A SEPARABLE SPACE WITH NO REMOTE POINTS

ALAN DOW

ABSTRACT. In the model obtained by adding ω_2 side-by-side Sacks reals to a model of CH, there is a separable nonpseudocompact space with no remote points. To prove this it is also shown that in this model the countable box product of Cantor sets contains a subspace of size ω_2 such that every uncountable subset has density ω_1. Furthermore assuming the existence of a measurable cardinal κ with $2^\kappa = \kappa^+$, a space X is produced with no isolated points but with remote points in $\upsilon X - X$. It is also shown that a pseudocompact space does not have remote points.

0. INTRODUCTION

All spaces are completely regular Hausdorff. For a space X, βX is the Stone-Čech compactification of X and $\upsilon X \subset \beta X$ is the Hewitt-Nachbin real-compactification. A point $p \in \beta X - X$ is a *remote point* of X if p is not in the closure of any nowhere dense subset of X. A space X is pseudocompact if each locally finite family of open sets is finite (equivalently: if $\upsilon X = \beta X$ or if each real-valued continuous function is bounded).

In this article we answer questions of van Douwen's about remote points which are in a sense at opposite ends of the spectrum. The first is whether or not each separable nonpseudocompact space has remote points. In 1960, Fine and Gillman introduced remote points and proved, assuming CH, that each separable nonpseudocompact space has remote points. Van Douwen (and independently Chae and Smith) proved that CH was not needed if one assumed, in addition, that the space had a countable π-base (e.g. if it is metrizable). It is asked in [vDvM] if all nonpseudocompact separable spaces have remote points. Various partial results have been obtained; i.e. Martin's Axiom (MA) can replace CH, countable π-weight can be weakened to π-weight at most ω_1 and in each of these cases separable could be replaced by ccc (no uncountable pairwise disjoint collections of open sets) plus weight at most $\mathfrak{c}$. Also any product (nonpseudocompact) of metric spaces has remote points [DP]. It has also been shown that if CH is assumed, then separable can be weakened to the space

having only c real-valued continuous functions [KvMM]. Examples of nonpseudocompact spaces without remote points were produced [vDvM, KvMM and D3] but these were all far from being separable. In §5, we show that it is consistent (with ZFC) to have a separable nonpseudocompact space with no remote points.

In each of the above results we are assuming that the space in question is nonpseudocompact. In 1979, Terada showed (with an unnecessary assumption about measurable cardinals) that a pseudocompact space does not have remote points. In fact Terada showed that if the cellularity of X is less than the first measurable cardinal then no point of $\upsilon X - X$ is a remote point of X (this was improved by Blair to include those X whose closed screenable subspaces had size less than the first measurable cardinal). The question of the existence of remote points is really only interesting for those X with no isolated points since every point of $\beta D - D$ is remote for discrete D (see 6.5 below). In [vD2], van Douwen asks if there is a space with no isolated points having a remote point in $\upsilon X - X$. We show that this statement is equiconsistent with the existence of a measurable cardinal. We also substantially improve Terada's and Blair's results by finding a much wider class of spaces which do not have remote points in $\upsilon X - X$.

1. WHY STUDY REMOTE POINTS?

The study of remote points began, most probably, simply as an effort to better understand βX. The property of a remote point at first encounter is a bit mystifying (at least to the author) in that you have a point in the closure of a set and yet its neighborhood trace never really settles down on any "topologically small" set of points. It was then noticed that a remote point can also be thought of as a point in $E(\beta X)$ (see [W5]) and this led to their usage in the study of coabsolutes [G1,2,3, PoW and W2,3,4]. It was van Douwen who first discovered their role in the study of the nonhomogeneity of βX and $\beta X - X$ and van Mill applied the techniques of their construction to constructing other points in βX [vD1,2, vM1,2, VWa]. Frolík had proven that βX (for nonpseudocompact X) is never homogeneous but his proof does not give any insight as to why, in that it does not produce topologically interesting points. Remote points are such points, as are the better known P-points [R] and weak P-points [K2]. In [vM2], van Mill, relying on earlier work in [vM1, vD2, K2, Be, DvM and D1], proved, assuming MA, that every nonpseudocompact space X has a point in βX which is not the limit of any countable nowhere dense subset of βX (call such a point of weak remote point). MA was actually only required for the "small" spaces (ccc plus weight at most c), in particular the separable spaces. The example we produce is a separable space with no remote points but we do not know if it has weak remote points (see 2.4).

Besides having applications in topology, the study of remote points has been both a consumer and producer of combinatorial set theory and consistency results. This is not so surprising given that the translation of the notion of

a remote point to that of an ultrafilter on a Boolean algebra (2.1) looks very much like the definition of a P-point ultrafilter on $\mathscr{P}(\omega)/\mathrm{fin}$. As van Mill observes in [vM2], a technique developed for constructing remote points and weak P-points is to construct, say, n-linked remote collections (e.g. see 2.2 below), rather than a filter, and then piece these collections together to build a filter (2.3). However it is precisely these n-linked remote collections which have found an application in forcing (see [DTW, F1, 2, T] or 2.7). As an example of it being a producer of consistency results see §4 where we prove a result about the box product of Cantor sets.

So what is left to do? It will undoubledly turn out to be a very interesting solution if one ever determines whether or not it is consistent (with or without large cardinals) that every nonpseudocompact ccc space (or even countable space) has remote points. It is also still open whether or not every nonpseudocompact space has a weak remote point or an ω-far point [vD2]. Furthermore the applications to forcing should be explored where remote collection is generalized to that of a lynx or endowment (see [DTW, F1, 2, T] or 2.2 below), including finding posets possessing various lynxes [D5].

2. Preliminaries

Our reference for facts about βX and υX is [GJ]. Recall that $C(X)$ $(C^*(X))$ is the ring of all real-valued continuous (bounded) functions on X. A subset Z of X is a zero set if $Z = Z(f) = f^{-1}(0)$ for some $f \in C(X)$; a cozero set is the complement of a zero set. Let $\mathscr{Z}(X)$ be the lattice of all zero sets of X. The topological space βX is the space of $\mathscr{Z}$-ultrafilters on X and we identify X with the set of fixed ultrafilters. The topology is given by $\mathrm{cl}_{\beta X} Z = Z \cup \{p \in \beta X - X : Z \in p\}$ for each $Z \in \mathscr{Z}(X)$; in particular disjoint zero sets have disjoint closures. The space υX is $\{p \in \beta X : p$ is countably complete$\}$. For any $f \in C^*(X)$ $(C(X))$ there is a unique function $\beta f \in C^*(\beta X)$ $(\upsilon f \in C(\upsilon X))$ such that $f \subset \upsilon f \subset \beta f$. A pair of subsets of X is said to be completely separated if there is a function in $C(X)$ sending all members of one set to 0 and the other set to 1 (equivalently if they are contained in disjoint zero sets or if they have disjoint closures in βX).

A collection $\mathscr{F} \subset \mathscr{Z}(X)$ is called *remote* if for each nowhere dense set $D \subset X$ there is a $Z \in \mathscr{F}$ such that Z and D are completely separated. Hence a remote point in $\beta X - X$ is a remote free ultrafilter on $\mathscr{Z}(X)$. The *Souslin number* of X, $\hat{c}(X)$, is the least cardinal such that all families of pairwise disjoint open sets have smaller cardinality. If $\hat{c}(X) \leq \omega_1$ then X is ccc. For a family $\{X_\alpha : \alpha \in I\}$ of spaces, let $\sum_{\alpha \in I} X_\alpha$ denote the space $\bigcup\{\{\alpha\} \times X_\alpha : \alpha \in I\}$ where $\{\alpha\} \times X_\alpha$ is open and homeomorphic to X_α for each $\alpha \in I$. A space X is 0-dimensional if $CO(X)$, the collection of clopen sets, forms a basis for the topology. If B is any Boolean algebra then the Stone space of B, or the space of ultrafilters, $S(B)$, is a compact 0-dimensional space and $CO(S(B)) \cong B$ (where $b^* = \{\mathscr{U} \in S(B) : b \in \mathscr{U}\} \in CO(S(B))$ for $b \in B$). If $\{B_\alpha : \alpha \in I\}$ is

a collection of Boolean algebras then let $\sum_{\alpha \in I} B_\alpha$ denote the Boolean algebra $CO(\sum_{\alpha \in I} S(B_\alpha))$ or the corresponding algebra generated by $\bigcup\{\{\alpha\} \times B_\alpha : \alpha \in I\}$, and if P is a separative poset let $\sum_I P$ denote $\sum_{\alpha \in I} B_\alpha$ where each B_α is the unique complete Boolean algebra containing P densely. A set $A \subset P$, P a poset, is said to be predense if for each $p \in P$ there is an $a \in A$ compatible with p (equivalently the set $\{p \in P : \exists a \in A \ p \leq a\}$ is dense in P).

Definition 2.1. For any collection $\{B_\alpha : \alpha \in I\}$ of Boolean algebras, we call $\mathscr{F} \subset \sum_{\alpha \in I} B_\alpha$ *remote* if for each sequence $\{A_\alpha : \alpha \in I\}$ with each A_α a predense subset of B_α, there are $A'_\alpha \in [A_\alpha]^{<\omega}$ such that $\bigvee_{\alpha \in I} \bigvee A'_\alpha \in \mathscr{F}$.

Lemma 2.2. *For any collection* $\{B_\alpha : \alpha \in I\}$ *of Boolean algebras,* $X = \sum_{\alpha \in I} S(B_\alpha)$ *has a remote point iff* $\sum_{\alpha \in I} B_\alpha$ *has a remote filter.*

Proof. If $\mathscr{F} \subset \sum_{\alpha \in I} B_\alpha$ is remote as in 2.1, then the collection $\mathscr{F}^* = \{b^* : b \in \mathscr{F}\} \subset CO(X) \subset \mathscr{Z}(X)$ is remote. Hence if $\mathscr{F}$ is a filter then $\mathscr{F}^*$ is a filter base on $\mathscr{Z}(X)$. Now suppose $p \in \beta X - X$ is a remote filter. Let $\mathscr{F}^* = p \cap CO(X)$ and define $\mathscr{F} = \{b \in \sum_{\alpha \in I} B_\alpha : b^* \in \mathscr{F}^*\}$. Clearly $\mathscr{F}$ is a filter so let us check that it is remote. Let $\{A_\alpha : \alpha \in I\}$ be a sequence of predense sets as in 2.1. Let $D \subset X$ be defined as $\bigcup_{\alpha \in I}[S(B_\alpha) - \bigcup\{a^* : a \in A_\alpha\}]$. Since each A_α is predense in B_α, D is a nowhere dense subset of X. Choose $Z \in p$ with $Z \cap D = 0$ and note that for each $\alpha \in I$, $Z \cap S(B_\alpha)$ is a compact subset of $\bigcup\{a^* : a \in A_\alpha\}$. Therefore we may choose $A'_\alpha \in [A_\alpha]^{<\omega}$ for $\alpha \in I$ so that $Z \subset \bigcup_{\alpha \in I} \bigcup\{a^* : a \in A'_\alpha\} \in \mathscr{F}^*$.

There are interesting generalizations of the notion of remote filter. Indeed let us say that $\mathscr{F} \subset \mathscr{Z}(X)$ is a *weak remote* collection if it satisfies the definition of remote after "nowhere dense" is replaced by "separable nowhere dense". A collection $\mathscr{F} \subset \mathscr{P}(X)$ is *n-linked* for $n \in \omega$ if $\bigcap \mathscr{F}' \neq 0$ for any $\mathscr{F}' \in [\mathscr{F}]^{\leq n}$. Now we have the notions (weak) remote n-linked collection on X and weak remote filter on X. Similarly for a Boolean algebra B we define $\mathscr{F} \subset B$ as a (weak) remote n-linked collection if $\mathscr{F}^* = \{b^* : b \in \mathscr{F}\}$ is the corresponding thing on $S(B)$. A collection $\mathscr{F}$ of subsets of $\sum_{n \in \omega} X_n$ is called *nice* [vM] if $\{n \in \omega : F \cap X_n = 0\}$ is finite for each $F \in \mathscr{F}$. A generalization of remote that has its origins in forcing is that of a lynx [F2]. For a complete Boolean algebra B, $\mathscr{L} \subset B$ is a (κ, λ)-n-lynx if for each predense $A \subset B$ with $|A| < \kappa$ there is an $A' \in [A]^{<\lambda}$ with $\bigvee A' \in \mathscr{L}$ and $\mathscr{L}$ is n-linked. The notion of n-linked arose in the context of remote points (e.g. [vD2 and CS]) because as van Mill [vM2] points out it is easier to construct n-linked collections than filters and the following holds.

Theorem 2.3 [vM2]. *If, for* $n \in \omega$, X_n *has an* n-linked (weak) remote collection, then $\sum_{n \in \omega} X_n$ *has a nice (weak) remote filter.*

Remark 2.4. Van Mill asks if $X = \sum_{n \in \omega} X_n$ always has a nice weak remote filter (since if it does it then follows from [vM2] that βX has a weak remote point). It appears that the space in [D2,4] can be modified slightly to prove that

the answer is consistently no. However the really interesting question in this case is whether βX has a weak remote point; i.e. one would like to produce an example with no weak remote filter.

Let us now record some known results about remote collections.

Proposition 2.5. (a) *Each metric space has remote n-linked collections for any $n \in \omega$ [vD2, CS], but there is a separable space with no remote 2-linked collection [D2].*

(b) *MA implies $B = \sum_{n\in\omega} B_n$ has a nice remote filter providing B is ccc and $|B| = \mathfrak{c}$ [D1].*

(c) *The space $\omega \times U(\omega_2)$ ($U(\kappa)$ is the subspace of $\beta\kappa$ consisting of all uniform ultrafilters) has no remote points and there is a space of weight ω_2 with no remote points [vDvM, KvMM].*

(d) *Any nonpseudocompact ccc space with π-weight less than κ_0 (defined below) has remote points (providing the covering lemma holds) [D4].*

For any $p \in \omega^*$, $(^\omega\omega, <_p)$ is a linear order (modulo equivalence classes) where for $f, g \in {}^\omega\omega$, $f <_p g$ iff $\{n \in \omega: f(n) < g(n)\} \in p$. Let κ_0 be the least cardinal such that $(^\omega\omega, <_p)$ has a cofinal sequence of cardinality less than κ_0 for all $p \in \omega^*$. The covering lemma is not needed in (d) for π-weight ω_1 (and κ_0 is always greater than ω_1). We include (d) because it seems to indicate that in order to produce a model by forcing in which there is a separable space with no remote points we must use an $^\omega\omega$-bounding poset (see 3.2 below).

We end this section by proving a result (2.7) which show how lynxes can be used in forcing arguments which has as a consequence that certain spaces have no remote points (in fact $\omega \times U(\omega_1)$, $\omega \times U(\omega_2)$ and even a space with a dense metrizable subspace). We begin with a trivial observation.

Lemma 2.6. *If B is a Boolean algebra such that $\sum_\omega B$ has remote points then B has an (∞, ω_1)-2-lynx (where ∞ can be replaced by $|B|^+ \cdot \omega$).*

We say that a poset P has a (κ, λ)-2-lynx below $p \in P$ if the Boolean algebra generated by $\{q \in P: q \leq p\}$ has a (κ, λ)-2-lynx.

Proposition 2.7. *Let P be a poset which has an (∞, ω_1)-2-lynx below p for each $p \in P$ and let X be a space with a subset D consisting of P-points in X. If, after forcing with P, the points of D can be surrounded by disjoint neighborhoods (i.e. separated) then D is separated in the ground model. (A point is a P-point if the intersection of countably many neighborhoods is still a neighborhood.)*

Proof. For $d \in D$, let $\{B(d, \alpha): \alpha < \lambda\}$ be a neighborhood base for d. Let $\dot{g}$ be a forcing term and let $p \in P$ be chosen so that $p \Vdash$ "for each $d \in \check{D}$, $\dot{g}(d) \in \check{\lambda}$ and $\check{B}(d, \dot{g}(d)) \cap \check{B}(e, \dot{g}(e)) = 0$ for $d \neq e \in \check{D}$". Fix $\mathscr{L} \subset [\{q \in P: q < p\}]^{\leq\omega}$ such that $\{\bigvee A: A \in \mathscr{L}\}$ generates an (∞, ω_1)-2-lynx below p (i.e. for each A predense below p there is an $A' \in [A]^{\leq\omega}$ such that $A' \in \mathscr{L}$ and for $A_0, A_1 \in \mathscr{L}$ there are $a_0 \in A_0$ and $a_1 \in A_1$ such that a_0 and

a_1 are compatible). Now, for each $d \in D$, the set $A_d = \{q < p : \exists \alpha < \lambda \ q \Vdash$ "$\dot{g}(d) = \alpha$"$\}$ is predense below q, hence there is a countable $A'_d \in \mathscr{L} \cap [A_d]^{\leq \omega}$. For $d \in D$, let $f(d) \in \lambda$ be chosen so that $q \Vdash$ "$\check{B}(d, f(d)) \subset \check{B}(d, \dot{g}(d))$" for all $q \in A'_d$ (since d is a P-point). Now $B(d, f(d)) \cap B(e, f(e)) = 0$ for all $d \neq e \in D$ since we may choose $q \in A'_d$, $q' \in A'_e$ and an $r < q$, q'. Hence $r \Vdash$ "$\check{B}(d, f(d)) \cap \check{B}(e, f(e)) \subset \check{B}(d, \dot{g}(d)) \cap \check{B}(e, \dot{g}(e)) \neq 0$".

Corollary 2.8. *Let X be any one of the spaces $U(\omega_1), U(\omega_2)$ or $(D(\kappa) + 1)^\omega$ (where $D(\kappa) + 1$ is the one-point compactification of the discrete space $\kappa \geq \omega_2$). Then $\omega \times X$ has no remote points.*

Proof. Clearly $\omega \times X$ has remote points iff $\sum_{n \in \omega} CO(X)$ has a remote filter. Hence to prove the result, it suffices by 2.6 and 2.7 to prove that $P = CO(X) - \{0\}$ does not satisfy the conclusion of 2.7. Now, it is known that forcing with P introduces a function from ω onto ω_2. Therefore, by some simple topology it suffices to find a P-space with a closed discrete set of size ω_2 which cannot be separated. Finally $\omega_1 \cup \mathscr{A}$ is such a space where $\mathscr{A}$ is a maximal family of uncountable subsets of ω_1 with pairwise countable intersection and a neighborhood base for $A \in \mathscr{A}$ is $\{\{A\} \cup A - \alpha : \alpha < \omega_1\}$ and ω_1 is open and discrete.

3. THE MODEL

In this section we introduce the forcing and the model that we will use in §5 to produce our space. The forcing is side-by-side Sacks forcing over a model of CH, which was introduced by Baumgartner [B1,2]. For the basics of forcing see [K1 or B3] and [ChKe] for model theory. We shall adopt the notation in [B1,2] and in this section list the basic results about the forcing found in [B2] or 6.4 of [B1]. Although [B2] has not yet been published the techniques involved are very similar to those found in [BL or S] concerning ω-proper, $^\omega\omega$-bounding iterations, hence we do not see any need to include the proofs of 3.1 and 3.2.

Let $Sq = \{^n2 : n < \omega\}$. A nonempty $p \subset Sq$ is called a *perfect tree* iff

(1) $\forall s \in p \ \forall n < \omega \ s|n \in p$ and

(2) $\forall s \in p \ \exists t, u \in p \ s \subset t, s \subset u$ and t, u are incomparable.

PS is the poset of all perfect trees ordered by $p \leq q$ iff $p \subset q$. If G is PS-generic over V (the ground model) then $f_G = \bigcup\{s \in Sq : \forall p \in G \ s \subset p\} \in {}^\omega2$ and $f_G \notin V$. We call f_G the Sacks real added by PS. Let $PS(\omega_2)$ be the set of functions p with domain a countable subset of ω_2 and range PS. Define $p \leq q$ for $p, q \in PS(\omega_2)$ if domain$(q) \subset$ domain(p) and, for all $\alpha \in$ domain(q), $p(\alpha) \leq q(\alpha)$. We shall let x_α be the Sacks real added (in the above sense) by the αth copy of PS; i.e. $x_\alpha = \bigcup\{s \in Sq : \forall p \in G \ \alpha \in$ domain(p) implies $s \subset p(\alpha)\}$.

Theorem 3.1 (CH). *$PS(\omega_2)$ has the ω_2-chain condition.*

Theorem 3.2. *If $p \in PS(\omega_2)$ and $p \Vdash$ "$\dot{f} : \omega \to V$", then there are $q \leq p$ and a sequence of finite sets $\{F_n : n < \omega\}$ such that $q \Vdash$ "$\forall n \ \dot{f}(n) \in \check{F}_n$".*

Both 3.1 and 3.2 are proven in [B1,2] and the proof of 3.2 is quite lengthy. A forcing poset P is said to be ${}^{\omega}\omega$-bounding if for each $f \in {}^{\omega}\omega \cap V[G]$ there is a $g \in {}^{\omega}\omega \cap V$ such that $f(n) < g(n)$ for all $n < \omega$ (where G is any P-generic filter over V).

Corollary 3.3 (CH). $PS(\omega_2)$ *preserves all cardinalities, cofinalities and is* ${}^{\omega}\omega$-*bounding. Furthermore, if* G *is* $PS(\omega_2)$-*generic over* V, *then* $V[G]$ *satisfies* $\mathbf{c} = \omega_2^V$.

4. A RESULT ON BOX PRODUCTS

We prove that in the model from §3 the countable box product of Cantor sets has a property we shall need for the example produced in §5. Let us let $\square C$ denote the *box topology* on ${}^{\omega}C$ (see [vD3, Ru or Wi]). For any $s \in Sq$, let $[s] = \{x \in C = {}^{\omega}2 : s \subset x\}$. Hence for any $f \in {}^{\omega}Sq$, $[f] = \prod_{n \in \omega}[f(n)]$ is a clopen subset of $\square C$. For any $y \in {}^{\omega}C$ and $f \in {}^{\omega}\omega$, let $[y^f] = [f']$ where $f' \in {}^{\omega}Sq$ is defined by $f'(n) = y \upharpoonright f(n)$.

Let V be a model satisfying CH and let G be $PS(\omega_2)$-generic over V. We shall prove that the following statement holds in $V[G]$:

(∗)   "there is a subspace Y of $\square C$ such that $|Y| = \mathbf{c}$ and for all $Z \subset Y$, the density of Z and the Lindelöf degree of Z are both equal $\min\{|Z|, \omega_1\}$ ".

Recall that the density of a space is the minimum cardinality of a dense subset and the Lindelöf degree is the minimum cardinal κ such that every open cover has a subcover of size at most κ.

It is trivial that (∗) follows from CH since the subspace of $\square C$ consisting of constant functions is closed and discrete. Furthermore if (∗) holds then $\mathbf{d} = \omega_1$ (recall that $\mathbf{d}$ is the minimum cardinality of a dominating family in ${}^{\omega}\omega$). To see this note first of all that if Y exhibits that (∗) holds and $y \in Y$ then $\{x \in Y : \{n : x(n) \neq y(n)\}$ is finite$\}$ is a separable hence countable subset of Y. Next we observe that if $Y \subset \square C$, $|Y| < \mathbf{d}$ and $x \in \square C - Y$ is such that for each $y \in Y$ $\{n : y(n) \neq x(n)\}$ is infinite then x is not in the closure of Y. Indeed, we may choose $h \in {}^{\omega}\omega$ so that for each $y \in Y$ $\{n : x(n)|_{h(n)} \neq y(n)|_{h(n)}\}$ is infinite and therefore $[x^f]$ is disjoint from Y. Now $d = \omega_1$ follows from (∗) since a dense subset of Y must have cardinality $\mathbf{d}$. However $\mathbf{d} = \omega_1$ does not imply (∗) (see 4.4). For $\lambda \in \lim(\omega_2)$, define $y_\lambda \in \square C$ by $y_\lambda(n) = x_{\lambda+n}$ (in $V[G]$) and let $Y = \{y_\lambda : \lambda \in \lim(\omega_2)\}$. For the rest of this section we must work in V.

Theorem 4.1. *Let* $\dot{Z}$ *be a* $PS(\omega_2)$-*term such that* $1 \Vdash$ "$\dot{Z} \subset \dot{Y} = \{\dot{y}_\lambda : \lambda \in \lim(\omega_2)\}$ *and* $|\dot{Z}| = \omega_2$ ". *Then there is a* $\alpha \in \omega_2$ *such that, for all* $\delta \in \omega_2 - \alpha$, $1 \Vdash$ "$\dot{Z} \cap \{\dot{y}_\lambda : \lambda \in \lim(\delta)\}$ *is dense and not open in* $\dot{Z}$ ".

Proof. Let $\alpha \in \omega_2$ with $\mathrm{cf}(\alpha) = \omega_1$ be such that there is an elementary submodel M of $H(\omega_3)$ (see [K1]) such that $|M| = \omega_1$, $\{PS(\omega_2), \dot{Z}, C\}$ and

$PS(\omega)$ are all subsets of M, and $M \cap \omega_2 = \alpha$. Let M be such a model, $\alpha = M \cap \omega_2$, and note that $PS(\alpha) \subset M$. Suppose $\lambda \in \lim(\omega_2)$ and $p \in PS(\omega_2)$ are such that $p \Vdash$ "$\dot{y}_\lambda \in \dot{Z}$ and $\dot{y}_\lambda$ is not a limit point of $\dot{Z} \cap \{\dot{y}_\beta : \beta \in \lim(\alpha)\}$". Since $PS(\omega_2)$ is ${}^\omega \omega$-bounding, there is an $f \in {}^\omega \omega$ (in V) and a $q \leq p$ such that $q \Vdash$ "$[\dot{y}_\lambda^f] \cap \dot{Z} \cap \{\dot{y}_\beta : \beta \in \lim(\alpha)\} = 0$". Let $q_0 = q \cap M$ and note that q_0 has the property "$\exists q \leq q_0 \; \exists \lambda \in \lim(\omega_2) \; q \Vdash$ "$\dot{y}_\lambda \in \dot{Z}$ and $\dot{y}_\lambda \in [f']$"" where we may assume $f' \in {}^\omega Sq$ and $q \Vdash$ "$[\dot{y}_\lambda^f] = [f']$" since this only requires that q "decides" $x_{\lambda+n} \upharpoonright f(n)$ for $n < \omega$. Therefore, by elementarity and the fact that $\{q_0, f'\} \subset M$, there is a $q_1 \in M$ with $q_1 \leq q_0$ and a $\beta \in M$ with $\beta \in \lim(\alpha)$ such that $q_1 \Vdash$ "$\dot{y}_\beta \in \dot{Z}$ and $\dot{U}_\beta \in [f']$". However, since $PS(\omega_2)$ is just product forcing, q_1 is compatible with q and $q_1 \wedge q \Vdash$ "$[\dot{y}_\lambda^f] = [f']$ and $[f'] \cap \dot{Z} \cap \{\dot{y}_\beta : \beta \in \lim(\alpha)\} \neq 0$", which contradicts our assumptions on p and λ. Therefore $1 \Vdash$ "$\dot{Z} \cap \{\dot{y}_\beta : \beta \in \lim(\alpha)\}$ is dense in $\dot{Z}$".

Now we show that $1 \Vdash$ "$\dot{Z} - \{\dot{y}_\beta : \beta \in \lim(\delta)\}$ is not closed in $\dot{Z}$" for any $\delta \in \omega_2 - \alpha$. Let $J = \{\lambda \in \lim(\omega_2) - \delta : \exists q \in PS(\omega_2) \; q \Vdash$ "$\dot{y}_\lambda \in \dot{Z}$"$\}$ and for each $\lambda \in J$ fix a q_λ exhibiting that $\lambda \in J$. Since CH holds in V, there are $R \in [\omega_2]^\omega$, $r_0 \in PS(\omega)$, $r_1 \in PS(R)$ and $J_0 \in [J]^{\omega_2}$ such that R is the root of the Δ-system $\{\mathrm{dom}(q_\lambda) : \lambda \in J_0\}$ and for each $\lambda \in J_0 \; q_\lambda \cap PS(R) = r_1$ and $q_\lambda \cap PS([\lambda, \lambda+\omega))$ is isomorphic to r_0. Let $q_0 = r_1 \cap M$. As above, using elementarity, we may choose $q_1 \in PS(\omega_2) \cap M$, $q_1 \leq q_0$ and $\beta \in M$ such that $q_1 \Vdash$ "$\dot{y}_\beta \in \dot{Z}$" and $q_1 \cap PS([\beta, \beta+\omega))$ is isomorphic to r_0.

Now suppose $f \in {}^\omega \omega \cap V$ and $q_2 \in PS(\omega_2)$ are such that $q_2 < q_1 \wedge r_1$ and $q_2 \Vdash$ "$[\dot{y}_\beta^f] \cap \dot{Z} - \{\dot{y}_\gamma : \gamma \in \lim(\delta)\} = 0$". Without loss of generality we may also assume that there is an $f' \in {}^\omega Sq \cap V$ such that $q_2 \Vdash$ "$[\dot{y}_\beta^f] = [f']$". Choose $\lambda \in J_0$ such that $\mathrm{dom}(q_\lambda) \cap \mathrm{dom}(q_2) \subset R$. Since both $q_1 \cap PS([\beta, \beta+\omega))$ and $q_\lambda \cap PS([\lambda, \lambda+\omega))$ are isomorphic to r_0 we may choose $q_3 \leq q_\lambda \wedge q_2$ so that $q_3 \Vdash$ "$[\dot{y}_\lambda^f] = [f']$". In which case $q_3 \Vdash y_\lambda \in$ "$[\dot{y}_\beta^f] \cap \dot{Z}$".

Corollary 4.2. *In $V[G]$, the space $Y = \{y_\lambda : \lambda \in \lim(\omega_2)\}$ exhibits that $(*)$ holds.*

Proof. First observe that countable subsets of Y are closed in Y since we are using the box topology and distinct elements of Y differ on every coordinate. Therefore it suffices to show that Y has hereditary density and Lindelöf degree $\leq \omega_1$. Now for density this follows immediately from 4.1. Also if there were a subspace Z with Lindelöf degree ω_2, then there would be such a Z which had a cover by open subsets with cardinality ω_1. In which case we should have that $Z \cap \{y_\lambda : \lambda \in \lim(\alpha)\}$ is open in Z for each $\alpha \in A'$ for some cub A'. Therefore by 4.1 there is no such Z.

Corollary 4.3. *Let $g \in {}^\omega \omega \cap V$ be a 1-1 function and redefine $\dot{y}_\lambda$ so that $\dot{y}_\lambda(n) = \dot{x}_{\lambda+g(n)}$. For any $h \in {}^\omega Sq \cap V \; 1 \Vdash$ "$\dot{Y} \cap [h]$ has cardinality ω_2 and for any $Z \subseteq \dot{Y}$ with $|Z| = \omega_2$, $\overline{Z} \cap \{\dot{y}_\lambda : \lambda \in \lim(\omega_1)\} \neq 0$".*

Proof. The redefinition of $\dot{Y}$ makes no difference in the proof of 4.1 since g is in V. The proof that $1 \Vdash "|\dot{Y} \cap [h]| = \omega_2"$ for $h \in {}^{\omega}Sq \cap V$ is trivial. For the final fact we observe that the role of M (hence α) besides being uncountable was to find a β so that $q_1 \Vdash "\dot{y}_{\beta} \in \dot{Z}"$. However in this case with $\{q_{\lambda} : \lambda \in J_0\}$, R, r_0 and r_1 as in 4.1 (with the new $\dot{Y}$) we may choose any $\beta \in \lim(\omega_1)$ so that $[\beta, \beta + \omega) \cap R = 0$ and find $q_1 < r_1$ with $q_1 \cap PS([\beta, \beta + \omega))$ isomorphic to r_0. Then proceed as in 4.1.

Remark 4.4. It can be shown that $(*)$ does not hold if one starts with a model of CH and adds random reals or uses any proper iteration of the usual kind of reals (e.g. random, Cohen, Sacks, Laver, Mathias, etc.) (see [S]). Let us sketch a proof for the case when Sacks reals are iterated. Assume that Y exhibits that $(*)$ holds and let D be a size ω_1 dense subspace of Y. In each of the above forcings we may assume that D is in the ground model, V. For each $y \in Y$ the set of members of Y which equal d on all but finitely many coordinates is separable hence countable. Therefore for any inner model of CH the set of members of Y which agree with some member of this model on all but finitely many coordinates has size at most ω_1. Furthermore, since $\mathbf{d} = \omega_1$, we can assume that the members of ${}^{\omega}\omega$ in the ground model are dominating. Let $y \in Y$ be chosen so that y disagrees with any member of the ground model on infinitely many coordinates. Let α be the minimum ordinal such that there is a P_{α}-name for y and we may assume that we are working in a model after adding some more reals to V. Let M be a countable elementary submodel with y etc. in M and let $f \in {}^{\omega}\omega \cap V$ dominate ${}^{\omega}\omega \cap M$. Recursively build a tree of conditions $\{p_s : s \in {}^{\omega}2\} \subset P_{\alpha} \cap M$ so that p_s decides $y(|s|)|_{f(n)}$ and, if possible, $p_{s^\frown 0}$ and $p_{s^\frown 1}$ decide differently. Also if p_s can be chosen so that for some k and any $n > |s|$ $y(n)|_{f(n)}$ can be decided without strengthening $p_s|_{p_{\beta_k}}$ then choose it so for the next possible k (where $\{\beta_n : n \in \omega\}$ is cofinal in α). Now if for some p_s there is no k as above then by genericity of $p_s|_{p_{\beta_k}}$ $[y^f]$ will not be in V. Furthermore in case such a k can always be found then for each $h \in {}^{\omega}2$ the meet of $\{p_s : s \subset h\}$ will be a condition. In this case we can assume for some h the corresponding condition will also force that $[y^f]$ is not in V (i.e. we are working in a model in which there are no perfect sets consisting entirely of reals from V). Therefore in either case $[y^f]$ is a neighborhood of y which misses D.

5. THE SPACE

In this section we shall show that in the model produced in §3 there is a separable nonpseudocompact space with no remote points. In fact we shall produce a poset P which is σ-centered and such that $\sum_{\omega} P$ does not have a remote filter as in 2.1.

Definition 5.1. For any set $X \subset C = {}^{\omega}2$, define $\mathscr{A}(X) = \{a \subset Sq \cup X : a$ is a finite nonmaximal antichain in the binary tree $^{<\omega+1}2\}$. $\mathscr{A}(X)$ is ordered by reverse inclusion.

We shall refer to the ground model (before adding the Sacks reals) by V and use $V[G]$ to denote the extension. Fix, in V, an indexing $\{a_m : m < \omega\}$ of $\mathscr{A}(0)$. For any $a \in \mathscr{A}(0)$, let $[a] = \bigcup\{[s] : s \in a\}$ and note that $[a]$ is a clopen subset of C. For any $X \subset C$ and $a \in \mathscr{A}(X)$ let $[a] = [a - X]$. Observe that for any $x \in X$ and $a \in \mathscr{A}(X)$, $a \cup \{x\} \in \mathscr{A}(X)$ iff $x \notin [a]$.

Proposition 5.2. *For any* $X \subset C$, $\mathscr{A}(X)$ *is σ-centered.*

Proof. Indeed by the above remark, for each $m \in \omega$, the collection $\{a \in \mathscr{A}(X) : a - X = a_m\}$ is centered.

The posets $\mathscr{A}(X)$ for various $X \subset C$ are also investigated in [D5] and the following is proved there.

Proposition 5.3. (a) *For any* $n \in \omega$, $\mathscr{A}(C)$ *has a remote n-linked collection.*

(b) *If* $X \subset C$ *is such that* $|X| > \omega$, $x + q \in X$ *for each* $x \in X$ *and dyadic rational* $q \in C$, *and* X *contains no perfect subset then* $\mathscr{A}(X)$ *has no remote 2-linked collection.*

Although we do not require 5.3(a) here, it will be useful to sketch the ideas in the proof of 5.3(b) since a more general version is required for our main result.

The basic idea of the property of X in (b) is that any uncountable subset of X must have a limit point in $C - X$. The additional restriction (that X is closed under translates by rationals) is a device which allows us to show that any uncountably many specially chosen n-tuples in X^n will have limit points in $(C - X)^n$. Finally Proposition 4.3 is designed to guarantee this same phenomenon will occur with countable sequences in $\square C$.

Indeed suppose that $\mathscr{L}$ is a remote collection on $\mathscr{A}(X)$ (i.e. $\mathscr{L} \subset [\mathscr{A}(X)]^{<\omega}$). Let us first suppose that $\mathscr{L} \cap [\mathscr{A}(X)]^1$ is uncountable. Then $\{x \in X : \exists a \in \mathscr{A}(X)$ with $x \in a$, $a \in \mathscr{L}\}$ is uncountable, hence there is some $y \in C - X$ which is a limit point of this set. Also since $y \notin X$ the collection $\{\{y \restriction n\} : 0 < n < \omega\}$ is predense in $\mathscr{A}(X)$. It follows without loss of generality that, for some $n < \omega$, the collection $\{\{y \restriction m\} : 0 < m < n\} \in \mathscr{L}$. Now since y is a limit point of the above set, there is an $x \in [y \restriction n]$ and an $\{a\} \in \mathscr{L}$ with $x \in a$. Therefore $\mathscr{L}$ is not 2-linked since $\{y \restriction m\} \cup a \notin \mathscr{A}(X)$ for any $m < n$. More generally, we can find an $n < \omega$ so that there are ω_1 distinct sequences $\langle x_0^\alpha, \ldots, x_{n-1}^\alpha \rangle \in X^n$ such that $\{a_m \cup \{x_m^\alpha\} : m < n\} \in \mathscr{L}$ for $\alpha < \omega_1$. We then consider $\langle y_0, \ldots, y_{n-1} \rangle \in (C - X)^n$ such that $\langle y_0, \ldots, y_{n-1} \rangle$ is a limit point of $\{\langle x_0^\alpha, \ldots, x_{n-1}^\alpha \rangle : \alpha < \omega_1\}$. In a similar fashion we show that the predense set $\{a \in \mathscr{A}(X) : \{y_0, \ldots, y_{n-1}\} \subset [a]\}$ cannot have a finite subset in $\mathscr{L}$ making $\mathscr{L}$ 2-linked.

Now let us prove the main theorem.

Theorem 5.4. *For* $\alpha \in \omega_2$, *let* $x_\alpha \in C$ *be as in* §3 *and let* $X = \{x_\alpha : \omega_1 \le \alpha < \omega_2\}$. *Then* $\sum_\omega \mathscr{A}(X)$ *has no remote filters (in fact, no remote 2-linked collection).*

Proof. Let g be any injection (in V) of $\omega \times \omega$ into ω. Fix $\{f_\alpha : \alpha < \omega_1\} \subset {}^\omega \omega \cap V$ such that for all $f \in {}^\omega \omega \cap V[G]$ there is an $\alpha < \omega_1$ with $f(n) < f_\alpha(n)$ for all n (by 3.3). In $V[G]$, let $J = \{\lambda \in \omega_2 - \omega_1 : a_m \cup \{x_{\lambda+g(n,m)}\} \in \mathscr{A}(X)$ for each $n, m \in \omega\}$. We claim that $|J| = \omega_2$. Indeed, for each $n, m \in \omega$, we may choose $s_{g(n,\omega)} \in Sq$ such that $a_m \cup \{s_{g(n,m)}\} \in \mathscr{A}(0)$ and the function taking (n, m) to $s_{g(n,m)}$ for $n, m \in \omega$ is in V. Clearly $J \supset \{\lambda \in \omega_2 - \omega_1 : x_{\lambda+g(n,m)} \supset s_{g(n,m)}$ for each $n, m \in \omega\}$, hence $|J| = \omega_2$ by 4.3.

For each $\lambda \in J$ and $n, m \in \omega$, let $A(\lambda, n) = \{a_m \cup \{x_{\lambda+g(n,m)}\} : m \in \omega\}$. Clearly $A(\lambda, n)$ is predense in $\mathscr{A}(X)$, hence $\bigcup\{\{n\} \times A(\lambda, n) : n < \omega\}$ is predense in $\sum_\omega \mathscr{A}(X)$. Suppose $\mathscr{F}$ is a remote collection on $\sum_\omega \mathscr{A}(X)$. Then for each $\lambda \in J$, there is an $\alpha < \omega_1$ such that $\bigcup\{\{n\} \times \{a_m \cup \{x_{\lambda+g(n,m)}\} : m < f_\alpha(n)\} \in \mathscr{F}$ (we may assume $\mathscr{F}$ is 'closed upwards'). Therefore there are an $\alpha < \omega_1$ and a $J_0 \subset J$ with $|J_0| = \omega_2$ such that for all $\lambda \in J_0$ $\bigcup\{\{n\} \times \{a_m \cup \{x_{\lambda+g(n,m)}\} : m < f_\alpha(n)\} : n < \omega\} \in \mathscr{F}$. For each $\lambda \in \lim(\omega_2)$, define $y_\lambda \in \prod_{n \in \omega} C^{f_\alpha(n)}$ by $y_\lambda(n) = \langle x_{\lambda+g(n,0)}, \ldots, x_{\lambda+g(n,f_\alpha(n)-1)}\rangle$. Let $Z = \{y_\lambda : \lambda \in J_0\}$ and consider Z as a subspace of $\prod_{n \in \omega} C^{f_\alpha(n)}$ with the box topology, which is clearly homeomorphic to $\square C$. By 4.3, there is a $\delta \in \lim(\omega_1)$, such that y_δ is a limit point of Z (in the box topology). For each $n \in \omega$, define $A_n \subset \mathscr{A}(X)$ by

$$A_n = \{a \in \mathscr{A}(X) : [a] \supset \{x_{\delta+g(n,m)} : m < f_\alpha(n)\}\}.$$

Let us check that A_n is predense. Indeed suppose $a' \in \mathscr{A}(X)$ and $m < f_\alpha(n)$ is the minimum such that $x_{\delta+g(n,m)} \notin [a']$. Since $x_{\delta+g(n,m)} \notin X$ there is a $k < \omega$ such that $[x_{\delta+g(n,m)} \upharpoonright k] \cap ([a'] \cup (a' \cap X)) = 0$. It follows that $a' \cup \{x_{\delta+g(n,m)} \upharpoonright k\} \in \mathscr{A}(X)$. Hence, by induction, A_n is predense.

Now, suppose $A'_n \in [A_n]^{<\omega}$ for $n \in \omega$ are such that $\bigcup\{\{n\} \times A'_n : n < \omega\} \in \mathscr{F}$. For each $n < \omega$, let $U_n = \bigcap\{[a] : a \in A'_n\}$ and consider the box neighborhood $W = \prod_{n \in \omega} U_n^{f_\alpha(n)}$ of y_δ. By assumption on y_δ there is a $\lambda \in J_0$ such that $y_\lambda \in W$. Finally, for each $n \in \omega$, $a \in A'_n$, and $m < f_\alpha(n)$, $a \cup \{x_{\lambda+g(n,m)}\} \notin \mathscr{A}(X)$ (since $x_{\lambda+g(n,m)} \in [a]$) hence $\mathscr{F}$ is not 2-linked.

6. NECESSARY CONDITIONS FOR REMOTE POINTS IN υX

As mentioned in the introduction Terada has shown that if X has remote points in υX then the cellularity of X must be at least as large as the first measurable cardinal, μ_0. This was improved by Blair who proved that there must also be a closed discrete set of cardinality μ_0 which can be separated by disjoint open sets. In this section we shall improve these results to obtain a much more complete picture of which spaces X do not have remote points

in vX. For instance we show that if X has remote points in vX then X is not pseudo-μ_0-compact (recall that X is pseudo-λ-compact if each locally finite family of open sets has cardinality less than λ). An immediate corollary is that a pseudocompact space does not have remote points (since $vX = \beta X$) (Terada showed this for X with cellularity less than μ_0).

For the remainder of this section, let X be a space and let $p \in vX - X$ be a remote point of X. We shall derive some properties of X in relation to the ultrafilter p.

Lemma 6.1. *For each* $Z \in p$, *$p \in \mathrm{int}_{\beta X}\, \mathrm{cl}_{\beta X}\, Z$.*

Proof. The set $D = Z \cap \mathrm{cl}_X(X - Z)$ is a nowhere dense subset of X, hence $p \notin \mathrm{cl}_{\beta X} D$. Choose $h \in C^*(\beta X)$ such that $h(p) = 0$ and $D \subset \mathrm{int}_{\beta X} h^{-1}(1)$. Define $h_0 \in C^*(X)$ as follows: $h_0(x) = h(x)$ for $x \in Z$ and $h_0(x) = 1$ for $x \in X - Z$. Since h is identically 1 on a neighborhood of D, h_0 is continuous. Now clearly $\beta h_0^{-1}((-\infty, 1/2))$ is a βX-neighborhood of p entirely contained in $\mathrm{cl}_{\beta X} Z$.

Recall that p is a P_κ-point (P_{ω_1}-point = P-point) of βX if $p \in \mathrm{int} \cap \mathcal{U}$ for any family $\mathcal{U}$ of fewer than κ many neighborhoods of p.

Corollary 6.2. *p is a P-point of βX.*

Proof. Since $p \in vX - X$, as a zero-set ultrafilter, p is ω_1-complete. Therefore by 6.1, p is a P-point.

The next result is really the main result of this section and it shows that, as far as the point p is concerned, the space X is really just a free union of an infinite collection of cozero sets.

Lemma 6.3. *For any maximal pairwise disjoint family $\mathcal{A}$ of cozero subsets of X such that $p \notin \mathrm{cl}_{\beta X} A$ for $A \in \mathcal{A}$, there is a zero set $Z \in p$ such that the family $\{A \cap Z : A \in \mathcal{A}\}$ is locally finite, $Z = \bigcup \{A \cap Z : A \in \mathcal{A}\}$ and $A \cap Z \in \mathcal{Z}(X)$ for each $A \in \mathcal{A}$.*

Proof. Since $p \notin \mathrm{cl}_{\beta X}(X - \bigcup \mathcal{A})$ (by maximality of $\mathcal{A}$), we may choose $f : \beta X \to [0, 1]$ such that $p \in \mathrm{int}_{\beta X} f^{-1}(0)$ and $X - \bigcup \mathcal{A} \subset \mathrm{int}_{\beta X} f^{-1}(1)$. Let $Z = f^{-1}(0) \cap X$. For $A \in \mathcal{A}$, choose $g_A : X \to [0, 1]$ such that $A = X - g_A^{-1}(0)$ and define h_A to be identically 0 on $X - A$ and equal to $g_A/(g_A + f \restriction A)$ on A. Again, $h_A \in C^*(X)$ since f is identically 1 on the boundary of A and $A \cap Z = h_A^{-1}(1) \in \mathcal{Z}(X)$. The family $\mathcal{A}$ exhibits the local finiteness of $\{A \cap Z : A \in \mathcal{A}\}$ since $\mathrm{cl}_X Z = Z \subset \bigcup \mathcal{A}$.

Corollary 6.4. *X is not pseudocompact.*

To further explore the properties of X in relation to p we introduce some definitions. For any space Y, define wn(Y) (the weak Novak number) to be $\min\{|\mathcal{D}| : \mathcal{D}$ is a family of nowhere dense sets whose union is dense$\}$ (if Y has isolated points, let wn$(Y) = \infty$ where $\kappa < \infty$ for all cardinals κ). Recall that $\pi w(Y) = \min\{|\mathcal{B}| : \mathcal{B}$ is a π-base for $Y\}$ (a collection $\mathcal{B}$ of nonempty open

subsets of Y is a π-base for Y if for each nonempty open $U \subset Y$ there is a $B \in \mathscr{B}$ with $B \subset U$).

Fix a maximal pairwise disjoint family $\mathscr{A}'$ of cozero sets so that $p \notin \text{cl}_{\beta X} A$ for $A \in \mathscr{A}'$. Further, suppose that $\text{wn}(U) = \text{wn}(A)$ and $\pi w(U) = \pi w(A)$ for each $A \in \mathscr{A}'$ and nonempty open $U \subset A$ (this is no loss of generality since $\text{wn}(U) \leq \text{wn}(A)$ and $\pi w(U) \leq \pi w(A)$ for each nonempty open $U \subset A$ for any open A). Let $\mathscr{A} \subset \mathscr{A}'$ be of minimum cardinality such that $p \in \text{cl}_{\beta X} \bigcup \mathscr{A}$; let $\lambda = |\mathscr{A}|$ and fix an indexing $\{A_\alpha : \alpha < \lambda\} = \mathscr{A}$. Let us observe the following.

Lemma 6.5. *Let $Z \in p$ be chosen as in 6.3 for the family $\mathscr{A}'$. For any $\mathscr{V} \subset \{A \cap Z : A \in \mathscr{A}'\}$, $\bigcup \mathscr{V} \in \mathscr{Z}(X)$.*

Proof. Let f be chosen as in 6.3 such that $Z = f^{-1}(0) \cap X$ and $\text{int}_{\beta X} f^{-1}(1) \supset X - \bigcup \mathscr{A}'$. Let $\hat{h}_A = h_A \cdot (1 - f)$ where h_A is as in 6.3. It follows now that $A \cap Z = \hat{h}_A^{-1}(1) = h_A^{-1}(1)$ and the family $\{X - \hat{h}_A^{-1}(0) \ (= A - h_A^{-1}(0)) : A \in \mathscr{A}'\}$ is locally finite. Therefore the function $h_{\mathscr{V}} = \sum\{\hat{h}_A : A \in \mathscr{A}'$ and $A \cap Z \in \mathscr{V}\} \in C^*(X)$ and $h_{\mathscr{V}}^{-1}(1) = \bigcup \mathscr{V} \in \mathscr{Z}(X)$.

Note that by 6.5 and 6.1, $p \notin \text{cl}_{\beta X}(X) - \bigcup\{A \cap Z : A \in \mathscr{A}\}$. Furthermore, let $I(X)$ be the isolated points of X and assume without loss of generality that $\{\{x\} : x \in I(X)\} \subset \mathscr{A}'$. By 6.5, either p is essentially a member of $\beta(I(X))$ or $p \in \beta X - \text{cl}_{\beta X} I(X)$ and $I(X)$ has no effect on p. This is why we assumed in the introduction that X has no isolated points rather than simply that $p \in \text{cl}_{\beta X}(X - I(X))$.

Definition 6.6. (i) Let $\kappa_p = \min\{|\mathscr{Z}| : \mathscr{Z} \subset p \cap \mathscr{Z}(X)$ and $\bigcap \mathscr{Z} = 0\}$.

(ii) Let $\mathscr{U}_p \subset P(\lambda)$ be defined by $I \in \mathscr{U}_p$ iff $p \in \text{cl}_{\beta X} \bigcup\{A_\alpha : \alpha \in I\}$, and let $\lambda_p = \lambda$.

(iii) Let $\gamma_p = \min\{|\mathscr{U}| : \mathscr{U} \subset \mathscr{U}_p$ and $\bigcap \mathscr{U} = 0\}$.

(iv) Let $w_p = \inf\{\sup\{\text{wn}(A_\alpha) : \alpha \in I\} : I \in \mathscr{U}_p\}$.

We will now show that $\mu_0 \leq \min\{\kappa_p, \lambda_p, \gamma_p, w_p\}$. Let us begin with what is essentially Terada's result.

Proposition 6.7. *$\mathscr{U}_p$ is an ultrafilter on λ_p and $\kappa_p \leq \gamma_p$ and γ_p is measurable.*

Proof. That $\mathscr{U}_p$ is an ultrafilter follows from 6.5 and the fact that disjoint zero sets have disjoint closures in βX. Furthermore 6.5 implies that $\kappa_p \leq \gamma_p$. Indeed let $\{I_\beta : \beta < \gamma_p\} \subset \mathscr{U}_p$ be such that $\bigcap\{I_\beta : \beta < \gamma_p\} = 0$. Then for each $\beta < \gamma_p$, $Z_\beta = \bigcup\{A_\alpha \cap Z : \alpha \in I_\beta\} \in p$ and $\{Z_\beta : \beta < \gamma_p\} = 0$. Finally, by 6.2, $\omega < \kappa_p \leq \gamma_p$ hence γ_p is measurable by the definition of a measurable cardinal.

Theorem 6.8. *p is a P_{κ_p}-point of βX and κ_p is measurable.*

Proof. Suppose $\mathscr{Z}$, $|\mathscr{Z}| < \kappa_p$, is a collection of zero set neighborhoods in βX of p. Let $U = \text{int}_X \bigcap \mathscr{Z}$ and $D = X - U \cap \bigcap \mathscr{Z}$. Since D is nowhere dense in X, we may choose $W \in p$ with $W \cap D = 0$. Since $|\mathscr{Z}| < \kappa_p$,

$W \cap U = \bigcap\{W \cap Z: Z \in \mathscr{Z}\} \neq 0$ and it follows that $p \in \mathrm{cl}_{\beta X} U$. To prove that p is a P_{κ_p}-point of βX it suffices to show that $p \notin \mathrm{cl}_{\beta X}(X - U)$, since this implies $p \in \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} U \subset \bigcap \mathscr{Z}$. This fact follows from results in [vD2] or by choosing $\mathscr{A}$ as in 6.3 to refine $\{U, X - \overline{U}\}$ and applying 6.5. To show that κ_p is measurable we show there is a κ_p-complete ultrafilter on κ_p. Choose $\{Z_\alpha: \alpha < \kappa_p\} \subset p$ so that $W_\alpha \subset \mathrm{int}_{\beta X} Z_\alpha \cap \bigcap\{\mathrm{cl}_{\beta X}(W_\beta \cap Z_\beta): \beta < \alpha\}$, which we may do by 6.1 and the above. Let, for $\alpha < \kappa_p$, $A_\alpha' = \mathrm{int}_X \bigcap\{W_\beta - W_\alpha: \beta < \alpha\}$ with $A_0' = X - W_0$. Choose any $f \in C^*(\beta X)$ such that $f(p) = 0$ and $f^{-1}(1) \supset X - \bigcup\{A_\alpha': \alpha < \kappa_p\}$. For each $\alpha < \kappa_p$, let $A_\alpha = A_\alpha' \cap f^{-1}((-1/2, 1/2))$. It follows that $p \in \mathrm{cl}_{\beta X} \bigcup\{A_\alpha: \alpha < \kappa_p\}$ and each A_α is a cozero set in X (i.e. just use the function $|f|$ on A_α and constant $1/2$ on $X - A_\alpha$). Now apply 6.3 and 6.7 with $\lambda_p = \kappa_p$ and a new $\mathscr{U}_p$.

Perhaps the most surprising and strongest result that we have is the following.

Theorem 6.9. $\kappa_p \leq w_p$, hence $\mu_0 \leq w_p$.

Proof. Choose $U \in \mathscr{U}_p$ such that $\sup\{\mathrm{wn}(A_\alpha): \alpha \in U\} = w_p$. We may as well assume $w_p < \infty$. For each $\alpha \in U$, fix $\mathscr{D}_\alpha = \{D_\gamma^\alpha: \gamma < w_p\}$ exhibiting that $\mathrm{wn}(A_\alpha) \leq w_p$. Define, for $\gamma < w_p$, $D_\gamma = \bigcup\{D_\gamma^\alpha: \alpha \in U\}$ and observe that D_γ is a nowhere dense subset of X (and βX). Furthermore, $\mathrm{cl}_{\beta X} \bigcup\{D_\gamma: \gamma < w_p\} = \mathrm{cl}_{\beta X} \bigcup\{A_\alpha: \alpha \in U\}$ from which it follows that $p \notin \mathrm{int}_{\beta X} \bigcap\{\beta X - D_\gamma: \gamma < w_p\}$. Since $p \in \mathrm{int}_{\beta X} \beta X - D_\gamma$ for each $\gamma < w_p$, it follows, from 6.8, that $\kappa_p \leq w_p$. ∎

The upshot of 6.9, which is new, is that if X has remote points in υX then X is not constructed as a free union (however large) of small spaces. For example let us state the following corollary.

Corollary 6.10. *The space $X = \lambda \times R$ does not have remote points in υX for any λ.*

Remark 6.11. Clearly the definition of λ_p, γ_p and w_p depend on $\mathscr{A}$ (making the restriction of "homogeneous" π-weight and weak Novak number for $A \in \mathscr{A}$ was just an attempt to make $\mathscr{A}$ canonical). In fact with these restrictions on $\mathscr{A}$, w_p does not depend on $\mathscr{A}$ and we could define λ_p to be minimum possible with respect to such homogeneous $\mathscr{A}$. We have shown that $\kappa_p \leq \gamma_p \leq \lambda_p$ and $\kappa_p \leq w_p$. In the next section we shall see that these inequalities can be strict. Furthermore we can have $w_p < \gamma_p$ or $\gamma_p < w_p$, but I do not know if $\lambda_p < w_p$ is possible.

7. Some examples with remote points in υX

In order to produce an example of a space X with a remote point in υX we shall obviously have to assume the existence of measurable cardinals. In fact we shall work with a supercompact cardinal.

Definition 7.1. A cardinal κ is *supercompact* if for each $\lambda, \lambda \geq \kappa$, there is a κ-complete normal fine ultrafilter $\mathscr{U}$ on $[\lambda]^{<\kappa}$. (Recall that $[\lambda]^{<\kappa} = \{a \subset \lambda: |a| < \kappa\}$; $\mathscr{U}$ is called *normal* if for each regressive function f on a member

of $\mathscr{U}$ (i.e. $f(a) \in a$ for all $a \in U$ for some $U \in \mathscr{U}$) there is an $\alpha \in \lambda$ such that $f^{-1}(\alpha) \in \mathscr{U}$; $\mathscr{U}$ is *fine* if $\langle \alpha \rangle = \{a \in [\lambda]^{<\kappa} : \alpha \in a\} \in \mathscr{U}$ for all $\alpha \in \lambda$).

Facts 7.2 [KaMa]. (a) If κ is measurable then there is a κ-complete normal fine ultrafilter on $[\kappa]^{<\kappa}$.

(b) If κ is supercompact and $\lambda^{<\kappa} = \lambda$ then $|U| = \lambda$ for each $U \in \mathscr{U}$ where $\mathscr{U}$ is as in 7.1.

We shall show, for certain spaces X of the form $\sum\{X_a : a \in [\lambda]^{<\kappa}\}$, where κ is measurable, that X has a remote point p in υX with $\kappa_p = \kappa$. We know from §6 that it is necessary that $w_p \geq \kappa$, however for our induction to work we will actually require a stronger completeness property on X. Recall that a poset is said to be λ *directed closed* if every directed subset of size less than λ has a lower bound. For a space Y let $G(Y)$ be the supremum of all λ such that Y has a π-base which is λ directed closed when ordered by inclusion. The relationship between $G(Y)$ and $\text{wn}(Y)$ is that $G(Y) \leq \text{wn}(U)$ for all nonempty open $U \subset Y$.

Theorem 7.3. *Let $\kappa \leq \lambda$ be cardinals such that $2^\lambda = \lambda^+$ and there is a κ-complete normal fine ultrafilter $\mathscr{U}$ on $[\lambda]^{<\kappa}$. Let $\{X_a : a \in [\lambda]^{<\kappa}\}$ be a family of spaces such that $\pi w(X_a) \leq \lambda^+$, $c(X_a) \leq \lambda^+$ and $G(X_a) > |a|$ for all $a \in [\lambda]^{<\kappa}$. Then the space $X = \sum\{X_a : a \in [\lambda]^{<\kappa}\}$ has a remote point $p \in \upsilon X$ such that $\mathscr{U}_p = \mathscr{U}$ where $\mathscr{U}_p$ and $\mathscr{A} = \{X_a : a \in [\lambda]^{<\kappa}\}$ are as in 6.3 and 6.6.*

Proof. For each $a \in [\lambda]^{<\kappa}$, fix an $|a|^+$ directed closed π-base $\mathscr{B}_a$ for X_a. Since $\pi w(X_a) \leq \lambda^+$ and $2^\lambda = \lambda^+$ we may suppose that $|\mathscr{B}_a| \leq \lambda^+$ for each $a \in [\lambda]^{<\kappa}$ (i.e. choose a π-base $\mathscr{B}_a'$ of size at most λ^+ and choose an $|a|^+$ directed closed subset of $\mathscr{B}_a$ which refines $\mathscr{B}_a'$). Under our current assumptions X has cellularity λ^+ hence the family of all maximal pairwise disjoint subsets of $\bigcup\{\mathscr{B}_a : a \in [\lambda]^{<\kappa}\}$ has cardinality $(\lambda^+)^\lambda = \lambda^+$. Let $\{\mathscr{C}_\xi : \xi < \lambda^+\}$ index this family. Let $\xi < \lambda^+$ and suppose that for $\gamma < \xi$ we have chosen f_γ a function from $[\lambda]^{<\kappa}$ into $\bigcup\{\mathscr{B}_a : a \in [\lambda]^{<\kappa}\}$ so that

(i) for each $a \in [\lambda]^{<\kappa}$ $f_\gamma(a) \in \mathscr{B}_a$ and there is a $B \in \mathscr{C}_\gamma$ such that $f_\gamma(a) \subset B$ and $f_\gamma(a)$ is completely separated from $X_a - B$, and

(ii) for any $F \subset X$ which is a finite intersection of members of the family $\{\bigcup\{f_\gamma(a) : a \in [\lambda]^{<\kappa}\} : \gamma < \xi\}$, $\{a \in [\lambda]^{<\kappa} : F \cap X_a \neq 0\} \in \mathscr{U}$.

Let us choose f_ξ. Let $\{F_\alpha : \alpha < \lambda\}$ list all finite intersections from the family $\{\bigcup\{f_\gamma(a) : a \in [\lambda]^{<\kappa}\} : \gamma < \xi\}$. Let us show that the set $U = \{a \in [\lambda]^{<\kappa} : \{F_\alpha \cap X_a : \alpha \in a\}$ is directed$\}$ is in $\mathscr{U}$. Suppose not and for each $a \in [\lambda]^{<\kappa} - U$ choose $m_a \in \omega$ and $\{\alpha_n(a) : n < m_a\} \subset a$ so that $\bigcap\{F_{\alpha_n(a)} \cap X_a : n < m_a\} = 0$. By the κ-completeness of $\mathscr{U}$ there is an $m < \omega$ and a $V \in \mathscr{U}$ such that $m_a = m$ for all $a \in V$. Next, by the normality of $\mathscr{U}$ used m times, there is a $V' \in \mathscr{U}$ and a sequence $\{\alpha_n : n < m\} \subset \lambda$ such that $\alpha_n(a) = \alpha_n$ for each $n < m$ and $a \in V'$. However, by (ii), $W = \{a \in [\lambda]^{<\kappa} : X_a \cap \bigcap\{F_{\alpha_n} : n < m\} \neq 0\} \in \mathscr{U}$

and $V \cap V' \cap W = 0$. Therefore $U \in \mathcal{U}$. For $a \notin U$, choose $f_\xi(a) \in \mathcal{B}_a$ so as to satisfy (i). For $a \in U$, we have that $\bigcap\{F_\alpha \cap X_a : \alpha \in a\}$ has nonempty interior since $\mathcal{B}_a$ is $|a|^+$ directed closed, $\{F_\alpha \cap X_a : \alpha \in a\}$ is directed and each $F_\alpha \cap X_a$ is a finite intersection of members of $\mathcal{B}_a$. Since $\mathcal{C}_\xi$ is a maximal almost disjoint family of open sets we may choose $f_\xi(a) \in \mathcal{B}_a$ so that (i) holds and $f_\xi(a) \subset \bigcap\{F_\alpha \cap X_a : \alpha \in a\}$. To see that (ii) holds note that for $\alpha < \lambda$ $\{a \in [\lambda]^{<\kappa} : F_\alpha \cap f_\xi(a) \neq 0\} \supset \langle\alpha\rangle \cap U \in \mathcal{U}$.

Let $\mathcal{F} \subset \mathcal{Z}(X)$ be the filter generated by those $Z \in \mathcal{Z}(X)$ such that for some $U \in \mathcal{U}$ and $\xi < \lambda^+$ $\bigcup\{f_\xi(a) : a \in U\} \subset Z$ (that $\mathcal{F}$ is a filter follows from property (ii)). Note that for each $\xi < \lambda^+$ there is a $Z_\xi \in \mathcal{Z}(X)$ and a cozero set C_ξ of X such that for each $a \in [\lambda]^{<\kappa}$ there is a unique $B \in \mathcal{C}_\xi$ such that $f_\xi(a) \subset Z_\xi \cap X_a \subset C_\xi \cap X_a \subset B$ (by property (i)). Furthermore observe that from the proof that (ii) holds in the induction step above it follows easily that for any $\{\xi_n : n < \omega\} \subset \lambda^+$, $\{a \in [\lambda]^{<\kappa} : \{f_{\xi_n}(a) : n < \omega\}$ is directed $\} \in \mathcal{U}$. Now let us show that $\mathcal{F}$ is a remote ultrafilter and that $\mathcal{F} \in \upsilon X$. For any nowhere dense set $D \subset X$, there is a $\xi < \lambda^+$ such that $\bigcup\mathcal{C}_\xi \cap D = 0$. Since $Z_\xi \subset C_\xi$ and $D \subset X - C_\xi$, Z_ξ is completely separated from D, hence $\mathcal{F}$ is remote. To see that $\mathcal{F}$ is an ultrafilter, let $W \in \mathcal{Z}(X)$. Choose $\xi < \lambda^+$ such that $\mathcal{C}_\xi$ refines $\{\text{int } W, X - W\}$. Therefore, for each $a \in [\lambda]^{<\kappa}$ either $Z_\xi \cap X_a \subset W$ or $Z_\xi \cap X_a \cap W = 0$. If $U = \{a \in [\lambda]^{<\kappa} : Z_\xi \cap X_a \subset W\} \in \mathcal{U}$ then $W \in \mathcal{F}$ while if $U \notin \mathcal{F}$ then $Z'_\xi = Z_\xi \cap \sum\{X_a : a \notin U\} \in \mathcal{F}$ and $Z'_\xi \cap W = 0$. Note that in fact for each $Z \in \mathcal{F}$ there is a $\xi < \lambda^+$ and a $U \in \mathcal{U}$ such that $\bigcup\{f_\xi(a) : a \in U\} \subset Z$. Now to show that $\mathcal{F} \in \upsilon X$ it suffices to show that $\bigcap\{Z_n : n < \omega\} \neq 0$ for any $\{Z_n : n < \omega\} \subset \mathcal{F}$. So let $\{Z_n : n < \omega\} \subset \mathcal{F}$ and choose $\{\xi_n : n < \omega\} \subset \lambda^+$ and $\{U_n : n < \omega\} \subset \mathcal{U}$ so that $\bigcup\{f_{\xi_n}(a) : a \in U_n\} \subset Z_n$. Let $U = \bigcap\{U_n : n < \omega\} \cap \{a \in [\lambda]^{<\omega} : \{f_{\xi_n}(a) : n < \omega\}$ is directed $\} \in \mathcal{U}$. Therefore for any infinite $a \in U$, $\bigcap\{Z_n \cap X_a : n < \omega\} \supset \bigcap\{f_{\xi_n}(a) : n < \omega\} \neq 0$ since $\{f_{\xi_n}(a) : n < \omega\}$ is a directed subset of the $|a|^+$-directed closed π-base $\mathcal{B}_a$.

To apply 7.3 we need a class of spaces satisfying the hypotheses. For a cardinal α let $(2^\alpha)_\alpha$ be the refinement of the usual topology on 2^α which has $\{[s] : s \in {}^{<\alpha}2\}$ as a base where $[s] = \{f \in 2^\alpha : s \subset f\}$ for $s \in {}^{<\alpha}2 = \{{}^\beta 2 : \beta < \alpha\}$. Let Y_α be the Stone-Čech compactification of $(2^\alpha)_\alpha$. If α is regular then $\pi w(Y_\alpha) = \pi w((2^\alpha)_\alpha) = 2^{<\alpha}$, $G(Y_\alpha) = G((2^\alpha)_\alpha) = \alpha$ and $c(Y_\alpha) = c((2^\alpha)_\alpha) = 2^{<\alpha}$.

Let us now list some examples.

Example 7.4. If κ is measurable and $2^\kappa = \kappa^+$ then there is a space X with no isolated points which has a remote point p in υX. We can also have $X = \sum\{X_\alpha : \alpha \in \kappa\}$ such that $wn(X_\alpha) < \kappa$ for all $\alpha < \kappa$. Indeed, let $\mathcal{U}$ be a normal κ-complete ultrafilter on κ (think of κ as a subset of $[\kappa]^{<\kappa}$) and for each $\alpha < \kappa$ let $X_\alpha \in \{Y_{\alpha^+}, Y_\kappa\}$ and apply 7.3.

Example 7.5. Let κ be supercompact and let $\kappa \leq \lambda$ such that $2^\lambda = \lambda^+$.

(a) There are spaces $\{X_\alpha : \alpha < \lambda\}$ such that $X = \sum\{X_\alpha : \alpha < \lambda\}$ has a remote point $p \in \upsilon X$. We can have $\kappa_p = \kappa$, $\lambda_p = \lambda$ and w_p any regular cardinal between κ and λ.

(b) For each $\lambda' \geq \lambda$, there are spaces $\{X_\alpha : \alpha < \lambda'\}$ such that $X = \sum\{X_\alpha : \alpha < \lambda'\}$ has a remote point $p \in \upsilon X$ such that $\kappa_p = \kappa$, $\lambda_p = (\lambda')^{<\kappa}$ and $\kappa \leq w_p \leq \lambda$.

The example in (a) is obviously the same as that for 7.4. For (b) we do not assume that $2^{\lambda'} = (\lambda')^+$ hence we cannot use 7.3 directly. Let $\mathscr{U}$ be any κ-complete normal fine ultrafilter on $[\lambda']^{<\kappa}$. For each $a \in [\lambda']^{<\kappa}$, let X_a be a space with a remote point $p_a \in \upsilon X_a$ (by part (a)). We let $p \in \upsilon X$ be the "$\mathscr{U}$-limit" of the collection $\{p_a : a \in [\lambda']^{<\kappa}\}$. That is, for $Z \in \mathscr{Z}(X)$, $Z \in p$ iff $\{a \in [\lambda']^{<\kappa} : Z \cap X_a \in p_a\} \in \mathscr{U}$.

Remark 7.6. The result that $\kappa_p \leq w_p$ can be improved to $\{a \in [\lambda]^{<\kappa} : |a| < \mathrm{wn}(X_a)\} \in \mathscr{U}$ in 7.3 where each X_a is pseudo-μ_0-compact (this is needed by 7.5(b)). However I do not know if this suffices (i.e. replace $G(X_a) > |a|$ by $\mathrm{wn}(X_a) > |a|$ in 7.3).

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ONTARIO M5S 1A1, CANADA

Current address: Department of Mathematics, York University, North York, Ontario M3J 1P3, Canada