

LOCAL PROPERTIES OF SECANT VARIETIES IN SYMMETRIC PRODUCTS. PART I

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ABSTRACT. Let L be a line bundle on an abstract nonsingular curve C , let $V \subset H^0(C, L)$ be a linear system, and denote by $C^{(d)}$ the symmetric product of d copies of C . There exists a canonically defined $C^{(d)}$ -bundle map:

$$\sigma : V \otimes \mathcal{O}_{C^{(d)}} \rightarrow E_L,$$

where E_L is a bundle of rank d obtained from L by a so-called symmetrization process. The various degeneracy loci of σ can be considered as subsecant schemes of $C^{(d)}$. Our main result, Theorem 4.2, is given in §4, where we obtain a local matrix description of σ valid (also) at points on the diagonal in $C^{(d)}$, and thereby we can determine the completions of the local rings of the secant schemes at arbitrary points. In §5 we handle the special case of giving a local scheme structure to the zero set of σ .

1. INTRODUCTION

For a curve $C \subset \mathbf{P}^n$ one is often interested in studying the linear subspaces of \mathbf{P}^n containing divisors on C of a certain degree. It is then convenient to find a variety (called a secant variety) which parametrizes the situations where an exceptional secancy by a linear subspace of some fixed dimension occurs. A typical example is to describe the trisecant lines for a curve in \mathbf{P}^3 ; another example may be to describe the 4-secant planes for a curve in \mathbf{P}^4 .

When dealing with these problems there are at least two main strategies at hand: One can work in the Grassmannian parametrizing linear subspaces of \mathbf{P}^n of some fixed dimension. This was done in [G-P] in the case $C \subset \mathbf{P}^3$. One can also work in $C^{(d)}$, the d th symmetric product of C , parametrizing effective divisors on C of the fixed degree d . For an example of this, see [A-C-G-H, Chapter VIII]. In addition there are other methods and setups, like the one in [L], which was applied in [WL] to obtain local results. See also [LB].

In our paper we will use the second approach and work with the symmetric product $C^{(d)}$. Our goal is to give local results about secant varieties, which will be subvarieties of $C^{(d)}$, and we will not take up any global questions. Roughly speaking we will show how the local geometry of C at points of secancy determines the local geometry of our secant varieties at the divisors in question.

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A lot of problems like this are completely trivial at points off the weak diagonal in $C^{(d)}$, that is, at points representing divisors without repeated points. We will show how such problems can be solved at points *on* the weak diagonal in $C^{(d)}$.

Like other authors working with secant varieties on $C^{(d)}$, we will define these varieties as degeneration loci (schemes) of a certain $C^{(d)}$ -bundle map:

$$\sigma: V \otimes \mathcal{O}_{C^{(d)}} \rightarrow E_L.$$

Here $V \subset H^0(C, L)$ is the linear system on C defining the map $C \rightarrow \mathbf{P}^n$, and E_L is a $C^{(d)}$ -bundle of rank d on $C^{(d)}$ obtained from the line bundle L on C by a so-called symmetrization process. C is regarded as an abstract nonsingular curve.

Part I of this paper is mainly devoted to giving a local matrix description of the map σ at points on as well as off the weak diagonal in $C^{(d)}$, for arbitrary d . This matrix description will give us enough information to describe power series that determine the completions of the local rings of our secant varieties at the points in question. We will not need the assumption that V is very ample or even base point free in order to describe the map σ this way, and we need no assumptions on the characteristic of our ground field. The geometric problems mentioned in the beginning can thus be seen as a motivation for, rather than the essence of, our work, which is purely algebraic. Our treatment is based on the approach and the methods in [Ma-Ma and Ma]

In §2 we list some standard facts about the symmetrization map:

$$\sigma: V \otimes \mathcal{O}_{C^{(d)}} \rightarrow E_L.$$

In §3 we treat the special case $L = \Omega$, the canonical sheaf of differentials on C . We show how a local matrix representation of σ can be obtained in this special case. The key point here is the differential analogue of the Newton identities; see [Ma-Ma, p. 225].

In §4 we show how the results for $L = \Omega$ can be generalized to be valid for any line bundle on C . We state our main result, Theorem 4.2, which gives the desired matrix description of the map σ and determines the formal completion of the local rings of our secant varieties.

In §5 we apply Theorem 4.2 to determine local multiplicities of the zero scheme of σ .

At last we use our step from §4 to reproduce two well-known formulas:

In §6 we calculate the contribution of a cusp singularity to the total number $\frac{1}{2}(d-1)(d-2) - g$ of singularities of a plane curve of degree d and genus g . In §7 we give the multiplicity of a Weierstrass point of an arbitrary linear system.

Part II is devoted to applications of Theorem 4.2 to various geometric problems. For more details, see the introduction to that part.

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2. SYMMETRIZATION OF LINE BUNDLES

Let C be an abstract nonsingular curve over some field K of arbitrary characteristic. Denote by C^d the d -fold Cartesian product of C and by $C^{(d)}$ the d -fold symmetric product.

Let $V \subset H^0(C, L)$ be a linear system on C , where L is some line bundle. We will define and study the symmetrized bundle E_L and the canonical $C^{(d)}$ -bundle map

$$\sigma: V \otimes_K \mathcal{O}_{C^{(d)}} \rightarrow E_L$$

which was mentioned in the introduction. In particular we will find a local analytic matrix description of σ at an arbitrary effective divisor D in $C^{(d)}$.

Set $D = \sum_{i=1}^k d_i P_i$, where $\sum_{i=1}^k d_i = d$ and the P_i are distinct points on C . Our matrix description will depend only on local analytic parametrizations at P_1, \dots, P_k of some chosen set of sections spanning V .

A description of $\sigma: V \otimes C^{(d)} \rightarrow E_L$. (See also [A-C-G-H, p. 340].) Consider the diagrams

$$(2.1) \quad C^{(d)} \xleftarrow{p} F \subset C^{(d)} \times C \xrightarrow{q} C$$

where F is the universal divisor $\{(D, P) \mid D \text{ contains } P\}$ and

$$(2.2) \quad C^{(d)} \xleftarrow{\chi} C^d \xrightarrow{\pi_i} C$$

where π_i is the i th projection from C^d to C , and χ is the natural map onto $C^{(d)}$.

Set $L[d] = \bigoplus_{i=1}^d \pi_i^* L$. Clearly $L[d]$ is a locally free sheaf on C^d of rank d . Let ε_L be the sheaf on $C^{(d)}$ whose sections on an open set U are the G -invariant sections of $L[d]$ on $\chi^{-1}(U)$, where G is the Galois covering-map group of χ .

Set $E_L = p_* q^* L$. It can be shown that $E_L = \varepsilon_L$, and it is a locally free sheaf of rank d . This is essentially Proposition 1, p. 781, in [Ma]. In particular each section S of L gives rise to a section $S[d] = \sum_{i=1}^d \pi_i^* S$ of $L[d]$ on C^d . The section $S[d]$ is G -invariant since G acts by rearranging the summands of $S[d]$. Hence S gives rise to a section S_d of $\varepsilon_L = E_L$ on $C^{(d)}$.

Definition. The map

$$\sigma: V \otimes_K \mathcal{O}_{C^{(d)}} \rightarrow E_L$$

is defined by letting S be mapped to S_d for each section S in V .

Remark. When D consists of d distinct points, σ “simply evaluates” S at these points.

A local simplification. When $D = \sum_{i=1}^k d_i P_i$, where the P_i are distinct, there are $d!/\pi_{i=1}^k d_i!$ points in C^d that are mapped to D by the canonical map χ . Let D'' in C^d be $(P_1, \dots, P_1, \dots, P_k, \dots, P_k)$, where the point P_i is taken d_i times, for $i = 1, \dots, k$. Clearly D'' is in the fiber of χ over D . Let χ_{d_1, \dots, d_k} be the partial symmetrization map which maps $(Q_1, \dots, Q_d) \in C^d$ to $(\sum_{i=1}^{d_1} Q_i, \dots, \sum_{i=d_{k-1}+1}^d Q_i) \in C^{(d_1)} \times \dots \times C^{(d_k)}$.

Definition. The sheaf L_{d_1, \dots, d_k} on $C^{(d_1)} \times \dots \times C^{(d_k)}$ is the sheaf whose sections on an open set U are the invariant sections of $L[d]$ on $\chi_{d_1, \dots, d_k}^{-1}(U)$ under the Galois group associated to χ_{d_1, \dots, d_k} .

We see that each (global) section S of L gives rise to a section S_{d_1, \dots, d_k} of L_{d_1, \dots, d_k} .

Definition. The map

$$\sigma' : V \otimes \mathcal{O}_{C^{(d_1)} \times \dots \times C^{(d_k)}} \rightarrow L_{d_1, \dots, d_k}$$

is defined by setting

$$\sigma'(S) = S_{d_1, \dots, d_k} \quad \text{for each section } S \text{ in } V.$$

We now make the following observation:

Observation 2.1. Study the following commutative diagram:

$$\begin{array}{ccc} C^d & \xrightarrow{\text{id}} & C^d \\ \chi_{d_1, \dots, d_k} \downarrow & & \downarrow \chi \\ C^{(d_1)} \times \dots \times C^{(d_k)} & \xrightarrow{\eta} & C^{(d)}. \end{array}$$

Denote by D' the point $(d_1 P_1, \dots, d_k P_k)$ of $C^{(d_1)} \times \dots \times C^{(d_k)}$. The natural map η is a local analytic isomorphism at D' , D . Moreover, η induces a local analytic isomorphism of the bundle maps σ' and σ at D' and D , respectively.

Explanation. The first statement is a standard fact. See [Ma-Ma, p. 226]. Since σ' and σ are derived from the natural C^d -bundle map

$$V \otimes_K \mathcal{O}_{C^d} \rightarrow L[d]$$

via χ_{d_1, \dots, d_k} and χ , respectively, the last statement also holds.

In §3 we will use Observation 2.1 to obtain a local matrix representation of σ in the case where L is the canonical sheaf on C .

3. THE SYMMETRIZATION OF DIFFERENTIALS

We will first consider the process of symmetrization of differentials; hence we will assume that $V \subset H^0(C, \Omega)$, where Ω is the sheaf of differentials on

C . In §4 we will explain how this special case gives us enough information to describe the map σ locally for a general line bundle L on C .

Let φ be a holomorphic differential on C , and set $D = \sum_{i=1}^k d_i P_i$ as usual. To φ there corresponds a section S^φ of Ω , which gives rise to a section $S^\varphi[d]$ of $\Omega[d]$ on C^d . The section $S^\varphi[d]$ can be interpreted as a 1-form $\varphi[d]$ on C^d , which is invariant with respect to any subgroup of the permutation group acting on C^d . Let t_i be a local parameter of C at P_i , and let φ be given locally at P_i as

$$\left(\sum_{j \geq 0} a_{i,j} t_i^j \right) dt_i, \quad i = 1, \dots, k.$$

Then the 1-form $\varphi[d]$ is given locally at $D'' = (P_1, \dots, P_1, \dots, P_k, \dots, P_k)$ as

$$(3.1) \quad \sum_{i=1}^k \sum_{j \geq 0} a_{i,j} \left(\sum_{l=1}^{d_i} t_{i,l}^j dt_{i,l} \right).$$

In (3.1) the symbol $t_{i,l}$ is short for $\pi_{d_1 + \dots + d_{i-1} + l}^*(t_i)$; see diagram (2.2). The $t_{i,l}$ can be viewed as formal replicas of t_i , for $l = 1, \dots, d_i$, $i = 1, \dots, k$.

A partially symmetrized section $S_{d_1, \dots, d_k}^\varphi$ of Ω_{d_1, \dots, d_k} corresponds to a partially symmetrized 1-form $\varphi_{d_1, \dots, d_k}$ on $C^{(d_1)} \times \dots \times C^{(d_k)}$. A set of regular parameters for $C^{(d_1)} \times \dots \times C^{(d_k)}$ at $D' = (d_1 P_1, \dots, d_k P_k)$ is

$$\bigcup_{i=1}^k \{s_{i,1}, \dots, s_{i,d_i}\}$$

where $s_{i,j}$ is the j th elementary symmetric function in the formal variables $t_{i,1}, \dots, t_{i,d_i}$. See [Ma-Ma, p. 226].

The sums $(\sum_{l=1}^{d_i} t_{i,l}^j dt_{i,l})$ appearing in (3.1) are symmetric in $\{t_{i,1}, \dots, t_{i,d_i}\}$. Hence (3.1) can be expressed in terms of $\{s_{i,1}, \dots, s_{i,d_i}\}$ and the differential symbols $\{ds_{i,1}, \dots, ds_{i,d_i}\}$ for $i = 1, \dots, k$. Clearly this will be the expression of $\varphi_{d_1, \dots, d_k}$ locally at D' . In the next paragraph we will study these partially symmetrized 1-forms more closely.

Partially symmetrized differentials in terms of the elementary symmetric functions. We will now express (3.1) in terms of $\bigcup_{i=1}^k \{s_{i,1}, \dots, s_{i,d_i}\}$, and we will treat each of the k "main summands" in (3.1) separately. Hence we may assume $D = dP$, and we have one set $\{s_1, \dots, s_d\}$ of elementary symmetric functions in $\{t_1, \dots, t_d\}$. The t_i are formal replicas of the local parameter t of C at P . In this case (3.1) reduces to

$$\sum_{j \geq 0} a_j \sum_{l=1}^d t_l^j dt_l.$$

Set $T'_j = \sum_{l=1}^d t_l^j dt_l$, and let T_j be the corresponding 1-form on $C^{(d)}$, which means that T_j is T'_j expressed in terms of the s_i . We have the following key formula, given for example in [Ma-Ma p. 225]:

$$(3.2) \quad s_k T_0 - s_{k-1} T_1 + \cdots + (-1)^k s_0 T_k = ds_{k+1}.$$

Formula (3.2) is valid in all characteristics and for all integers k with the convention that $s_0 = 1$, $s_i = T_i = 0$ when $i < 0$, $s_i = 0$ when $i > d$. Using Formula (3.2) recursively one gets

$$\begin{aligned} T_0 &= ds_1, \\ T_1 &= s_1 ds_1 - ds_2, \\ T_2 &= (s_1^2 - s_2) ds_1 - s_1 ds_2 + ds_3 \quad \text{and so on.} \end{aligned}$$

Let the symbol (\underline{s}) be short for (s_1, \dots, s_d) . In general one has

Proposition 3.1.

$$T_j = W_j(\underline{s}) ds_1 - W_{j-1}(\underline{s}) ds_2 + \cdots + (-1)^{d-1} W_{j-d+1}(\underline{s}) ds_d$$

where

$$W_j(\underline{s}) = \det \begin{bmatrix} s_1 & s_2 & \cdots & & s_j \\ 1 & s_1 & \cdots & & s_{j-1} \\ 0 & 1 & \cdots & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & s_1 \end{bmatrix}$$

when $j \geq 1$ and $W_0(\underline{s}) = 1$, $W_j(\underline{s}) = 0$, $j < 0$.

Proof. Set $T_j = \sum_{i=1}^d W_{i,j}(\underline{s}) ds_i$, where $W_{i,j}(\underline{s}) \in k[(\underline{s})]$. Since the s_i , for $i = 1, \dots, d$, are local parameters at D , the $W_{i,j}(\underline{s})$ are uniquely determined, and we see that for fixed i the $W_{i,k}(\underline{s})$ satisfy the following recursion formula, which is derived from formula (3.2):

$$(3.3) \quad W_{i,k}(\underline{s}) = (-1)^{k-1} (s_k W_{i,0}(\underline{s}) - s_{k-1} W_{i,1}(\underline{s}) + \cdots + (-1)^{k-1} s_1 W_{i,k-1}(\underline{s})).$$

We define $W_j(\underline{s}) = W_{1,j}(\underline{s})$.

Observe that for $i = 1$ the recursion formula (3.3) “matches” expanding the determinant in Proposition 3.1 along the last column. Hence the $W_{1,j}(\underline{s})$ are as stated in Proposition 3.1.

From formula (3.2) we see that

$$W_{i,i-1}(\underline{s}) = (-1)^{i-1} \quad \text{and} \quad W_{i,j}(\underline{s}) = 0 \quad \text{for } j \leq i-2.$$

This implies that

$$W_{i+1,j+1}(\underline{s}) = -W_{i,j}(\underline{s})$$

for $1 \leq i \leq d-1$ and arbitrary j , since the recursion using formula (3.3) starts one step later for $i+1$ than for i and with opposite sign.

The last identity gives that the $W_{i,j}(\underline{s})$ for arbitrary i are as stated in Proposition 3.1, since we know that the $W_{1,j}$ are. This completes our proof.

Remark. We have

$$(3.4) \quad W_k(\underline{s}) = \sum s_1^{i_1} \cdots s_d^{i_d} \cdot (-1)^{\sum_{j=1}^d i_j(j-1)} \cdot \frac{(i_1 + \cdots + i_d)!}{i_1! \cdots i_d!}$$

where the first sum is taken over those (i_1, \dots, i_d) such that $\sum_{j=1}^d j i_j = k$.

We also have

$$(3.5) \quad W_j(s_1(t_1, \dots, t_d), \dots, s_d(t_1, \dots, t_d))$$

is the sum of all (monic) monomials of degree j in $\{t_1, \dots, t_d\}$.

Furthermore we have

$$W_j(\underline{s}) = \Delta_{1, \dots, 1}(\underline{s})$$

in the sense of [F, p. 264].

Formulas (3.4), (3.5) can be proved, for example, by using formula (3.3).

Corollary 3.2. *If φ is a holomorphic differential on C given locally at P as $(\sum_{j \geq 0} a_j t^j) dt$, then the symmetrized 1-form φ_d on $C^{(d)}$ is given locally at $D = dP$ as*

$$(3.6) \quad \sum_{j \geq 0} a_j T_j = \sum_{j \geq 0} a_j [W_j(\underline{s}) ds_1 - W_{j-1}(\underline{s}) ds_2 + \cdots + (-1)^{d-1} W_{j-d+1}(\underline{s}) ds_d].$$

If $D = \sum_{i=1}^k d_i P_i$, where $\sum_{i=1}^k d_i = d$, then $\varphi_{d_1, \dots, d_k}$ is given locally at $D' = (d_1 P_1, \dots, d_k P_k)$ as the sum of k sums of the form (3.6). Choose the ds_i , for $i = 1, \dots, d$, as a basis of the fiber of E_Ω over dP . A local parametrization of the section S_d^φ of E_Ω on $C^{(d)}$ at dP in terms of coordinates Y_1, \dots, Y_d relative to this basis is

$$(3.7) \quad \left\{ Y_1 = \sum_{j \geq 0} a_j W_j(\underline{s}), \dots, Y_d = \sum_{j \geq 0} a_j W_{j-d+1}(\underline{s}) \right\}.$$

Proof. Everything but the last assertion is just a restatement of Proposition 3.1 using the linearity of the (partial) symmetrization map.

The last assertion holds since $\{s_1, \dots, s_d\}$ is a set of regular parameters of $C^{(d)}$ at dP . Hence the 1-forms ds_1, \dots, ds_d form a local basis for the 1-forms on $C^{(d)}$ at dP .

If S is a basis element of V , we see from Corollary 3.2 that the entries

$$\sum_{j \geq 0} a_j W_j(\underline{s}), \dots, \sum_{j \geq 0} a_j W_{j-d+1}(\underline{s})$$

can be taken to form one of the columns of a local matrix description of $C^{(d)}$ -bundle map

$$\sigma : V \otimes \mathcal{O}_{C^{(d)}} \rightarrow E_\Omega$$

at dP . The generalization to the case $D = \sum_{i=1}^k d_i P_i$ is obvious when working with partially symmetrized 1-forms on $C^{(d_1)} \times \cdots \times C^{(d_k)}$ and using Observation 2.1.

4. A LOCAL DESCRIPTION OF SECANT VARIETIES

In §2 we defined the canonical bundle map

$$\sigma : V \otimes \mathcal{O}_{C^{(d)}} \rightarrow E_L.$$

In §3 we assumed $L = \Omega$, and we described how local parametrizations of the sections of V at P_1, \dots, P_k gives rise to a matrix description of σ in terms of local analytic coordinates of $C^{(d_1)} \times \dots \times C^{(d_k)}$ at $D' = (d_1 P_1, \dots, d_k P_k)$.

It is clear that this matrix description depends only on the parametrizations of the sections of V and, once the parametrizations are given, is independent of whether the sections of V correspond to differentials or not, that is, whether $L = \Omega$ or not.

This was explained in [Ma, p. 783], the proof of Proposition 2 of that paper. Hence our local matrix description of σ is valid for any linear system V , where the local parametrizations of the sections (divisors) of V are equal to those of some globally holomorphic differentials on C .

The last condition does not necessarily hold for an arbitrary linear system V . However, we give the following remark. Set $D = \sum_{i=1}^k d_i P_i$.

Remark 4.1. (a) In the symmetrization process described in §3 we treated each point P_i separately.

(b) The symmetrization process can be carried out even if the differentials we treat are not globally holomorphic. For each point P_i contained in D we can describe an analogous symmetrization process for any differential which is holomorphic on a neighborhood U_i of P_i . We can produce a symmetrized 1-form on $U_i^{(d_i)}$ which is a neighborhood of $d_i P_i$ on $C^{(d_i)}$.

Remark 4.1 and the comment in [Ma, p. 783] imply together that for a section S of an arbitrary line bundle L , where S is given locally at P_i as $\sum_{j \geq 0} a_{i,j} t_i^j$, we can describe the map

$$V \otimes \mathcal{O}_{C^{(d_i)}} \rightarrow E_L$$

locally simply by pretending that we are symmetrizing a differential given locally at P_i as

$$\left(\sum_{j \geq 0} a_{i,j} t_i^j \right) dt_i.$$

Hence it is clear that our local matrix description of σ in §3 holds for $V \subseteq H^0(C, L)$ for any line bundle L on C .

We then obtain

Theorem 4.2. Let X_0, \dots, X_n be sections spanning the linear system $V \subset H^0(C, L)$ for a line bundle L . Choose t_i as a local parameter for C at P_i , for $i = 1, \dots, k$, and let

$$\sum_{j \geq 0} a_{r,i,j} t_i^j$$

be local parametrizations of X_r at P_i , for $i = 1, \dots, k$ and $r = 0, \dots, n$. Then a local matrix description of the map $\sigma : V \otimes \mathcal{O}_{C^{(d)}} \rightarrow E_L$ at (the point) $D = \sum_{i=1}^k d_i P_i$ is

$$(4.1) \quad M = \begin{bmatrix} \sum_{j \geq 0} a_{0,1,j} W_j(\underline{s}^{(1)}) & \cdots & \sum_{j \geq 0} a_{n,1,j} W_j(\underline{s}^{(1)}) \\ \vdots & & \vdots \\ \sum_{j \geq 0} a_{0,1,j} W_{j-d_1+1}(\underline{s}^{(1)}) & \cdots & \sum_{j \geq 0} a_{n,1,j} W_{j-d_1+1}(\underline{s}^{(1)}) \\ \vdots & & \vdots \\ \sum_{j \geq 0} a_{0,k,j} W_j(\underline{s}^{(k)}) & \cdots & \sum_{j \geq 0} a_{n,k,j} W_j(\underline{s}^{(k)}) \\ \vdots & & \vdots \\ \sum_{j \geq 0} a_{0,k,j} W_{j-d_k+1}(\underline{s}^{(k)}) & \cdots & \sum_{j \geq 0} a_{n,k,j} W_{j-d_k+1}(\underline{s}^{(k)}) \end{bmatrix}.$$

The $\underline{s}^{(i)}$ indicate d_i -tuples $(s_{i,1}, \dots, s_{i,d_i})$ of elementary symmetric functions in d_i formal replicas of t_i . The W_l , for $l \in \mathbb{Z}$, were defined in Proposition 3.2.

Definition 4.3. For a map f of $C^{(d)}$ -bundles denote by $Z(f)$ the zero scheme of the map. We denote by V_d^r or $D_{d-r}(\sigma)$ the zero scheme $Z(\Lambda^{d-r+1} \sigma)$, where σ is the canonical map $V \otimes \mathcal{O}_{C^{(d)}} \rightarrow E_L$.

Set-theoretically we have

$$V_d^r = \{D \in C^{(d)} \mid \text{rk } \sigma \leq d - r \text{ at } D\}.$$

Scheme-theoretically V_d^r is given locally at D by (the vanishing of) the $d - r + 1$ -minors of some matrix representation of σ at D .

From Proposition 4.2 we have when $D = \sum_{i=1}^k d_i P_i$:

Theorem 4.4.

$$\hat{\mathcal{O}}_{V_d^r, D} \simeq K[[s_{1,1}, \dots, s_{1,d_1}, \dots, s_{k,1}, \dots, s_{k,d_k}]]/J,$$

where J is the ideal in $K[[\underline{s}]]$ generated by the $d - r + 1$ -minors of M , and the $s_{i,j}$ are regarded as algebraically independent (formal) variables.

We see that $D \in V_d^r$ iff all $d - r + 1$ -minors of M vanish modulo $\underline{m} = (s_{i,j})$. Modulo \underline{m} the matrix M is

$$(4.2) \quad BN = \begin{bmatrix} a_{0,1,0} & \cdots & a_{n,1,0} \\ \vdots & & \vdots \\ a_{0,1,d_1-1} & \cdots & a_{n,1,d_1-1} \\ \vdots & & \vdots \\ a_{0,k,0} & \cdots & a_{n,k,0} \\ \vdots & & \vdots \\ a_{0,k,d_k-1} & \cdots & a_{n,k,d_k-1} \end{bmatrix}.$$

We will sometimes refer to BN as the Brill-Noether matrix. The entries of BN consist of the coefficients of $W_0 = 1$ in the infinite sums in each of the entries in M , since $W_l(\underline{s}^{(i)})$ is contained in \underline{m} when $l \geq 1$.

5. THE SYMMETRIZED BASE POINT SCHEME OF A LINEAR SYSTEM

In this section we will give an application of Theorem 4.2. Assume that V is a linear system of rank 1 on C , where $V \subset H^0(C, L)$, $\deg L = m$, and where V is generated by a global section S . We will study the symmetrized zero (base-point) scheme V_d^d .

Let $\sum_{j \geq 0} a_{i,j} t_i^j$ be local parametrizations of S at P_i , $i = 1, \dots, k$. From the Brill-Noether (column) matrix we see that $D = \sum_{i=1}^k d_i P_i$ is contained in V_d^d iff

$$a_{1,0} = \dots = a_{1,d_1-1} = \dots = a_{k,0} = \dots = a_{k,d_k-1} = 0.$$

Hence $D \in V_d^d$ iff $D \leq D_S$, where D_S is the element of $C^{(m)}$ corresponding to S . (Hence V_d^d is finite.) This is well known; see, e.g., [Ma, Proposition 2, p. 783]. We will study the local lengths (multiplicities) of V_d^d . The total length is always $\binom{m}{d}$, roughly speaking, since D_S contains $\binom{m}{d}$ subdivisors of degree d when D_S is off the weak diagonal in $C^{(m)}$, and since moving S in $H^0(C, L)$ does not change total lengths. We will, however, not use any global argument like this.

We have when $D_S = \sum_{i=1}^k m_i P_i$ and $D = \sum_{i=1}^k d_i P_i$:

Result 5.1. (a) The local length of V_d^d at D is

$$\prod_{i=1}^k \binom{m_i}{d_i}.$$

(Observe that

$$\sum \prod_{i=1}^k \binom{m_i}{d_i} = \binom{m}{d},$$

where the sum is taken over all divisors D such that $D \leq D_S$.)

(b) $\hat{\mathcal{O}}_{V_d^d, D} = k[[\underline{s}]]/I$, where

$$I = (W_{m_1}(\underline{s}^{(1)}), \dots, W_{m_1-d_1+1}(\underline{s}^{(1)}), \dots, W_{m_k}(\underline{s}^{(k)}), \dots, W_{m_k-d_k+1}(\underline{s}^{(k)})).$$

The d variables $s_{i,j}$ are algebraically independent, and the W_j were defined in Proposition 3.1.

Proof. We will prove (b) and then use (b) to prove (a). Clearly we may assume $D_S = mP$, $D = dP$, and $m \geq d$ when proving (b). The ideal cutting out V_d^d

locally at D is generated by the entries of M . The entries are

$$\begin{array}{ccccccc} a_m W_m(\underline{s}) & + & a_{m+1} W_{m+1}(\underline{s}) & + \cdots + & \cdots & + \cdots \\ a_m W_{m-1}(\underline{s}) & + & a_{m+1} W_m(\underline{s}) & + \cdots + & \cdots & + \cdots \\ \vdots & & & & & \\ a_m W_{m-d+1}(\underline{s}) & + & a_{m+1} W_{m-d+2}(\underline{s}) & + \cdots + & a_{m+d-1} W_m(\underline{s}) & + \cdots \end{array}$$

We see that modulo $(W_{m+1}(\underline{s}), W_{m+2}(\underline{s}), \dots)$ these relations give a rise to the following coefficient matrix in the "variables" $W_{m-d+1}(\underline{s}), \dots, W_m(\underline{s})$:

$$N = \begin{bmatrix} 0 & \cdots & 0 & a_m \\ & & a_m & a_{m+1} \\ \vdots & \ddots & & \vdots \\ a_m & a_{m+1} & \cdots & a_{m+d-1} \end{bmatrix}.$$

By the recursion formula (3.3) we have

$$W_r(\underline{s}) = \sum_{i=m-d+1}^m P_i(\underline{s}) W_i(\underline{s})$$

for suitable polynomials $P_i(\underline{s}) \in \underline{m} = (s_1, \dots, s_d)$ when $r \geq m+1$. The summation starts at $i = m-d+1$ since in formula (3.3) we have $s_j = 0$ when $j > d$. Hence the d generators of our ideal can be expressed as linear combinations in $W_{m-d+1}(\underline{s}), \dots, W_m(\underline{s})$ with coefficients in $k[[s]]$. Since the coefficient matrix thus obtained is equal to N modulo $\underline{m} = (s_1, \dots, s_d)$, we see that (b) holds since $\det N = \pm a_m^d \neq 0$.

We will now use (b) to prove (a). Let I_i be the ideal $(W_{m_i}(\underline{s}^{(i)}), \dots, W_{m_i-d_i+1}(\underline{s}^{(i)}))$ in $k[[s_{i,1}, \dots, s_{i,d_i}]]$ for $i = 1, \dots, k$. We see that

$$\text{colength } I = \prod_{i=1}^k \text{colength } I_i,$$

since the I_i are generated by functions in mutually disjoint set of variables $\{\underline{s}^{(i)}\}$. This means that we can assume again that $D_S = mP$, $D = dP$, and we must show that $\text{colength } I = \binom{m}{d}$.

We proceed by induction on m and we may start with $m = d = 1$. In this case $I = (\sum_{j \geq 1} a_j s_1^j)$, where $a_1 \neq 0$, so $\text{colength } I = 1 = \binom{1}{1}$.

Set $W_j = W_j(\underline{s})$ for all j . For general m, d we have

$$\begin{aligned} I &= (W_{m-d+1}, \dots, W_m) \\ &= (W_{m-d+1}, \dots, W_{m-1}, s_1 W_{m-1} - s_2 W_{m-2} + \cdots + (-1)^{d-1} s_d W_{m-d}) \\ &= (W_{m-d+1}, \dots, W_{m-1}, s_d W_{m-d}) \end{aligned}$$

by formula (3.3).

Hence

$$\begin{aligned} \text{colength } I &= \text{colength}(W_{m-d}, W_{m-d+1}, \dots, W_{m-1}) \\ &\quad + \text{colength}(W_{m-d+1}, \dots, W_{m-1}, s_d). \end{aligned}$$

The first summand is $\binom{m-1}{d}$ by the induction hypothesis. The second summand is the colength of

$$(W_{m-d+1}(s_1, \dots, s_{d-1}), \dots, W_{m-1}(s_1, \dots, s_{d-1}))$$

in $K[[s_1, \dots, s_{d-1}]]$. By the induction hypothesis this is $\binom{m-1}{d-1}$.

Hence colength $I = \binom{m-1}{d} + \binom{m-1}{d-1} = \binom{m}{d}$. This proves (a).

We will now give a generalization of Result 5.1.

Corollary 5.2. *Assume that $\text{rank } V = n + 1$ and that V is generated by sections corresponding to divisors D_0, \dots, D_n on C , where*

$$D_j = \sum_{i=1}^k m_{i,j} P_i, \quad j = 0, \dots, n.$$

Set $D_W = \sum_{i=1}^k m_i P_i$, where $m_i = \min\{m_{i,j}\}_{j=0}^n$, for $i = 1, \dots, k$. Let W be the linear system of rank 1 spanned by the section corresponding to D_W . Then

(a) $V_d^d = W_d^d$ for all d .

(b) The multiplicity of V_d^d at $D = \sum_{i=1}^k d_i P_i$ is $\prod_{i=1}^k \binom{m_i}{d_i}$.

Proof. By Result 5.1 we see that (b) is a consequence of (a).

To prove (a) we may assume that $D_W = mP$, $D = dP$, and $D_j = m_j P$, $j = 0, \dots, n$. By the proof of Result 5.1 the entries in the column corresponding to D_j form the ideal $(W_{m_j}(\underline{s}), \dots, W_{m_j-d+1}(\underline{s}))$. By formula (3.3) $W_r(\underline{s})$ is contained in the ideal $J = (W_m(\underline{s}), \dots, W_{m-d+1}(\underline{s}))$ for all $r \geq m - d + 1$. Hence

$$\hat{\mathcal{O}}_{V_d^d, D} = \hat{\mathcal{O}}_{W_d^d, D} = K[[s_1, \dots, s_d]]/J.$$

Since both schemes V_d^d and W_d^d are zero-dimensional, this is enough to prove (a). This completes our proof.

6. SINGULARITIES OF PLANE CURVES

Here we apply Theorem 4.2 to study the well-known case of plane cusps. Assume $\text{rk } V = 3$ and that V is base point free. Thus V defines a map

$$\phi: C \rightarrow \overline{C} \subseteq \mathbf{P}^2.$$

We can “measure” the singularities of \overline{C} by studying the scheme V_2^1 in $C^{(2)}$. This scheme may consist of two kinds of points:

- (1) Divisors $P_1 + P_2$, where $P_1 \neq P_2$.
- (2) Divisors $2P$.

The first ones correspond to nodes of C , the latter ones to cusps. If V_2^1 is finite, it is well known that its total length is $\frac{1}{2}(d-1)(d-2) - g$, where $d = \deg \overline{C} = \deg L$ and $g = \text{genus}(C)$.

The multiplicity of V_2^1 at $D = 2P$. From now on we will concentrate on divisors of the type $2P$. We will not prove anything essentially new, but we will show how our setup fits in well with traditional results.

Denote by $\text{Mult}_D V_2^1$ the multiplicity or local length of V_2^1 at a divisor D . Clearly the δ -invariant of \overline{C} at $Q \in P^2$ is $\sum \text{Mult}_D V_2^1$, where the sum is taken over those divisors $P_1 + P_2$ and $2P$ such that $\phi(P_1) = \phi(P_2) = Q$ and $\phi(P) = Q$. We will show how to find $\text{Mult}_{2P} V_2^1$, when $\text{Char } K = 0$ and $K = \overline{K}$.

Choose

$$X_r = \sum_{j=0}^{\infty} \alpha_{r,j} t^j \quad \text{for } r = 0, 1, 2$$

as local parametrizations at P of the sections spanning V . We may assume $X_0 \equiv 1$. The matrix M from Theorem 4.2 is

$$\begin{bmatrix} 1 & \sum_{j \geq 0} \alpha_{1,j} W_j(s_1, s_2) & \sum_{j \geq 0} \alpha_{2,j} W_j(s_1, s_2) \\ 0 & \sum_{j \geq 1} \alpha_{1,j} W_{j-1}(s_1, s_2) & \sum_{j \geq 1} \alpha_{2,j} W_{j-1}(s_1, s_2) \end{bmatrix}.$$

We assume $P = (1, 0, 0)$ and obtain

$$\hat{\mathcal{O}}_{V_2^1, 2P} = K[[s_1, s_2]]/I$$

where

$$I = \left(\sum_{j \geq 2} \alpha_{1,j} W_{j-1}(s_1, s_2), \sum_{j \geq 2} \alpha_{2,j} W_{j-1}(s_1, s_2) \right).$$

We have used that $\alpha_{1,1} = \alpha_{2,1} = 0$ by assumption. When $\text{Char } K = 0$, it is a standard fact that we may simplify our local parametrizations:

$$X_0 = 1, \quad X_1 = t^n, \quad X_2 = \sum_{j \geq n+1} \alpha_{2,j} t^j,$$

where $n \geq 2$ is the multiplicity at $\phi(P)$ of the branch of \overline{C} in question. The ideal I reduces to

$$\left(W_{n-1}(s_1, s_2), \sum_{j \geq n+1} \alpha_{2,j} W_{j-1}(s_1, s_2) \right).$$

We see that $\text{Mult}_{2P} V_2^1 = \text{colength } I$ is equal to the intersection number of two algebroid curves at the origin in the s_1, s_2 -plane. We will compute this number (Result 6.1).

Considering s_1, s_2 as elementary symmetric functions in two formal replicas t_1, t_2 , we have by formula (3.5)

$$W_{n-1}(s_1(t_1, t_1), s_2(t_1, t_2)) = \prod_{r=1}^{n-1} (t_1 - \varepsilon_{n,r} t_2)$$

where $\varepsilon_{n,r} = e^{2\pi ri/n}$. By standard arithmetic this gives

$$(6.1) \quad \begin{aligned} W_{n-1}(s_1, s_2) &= \prod_{r=1}^{(n-1)/2} (s_1^2 - k_{n,r} s_2) \quad \text{when } n \text{ is odd,} \\ W_{n-1}(s_1, s_2) &= s_1 \cdot \prod_{r=1}^{(n-2)/2} (s_1^2 - k_{n,r} s_2) \quad \text{when } n \text{ is even,} \end{aligned}$$

where $k_{n,r} = 2 + \varepsilon_{n,r} + \varepsilon_{n,r}^{-1}$.

In any case $\text{Mult}_{2P} V_2^1$ is the sum of the intersection numbers obtained by intersecting the algebroid curve with the equation

$$\sum_{j \geq n+1} \alpha_{2,j} W_{j-1}(s_1, s_2)$$

with each of the curves corresponding to the factors of $W_{n-1}(s_1, s_2)$ (at the origin).

Formula (6.1) implies:

(a) $W_{j-1}(s_1, s_1^2/k_{n,r}) \equiv 0$ iff $\varepsilon_{n,r}$ is a primitive m th root of unity for an m dividing j ;

(b) $W_{j-1}(0, s_2) \equiv 0$ iff j is even.

For each $m \geq 2$ we define

$$\begin{aligned} B_m &= \min\{l \mid m \text{ does not divide } l, \text{ and } \alpha_{2,l} \neq 0\}, \\ r_m &= \#\{\text{primitive } m\text{th roots of unity}\} \end{aligned}$$

or recursively $r_m = m - 1 - \sum r_{m_i}$, where the sum is taken over all m_i that divide m , except 1 and m . We then obtain

Result 6.1.

$$\text{Mult}_{2P} V_2^1 = \sum_{i=1}^s \frac{r_{m_i}(B_{m_i} - 1)}{2}$$

where m_1, \dots, m_s are the positive integers (except 1) dividing n .

7. A NOTE ON WEIERSTRASS POINTS

Let V be a linear system of rank $r+1$ and degree d on a curve C . We will use Theorem 4.2 to prove a well-known formula for the weight (multiplicity) of a rank $l+1$ Wronskian point of V , $0 \leq l \leq r$. A rank $r+1$ Wronskian point is a Weierstrass point.

First we will define our terms, without making any assumptions on the characteristic of K . Consider the map

$$\phi_l: C \rightarrow C^{(l+1)}$$

where $\phi_l(P) = (l+1)P$, for $P \in C$.

Definition 7.1. (a) We say that V is classical if $\phi_l^{-1}(V_{l+1}^1)$ is a finite set for $0 \leq l \leq r$.

(b) Assume V is classical. We define the (finite) rk $l+1$ Wronskian scheme of V as $T_l = \phi_l^{-1}(V_{l+1}^1)$.

(c) We define the (finite) Weierstrass scheme of V as T_r . The points of T_r are denoted by Weierstrass points of V .

Let P be an arbitrary point of C , and let t be a local parameter of C at P . Then there are uniquely determined integers (not depending on the choice of t) $h_0 < h_1 < \dots < h_r$ such that there are sections X_0, \dots, X_r spanning V with local parametrizations

$$X_0 = \sum_{j \geq h_0} \alpha_{0,j} t^j, \dots, X_r = \sum_{j \geq h_r} \alpha_{r,j} t^j,$$

with $\alpha_{i,h_i} \neq 0$, for $i = 0, \dots, r$. The integers h_0, \dots, h_r are called the Hermite invariants of V at P . If V is classical, then $h_i = i$, for $i = 0, \dots, r$, for all but a finite set of points on C .

We now give our result:

Proposition 7.2. Assume $\text{char } K = 0$ or $\text{char } K \geq l+1$, and that V is classical. Then the multiplicity of T_l at P is $\sum_{i=1}^l (h_i - i) = \sum_{i=0}^l h_i - l(l+1)/2$.

Comment. This is essentially [L2, Theorem 15, ii]. In [L2, Theorem 15, i] one proves that if $\text{char } K = 0$, or $\text{char } K \geq d+1$, then V is classical.

Proof. By Theorem 4.2 we have

$$\hat{\mathcal{O}}_{V_{l+1}^1, (l+1)P} \approx K[[s_1, \dots, s_{l+1}]]/J$$

where J is generated by the $l+1$ -minors of

$$M = \begin{bmatrix} \sum_{j \geq h_0} \alpha_{0,j} W_j(\underline{s}) & \dots & \sum_{j \geq h_r} \alpha_{r,j} W_j(s) \\ \vdots & & \vdots \\ \sum_{j \geq h_0} \alpha_{0,j} W_{j-l}(\underline{s}) & \dots & \sum_{j \geq h_r} \alpha_{r,j} W_{j-l}(s) \end{bmatrix}.$$

The map $\phi_l : C \rightarrow C^{(l+1)}$ induces a map

$$\phi_l^* : \hat{\mathcal{O}}_{C^{(l+1)}, (l+1)P} \approx K[[s_1, \dots, s_{l+1}]] \rightarrow \hat{\mathcal{O}}_{C,P} \simeq K[[t]]$$

such that for $k = 1, \dots, l+1$ we have $\phi_l^*(s_k) = s_k(t, \dots, t)$ where s_k is the k th elementary symmetric function in $l+1$ variables. From formula (3.5) we have

$$\phi_l^*(W_j(\underline{s})) = W_j(\phi_l^* s_1, \dots, \phi_l^* s_{l+1}) = \binom{j+l}{l} \cdot t^j \quad \text{for all } j.$$

This implies that

$$\hat{\mathcal{O}}_{T_l, P} \simeq K[[t]]/\phi_l^*(J)$$

where $\phi_l^*(J)$ is generated by the $l+1$ -minors of the matrix

$$(7.1) \quad \begin{bmatrix} \sum_{j \geq h_0} \alpha_{0,j} \binom{j+l}{l} t^j & \cdots & \sum_{j \geq h_r} \alpha_{r,j} \binom{j+l}{l} t^j \\ \vdots & & \vdots \\ \sum_{j \geq h_0} \alpha_{0,j} \binom{j}{l} t^{j-l} & \cdots & \sum_{j \geq h_r} \alpha_{r,j} \binom{j}{l} t^{j-l} \end{bmatrix}.$$

The multiplicity of T_l at P is the lowest number m such that there is a term t^m in one of the minors generating $\phi_l^*(J)$.

The $l+1$ -minor consisting of the $l+1$ first columns of (7.1) can be written as $\sum_{j \geq m} c_j t^j$, where $m = h_0 + (h_1 - 1) + \cdots + (h_l - l) = \sum_{i=0}^l (h_i - i)$. Clearly no term t^n , with $n < m$, is contained in any of the generators of $\phi_l^* J$. Hence we have proved the proposition if we can show that c_m is nonzero. We have

$$c_m = \alpha_{0,h_0} \cdots \alpha_{l,h_l} \cdot \begin{vmatrix} \binom{h_0+l}{l} & \cdots & \binom{h_l+l}{l} \\ \binom{h_0-1+l}{l} & \cdots & \binom{h_l-1+l}{l} \\ \vdots & & \vdots \\ \binom{h_0}{l} & \cdots & \binom{h_l}{l} \end{vmatrix}.$$

The proposition follows from the following lemma:

Lemma 7.3.

$$\begin{vmatrix} \binom{h_0+l}{l} & \cdots & \binom{h_l+l}{l} \\ \vdots & & \vdots \\ \binom{h_0}{l} & \cdots & \binom{h_l}{l} \end{vmatrix} = \begin{vmatrix} \binom{h_0}{0} & \cdots & \binom{h_l}{0} \\ \binom{h_0}{1} & \cdots & \binom{h_l}{1} \\ \vdots & & \vdots \\ \binom{h_0}{l} & \cdots & \binom{h_l}{l} \end{vmatrix}.$$

Comment. In [L1, Lemma 9] one shows that the determinant to the right is $\prod_{0 \leq j < i \leq l} (h_i - h_j) \cdot 1 / \prod_{i=1}^l i!$, which is nonzero.

Proof of Lemma 7.3. In the first row set

$$\binom{h_j+l}{l} = \binom{h_j+l-1}{l-1} + \binom{h_j+l-1}{l} \quad \text{for } 0 \leq j \leq l.$$

Since the entries in the second row are $\binom{h_j+l-1}{l-1}$, these terms can be deleted in the first row. In this way the entries in row $k+1$ can be changed from $\binom{h_j+l-k}{l-k}$ to $\binom{h_j+l-k-1}{l-k-1}$ for $k = 0, \dots, l-1$ and $j = 0, \dots, l$. Then start at the top again and treat all but the two last rows the same way once more. When the top row has been treated this way l times, we end up with the desired determinant. This completes the proof of Lemma 7.3 and also of Proposition 7.2.

Corollary 7.4. Assume $\text{char } K = 0$, or $\text{char } K \geq r+1$. Then the multiplicity of P as a Weierstrass point is $\sum_{i=0}^r (h_i - i)$.

Remark. It is a well-known fact that the total length of T_r , that is, the sum of the multiplicities of the Weierstrass points, is

$$((g-1)r+d)(r+1), \quad \text{where } g \text{ is the genus of } C.$$

This follows from [A-C-G-H, pp. 345 and 358] where $k = C$.

Nonclassical linear systems. What happens if we impose no restrictions on $\text{char } K$? This question has been answered in a very satisfactory way in [L2], and we would be happy to reproduce some of the results in [L2] using our setup. It seems, however, that our methods are too crude when $0 < \text{char } K < d$. Still, we will add a few words about this case.

Let h_0, \dots, h_r be the Hermite invariants of V at a point P of C . On an open set of C the Hermite invariants are constant with values b_0, \dots, b_r . When $\text{char } K = 0$ or $\text{char } K > d$, we have $b_i = i$ for $i = 0, \dots, r$. When $2 \leq \text{char } K \leq d$, we have $i \leq b_i < b_{i+1}$ for $0 \leq i \leq r-1$, and b_i might or might not be equal to i for all i . In this case we have

$$\phi_{b_l}^{-1}(V_{b_{l+1}}^{b_l-l+1}) \text{ is a finite set for } 0 \leq l \leq r,$$

in analogy with Definition 7.1(a).

$\phi_{b_l}^{-1}(V_{b_{l+1}}^{b_l-l+1})$ is also defined as a finite scheme, which we denote by T_l , and

$$R = \hat{\mathcal{O}}_{T_l, P} \simeq K[[t]]/\phi_{b_l}^*(J),$$

where $\phi_{b_l}^*(J)$ is generated by the $l+1$ -minors of the following $(b_l+1) \times (r+1)$ matrix:

$$\begin{bmatrix} \sum_{j \geq h_0} \alpha_{0,j} \binom{j+b_l}{b_l} t^j & \cdots & \sum_{j \geq h_r} \alpha_{r,j} \binom{j+b_l}{b_l} t^j \\ \vdots & & \vdots \\ \sum_{j \geq h_0} \alpha_{0,j} \binom{j}{b_l} t^{j-b_l} & \cdots & \sum_{j \geq h_r} \alpha_{r,j} \binom{j}{b_l} t^{j-b_l} \end{bmatrix}.$$

(As usual $\binom{a}{b} = 0$ if $b > a$.)

The multiplicity of T_l at P is the length of the ring R . One sees that $P \in T_l$ iff $\alpha_{l,j} = 0$ for $j \leq b_l$, that is, iff $h_l \geq b_l + 1$.

Set-theoretically we have: P is a rank $l+1$ Wronskian point in the sense of [L2] iff $P \in \bigcup_{k=0}^l T_k$. In [L2, Example 1, p. 64] one shows that it is possible that $P \notin T_l$ but $P \in T_k$ for some $k < l$. Hence the multiplicity we have described for a point of T_l is different from the multiplicity described in [L2] for a rank $l+1$ Wronskian point.

REFERENCES

- [A-C-G-H] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris, *Geometry of algebraic curves*, Vol. I, Springer-Verlag, New York, 1985.
- [F] W. Fulton, *Intersection theory*, Springer-Verlag, New York, 1984.
- [G-P] L. Gruson and C. Peskine, *Courbes de l'espace projectif, variétés de sécantes*, Enumerative Geometry and Classical Algebraic Geometry, Progress in Math., Vol. 24, Birkhäuser, Boston, Mass., 1982, pp. 1–31.
- [L] O. A. Laudal, *Formal moduli of algebraic structures*, Lecture Notes in Math., Vol. 754, Springer-Verlag, Berlin and New York, 1979.
- [L1] D. Laksov, *Weierstrass points on curves*, Astérisque **87–88** (1981), 221–247.

- [L2] —, *Wronskians and Plücker formulas for linear systems on curves*, Ann. Sci. Norm. Sup. (4) **17** (1984), 45–66.
- [LB] P. Le Barz, *Formules multisecantes pour les courbes quelconques*, Enumerative Geometry and Classical Algebraic Geometry, Progress in Math., Vol. 24, Birkhäuser, Boston, Mass., 1982, pp. 165–197.
- [Ma] A. Mattuck, *Secant bundles on symmetric products*, Amer. J. Math. **87** (1965), 779–797.
- [Ma-Ma] A. Mattuck and A. Mayer, *The Riemann-Roch theorem for algebraic curves*, Ann. Scuola Norm. Sup. Pisa (3) **17** (1963), 223–237.
- [WL] T. Wentzel-Larsen, *Deformation theory of trisecant varieties*, Preprint No. 1, Univ. of Oslo, 1985.

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