \varepsilon_n \alpha_n = \sum_{i=0}^{n-1} \varepsilon_i \alpha_i \in S_d(\alpha)$, a contradiction. □

Lemma 4.4. *Suppose $b \in T_d(\alpha)$ and, for some $r \geq 1$, the following two inequalities hold*:

$$\alpha_{r+1} - (d-1)\alpha_r > b > (d-1)\sum_{i=0}^{r-1} \alpha_i - \alpha_r, \tag{4.5}$$

$$\alpha^r(\alpha^2 - (d-1)\alpha - 1) > 1. \tag{4.6}$$

Define the sequence (b_k) by $b_0 = b$ and

$$b_k = b + (d-1)\sum_{i=0}^{k-1} \alpha_{r+2i}, \qquad k \geq 1. \tag{4.7}$$

Then $b_k \in T_d(\alpha)$ for all $k \geq 0$.

Proof. We apply Lemma 4.1 with $m = b_k$, $m' = b_{k+1}$ and $n = r + 2k$; let

$$E(k) = \alpha_{r+2k+1} - (d-1)\alpha_{r+2k} - b_k, \tag{4.8}$$

$$F(k) = b_k - (d-1)\sum_{i=0}^{r+2k-1} \alpha_i + \alpha_{r+2k}. \tag{4.9}$$

We want to show that $E(k) > 0$ and $F(k) > 0$ for all $k \geq 0$. For $k = 0$, this is (4.5). Observe that

$$E(k+1) - E(k) = \alpha_{r+2k+3} - (d-1)\alpha_{r+2k+2} - \alpha_{r+2k+1}, \tag{4.10}$$

$$F(k+1) - F(k) = \alpha_{r+2k+2} - (d-1)\alpha_{r+2k+1} - \alpha_{r+2k}, \tag{4.11}$$

so we are done if we can show that

$$D(s) = \alpha_{s+2} - (d-1)\alpha_{s+1} - \alpha_s \geq 0 \tag{4.12}$$

for $s \geq r$. Using $\alpha^m \geq \alpha_m > \alpha^m - 1$ and (4.6), we have

$$\begin{aligned} D(s) &= \alpha_{s+2} - (d-1)\alpha_{s+1} - \alpha_s \geq \alpha^{s+2} - 1 - (d-1)\alpha^{s+1} - \alpha^s \\ &= \alpha^{s-r}\alpha^r(\alpha^2 - (d-1)\alpha - 1) - 1 > \alpha^{s-r} - 1 \geq 0, \end{aligned} \tag{4.13}$$

which establishes (4.12), and completes the proof. □

The first three cases are proved by a direct application of Lemma 4.4; the fourth, involving the parameter d, is more complicated.

Proofs of Theorems B(e), B(f), *and* B(h). As with Theorem A, the proofs in these three cases are virtually identical. In each case we use the element of $T_d(\alpha)$ cited in Theorem A. We only prove case (e) in detail. Suppose $d = 2$ and $\alpha \in [13^{1/4}, 7^{1/3})$, and let $b = 12$ and $r = 5$ as in Lemma 4.4. Then $\underline{\alpha} = (1, 1, 3, 6, 13, \alpha_5, \alpha_6, \ldots)$, where $\alpha_5 = 24$ or 25 and $\alpha_6 \geq 46$. Then (4.5) becomes

$$\alpha_6 - \alpha_5 > 12 > 24 - \alpha_5, \tag{4.14}$$

which is clearly true. For (4.6), note that $\alpha^5(\alpha^2-\alpha-1)$ is a product of increasing functions on $[1,\infty)$, and its value at $\alpha = 13^{1/4}$ is $17.445\ldots > 1$. It follows from Lemma 4.4 that the sequence

$$(4.15)\qquad b_0 = 12, \qquad b_k = 12 + \sum_{i=0}^{k-1} \alpha_{5+2i}, \qquad k \geq 1,$$

lies in $T_2(\alpha)$.

For case (f), take $d = 2$, $\alpha \in [7^{1/3}, 2)$, $b = 6$ and $r = 4$. Then $\underline{\alpha} = (1, 1, 3, 7, \alpha_4, \alpha_5, \ldots)$ with $13 \leq \alpha_4 \leq 15$ and $\alpha_5 \geq 25$. The same reasoning shows that $6 + \sum_{i=0}^{k-1} \alpha_{4+2i} \in T_2(\alpha)$. For case (h), take $d = 3$, $\alpha \in [22^{1/3}, 8^{1/2})$ and $b = 21$ and $r = 4$. Then $\underline{\alpha} = (1, 2, 7, 22, \alpha_4, \alpha_5, \ldots)$ with $61 \leq \alpha_4 \leq 63$ and $\alpha_5 \geq 172$, and $21 + 2\sum_{i=0}^{k-1} \alpha_{4+2i} \in T_3(\alpha)$. □

Proof of Theorem B(k). In this case, $\alpha \in [(d^2-d+2)^{1/2}, d)$ with $d \geq 3$; choose $b = d^2 - d + 1$ and $r = 3$. Then (4.5) and (4.6) are:

$$(4.16)\qquad \alpha_4 - (d-1)\alpha_3 > d^2 - d + 1 > (d-1)(\alpha_0 + \alpha_1 + \alpha_2) - \alpha_3,$$

$$(4.17)\qquad \alpha^3(\alpha^2 - (d-1)\alpha - 1) > 1.$$

Using $\alpha_0 = 0$, $\alpha_1 = d - 1$, $\alpha_i > \alpha^i - 1$ and $\alpha \geq \max(d - \frac{1}{2}, 8^{1/2})$, we have:

$$(4.18)\qquad \begin{aligned} \alpha_4 - (d-1)\alpha_3 - (d^2 - d + 1) &> \alpha^4 - (d-1)\alpha^3 - (d^2 - d + 2) \\ &\geq \alpha^2(\alpha^2 - (d-1)\alpha - 1) \geq \alpha^2(\alpha \cdot \tfrac{1}{2} - 1) > 0, \end{aligned}$$

$$(4.19)\qquad \begin{aligned} d^2 - d + 1 - (d-1)(d + \alpha_2) + \alpha_3 &\geq 1 - (d-1)\alpha^2 + \alpha^3 - 1 \\ &= \alpha^2(\alpha - (d-1)) \geq 0, \end{aligned}$$

$$(4.20)\qquad \alpha^3(\alpha^2 - (d-1)\alpha - 1) \geq \alpha^3(\alpha \cdot \tfrac{1}{2} - 1) \geq 8^{3/2}(2^{1/2} - 1) > 1.$$

Thus, $d^2 - d + 1 + (d-1)\sum_{i=0}^{k-1} \alpha_{3+2i} \in T_d(\alpha)$. □

5. The greedy representation and the proofs of Theorems C and D

In this section we try to represent integers in the form $\sum \varepsilon_i \alpha_i$, $\varepsilon_i \in [[0, d-1]]$ when $\alpha < d$ using the greedy algorithm. If $\alpha \notin I_d$, this cannot be done for infinitely many integers by Theorem B. Nevertheless, a truncated version of the greedy algorithm is possible, and leads to the proofs of Theorems C and D. (The index $t = t(\alpha, d)$ below is called the *cutoff* value for (α, d).)

Lemma 5.1 (The greedy representation). *Suppose $\alpha < d$ and t is the smallest integer so that $(d-\alpha)\alpha_t \geq \alpha - 1$. Then every integer m can be written*:

$$(5.2)\quad m = \sum_{i=t}^{\infty} \varepsilon_i(m)\alpha_i + u(m), \qquad \varepsilon_i(m) \in [[0, d-1]], \ u(m) \in [[0, \alpha_t - 1]],$$

and, for all $m \geq 0$,

$$u(m+1)=0 \quad or \quad u(m+1)=u(m)+1. \tag{5.3}$$

Proof. By Lemma 2.11, $s \geq t$ and $\alpha_s \geq \alpha_t \geq (\alpha-1)/(d-\alpha)$ imply that $d\alpha_s - \alpha_{s+1} > -1$, so $d\alpha_s \geq \alpha_{s+1}$. Let $I_0 = [[0, \alpha_t - 1]]$ and $I_s = [[\alpha_s, \alpha_{s+1} - 1]]$ for $s \geq t$. We define $(\underline{\varepsilon}(m), u(m))$ first for $m \in I_0$ and I_t, and then, recursively, for $m \in I_s$ when $s > t$.

For $m \in I_0$, let $(\underline{\varepsilon}(m), u(m)) = (\underline{0}, m)$; this satisfies (5.2). For $m \in I_t$, let $\varepsilon_i(m) = 0$ for $i \geq t+1$, $\varepsilon_t(m) = [m/\alpha_t]$ and $u(m) = m - \varepsilon_t(m)\alpha_t$. Then $u(m) \in I_0$ and

$$0 \leq \varepsilon_t(m) \leq [(\alpha_{t+1}-1)/\alpha_t] \leq [(d\alpha_t - 1)/\alpha_t] < d, \tag{5.4}$$

so (5.2) is verified. Suppose that every integer less than $\alpha_s - 1$ has been written in the form (5.2) with $\varepsilon_i(m) = 0$ for $i \geq s$. For $m \in I_s$, let $R(m) = m - [m/\alpha_s]\alpha_s$, so $R(m) \leq \alpha_s - 1$ and $\varepsilon_i(R(m)) = 0$ for $i \geq s$. Define $\underline{\varepsilon}(m)$ as follows: $\varepsilon_i(m) = 0$ for $i \geq s+1$, $\varepsilon_s(m) = [m/\alpha_s]$ $(\leq d-1)$, $\varepsilon_i(m) = \varepsilon_i(R(m))$ for $i \leq s-1$ and $u(m) = u(R(m))$. This verifies (5.2) for $m \in I_s$.

We must still check (5.3). Since $u(\alpha_t) = 0$, (5.3) holds for $m \leq \alpha_t - 1$. Suppose it holds for $m \leq \alpha_s - 1$, and $m \in I_s$. If $m = \alpha_{s+1} - 1$, then $m+1 = \alpha_{s+1}$ and $u(m+1) = 0$; otherwise, m, $m+1 \in I_s$. If $\varepsilon_s(m) = \varepsilon_s(m+1)$, then $R(m+1) = R(m) + 1$, so (5.3) holds by induction. If $\varepsilon_s(m) < \varepsilon_s(m+1)$, then α_s divides $m+1$, so $R(m+1) = 0$ and $u(m+1) = 0$. □

Proof of Theorem C. Since $\alpha < d$, $\alpha \in I_{d+1}$ by Theorem A(c); thus $u(m) = \sum \eta_i \alpha_i \in S_{d+1}(\alpha)$ with $\eta_i \in [[0, d]]$. As $u(m) \leq \alpha_t - 1$, $\eta_i = 0$ for $i \geq t$. After replacing $u(m)$ by $\sum \eta_i \alpha_i$ in (5.2), we obtain the desired representation. □

Theorem B states that $|T_d(\alpha)| = \infty$ if $\alpha < d$ and $\alpha \in I_d$, but the set (b_k) constructed in the proof is quite sparse in $\mathbf{Z}^+$. The following notation is standard. For $A \subseteq \mathbf{Z}^+$, let $N(A; n)$ denote the number of integers $\leq n$ which belong to A. We define the *lower density* of A, $D(A)$:

$$D(A) = \liminf_{n\to\infty} \frac{N(A;n)}{n+1}. \tag{5.5}$$

Theorem D asserts that $D(S_d(A)) > 0$ when $\alpha < d$.

Proof of Theorem D. Fix $\alpha < d$ with $\alpha \notin I_d$, and define t and $u(m)$ as above. Since $u(m) \leq \alpha_t - 1$, any representation $u(m) = \sum \varepsilon_i \alpha_i$ has $\varepsilon_i = 0$ for $i \geq t$. Thus, if $u(m) \in S_d(\alpha)$, then (5.2) shows that $m \in S_d(\alpha)$. It therefore suffices to show that $\{m\colon u(m) \in S_d(\alpha)\}$ has positive lower density. By (5.3), every interval $[[0, n]]$ can be written as a disjoint union of blocks $W_i = [[w_i, w_{i+1} - 1]]$ so that $u(j) = j - w_i$ for $j \in W_i$. Let

$$\lambda = \lambda(\alpha, d) = \min\{(k+1)^{-1} N(S_d(\alpha), k)\colon k \in [[0, \alpha_t - 1]]\}; \tag{5.6}$$

since $[[0, d-1]] \subseteq S_d(\alpha)$, $\lambda > 0$. As W_i contains $\geq \lambda(w_{i+1} - w_i)$ elements of $S_d(\alpha)$, $N(S_d(\alpha), n) \geq \lambda(n+1)$ for all n, so $D(S_d(\alpha)) \geq \lambda > 0$. □

For any fixed $(\alpha, d), \lambda(\alpha, d)$ can be readily computed. For example, if $\alpha \in [13^{1/4}, 7^{1/3})$ and $d = 2$, then $t = 4$ and $\lambda = 12/13$. Since $t \to \infty$ as $\alpha \to d$, the computations become extremely unpleasant. If Conjecture F is true, then the question is moot.

This greedy representation of m may not be the only one satisfying (5.2). Fix (α, d) with $\alpha < d$, and for $r \geq t(\alpha, d)$, let $E_r = E_r(\alpha, d)$ denote the set of strings $(\underline{\varepsilon}, u) = (\varepsilon_r, \dots, \varepsilon_t, u)$ with $\varepsilon_i \in [[0, d-1]]$ and $u \in [[0, \alpha_t - 1]]$. Let

$$f(\underline{\varepsilon}, u) = \sum_{i=t}^{r} \varepsilon_i \alpha_i + u. \tag{5.7}$$

Then the greedy representation of m gives $m = f(\underline{\varepsilon}(m), u(m))$. Since there are $\alpha_t d^{r-t+1}$ strings in E_r and $0 \leq f(\underline{\varepsilon}, u) \leq \alpha_t - 1 + (d-1)\sum_{i=t}^{r} \alpha_i < c_0 \alpha^r$, an integer in $[[0, \alpha^r]]$ has $\approx c_1 (d/\alpha)^r$ representations on average. The greedy representations within E_r can be characterized as the maximal elements of the level sets $\{f(\underline{\varepsilon}, u) = m : (\underline{\varepsilon}, u) \in E_r\}$ under the lexicographic ordering.

Lemma 5.8. *Suppose* $m = f(\underline{\varepsilon}(m), u(m)) = f(\underline{\eta}, v)$, *where* $(\underline{\eta}, v) \in E_r$, *and suppose* $\eta_i = \varepsilon_i(m)$ *for* $i \geq k+1$. *Then* $\varepsilon_k(m) \geq \eta_k$.

Proof. First suppose $k < r$. We have

$$m = \sum_{i=t}^{r} \varepsilon_i(m)\alpha_i + u(m) = \sum_{i=t}^{r} \eta_i \alpha_i + v. \tag{5.9}$$

As usual, let $R(m) = m - \varepsilon_r(m)\alpha_r$; $\varepsilon_i(m) = \varepsilon_i(R(m))$ for $i \leq r-1$. Thus, we may cancel $\varepsilon_r(m)\alpha_r = \eta_r \alpha_r$ from both sides of (5.9) while preserving the fact that the first representation is greedy. This process may be repeated so that we assume that $m \leq \alpha_{k+1} - 1$, and

$$m = \sum_{i=t}^{k} \varepsilon_i(m)\alpha_i + u(m) = \sum_{i=t}^{k} \eta_i \alpha_i + v. \tag{5.10}$$

But $\eta_k \leq m/\alpha_k$ and $\varepsilon_k(m) = [m/\alpha_k]$, so $\varepsilon_k(m) \geq \eta_k$. □

6. The proof of Theorem E

A PV (or Pisot-Vijayoragnavan) number is an algebraic integer $\alpha = \alpha_1$ whose conjugates, $\alpha_2, \dots, \alpha_m$, all lie in $|z| < 1$. These numbers have been extensively studied for their algebraic and analytic properties (see §§K25 and R06 in [9] and [14].) Suppose α is a PV number with minimal polynomial $p(x) = x^m + \sum b_i x^i \in \mathbf{Z}[x]$, and let $a_n = \sum \alpha_j^n$. By the elementary theory of symmetric polynomials, a_n is integral for $n \geq 0$; moreover, (a_n) satisfies the linear recurrence called Newton's identity:

$$a_{n+m} + \sum_{i=0}^{m-1} b_i a_{n+i} = \sum_{j=1}^{m} \alpha_j^n p(\alpha_j) = 0. \tag{6.1}$$

Since α is PV, $\alpha_j^n \to 0$ for $j \geq 2$, and so $a_n \approx \alpha^n$. If $m = 2$ and $\alpha_2 \in (-1, 0)$, then $a_n = [\alpha^n]$ or $[\alpha^n] + 1$, depending on the parity of $n \bmod 2$. Using (6.1), we can always obtain an explicit linear recurrence for $[\alpha^n] = \alpha_n$.

We are interested in one particular PV number. Fix $d \geq 4$ and let

$$\beta = \beta(d) = \tfrac{1}{2}(d - 1 + (d^2 + 2d - 3)^{1/2}). \tag{6.2}$$

As previously noted, β is the positive root of $H(x) = x^2 - (d-1)x - (d-1)$ (cf. (3.5)) and, as $d > \beta \geq (d^2 - d + 2)^{1/2}$, $\beta \notin I_d$. (It turns out that $\beta(d) \in I_d$ for $d \leq 3$.) Since the conjugate of β lies in $(-1, 0)$, β is a PV number. We show in Lemma 6.7 that $\beta_n = [\beta^n]$ satisfies the second-order recurrence:

$$\beta_{n+2} = (d-1)(\beta_{n+1} + \beta_n) + \delta_n, \tag{6.3}$$

where

$$\delta_n = d - \tfrac{3}{2} - \tfrac{1}{2}(-1)^n. \tag{6.4}$$

By Lemma 5.1, every integer m can be written greedily as (5.2), where $\varepsilon_i(m) \in [[0, d-1]]$ and $u(m) \in [[0, \beta_t - 1]]$. It turns out that $t = t(d, \beta)$ always equals 2 (Lemma 6.10) and $\beta_2 = d^2 - 2$. We shall write $[[0, \beta_2 - 1]] = J_1 \cup J_2$, where $J_1 = [[0, d^2 - d]]$ and $J_2 = [[d^2 - d + 1, d^2 - 3]]$. As $(d-1)(\beta_0 + \beta_1) = d^2 - d$, if $u(m) \in J_1$, then $u(m) = \varepsilon_0\beta_0 + \varepsilon_1\beta_1$ for $\varepsilon_i = [[0, d-1]]$ and $m \in S_d(\beta)$. If $u(m) \in J_2$, then m may still belong to $S_d(\beta)$, provided there is a nongreedy way to write m in the form $\sum \eta_k \beta_k$.

Using (6.3), we give a precise description of the strings $(\underline{\varepsilon}(m), u(m))$ which arise from greedy representations (Lemma 6.14). We also describe the alternative representations (Lemma 6.23). Suppose $u(m) \in J_2$ and $\varepsilon_{j+2}(m) \geq 1$ and $\varepsilon_{j+1}(m) = \varepsilon_j(m) = 0$ with $j \geq 3$. By (6.3) we can "trade" one β_{j+2} from the greedy representation for $d-1$ β_{j+1}'s and $d-1$ β_j's, with δ_j left over, which we add to $u(m)$ to obtain one more β_2 (if $u(m) \in J_2$, then $\varepsilon_2(m) \leq d-2$) and a smaller $u'(m)$ which now lies in J_1. It turns out that, if the pattern $(\geq 1, 0, 0, \geq 1)$ does not occur in $\underline{\varepsilon}(m)$, then $m \in T_d(\beta)$. This reduces to a purely combinatorial counting problem; we show that

$$N_r = N(T_d(\beta); \beta_{r+1} - 1) = c(d)\sigma^r + O(1), \tag{6.5}$$

where

$$\sigma = \sigma(d) = \tfrac{1}{2}(d - 1 + (d^2 + 2d - 7)^{1/2}). \tag{6.6}$$

Since $\beta_{r+1}/\beta_r \to \beta$, it follows that $c_1 n^{e(d)} \leq N(T_d(\beta); n) \leq c_2 n^{e(d)}$ for some $c_i > 0$, where $e(d) = \log_{\beta(d)} \sigma(d) < 1$. Since $e(d) < 1$, $N(T_d(\beta); n) = o(n)$ and $S_d(\beta)$ has density 1, which proves Theorem E.

We begin with two lemmas which give the recurrence for β_n and, with Lemma 5.1, establish that $t(\beta(d), d) = 2$ for all $d \geq 4$.

Lemma 6.7. (β_n) *satisfies* (6.3).

Proof. The minimal polynomial of β is $H(x) = x^2 - (d-1)x - (d-1)$ and $\overline{\beta} = -(d-1)/\beta \in (-1, 0)$; let $b_n = \beta^n + \overline{\beta}^n$ for $n \geq 0$. By (6.1),

$$b_{n+2} - (d-1)(b_{n+1} + b_n) = 0; \tag{6.8}$$

since $b_0 = 2$ and $b_1 = d-1$, $b_n \in \mathbf{Z}$. Since $\overline{\beta}^n$ lies alternately in $(-1, 0)$ and $(0, 1)$, $\beta_n = [\beta^n] = b_n - \frac{1}{2}(1 + (-1)^n)$, and (6.3) follows from (6.8). □

For later use, we record the first few values of β_i:

$$\beta_0 = 1, \qquad \beta_1 = d-1, \qquad \beta_2 = d^2 - 2, \qquad \beta_3 = d^3 - 3d + 2. \tag{6.9}$$

Lemma 6.10.

$$\beta_2 \geq (\beta - 1)/(d - \beta) > \beta_1. \tag{6.11}$$

Proof. A calculation shows that

$$(\beta - 1)/(d - \beta) = \tfrac{1}{2}\{(d^2 - 3) + (d-1)(d^2 + 2d - 3)^{1/2}\}. \tag{6.12}$$

The first inequality in (6.11) follows from putting $d^2 + 2d - 3 < (d+1)^2$ in (6.12), and (6.9); the second is immediate. □

We now analyze greedy strings. Recall that E_r is the set of strings $(\underline{\varepsilon}, u) = (\varepsilon_r, \dots, \varepsilon_2, u)$ with $\varepsilon_i \in [[0, d-1]]$ and $u \in J_1 \cup J_2 = [[0, d^2 - 3]]$, and that $f(\underline{\varepsilon}, u) = \sum \varepsilon_k \beta_k + u$. Let

$$\widetilde{E}_r = \{(\underline{\varepsilon}, u) \in E_r : (\underline{\varepsilon}, u) = (\underline{\varepsilon}(m), u(m)) \text{ for some } m < \beta_{r+1}\} \tag{6.13}$$

denote the set of strings which arise from greedy representations. We shall say that (i, j) *occurs at* k in $(\underline{\varepsilon}, u)$ if $\varepsilon_{k+1} = i$ and $\varepsilon_k = j$ and that (i, j) *does not occur* in $(\underline{\varepsilon}, u)$ if it occurs at no k.

Lemma 6.14. *For* $r \geq 2$, $\widetilde{E}_r$ *consists of all strings* $(\underline{\varepsilon}, u)$ *in* E_r *satisfying one of the following three disjoint conditions* (δ_j *has its previous meaning*):

(6.15)(i) $\varepsilon_2 < d-1$ *and* $(d-1, d-1)$ *does not occur,*

(6.15)(ii) $\varepsilon_2 = d-1$, $u \in [[0, d^2 - d - 1]]$ *and* $(d-1, d-1)$ *does not occur,*

(6.15)(iii) $(d-1, d-1)$ *does not occur at* $k > j$, $(d-1, d-1)$ *occurs at* j *and* $(\varepsilon_{j-1}, \dots, \varepsilon_2, u) = (0, \dots, 0, u)$, *where* $u \in [[0, \delta_j - 1]]$.

Proof. Let $\overline{E}_r$ denote the set of strings in E_r which satisfy one of the conditions in (6.15). We give a recursive description of $\overline{E}_r$, and then show that $\widetilde{E}_r$ satisfies the same recursion.

It is easy to check that

$$\overline{E_2} = \left(\bigcup_i \bigcup_u (i, u)\right) \cup \left(\bigcup_v (d-1, v)\right), \tag{6.16}$$

(6.17)
$$\overline{E_3} = \left(\bigcup_i (i, \overline{E}_2)\right) \cup \left(\bigcup_i \bigcup_u (d-1, i, u)\right) \cup \left(\bigcup_w (d-1, d-1, w)\right);$$

where the unions are taken over the following: $i \in [[0, d-2]]$, $u \in [[0, d^2-3]]$, $v \in [[0, d^2-d-1]]$ and $w \in [[0, \delta_2 - 1]]$. Fix $r \geq 4$. If $\varepsilon_r \leq d-2$, and $(\underline{\varepsilon}, u) \in E_{r-1}$, then $(\varepsilon_r, \underline{\varepsilon}, u) \in \overline{E}_r$ if and only if $(\underline{\varepsilon}, u) \in \overline{E}_{r-1}$. If $\varepsilon_r = d-1$, $\varepsilon_{r-1} \in [[0, d-2]]$ and $(\underline{\varepsilon}, u) \in E_{r-2}$, then $(\varepsilon_r, \varepsilon_{r-1}, \underline{\varepsilon}, u) \in \overline{E}_r$ if and only if $(\underline{\varepsilon}, u) \in \overline{E}_{r-2}$. Finally, if $\varepsilon_r = \varepsilon_{r-1} = d-1$, then $(\varepsilon_r, \varepsilon_{r-1}, \underline{\varepsilon}, v) \in \overline{E}_r$ if and only if $\underline{\varepsilon} = \underline{0}$ and $v \in [[0, \delta_{r-1} - 1]]$. In other words,

$$\begin{aligned}\overline{E}_r = &\left(\bigcup_i (i, \overline{E}_{r-1})\right) \cup \left(\bigcup_i (d-1, i, \overline{E}_{r-2})\right) \\ &\cup \left(\bigcup_z (d-1, d-1, 0, \dots, 0, z)\right),\end{aligned} \tag{6.18}$$

where the unions in (6.18) are taken over $i \in [[0, d-2]]$ and $z \in [[0, \delta_{r-1} - 1]]$.

Let $B_s(i, p)$ denote the block $[[i\beta_s, i\beta_s + p]]$, where $p \leq \beta_s - 1$; if $m \in B_s(i, p)$, then $\varepsilon_s(m) = i$ and $R(m) = p$. If $r = 2$, then $(d-1, d-1)$ cannot occur. By (6.9), $\beta_3 - 1 = d^3 - 3d + 1 = (d-1)\beta_2 + d^2 - d - 1$, so

$$[[0, \beta_3 - 1]] = \left(\bigcup_{i=0}^{d-2} B_2(i, \beta_2 - 1)\right) \cup B_2(d-1, d^2 - d - 1). \tag{6.19}$$

If $m \in B_2(i, \beta_2 - 1)$, then $(\underline{\varepsilon}(m), u(m)) = (i, R(m))$, where $i \in [[0, d-2]]$ and $R(m) \in [[0, \beta_2 - 1]]$. If $m \in B_2(d-1, d^2 - d - 1)$, then $(\underline{\varepsilon}(m), u(m)) = (d-1, R(m))$, where $R(m) \in [[0, d^2 - d - 1]]$. Comparing with (6.16), we see that $\widetilde{E}_2 = \overline{E}_2$.

Similarly, $\beta_4 - 1 = (d-1)\beta_3 + (d-1)\beta_2 + \delta_2 - 1$ by (6.3), so

$$[[0, \beta_4 - 1]] = \left(\bigcup_{i=0}^{d-2} B_3(i, \beta_3 - 1)\right) \cup B_3(d-1, (d-1)\beta_2 + \delta_2 - 1). \tag{6.20}$$

Since $\underline{\varepsilon}(m) = (\varepsilon_s(m), \underline{\varepsilon}(R(m)))$, with allowance for zeros at the left, the greedy representations for $m \in \bigcup_{i=0}^{d-2} B_3(i, \beta_3 - 1)$ simply amount to $\bigcup_{i=0}^{d-2}(i, \widetilde{E}_2)$. If $m \in B_3(d-1, (d-1)\beta_2 + \delta_2 - 1)$, then $\varepsilon_3(m) = d-1$ and $R(m) \leq (d-1)\beta_2 + \delta_2 - 1$. Thus (6.20) shows that

$$\widetilde{E}_3 = \left(\bigcup_i (i, \widetilde{E}_2)\right) \cup \left(\bigcup_i \bigcup_u (d-1, i, u_1)\right) \cup \left(\bigcup_w (d-1, d-1, u_2)\right), \tag{6.21}$$

where the unions are taken over $i \in [[0, d-2]]$, $u \in [[0, \beta_2 - 1]]$, and $w \in [[0, \delta_2 - 1]]$. That is, $\widetilde{E}_3 = \overline{E}_3$.

Since $\beta_{r+1} - 1 = (d-1)\beta_r + (d-1)\beta_{r-1} + \delta_{r-1} - 1$ by (6.3), the same reasoning applied to $[[0, \beta_{r+1} - 1]]$ for $r \geq 4$ gives

$$\begin{aligned}E_r = &\left(\bigcup_i (i, \widetilde{E}_{r-1})\right) \cup \left(\bigcup_i (d-1, i, \widetilde{E}_{r-2})\right) \\ &\cup \left(\bigcup_z (d-1, d-1, 0, \dots, 0, z)\right),\end{aligned} \tag{6.22}$$

where the unions are taken over $i \in [[0, d-2]]$ and $z \in [[0, \delta_{r-1}-1]]$. This is the same recurrence, (6.18), satisfied by $\overline{E}_r$, so $\widetilde{E}_r = \overline{E}_r$ for all $r \geq 2$. □

As noted earlier, if $u(m) \in J_1$, then the greedy representation shows directly that $m \in S_d(\beta)$. If $u(m) \in J_2$ and $m = \sum \eta_k \beta_k \in S_d(\beta)$, then Lemma 6.14 gives us enough information about $(\underline{\varepsilon}(m), u(m))$ to determine whether m has an alternative representation.

Lemma 6.23. *Suppose* $m \in [[0, \beta_{r+1}-1]]$ *and* $u(m) \in J_2$. *Then* $m \in S_d(\beta)$ *if and only if there exists* $j \geq 3$ *so that*

$$\varepsilon_j(m) = \varepsilon_{j+1}(m) = 0 \quad \textit{and} \quad \varepsilon_{j+2}(m) > 0. \tag{6.24}$$

Proof. Since $u(m) \geq d^2 - d$ for $m \in J_2$ and $\delta_j \leq d-1$, neither (6.15)(ii) nor (iii) holds for $(\underline{\varepsilon}(m), u(m))$, thus (6.15)(i) holds: $\varepsilon_2(m) < d-1$ and $(d-1, d-1)$ does not occur in $\underline{\varepsilon}(m)$. Suppose (6.24) holds for $j \geq 3$; we define $(\underline{\eta}, v)$ by:

$$\begin{aligned} &\eta_{j+2} = \varepsilon_{j+2}(m) - 1, \qquad \eta_{j+1} = \eta_j = d-1, \\ &\eta_2 = \varepsilon_2(m) + 1, \quad \eta_i = \varepsilon_i(m) \quad \text{otherwise}, \quad \text{and} \\ &v = u(m) - (d^2 - 2) + \delta_j. \end{aligned} \tag{6.25}$$

As $\delta_j = d-2$ or $d-1$ and $u(m) \in J_2$, $v \in [[1, d-2]] \subset J_1$. Since $\varepsilon_2(m) \leq d-2$ and $\varepsilon_{j+2}(m) \geq 1$, $\eta_i \in [[0, d-1]]$ for all i and $(\underline{\eta}, v) \in E_r$. Finally,

$$\begin{aligned} f(\underline{\eta}, v) &= f(\underline{\varepsilon}, u) - \beta_{j+2} + (d-1)(\beta_{j+1} + \beta_j) + \beta_2 + \delta_j - (d^2 - 2) \\ &= m - \delta_j + \beta_2 + \delta_j - \beta_2 = m, \end{aligned} \tag{6.26}$$

so $m \in S_d(\beta)$.

Suppose now that (6.24) holds for no $j \geq 3$: there are no "internal" consecutive 0's in $\underline{\varepsilon}(m)$. We shall show that $m \in T_d(\beta)$. If $r = 2$ and

$$m = \varepsilon_2(m)\beta_2 + u(m) = \eta_2\beta_2 + \eta_1\beta_1 + \eta_0\beta_0 \in S_d(\beta), \tag{6.27}$$

then $\eta_1\beta_1 + \eta_0\beta_0 \equiv u(m) \pmod{\beta_2}$. This is a contradiction, since $\eta_1\beta_1 + \eta_0\beta_0 \in J_1$ and $u(m) \in J_2$. Suppose $r \geq 3$ and

$$m = \sum_{i=2}^{r} \varepsilon_i(m)\beta_i + u(m) = \sum_{i=0}^{r} \eta_i\beta_i, \tag{6.28}$$

and suppose $\varepsilon_i(m) = \eta_i$ for $i \geq s+1$, but $\varepsilon_s(m) \neq \eta_s$. By Lemma 5.8, $\varepsilon_s(m) > \eta_s$; consider the truncated integer n:

$$n = (\varepsilon_s(m) - \eta_s)\beta_s + \sum_{i=2}^{s-1} \varepsilon_i(m)\beta_i + u(m) = \sum_{i=0}^{s-1} \eta_i\beta_i. \tag{6.29}$$

(By the preceding argument, $s \geq 3$.) By hypothesis, at least one of each pair $\{\varepsilon_{s-2k-1}, \varepsilon_{s-2k-2}\}$ is positive for $s - 2k - 2 \geq 3$. Since $\varepsilon_s(m) - \eta_s \geq 1$ and

$u(m) \geq d^2 - d + 1$, we obtain from the first expression for n the inequalities:

(6.30)(i) $\quad n \geq \beta_s + \beta_{s-2} + \cdots + \beta_4 + d^2 - d + 1 \qquad (s \text{ even}, s \geq 4)$

(6.30)(ii) $\quad n \geq \beta_s + \beta_{s-2} + \cdots + \beta_3 + d^2 - d + 1 \qquad (s \text{ odd}, s \geq 3)$.

A repeated application of (6.3) to (6.30) gives

(6.31)(i)
$$n \geq \sum_{i=2}^{s-1}(d-1)\beta_i + \left(\frac{s}{2} - 1\right)\delta_s + d^2 - d + 1 \qquad (s \text{ even}, s \geq 4),$$

(6.31)(ii)
$$n \geq \sum_{i=1}^{s-1}(d-1)\beta_i + \frac{1}{2}(s-1)\delta_s + d^2 - d + 1 \qquad (s \text{ odd}, s \geq 3).$$

But, by the second expression for n in (6.29),

$$n = \sum_{i=0}^{s-1} \eta_i \beta_i \leq (d-1)\sum_{i=1}^{s-1} \beta_i + (d-1), \tag{6.32}$$

which contradicts either inequality in (6.31). Thus (6.28) is impossible, so $m \in T_d(\beta)$. This completes the proof. □

We may summarize our discussion as follows.

Lemma 6.33. *Suppose* $m \leq \beta_{r+1}$. *Then* $m \in T_d(\beta)$ *if and only if* $(\underline{\varepsilon}(m), u(m)) \in E_r^*$, *where* E_r^* *is the set of* $(\underline{\varepsilon}, u)$ *in* E_r *for which* $u \in J_2$, $\varepsilon_2 < d-1$, *and neither* $(d-1, d-1)$ *nor* $(\geq 1, 0, 0, \geq 1)$ *occurs.*

Let $Y(r)$ denote the set of strings $(\varepsilon_r, \ldots, \varepsilon_3) \in [[0, d-1]]^{r-2}$ such that $(d-1, d-1)$ does not occur and $(0, 0)$ does not occur, except initially, and let $F(r) = |Y(r)|$. Then $(\underline{\varepsilon}, u) \in E_r^*$ if and only if $(\varepsilon_r, \ldots, \varepsilon_3) \in Y(r)$, $\varepsilon_2 \in [[0, d-2]]$ and $u \in J_2$. Since $|J_2| = d-3$, it follows that

$$N_r = N(T_d(\beta); \beta_{r+1} - 1) = (d-2)(d-3)F(r). \tag{6.34}$$

We compute N_r from (6.34), by describing $Y(r)$ recursively.

Lemma 6.35. *There are constants* c_i *so that*

$$F(r) = c_1\sigma^r + c_2\overline{\sigma}^r + c_3 + c_4(-1)^r, \tag{6.36}$$

where σ *(cf. (6.6)) and* $\overline{\sigma} = d - 1 - \sigma$ *are the roots of the quadratic* $\widetilde{H}(x) = x^2 - (d-1)x - (d-2)$.

Proof. We divide $Y(r) = \bigcup Y_i(r)$ into four subclasses. Let

$$\begin{gathered} Y_1(3) = \varnothing, \qquad Y_2(3) = \{(0)\}, \\ Y_3(3) = \{(1), \ldots, (d-2)\}, \qquad Y_4(3) = \{(d-1)\}, \end{gathered} \tag{6.37}$$

and, for $r \geq 4$, let

(6.38)(i) $$Y_1(r) = \{\underline{\varepsilon} \in Y(r) \colon \varepsilon_r = \varepsilon_{r-1} = 0\},$$

(6.38)(ii) $$Y_2(r) = \{\underline{\varepsilon} \in Y(r) \colon \varepsilon_r = 0,\ \varepsilon_{r-1} \neq 0\},$$

(6.38)(iii) $$Y_3(r) = \{\underline{\varepsilon} \in Y(r) \colon 1 \leq \varepsilon_r \leq d-2\},$$

(6.38)(iv) $$Y_4(r) = \{\underline{\varepsilon} \in Y(r) \colon \varepsilon_r = d-1\}.$$

For $r \geq 3$, the strings in $Y(r+1)$ arise by prefixing any $\varepsilon_{r+1} \in [[0, d-1]]$ to any string in $Y(r)$, with two exceptions. If $\underline{\varepsilon} \in Y_1(r)$, then $\varepsilon_{r+1} = 0$; if $\underline{\varepsilon} \in Y_4(r)$, then $\varepsilon_{r+1} \neq d-1$. Thus, $F_i(r) = |Y_i(r)|$ satisfies the following recurrence:

(6.39)(i) $F_1(r+1) = F_1(r) + F_2(r) +$
(6.39)(ii) $F_2(r+1) = F_3(r) + F_4(r)$
(6.39)(iii) $F_3(r+1) = (d-2)F_2(r) + (d-2)F_3(r) + (d-2)F_4(r)$
(6.39)(iv) $F_4(r+1) = F_2(r) + F_3(r)$

Each F_i (and so $F = \sum F_i$) satisfies the recurrence whose auxiliary equation is the characteristic polynomial of the matrix implicit in (6.39), namely $(\lambda^2 - 1)(\lambda^2 - (d-1)\lambda - (d-2))$. That is, for $r \geq 3$,

(6.40) $$F(r+4) = (d-1)\{F(r+3) + F(r+2) - F(r+1)\} - (d-2)F(r).$$

This can also be directly verified from (6.39). The conclusion follows from the standard theory of linear recurrences. The c_i's may be computed from the initial conditions. □

Proof of Theorem E. Since $|\overline{\sigma}| < 1$, (6.36) implies that

$$N_r = N(T_d(\beta); \beta_{r+1} - 1) = c_1\sigma^r + O(1).$$

For any n, choose r such that $n \in [[\beta_r - 1, \beta_{r+1} - 1]]$, then

$$N_r \leq N(T_d(\beta); n) \leq N_{r+1}.$$

Since $\beta_n \approx \beta^n$, it follows that $N(T_d(\beta); n) = O(n^{\log\sigma/\log\beta}) = O(n^{e(d)})$. Hence $N(S_d(\beta); n) = n - o(n)$, so $D(S_d(\beta)) = 1$. □

Observe that $T_d(\beta)$ contains no integers between β_{r+3} and $\beta_{r+3} + \beta_{r+1} - 1$, as $\varepsilon_{r+2}(m) \geq 1$ and $\varepsilon_{r+1}(m) = \varepsilon_r(m) = 0$ for m in this interval. Since $\beta_{r+3} + \beta_{r+1} - 1 \approx (1+\beta^{-2})\beta_{r+3}$, $n^{-e(d)}N(T_d(\beta); n)$ does not converge as $n \to \infty$.

7. Discussion of Conjecture F

One can imagine the method of the last section extended to a slightly larger set of α's. Suppose $\gamma \in [1, d]\backslash I_d$ is a PV number with minimal polynomial $H(x) = x^k - \sum c_i x^i$, where $c_i \in [[0, d-1]]$. Then, as before, $a_n = \sum \gamma_j^n$ satisfies the recurrence induced by H, and there ought to be a close relationship between a_n and $\gamma_n = [\gamma^n]$. We now argue probabilistically that $T_d(\gamma)$ "ought" to have density zero. If $\gamma < d$, then there exists r such that, for all s, $(d-1)(\gamma_{s+r} + \cdots + \gamma_s) > \gamma_{s+r+1}$. Thus, if $\varepsilon_i(m) = 0$ for $i \in [[s, s+r]]$ and

$\varepsilon_{s+r+1} \geq 1$, then we may "cash in" γ_{s+r+1} for smaller γ_i's, as in the last section, and in the ensuing rearrangement, the new u might lie in $S_d(\gamma)$. Since the probability that a string of length r contains a fixed pattern increases to 1 as $r \to \infty$, this argument gives a reason to believe that $D(T_d(\gamma)) = 0$.

As α gets close to d, the cutoff value t in the greedy representation increases; the explicit description of the strings $(\underline{\varepsilon}(m), u(m))$ seems to become a very difficult problem. However, if $\alpha = d - \delta$, where $\delta > 0$ is small, and r is sufficiently small, then we can determine both $S_d^r(\alpha)$ and N_r.

Lemma 7.1. *If* $\alpha \in [(d^s - 1)^{1/s}, d)$, *then* $\alpha_i = d^i - 1$ *for* $1 \leq i \leq s$.

Proof. Certainly $\alpha^i < d^i$; on the other hand,

$$(d^i - 1)^{1/i} = d(1 - d^{-i})^{1/i} \leq d(1 - d^{-s})^{1/i} \leq d(1 - d^{-s})^{1/s} \leq \alpha. \quad \square \tag{7.2}$$

For fixed d, $t((d^s - 1)^{1/s}, d) \approx s - 1 + \log_d(s(d - 1))$.

Lemma 7.3. *If* $r \geq 2$ *and* $\alpha \in [(d^r - 1)^{1/r}, d)$, *then* $N(T_d(\alpha); d^r - 1) = d^{r-1} - d^{r-2}$.

Proof. Suppose $m \leq d^r - 1$ and $m = \sum_{i=0}^{\infty} \varepsilon_i \alpha_i$; $d^r - 1 = \alpha_r \in S_d(\alpha)$, otherwise, $\varepsilon_i = 0$ for $i \geq r$. There are d^r formally distinct sums $\sum_{i=0}^{r-1} \varepsilon_i \alpha_i$, and any such sum is bounded above by

$$(d-1)\sum_{i=0}^{r-1} \alpha_i = (d-1)\left(1 + \sum_{i=1}^{r-1}(d^i - 1)\right) = d^r - (r-1)d + r - 2 \leq d^r - 2. \tag{7.4}$$

It follows that $N(T_d(\alpha); d^r - 1) = d^r - N(S_d(\alpha); d^r - 1)$ equals the number of "repeats" among the sums. Suppose

$$\sum_{i=0}^{r-1} \varepsilon_i \alpha_i = \sum_{i=0}^{r-1} \eta_i \alpha_i, \tag{7.5}$$

and $\varepsilon_k = \eta_k$ for $k > j$, but $\varepsilon_j > \eta_j$; clearly, $j \geq 1$. By pruning common higher terms, we obtain

$$(\varepsilon_j - \eta_j)\alpha_j = \sum_{i=0}^{j-1}(\eta_i - \varepsilon_i)\alpha_i. \tag{7.6}$$

If $j \geq 2$, then, as before,

$$d^j - 1 = \alpha_j \leq (\varepsilon_j - \eta_j)\alpha_j \leq (d - 1)\sum_{i=0}^{j-1} \alpha_i = d^j - (j - 1)d + (j - 2), \tag{7.7}$$

hence $(j - 1)d \leq (j - 1)$, a contradiction. If $j = 1$, then (7.6) implies that $(\varepsilon_1 - \eta_1)\alpha_1 = (\eta_0 - \varepsilon_0)\alpha_0$, hence $\varepsilon_1 - \eta_1 = 1$ and $\eta_0 - \varepsilon_0 = d - 1$. On restoring the pruned terms, we see that (7.5) occurs with $\underline{\varepsilon} \neq \underline{\eta}$ if and only if

$$\underline{\varepsilon} = (\varepsilon_{r-1}, \ldots, \varepsilon_2, 1 + \varepsilon_1, 0) \quad \text{and} \quad \underline{\eta} = (\varepsilon_{r-1}, \ldots, \varepsilon_2, \varepsilon_1, d - 1), \tag{7.8}$$

where $\varepsilon_1 \in [[0, d-2]]$ and $\varepsilon_i \in [[0, d-1]]$ for $i \geq 2$. There are $d^{r-2}(d-1)$ such pairs, and this completes the proof. □

This result shows that we cannot have a uniform bound of the form

$$N(T_d(\alpha), n) \leq f(d, n)$$

for all $\alpha \in [1, d]\backslash I_d$, where $f(d, n) = o(n)$. Of course, every fixed α is eventually less than $(d^s - 1)^{1/s}$ as $s \to \infty$.

Taking a broader view, we let $R(\alpha, d, m)$ denote the number of ways that m can be written in the form $m = \sum_{i=0}^{\infty} \varepsilon_i [\alpha^i]$, $\varepsilon_i \in [[0, d-1]]$. This paper has determined those (α, d) for which $R(\alpha, d, m) \geq 1$ for all m. For fixed (α, d), we have an infinite product formula for the generating function:

$$\sum_{m=0}^{\infty} R(\alpha, d, m) z^m = \prod_{i=0}^{\infty} \left(\sum_{k=0}^{d-1} z^{k\alpha_i} \right) = \prod_{i=0}^{\infty} \frac{(1 - z^{d\alpha_i})}{(1 - z^{\alpha_i})}. \tag{7.9}$$

When $\alpha \notin \mathbf{Z}$, the irregularity of (α_i) renders (7.9) unhelpful in understanding $R(\alpha, d, m)$. However, when α is an integer, cancellation can occur.

$$\sum_{m=0}^{\infty} R(k, k^r, m) z^m = \prod_{i=0}^{\infty} \frac{(1 - z^{k^{r+i}})}{(1 - z^{k^i})} = \prod_{i=0}^{r-1} \frac{1}{(1 - z^{k^i})}. \tag{7.10}$$

If $r = 2$, (7.10) leads to the explicit closed formula: $R(k, k^2, m) = [m/k] + 1$. The computation of $R(2, 4, m)$ was Problem B-2 on the 1983 Putnam Competition, see [12].

If $\alpha = 2$ and $d = 3$, then (7.9) becomes

$$\sum_{m=0}^{\infty} R(2, 3, m) z^m = \prod_{i=0}^{\infty} (1 + z^{2^i} + z^{2^{i+1}}). \tag{7.11}$$

Let $s(n)$ denote the Stern diatomic sequence (see, [6, 13, 15]), which is defined recursively by:

$$\begin{gathered} s(0) = 0, \qquad s(1) = 1, \\ s(2n) = s(n), \quad s(2n+1) = s(n) + s(n+1), \qquad n \geq 1. \end{gathered} \tag{7.12}$$

An easy argument shows that $R(2, 3, m) = s(m+1)$. The Stern sequence has had a rather fugitive history; see [17]. The exact value of $s(n)$ can be calculated from the pattern of 0's and 1's in the binary expansion of n, as the denominator of a certain continued fraction.

It is easy to see from (7.12) that $s(2^n) = 1$ for all n; it was already known in the 19th century that the maximum value of $s(m)$ for $m \in [[2^n, 2^{n+1}]]$ is F_{n+2}, the $(n+2)$nd Fibonacci number (and so is $\approx c\Phi^n$). This maximum is achieved twice in each such interval, at the integers closest to $\frac{4}{3}2^n$ and $\frac{5}{3}2^n$. We show in [18] that the growth of $R(2, k, m)$ depends on the parity of k mod 2. There exist positive $c_i = c_i(k)$ such that, for all m,

$$c_1 \leq m^{-(\log k)/(\log 2)} R(2, 2k, m) \leq c_2, \tag{7.13}$$

but $m^{-\tau}R(2, 2k+1, m)$ cannot be bounded in this way for any τ.

Finally, consider $\lim_{d\to\infty} R(k, d, m) = R(k, m)$, the number of partitions of m into powers of k; $R(2, m)$, usually written $b(m)$, was studied by Euler. The asymptotics of $R(k, m)$ have been studied by Mahler [16] and de Bruijn [3]. Mahler proved that, for fixed k,

$$(2\log k)(\log m)^{-2}\log R(k, m) \to 1. \tag{7.14}$$

In a beautiful analysis, deBruijn gave $\log R(k, m)$ up to $o(1)$. Let $h = m/k$, then

$$\begin{aligned}\log R(k, m) = {} & \frac{1}{2\log k}\left(\log\left(\frac{h}{\log h}\right)\right)^2 + \left(\frac{1}{2} + \frac{1+\log\log k}{\log k}\right)\log h \\ & - \left(1 + \frac{\log\log k}{\log k}\right)\log\log h + \psi\left(\frac{\log h - \log\log h}{\log k}\right) + o(1),\end{aligned} \tag{7.15}$$

where ψ is a certain periodic function with period one.

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