TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 312, Number 1, March 1989

THE SPECTRUM OF THE SCHRÖDINGER OPERATOR

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ABSTRACT. We describe the negative spectrum of the Schrödinger operator with a singular potential. We determine the exact value of the bottom of the spectrum and estimate it from above and below. We describe the dependence of a crucial constant on the eigenvalue parameter and discuss some of its properties. We show how recent results of others are simple consequences of a theorem proved by the author in 1972.

1. INTRODUCTION

For $V(x) \geq 0$ in $L^{\text{loc}}(\mathbf{R}^n)$, the smallest constant $C_\lambda(V)$ which satisfies

$$(Vu, u) \leq C_\lambda(V)(\|\nabla u\|^2 + \lambda^2\|u\|^2), \qquad u \in C_0^\infty, \tag{1.1}$$

is of importance in the study of the spectrum of the Schrödinger operator

$$H = -\Delta - V. \tag{1.2}$$

We shall show that $-\lambda_0^2$ is the smallest point of the spectrum of H if and only if, λ_0 is the smallest value of $\lambda \geq 0$ such that $C_\lambda(V) \leq 1$ (if $C_\lambda(V) > 1$ for all $\lambda \geq 0$, then the operator H is not bounded from below; the smallest point in the spectrum is $-\infty$). In 1972 the author obtained an expression determining the exact value of $C_\lambda(V)$ (cf. [1, p. 498]). It is given by

$$C_\lambda(V) = \inf_{\psi>0} \sup_x \psi(x)^{-1} \int_{\mathbf{R}^n} V(y)\psi(y)G_{2,\lambda}(x-y)\,dy \tag{1.3}$$

where $G_{2,\lambda}(x)$ is the Bessel potential of order 2. It is the kernel of the operator

$$G_{2,\lambda}f = (\lambda^2 - \Delta)^{-1}f, \qquad I_2 = G_{2,0}. \tag{1.4}$$

In (1.3) one obtains an upper bound for $C_\lambda(V)$ by picking a particular function $\psi(x) > 0$, e.g., $\psi(x) \equiv 1$. One can improve the estimate by varying ψ.

The cases $\lambda = 0$ and $\lambda = 1$ have received much attention. In 1962 Mazya [2] showed that for $n > 2$, $C_0(V) \leq 1$ if

$$\int_e V(x)\,dx \leq \frac{n-2}{4}\omega\,\text{cap}(e) \tag{1.5}$$

Received by the editors January 12, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 26D10, 35J10, 35P15; Secondary 42B25, 47F05, 47G05.

holds for all compact sets $e \subset \mathbf{R}^n$. Here ω is the surface area of the unit ball in $\mathbf{R}^n$ and $\operatorname{cap}(e)$ is the Green capacity of e. More recently Adams [3] showed that

$$\int \left(\int_e |x-y|^{2-n} V(y)\, dy \right)^2 dx \le C \int_e V(y)\, dy, \qquad e \subset \mathbf{R}^n, \tag{1.6}$$

implies a bound for $C_0(V)$. In [4], Fefferman and Phong show that

$$C_0(V) \le C_p \sup_{\delta, x} \left(\delta^{2p-n} \int_{|x-y|<\delta} V(y)^p\, dy \right)^{1/p} \tag{1.7}$$

if $p > 1$. The proof of (1.7) given in [4] is rather long and involved. In the next section we shall show that it is a simple consequence of (1.3). In fact, we shall give a direct easy proof of (1.7) without involving the ideas of [4]. In [1] Kerman and Sawyer show that

$$C_\lambda(V) \sim \sup_{|Q| \le \lambda^{-n}} \frac{\int \left(\int_Q G_{1,\lambda}(x-y) V(y)\, dy \right)^2 dx}{\int_Q V(y)\, dy} \tag{1.8}$$

where the supremum is taken over all dyadic cubes $Q \subseteq \mathbf{R}^n$. Previous to [1], sufficient conditions for (1.1) to hold for various values of λ were obtained by Kato, Rollnik, Schechter, Simon (cf. [14, 15] for references). Other sets of sufficient conditions were recently obtained in [12 and 16]. These authors were apparently unaware of the results of [1] where a condition which is both necessary and sufficient is obtained.

In §3 we show that there is a constant C_p depending only on n and p such that

$$C_\lambda(V) \le C_p \| M_{2p, 1/\lambda}[V^p] \|_\infty^{1/p}, \qquad \lambda \ge 0, \tag{1.9}$$

where

$$M_{\alpha,\delta}[V](x) = \sup_{r \le \delta} r^{\alpha-n} \int_{|y-x|<r} V(y)\, dy.$$

This allows us to show that the lowest point $-\mu^2$ of the spectrum of the operator (1.2) satisfies

$$\mu^2 \le \sup_{x,\delta} \left(2C_p \left(\delta^{-n} \int_{|y-x|<\delta} V(y)^p\, dy \right)^{1/p} - \delta^{-2} \right)$$

which is another estimate of Fefferman-Phong [4]. In our estimate only one constant appears (the one from (1.9)) and can be readily estimated. In proving (1.9) we show that there is a constant $C_{s,q}$ depending only on s, n and q such that

$$\| I_{s,\delta} f \|_q \le C_{s,q} \| M_{s,\delta} f \|_q \tag{1.10}$$

where

$$I_{s,\delta} f(x) = \int_{|y-x|<\delta} |y-x|^{s-n} f(y)\, dy.$$

The estimate (1.10) is of interest in its own right. Our proof extends a method of Muckenhoupt-Wheeden [5]. As a consequence of (1.10) we obtain

$$\|G_{s,\lambda}f\|_q \le C'_{s,q}\|M_{s,1/\lambda}f\|_q \tag{1.11}$$

where

$$G_{s,\lambda}f = (\lambda^2 - \Delta)^{-s/2} f.$$

In §4 we show that the constant $C_\lambda(V)$ is continuous in λ in the interval $[0,\infty)$. Moreover

$$\begin{aligned}\mu^2 &= \inf_{C_\lambda(V)\le 1} \lambda^2 C_\lambda(V) = \inf_{C_\lambda(V)\le 1} \lambda^2 \\ &= \sup_{C_\lambda(V)>1} \lambda^2 = \sup_{C_\lambda(V)>1} \lambda^2 C_\lambda(V).\end{aligned}$$

From this it follows easily that

$$\sup_\lambda \lambda^2[C_\lambda(V) - 1] \le \mu^2 \le \sup_\lambda \lambda^2[2C_\lambda(V) - 1].$$

Next we show that if V is in the Muckenhoupt-Wheeden class A_∞ (cf. [5]), then

$$C_\lambda(V) \le N_p\|M_{2,1/\lambda}\|_\infty. \tag{1.12}$$

In §5 we show that the essential spectrum of H is the same as that of $-\Delta$, i.e.,

$$\sigma_e(H) = [0,\infty) \tag{1.13}$$

provided

(a) $C_\lambda(V) \to 0$ as $\lambda \to \infty$;

(b) $C_{\lambda_0}(V^R) \to 0$ as $R \to \infty$

for some $\lambda_0 \ge 0$, where

$$V^R(x) = \begin{cases} 0, & |x| \le R, \\ V(x), & |x| > R. \end{cases}$$

2. A simple proof of the Fefferman-Phong estimate

We now show that (1.7) is a simple consequence of (1.3). Let

$$M_\alpha[V](x) = \sup_r r^{\alpha-n}\int_{|y-x|<r} V(y)\,dy, \qquad M = M_0, \tag{2.1}$$

denote the maximal function. The right-hand side of (1.7) is equivalent to

$$K_p\|M_{2p}[V^p]\|_\infty^{1/p}.$$

By Hölder's inequality

$$M_1[V^{1/2}u] \le M_q[V^{q/2}]^{1/q} M[|u|^{q'}]^{1/q'} \tag{2.2}$$

holds for any $q \geq 1$, where $1/q + 1/q' = 1$. If we take $q = 2p > 2$, we have

$$\|M_1[V^{1/2}u]\|_2 \leq K_p^{1/2}\|M[|u|^{q'}]^{1/q'}\|_2 = K_p^{1/2}\|M[|u|^{q'}]\|_{2/q'}^{1/q'} \tag{2.3}$$
$$\leq C'K_p^{1/2}\|\,|u|^{q'}\|_{2/q'}^{1/q'} = C'K_p^{1/2}\|u\|_2$$

since $q' < 2$. By a theorem of Muckenhoupt and Wheeden [5], this implies

$$\|I_1[V^{1/2}u]\|_2 \leq C''K_p^{1/2}\|u\|_2, \tag{2.4}$$

where $I_s = G_{s,0}$. Inequality (2.4) is equivalent to

$$\|V^{1/2}I_2[V^{1/2}u]\|_2 \leq C''^2K_p\|u\|_2.$$

If $C > C''^2K_p$ and $h > 0$ is in L^2, then there is a $\phi > 0$ in L^2 such that

$$\phi = h + C^{-1}V^{1/2}I_2[V^{1/2}\phi]. \tag{2.5}$$

This shows that the right-hand side of (1.3) is bounded by a constant times K_p. Hence, (1.7) holds.

Another approach is to note that (2.4) is equivalent to

$$\|V^{1/2}I_1\nu\|_2 \leq C''K_p^{1/2}\|\nu\|_2 \tag{2.6}$$

which in turn is equivalent to

$$(Vu, u) \leq C''^2K_p\|\nabla u\|^2 \tag{2.7}$$

which shows that (1.7) holds.

3. Estimates for arbitrary λ

For μ a locally finite Borel measure, we define

$$I_{s,\delta}\,d\mu(y) = \int_{|x-y|<\delta} |x-y|^{s-n}\,d\mu(x), \qquad 0 < s \leq n, \tag{3.1}$$

and

$$M_{s,\delta}\,d\mu(y) = \sup_{r\leq\delta} r^{s-n}\int_{|x-y|<r} d\mu(x), \qquad 0 \leq s \leq n, \tag{3.2}$$
$$M_s\,d\mu = M_{s,\infty}\,d\mu.$$

For $1 \leq q < \infty$ we let

$$\|u\|_q = \left(\int_{R^n} |u(x)|^q\,dx\right)^{1/q}$$

by the norm in $L^q(\mathbf{R}^n)$. Our first result is

Theorem 3.1. *There is a constant $C_{s,q}$ depending only on s, n and q such that*

$$\|I_{s,\delta}\,d\mu\|_q \leq C_{s,q}\|M_{s,\delta}\,d\mu\|_q. \tag{3.3}$$

Moreover

$$C_{s,q} \leq 2^{n-s+1} + (\omega/s)5^{n-s}n^{n/2}2^{(n+2-s)q+2s+2}.$$

Before proving Theorem 3.1 we state some consequences.

Theorem 3.2. *For each* $p > 1$ *there is a constant* C_p *depending only on* n *and* p *such that*

$$C_\lambda(V) \le C_p \sup_x (M_{2p,1/\lambda} V^p)^{1/p}, \qquad \lambda \ge 0. \tag{3.4}$$

Moreover, there is a constant C_1 *depending only on* n *such that*

$$C_\lambda(V) \ge C_1 M_{2,1/\lambda} V. \tag{3.5}$$

Corollary 3.3. *If* $-\mu^2$ *is the lowest point of the spectrum of the operator* (1.2), *then*

$$\begin{aligned}\mu^2 &\le \sup_{\delta>0} \left(2C_p \delta^{-2} \sup_x (M_{2p,\delta} V^p)^{1/p} - \delta^{-2}\right) \\ &= \sup_{x,\delta} \left(2C_p \left(\delta^{-n} \int_{|y-x|<\delta} V(y)^p \, dy\right)^{1/p} - \delta^{-2}\right)\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\mu^2 &\ge \sup_\delta \left(C_1 \delta^{-2} \sup_x M_{2,\delta} V - \delta^{-2}\right) \\ &= \sup_{x,\delta} \left(C_1 \delta^{-n} \int_{|y-x|<\delta} V(y) \, dy - \delta^{-2}\right).\end{aligned}$$

Corollary 3.4. *If* $C_p^p M_{2p} V^p \le 1$, *then* $\mu = 0$.

Corollaries 3.3 and 3.4 are proved by Fefferman and Phong [4]. Their proof is rather long and involved. They require two constants in (3.6) and do not provide a way of estimating them. Our proof is much shorter. They were unaware of the authors results in [1].

Proof of Theorem 3.1. Let

$$S_t = \{x \in \mathbf{R}^n | I_{s,\delta} \, d\mu(x) < t\} \tag{3.7}$$

for each $t > 0$. If $S_t \ne \mathbf{R}^n$, then

$$S_t = \bigcup_{j=1}^{\infty} Q_j, \tag{3.8}$$

where the cubes Q_j have sides parallel to the coordinate axes, have disjoint interiors and satisfy

$$d(Q_j, S_t^c) \le 3\sqrt{n}\, l(Q_j) \tag{3.9}$$

where M^c is the complement of M in $\mathbf{R}^n$ and $l(Q)$ is the edge length of Q (cf. [6, p. 10]). By subdividing Q_j if necessary, we may require that

$$\rho_j \equiv 4\sqrt{n}\, l(Q_j) \le \delta. \tag{3.10}$$

If (3.10) is achieved by subdivision, we lose (3.9). But in this case we can require

$$\delta \le 2\rho_j. \tag{3.11}$$

Thus we can make each Q_j satisfy (3.10). If it does not satisfy (3.11) as well, then it will satisfy (3.9).

Let b, d be positive numbers to be determined later. Define

$$E_j = \{x \in Q_j | I_{s,\delta/2}\, d\mu(x) > tb,\ M_{s,\delta}\, d\mu(x) \le td\}. \tag{3.12}$$

Let Q be one of the cubes Q_j, and let $E \subset Q$ be the set given by (3.12). Assume first that Q satisfies (3.10) and (3.11). Then we have

$$\begin{aligned} tb|E| &\le \int_Q I_{s,\delta/2}\, d\mu(x)\, dx \\ &= \int_Q \int_{|y-x|<\delta/2} |x-y|^{s-n}\, d\mu(y)\, dx \\ &= \int \int_{\substack{|x-y|<\delta/2 \\ x\in Q}} |x-y|^{s-n}\, dx\, d\mu(y) \\ &\le (\omega/s)(\delta/2)^s \int_{Q+\delta} d\mu(y) \end{aligned}$$

where ω is the surface area of the unit sphere in $\mathbf{R}^n$ and $Q+\delta$ is the cube having the same center as Q but edge length equal to $l(Q)+\delta$. Assume that E is not empty, and let x_0 be any point in E. The cube $Q+\delta$ is contained in the ball with center x_0 and radius $\sqrt{n}l(Q) + (\delta/2) \le (\rho/4) + (\delta/2) \le 3\delta/4$ by (3.10). Hence by (3.11)

$$\begin{aligned} tb|E| &\le (\omega/s)(\delta/2)^s(\rho/4 + \delta/2)^{n-s} M_{s,\delta}\, d\mu(x_0) \\ &\le (\omega/s)\rho^s(5\rho/4)^{n-s}\, td \\ &\le (\omega/s)4^s 5^{n-s}\, td\, n^{n/2}|Q|. \end{aligned}$$

Consequently,

$$|E| \le (\omega/s)4^s 5^{n-s} n^{n/2}(d/b)|Q|. \tag{3.13}$$

Note that (3.13) holds if E is empty. Next assume that (3.9) and (3.10) hold. Then there is a point x_1 not in S_t that

$$d(x_1, Q) \le 3\sqrt{n}l(Q).$$

If x is in Q, then

$$|x - x_1| < \rho. \tag{3.14}$$

Consequently, if y is any point such that

$$|y - x| > \rho, \tag{3.15}$$

then

$$|y-x_1| \le |y-x| + |x-x_1| < 2|y-x|. \tag{3.16}$$

Hence we have

$$\begin{aligned} I_{s,\delta/2}\,d\mu(x) &= \int_{|y-x|<\rho} + \int_{\rho<|y-x|<\delta/2} |y-x|^{s-n}\,d\mu(y) \\ &\le I_{s,\rho}\,d\mu(x) + 2^{n-s}\int_{|y-x_1|<\delta} |y-x_1|^{s-n}\,d\mu(y) \\ &\le I_{s,\rho}\,d\mu(x) + 2^{n-s} I_{s,\delta}\,d\mu(x_1) \\ &\le I_{s,\rho}\,d\mu(x) + 2^{n-s} t \end{aligned}$$

since x_1 is not in S_t. We now take $b = 2^{n+1-s}$. This implies that if $x \in E$, we have

$$tb \le I_{s,\rho}\,d\mu(x) + tb/2$$

and consequently

$$tb/2 \le I_{s,\rho}\,d\mu(x).$$

Thus E is contained in the set

$$\{x \in Q | I_{s,\rho}\,d\mu(x) > tb/2,\ M_{s,\delta}\,d\mu(x) \le td\}.$$

Hence, if $x \in E$

$$\begin{aligned} tb|E|/2 &\le \int_Q I_{s,\rho}\,d\mu(x)\,dx \\ &= \int\int_{\substack{|x-y|<\rho \\ x\in Q}} |x-y|^{s-n}\,dx\,d\mu(y) \\ &\le (\omega/s)\rho^s \int_{Q+2\rho} d\mu. \end{aligned}$$

Since $2\rho \le \delta$ and the cube $Q+2\rho$ is contained in a ball of radius $5\rho/4 < \delta$ about any point in Q, we see that

$$\begin{aligned} tb|E|/2 &\le (\omega/s)\rho^s (5\rho/4)^{n-s} M_{s,\delta}\,d\mu(x_0) \\ &\le (\omega/s) r^s 5^{n-s} n^{n/2} td|Q| \end{aligned}$$

or

$$|E| \le 2^{2s+1}(\omega/s)5^{n-s} n^{n/2}(d/b)|Q| \tag{3.17}$$

if we take $x_0 \in E$. If E is empty, (3.17) holds as well. Thus we see that (3.17) holds in all cases. If we sum over all the cubes Q_j, we see that

$$|\{I_{s,\delta/2}\,d\mu(x) \ge tb,\ M_{s,\delta}\,d\mu(x) \le td\}| \le C_{n,s}\,d|S_t|$$

where

$$C_{n,s} = \omega 5^{n-s} n^{n/2} 2^{3s-n}/s.$$

Hence

$$|\{I_{s,\delta/2}\,d\mu(x) > tb\}| \le C_{n,s}\,d|S_t| + |\{M_{s,\delta}\,d\mu(x) > td\}|.$$

This means that

$$\int_0^N |\{I_{s,\delta/2}\,d\mu(x) > tb\}|\,dt^q \le C_{n,s}\,d\int_0^N |S_t|\,dt^q + \int_0^N |\{M_{s,\delta}\,d\mu(x) > td\}|\,dt^q$$

or

$$b^{-q}\int_0^{Nb} |\{I_{s,\delta/2}\,d\mu(x) > \tau\}|\,d\tau^q \le C_{n,s}\,d\int_0^N |S_t|\,dt^q + d^{-q}\int_0^{Nd} |\{M_{s,\delta}\,d\mu(x) > \tau\}|\,d\tau^q.$$

Letting $N \to \infty$, we have

$$\|I_{s,\delta/2}\,d\mu\|_q^q \le C_{n,s}\,db^q\|I_{s,\delta}\,d\mu\|_q^q + (b/d)^q\|M_{s,\delta}\,d\mu\|_q^q$$

and consequently

$$\|I_{s,\delta/2}\,d\mu\|_q \le C_{n,s}^{1/q}\,d^{1/q}b\|I_{s,\delta}\,d\mu\|_q + (b/d)\|M_{s,\delta}\,d\mu\|_q.$$

Now

$$I_{s,\delta}\,d\mu(x) = I_{s,\delta/2}\,d\mu(x) + \int_{\delta/2<|y-x|<\delta} |x-y|^{s-n}\,d\mu(y) \le I_{s,\delta/2}\,d\mu + 2^{n-s}M_{s,\delta}\,d\mu.$$

Hence

$$\|I_{s,\delta}\,d\mu\|_q \le C_{n,s}^{1/q}\,d^{1/q}b\|I_{s,\delta}\,d\mu\|_q + (bd^{-1} + 2^{n-s})\|M_{s,\delta}\,d\mu\|_q.$$

Take $1/d = C_{n,s}2^q b^q$. Then

$$\|I_{s,\delta}\,d\mu\|_q \le b(2d^{-1}+1)\|M_{s,\delta}\,d\mu\|_q = (2^{n-s+1} + (\omega/s)5^{n-s}n^{n/2}2^{(n+2-s)q+2s+2})\|M_{s,\delta}\,d\mu\|_q.$$

This gives the theorem.

Next we shall prove

Theorem 3.5. *Under the same hypothesis,*

$$\|G_{s,\lambda}\,d\mu\|_q \le C'_{s,q}\|M_{s,1/\lambda}\,d\mu\|_q \tag{3.18}$$

where the constant depends only on s, n *and* q. *Here*

$$G_{s,\lambda}\,d\mu(x) = \int G_{s,\lambda}(x-y)\,d\mu(y), \tag{3.19}$$

$$(\lambda^2 - \Delta)^{-s/2}f(x) = \int G_{s,\lambda}(x-y)f(y)\,dy.$$

Proof. For each $s > 0$, the function $G_{s,\lambda}(x)$ has been studied extensively by Aronszajn-Smith [7]. In particular, it satisfies

$$G_{s,\lambda}(x) \le \begin{cases} c_0|x|^{s-n}, & \lambda|x| \le 1, \\ c_1\lambda^{n-s}|\lambda x|^{\gamma}e^{-\lambda|x|}, & \lambda|x| > 1, \end{cases} \tag{3.20}$$

where $\gamma = (n-s-1)/2$ and the c_j do not depend on λ. Let

$$\widetilde{G}_{s,\lambda}(x) = \begin{cases} 0, & \lambda|x| \le 1, \\ G_{s,\lambda}(x), & \lambda|x| > 1. \end{cases} \tag{3.21}$$

It suffices to show that

$$\|\widetilde{G}_{s,\lambda}\, d\mu\|_q \le C\|M_{s,1/\lambda}\, d\mu\|_q. \tag{3.22}$$

For by Theorem 3.1 and (3.20)

$$\|[G_{s,\lambda} - \widetilde{G}_{s,\lambda}]\, d\mu\|_q \le c_0\|I_{s,1/\lambda}\, d\mu\|_q \le c_0 C_{s,q}\|M_{s,1/\lambda}\, d\mu\|_q.$$

Now by (3.20) and (3.21)

$$\begin{aligned}\widetilde{G}_{s,\lambda}\, d\mu(y) &\le c_1\int_{\lambda|x-y|>1} \lambda^{n-s}|\lambda(x-y)|^{\gamma}e^{-\lambda|x-y|}\, d\mu(x) \\ &\le c_1\lambda^{n-s}\sum_{k=1}^{\infty}\int_{k<\lambda|x-y|<k+1}(k+1)^{\gamma}e^{-k}\, d\mu(x).\end{aligned}$$

The set $k < |x| < k+1$ can be covered by $N(k)$ balls of radius 1 and centers $z^{(1)}, \dots, z^{N(k)}$ with $N(k) \le c_2k^{n-1}$. Thus the set $k < \lambda|x| < k+1$ can be covered by $N(k)$ balls with centers $z^{(1)}/\lambda, \dots, z^{N(k)}/\lambda$ having radius $1/\lambda$. Hence

$$\begin{aligned}\widetilde{G}_{s,\lambda}\, d\mu(y) &\le c_1\lambda^{n-s}\sum_{k=1}^{\infty}\sum_{j=1}^{N(k)}(k+1)^{\gamma}e^{-k}\int_{|x-y-z^{(j)}/\lambda|<1/\lambda} d\mu(x) \\ &\le c_1\sum_{k=1}^{\infty}\sum_{j=1}^{N(k)}(k+1)^{\gamma}e^{-k}M_{s,1/\lambda}\, d\mu(y+z^{(j)}/\lambda).\end{aligned}$$

Consequently

$$\|\widetilde{G}_{s,\lambda}\, d\mu\|_q \le c_1\sum_{k=1}^{\infty}N(k)(k+1)^{\gamma}e^{-k}\|M_{s,1/\lambda}\, d\mu\|_q.$$

This gives (3.22).

We can now give the

Proof of Theorem 3.2. Let $\delta = 1/\lambda$ and put

$$K_p = \sup_x(M_{2p,\delta}V^p)^{1/p}.$$

If $q = 2p > 2$, then Hölder's inequality gives

$$M_{1,\delta}[V^{1/2}u] \le M_{q,\delta}(V^{q/2})^{1/q}M_{0,\delta}(|u|^{q'})^{1/q'} \le K_p^{1/2}(M|u|^{q'})^{1/q'}.$$

Hence

$$\|M_{1,\delta}[V^{1/2}u]\|_2 \le K_p^{1/2}\|(M|u|^{q'})^{1/q'}\|_2.$$

Since $q' < 2$, this is bounded by

$$K_p^{1/2}\|M|u|^{q'}\|_{2/q'}^{1/q'} \le c'K_p^{1/2}\|\,|u|^{q'}\|_{2/q'}^{1/q'} = c'K_p^{1/2}\|u\|_2.$$

By Theorem 3.5, this implies

$$\|G_{1,\lambda}[V^{1/2}u]\|_2 \le c'C'_{1,2}K_p^{1/2}\|u\|_2.$$

This implies by duality

$$\|V^{1/2}G_{1,\lambda}\nu\|_2 \le c'C'_{1,2}K_p^{1/2}\|\nu\|_2$$

which is equivalent to

$$(Vu, u) \le c'^2C'^2_{1,2}\ K_p(\|\nabla u\|^2 + \lambda^2\|u\|^2).$$

Thus

$$C_\lambda(V) \le c'^2C'^2_{1,2}\ K_p$$

which is precisely (3.4). To prove (3.5), let $\phi(x)$ be a test function which equals 1 for $|x| < 1$ and 0 for $|x| > 2$. Put $\phi_\lambda(x) = \phi(\lambda(x-z))$, where $z \in \mathbf{R}^n$ is fixed. Then

$$\begin{aligned}(V\phi_\lambda, \phi_\lambda) &\le C_\lambda(V)(\|\nabla\phi_\lambda\|^2 + \lambda^2\|\phi_\lambda\|^2)\\ &= C_\lambda(V)\lambda^{2-n}(\|\nabla\phi\|^2 + \|\phi\|^2) = C\lambda^{2-n}C_\lambda(V).\end{aligned}$$

Hence

$$\lambda^{n-2}\int_{\lambda|x-z|<1} V(x)\,dx \le C\,C_\lambda(V)$$

and consequently

$$M_{2,1/\lambda}V(z) \le C\,C_\lambda(V).$$

Remark 3.6. The constant C_p in (3.4) can be estimated readily from the proofs of Theorems 3.1, 3.2 and 3.5.

4. Properties of $C_\lambda(V)$

In this section we shall derive some properties of the constant $C_\lambda(V)$.

Theorem 4.1. *$C_\lambda(V)$ is continuous in λ in the interval* $[0, \infty)$.

Proof. Suppose

$$C_\nu(V) \le A, \qquad \nu > \lambda.$$

Then $C_\lambda(V) \le A$. For we have

$$(Vu, u) \le A(\|\nabla u\|^2 + \nu^2\|u\|^2), \qquad u \in C_0^\infty.$$

Let $\nu \to \lambda$. Then

$$(Vu, u) \le A(\|\nabla u\|^2 + \lambda^2\|u\|^2), \qquad u \in C_0^\infty.$$

Thus $C_\lambda(V) \le A$. Next, suppose $\lambda > 0$ and

$$C_\nu(V) \ge A, \qquad \nu < \lambda.$$

Then $C_\lambda(V) \ge A$. For if $C_\lambda(V) \le A - \varepsilon$, we can find for each $\nu < \lambda$ a function $u_\nu \in C_0^\infty$ such that

$$\|\nabla u_\nu\|^2 + \nu^2 \|u_\nu\|^2 = 1 \tag{4.1}$$

and

$$C_\nu(V) - \varepsilon/2 \le (Vu_\nu, u_\nu) \le C_\lambda(V)(\|\nabla u_\nu\|^2 + \lambda^2\|u_\nu\|^2).$$

Thus

$$A - \varepsilon/2 \le C_\lambda(V)(1 + (\lambda^2 - \nu^2)\|u_\nu\|^2) \le C_\lambda(V)\lambda^2/\nu^2$$

in view of (4.1). Let $\nu \to \lambda$. We have

$$A - \varepsilon/2 \le C_\lambda(V) \le A - \varepsilon$$

providing a contradiction. Since $C_\lambda(V)$ is a decreasing function of λ, it must be continuous.

Theorem 4.2. *If $-\mu^2$ is the lowest point of the spectrum of $-\Delta - V$, then*

$$\mu^2 = \inf_{C_\lambda(V) \le 1} \lambda^2 = \sup_{C_\lambda(V) > 1} \lambda^2$$
$$= \inf_{C_\lambda(V) \le 1} \lambda^2 C_\lambda(V) = \sup_{C_\lambda(V) > 1} \lambda^2 C_\lambda(V).$$

If the set $C_\lambda(V) \le 1$ is empty, then $\mu = \infty$. If the set $C_\lambda(V) > 1$ is empty, then $\mu = 0$.

Proof. Let H be the operator (1.2). If $C_\lambda(V) \le 1$, then (1.1) implies

$$-C_\lambda(V)\lambda^2\|u\|^2 \le (Hu, u).$$

Thus

$$\mu^2 \le C_\lambda(V)\lambda^2 \le \lambda^2, \qquad C_\lambda(V) \le 1. \tag{4.2}$$

If $C_\lambda(V) > 1$, then for every $\varepsilon > 0$ there is a $u \in C_0^\infty$ such that

$$(Vu, u) \ge (C_\lambda(V) - \varepsilon)(\|\nabla u\|^2 + \lambda^2\|u\|^2).$$

Thus

$$(Hu, u) + \lambda^2(C_\lambda(V) - \varepsilon)\|u\|^2 \le (1 + \varepsilon - C_\lambda(V))\|\nabla u\|^2.$$

For ε sufficiently small, this is ≤ 0. Thus

$$-\mu^2 \le -\lambda^2(C_\lambda(V) - \varepsilon) \quad \text{or} \quad \mu^2 \ge \lambda^2(C_\lambda(V) - \varepsilon).$$

Letting $\varepsilon \to 0$, we have

$$\mu^2 \ge \lambda^2 C_\lambda(V) \ge \lambda^2, \qquad C_\lambda(V) > 1. \tag{4.3}$$

In particular we see from this that

$$C_\mu(V) \le 1.$$

If $\mu \neq 0$, we see by (4.2) that

$$C_\mu(V) = 1. \tag{4.4}$$

By (4.2),

$$\mu^2 \le \inf_{C_\lambda(V)\le 1} \lambda^2 C_\lambda(V) \le \inf_{C_\lambda(V)\le 1} \lambda^2. \tag{4.5}$$

But by (4.4) we see that equality holds. Similarly, by (4.2) we see that

$$\mu^2 \ge \sup_{C_\lambda(V)} \lambda^2 C_\lambda(V) \ge \sup_{C_\lambda(V)} \lambda^2. \tag{4.6}$$

But there cannot be a positive ε such that $\mu^2 \ge \varepsilon + \lambda^2$ holds for all λ satisfying $C_\lambda(V) > 1$. For that would imply the existence of a $\nu < \mu$ such that $C_\nu(V) \le 1$, contradicting (4.5). Thus, equality holds throughout (4.6) as well.

Corollary 4.3.

$$\mu^2 \le \sup_\lambda \lambda^2[2C_\lambda(V) - 1], \tag{4.7}$$

$$\mu^2 \ge \sup_\lambda \lambda^2[C_\lambda(V) - 1]. \tag{4.8}$$

Proof. If $C_\lambda(V) > 1$, then

$$\lambda^2 \le \lambda^2[2C_\lambda(V) - 1].$$

Thus $\sup \lambda^2$ over the set $C_\lambda(V) > 1$ is bounded by the right-hand side of (4.7). Similarly, if $C_\lambda(V) > 1$, then

$$\lambda^2 C_\lambda(V) \ge \lambda^2[C_\lambda(V) - 1]. \tag{4.9}$$

On the other hand, the right-hand side of (4.9) is negative if $C_\lambda(V) < 1$. Thus $\sup \lambda^2 C_\lambda(V)$ over the set $C_\lambda(V) > 1$ is $\ge$ the right-hand side of (4.8).

Now we turn to the

Proof of Corollary 3.3. By (4.7) and (3.4)

$$\begin{aligned} \mu^2 &\le \sup_\lambda \lambda^2[2C_p \sup_x (M_{2p,1/\lambda} V^p)^{1/p} - 1] \\ &= \sup_{x,\delta}[2C_p \delta^{-2}(M_{2p,\delta} V^p)^{1/p} - \delta^{-2}]. \end{aligned} \tag{4.10}$$

This equals the last expression in (3.6). For let L be the latter expression. Then

$$\left(\delta^{-n} \int_{|y-x|<\delta} V(y)^p \, dy\right)^{1/p} \le (L + \delta^{-2})/2C_p, \qquad \delta > 0.$$

This implies

$$(M_{2p,\delta} V^p)^{1/p} \le (\delta^2 L + 1)/2C_p.$$

If we substitute this into (4.10), we obtain

$$\mu^2 \le \sup_{x,\delta}[\delta^{-2}(\delta^2 L + 1) - \delta^{-2}] = L.$$

The same reasoning works in reverse. The second estimate in Corollary 3.3 is proved in the same way using inequality (3.5).

Corollary 3.4 is an immediate consequence of (3.4) taking $\lambda = 0$.

A function $V(x)$ is said to satisfy the A_∞ condition if there is $p > 1$ such that

$$\left(|Q|^{-1}\int_Q V(x)^p\,dx\right)^{1/p} \le L_p|Q|^{-1}\int_Q V(x)\,dx$$

holds for all cubes Q, where $|Q|$ is the volume of Q (cf. [8]). We have

Corollary 4.4. *If $V(x)$ satisfies the A_∞ condition, then*

(4.11) $$C_\lambda(V) \le N_p\|M_{2,1/\lambda}V\|_\infty.$$

Proof. From the definition we see that there is a constant L'_p such that

$$(M_{2p,\delta}V^p)^{1/p} \le L'_p M_{2,\delta}V.$$

We now merely apply Theorem 3.2.

5. Invariance of the essential spectrum

For a closed operator A on a Banach space we define the essential spectrum of A as

$$\sigma_e(A) = \bigcap_K \sigma(A+K)$$

where the intersection is taken over all compact operators K. We give sufficient conditions for H to have the same essential spectrum as $-\Delta$.

Theorem 5.1. *Assume that*

(a) $C_\lambda(V) \to 0$ *as* $\lambda \to \infty$.

(b) *For some* $\lambda_0 \ge 0$,

$$C_{\lambda_0}(V^R) \to 0 \quad \text{as } R \to \infty$$

where

$$V^R(x) = \begin{cases} 0, & |x| \le R, \\ V(x), & |x| > R. \end{cases}$$

Then

(5.1) $$\sigma_e(H) = \sigma_e(-\Delta) = [0,\infty).$$

Proof. By (a) and (1.1), for each $\varepsilon > 0$ there is a constant C_ε such that

$$(Vu, u) \le \varepsilon\|\nabla u\|^2 + C_\varepsilon\|u\|^2.$$

Moreover, if $\phi(x) \in C_0^\infty$ is the function used in the proof of Theorem 3.2, then

$$C_{\lambda_0}(V(1-\phi_R)) \to 0 \quad \text{as } R \to \infty$$

by (b). These two conditions are necessary and sufficient for $V^{1/2}$ to be compact from $H^{1,2}$ to L^2 (cf. [14, p. 172]). This in turn is sufficient for H to have a $1/2$ extension satisfying (5.1) (cf. [14, p. 149]).

Bibliography

1. M. Schechter, *Hamiltonians for singular potentials*, Indiana Univ. Math. J. **22** (1972), 483–503.
2. V. G. Mazya, *On the negative spectrum of multidimensional Schrodinger operators*, Dokl. Akad. Nauk SSSR 1962, 721–722.
3. D. R. Adams, *On the existence of capacitary strong type estimates in* $\mathbf{R}^n$, Ark. Mat. **14** (1976), 125–140.
4. C. L. Fefferman, *The uncertainty principle*, Bull. Amer. Math. Soc. **9** (1983), 129–206.
5. B. Muckenhoupt and R. L. Wheeden, *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc. **192** (1974), 251–275.
6. M. de Guzman, *Differentiation of integrals in* $\mathbf{R}^n$, Lecture Notes in Math., vol. 481, Springer-Verlag, Berlin and New York, 1975.
7. N. Aronszajn and K. T. Smith, *Theory of Bessel potentials*, Ann. Inst. Fourier (Grenoble) **11** (1961), 385–475.
8. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. **51** (1974), 241–250.
9. R. Kerman and E. Sawyer, *Weighted norm inequalities for potentials with applications to Schroedinger operators, Fourier transforms and Carleson measures*, Bull. Amer. Math. Soc. **12** (1985), 112–116.
10. ——, *The trace inequality and eigenvalue estimates for Schrodinger operators*, Ann. Inst. Fourier (Grenoble) **36** (1986), 207–228.
11. E. B. Fabes, C. E. Kenig and R. P. Serapioni, *The local regularity of solutions of degenerate elliptic equations*, Comm. Partial Differential Equations **7** (1982), 77–116.
12. S. Chanillo and R. L. Wheeden, L^p *estimates for fractional integrals and Sobolev inequalities with applications to Schrodinger operators*, Comm. Partial Differential Equations **10** (1985), 1077–1116.
13. E. T. Sawyer, *A weighted inequality and eigenvalue estimates for Schrodinger operators*, Indiana Univ. Math. J. **35** (1986), 1–28.
14. M. Schechter, *Spectra of partial differential operators*, North-Holland, 1971; rev. ed. 1986.
15. T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin and New York, 1966.
16. S. Y. A. Chang, J. M. Wilson and T. H. Wolff, *Some weighted inequalities concerning the Schrodinger operators*, Comment Math. Helv. **60** (1985), 217–246.

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