

## THE SPACE OF HARMONIC MAPS OF $S^2$ INTO $S^4$

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**ABSTRACT.** Every branched superminimal surface of area  $4\pi d$  in  $S^4$  is shown to arise from a pair of meromorphic functions  $(f_1, f_2)$  of bidegree  $(d, d)$  such that  $f_1$  and  $f_2$  have the same ramification divisor. Conditions under which branched superminimal surfaces can be generated from such pairs of functions are derived. For each  $d \geq 1$  the space of harmonic maps (i.e. branched superminimal immersions) of  $S^2$  into  $S^4$  of harmonic degree  $d$  is shown to be a connected space of complex dimension  $2d + 4$ .

### INTRODUCTION

In a study of minimal surfaces in euclidean spheres, Calabi showed that every minimal immersion of  $S^2$  in  $S^n$  arises from an isotropic map to projective space [4], [5]. This work was used by Bryant who showed that every compact Riemann surface can be superminimally immersed in  $S^4$ . There exist Calabi-type theorems representing harmonic maps of  $S^2$  into other locally symmetric spaces in essentially algebro-geometric terms. These are of interest to people studying  $\sigma$ -models in physics. In this paper, we study the space of branched superminimal immersions of compact Riemann surfaces into  $S^4$ .

In §I, we characterize branched superminimal surfaces in  $S^4$  by pairs of meromorphic functions with the same ramification divisor. This is done by constructing a contact map between  $\tilde{\mathbb{P}}^3$  and  $PT(\mathbb{CP}^1 \times \mathbb{CP}^1)$  where  $\tilde{\mathbb{P}}^3$  is the blow-up of  $\mathbb{CP}^3$  along 2 skew lines. The bidegree of such a pair is related to the degree of the canonical lift of the surface in  $\mathbb{CP}^3$ . We then show that if in addition the surface is linearly full (i.e. not contained in any strict subspace of  $\mathbb{R}^5$ ) then the pair of meromorphic functions has bidegree  $(d, d)$  where  $d \geq 3$  and where the 2 functions do not differ by a Möbius transformation.

In §II, we analyze the space of harmonic maps of  $S^2$  into  $S^4$ . By examining the projective geometry of certain Grassmann varieties, we show that the space  $\mathcal{H}_d$  of harmonic maps of  $S^2$  into  $S^4$  of degree  $d$  is a *connected* space of complex dimension  $2d + 4$ . We also construct examples of *unbranched* superminimal surfaces of genus 0 in  $S^4$  of area  $4\pi d$  for  $d \geq 3$ .

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In §III, we consider branched superminimal surfaces of genus  $g$ . We discuss conditions under which a pair of meromorphic functions on a Riemann surface  $\Sigma$  can give rise to a branched superminimal immersion of  $\Sigma$  into  $S^4$ .

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### PRELIMINARIES

Let  $\Sigma$  be a compact Riemann surface and  $\psi : \Sigma \looparrowright S^4$  an immersion into the unit 4-sphere. Let  $B$  denote the second fundamental form of  $\psi$ . Then  $\psi$  is a *minimal immersion* if the mean curvature  $H := \text{trace } B$  vanishes identically. More generally,  $\psi$  is a *branched minimal immersion* if it is minimal away from the set of isolated singular points. These are precisely the nonconstant conformal harmonic maps. Observe that any harmonic map  $\psi : S^2 \rightarrow S^4$  is automatically conformal. Thus, branched minimal immersions of  $S^2$  in  $S^4$  are just the nonconstant harmonic maps from  $S^2$  to  $S^4$  (Eells-Lemaire [7]).

Let  $\psi : \Sigma \looparrowright S^4$  be a (branched) minimal immersion of a compact Riemann surface in  $S^4$ . Let  $x$  and  $y$  denote the local isothermal coordinates on  $\Sigma$ . Consider the holomorphic quartic form  $\Phi \in H^0(\Sigma; (\Omega^1)^4)$  defined by  $\Phi := \varphi \cdot \bar{\varphi} dz^4$  where

$$\varphi = \frac{1}{2} \left\{ B \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right) - i B \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \right\},$$

and where “ $\cdot$ ” is the complex bilinear extension of the dot product to  $\mathbb{C}^5$ . We say that  $\psi$  is a (*branched*) *superminimal immersion* if  $\Phi$  vanishes identically. This means that  $\psi$  has a holomorphic horizontal lift,  $\tilde{\psi}$ , to  $\mathbb{CP}^3$  (Bryant [3], Chern-Wolfson [6], Lawson [10]). Observe that since  $S^2$  has no nontrivial holomorphic quartic differentials, every branched minimal immersion (i.e. harmonic map) of  $S^2$  into  $S^4$  is automatically branched superminimal.

Consider the Calabi-Penrose fibration  $\pi : \mathbb{CP}^3 \rightarrow S^4 = \mathbb{HP}^1$ . This fibration can be obtained via a quotient of 2 Hopf maps. Choose homogeneous coordinates  $(z_0, z_1, z_2, z_3)$  for  $\mathbb{CP}^3$ . Consider  $\mathbb{C}^4 \cong \mathbb{H}^2$  as a quaternion vector space with left scalar multiplication, where the identification is given by  $(z_0, z_1, z_2, z_3) \mapsto (z_0 + z_1 j, z_2 + z_3 j)$ . The Kähler form of the Fubini-Study metric is given by  $\omega = \partial \bar{\partial} \log \|z\|^2$ . The Calabi-Penrose fibration is then given by the quotient

$$\begin{array}{ccc} \mathbb{C}^4 - \{0\} & \xlongequal{\quad} & \mathbb{H}^2 - \{0\} \\ \text{Hopf}_{\mathbb{C}} \downarrow & & \downarrow \text{Hopf}_{\mathbb{H}} \\ \mathbb{CP}^3 & \xrightarrow{\quad \pi \quad} & \mathbb{HP}^1 \end{array}$$

with fiber  $\mathbb{CP}^1$ . The horizontal 2-plane field  $\mathcal{H}$  for  $\pi$  is given by a 1-form whose lifting to  $\mathbb{C}^4 - \{0\}$  is

$$\Omega := \frac{1}{\|z\|^2} (z_0 dz_1 - z_1 dz_0 + z_2 dz_3 - z_3 dz_2).$$

Superminimal surfaces in  $S^4$  are just the projections to  $S^4$  of nonsingular holomorphic curves in  $\mathbb{CP}^3$  which are integral curves of  $\mathcal{H}$ . Unfortunately, it is difficult to find integral curves of  $\mathcal{H}$  directly. Our search for superminimal surfaces would be vastly simplified if we can find a contact manifold  $(M, \mathcal{F})$  birationally equivalent to  $\mathbb{CP}^3$ , where it is easy to find integral curves of the contact plane field  $\mathcal{F}$ . Robert Bryant has found a birational correspondence between  $\mathbb{CP}^3$  and the projectivized tangent bundle of  $\mathbb{CP}^2$  carrying  $\mathcal{H}$  to the contact plane field of  $PT(\mathbb{CP}^2)$ . Using that, he was able to prove the following result:

**Theorem** (Bryant [3]). *Every compact Riemann surface admits a superminimal immersion into  $S^4$ .*

In this paper, I will be using another contact manifold— $PT(\mathbb{P}^1 \times \mathbb{P}^1)$ . From now on, I will let  $\mathbb{P}^n$  denote  $\mathbb{CP}^n$ .

## I. SOME PROJECTIVE GEOMETRY

**1. Holomorphic contact structures.** Let  $V$  be a complex  $(2n+1)$ -manifold. A *holomorphic contact structure* on  $V$  is a nondegenerate holomorphic distribution  $\mathcal{F}$  of hyperplanes on  $V$  (i.e. the orthogonal spaces of some twisted holomorphic 1-form). (cf. Arnold [1], LeBrun [12]).

Let  $M$  be a complex  $n$ -manifold. Then the projectivized cotangent bundle of  $M$  has a canonical holomorphic contact structure. Now let  $\pi : PT^*M \rightarrow M$  denote the projection map onto the base space. A point  $\varphi \in PT^*M$  defines a hyperplane  $P_\varphi$  in  $T_{\pi(\varphi)}M$ . The contact hyperplane at  $\varphi$  is given by  $(\pi_*^{-1})_\varphi(P_\varphi)$ . Thus the canonical contact 2-plane field  $\mathcal{H}$  at a point  $y \in PT(\mathbb{P}^1 \times \mathbb{P}^1) \cong PT^*(\mathbb{P}^1 \times \mathbb{P}^1)$  is given by  $(\pi_*^{-1})_y(L_y)$  where  $L_y$  denotes the tangent line at  $\pi(y)$  corresponding to  $y$ .

The Calabi-Penrose fibration  $p : \mathbb{P}^3 \rightarrow S^4$  has a contact 2-plane field  $\mathcal{H}$  orthogonal to the fibers of  $p$  with respect to the Fubini-Study metric. The 2-plane field  $\mathcal{H}$  for  $p$  is given by a 1-form whose lifting to  $\mathbb{C}^4 - \{0\}$  is  $\Omega = \|z\|^{-2} (z_0 dz_1 - z_1 dz_0 + z_2 dz_3 - z_3 dz_2)$ . Let  $\omega := dz_0 \wedge dz_1 + dz_2 \wedge dz_3$  denote the standard holomorphic symplectic form on  $\mathbb{C}^4$ . Let

$$\xi := z_0 \frac{\partial}{\partial z_0} + z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} + z_3 \frac{\partial}{\partial z_3}.$$

Then  $\Omega = \|z\|^{-2} \xi \lrcorner \omega$ .

**2. Projection to  $\mathbf{P}^1 \times \mathbf{P}^1$ .** Consider the two distinguished skew lines in  $\mathbf{P}^3$  defined by  $L_1 := p^{-1}(N) = \{[0, 0, z_2, z_3] \mid [z_2, z_3] \in \mathbf{P}^1\}$  and  $L_2 := p^{-1}(S) = \{[z_0, z_1, 0, 0] \mid [z_0, z_1] \in \mathbf{P}^1\}$ , where  $N$  and  $S$  denote the north and south poles of  $S^4$  respectively.

**Lemma 1.1.** *There is a well-defined projection map  $\text{pr}: \mathbf{P}^3 - (L_1 \cup L_2) \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$  with  $\mathbf{P}^1$  as fiber.*

*Proof.* It suffices to show that there is a unique line  $L$  through each point  $x \in \mathbf{P}^3 - (L_1 \cup L_2)$  which intersects  $L_1$  and  $L_2$ . The intersection of  $L$  with  $L_1$  and  $L_2$  (identifying  $L_1 \times L_2$  with  $\mathbf{P}^1 \times \mathbf{P}^1$ ) gives us the desired projection map. For each  $x \in \mathbf{P}^3 - (L_1 \cup L_2)$  consider the planes  $P_1$  and  $P_2$  in  $\mathbf{P}^3$  defined by  $P_1 = \text{span}(x, L_1)$  and  $P_2 = \text{span}(x, L_2)$ . Since  $L_1$  and  $L_2$  are skew,  $P_1$  and  $P_2$  intersect in a line  $L$  which contains the point  $x$  and which intersects both  $L_1$  and  $L_2$ .  $\square$

**Proposition 1.2.** *The fibers of  $\text{pr}: \mathbf{P}^3 - (L_1 \cup L_2) \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$  are horizontal with respect to  $p$  (i.e. the fibers of  $\text{pr}$  are integral curves of  $\mathcal{H}$ ).*

*Proof.* Let  $(x, y) \in L_1 \times L_2$ . Let  $L$  denote the line through  $x$  and  $y$ , i.e.  $L = \text{pr}^{-1}(x, y)$ . Denote the inverse images of  $L$ ,  $L_1$ ,  $L_2$ ,  $x$  and  $y$  to  $\mathbb{C}^4 - \{0\}$  by  $P$ ,  $P_1$ ,  $P_2$ ,  $l_x$  and  $l_y$  respectively.

*Note.*  $P_1$  and  $P_2$  are orthogonal with respect to  $\omega$ . Let  $A \in P_1$  and  $B \in P_2$ . Then  $A = (0, 0, a, b)$  and  $B = (c, d, 0, 0)$  for some  $a, b, c, d \in \mathbb{C}$ . It is clear from the definition of  $\omega$  that  $\omega(A, B) = 0$ . Since  $\omega$  is skew, we also have  $\omega(A, A) = \omega(B, B) = 0$ .

Now pick nonzero vectors  $X \in l_x \subset P_1$  and  $Y \in l_y \subset P_2$ . Observe that  $P$  is spanned by  $X$  and  $Y$ . Now let  $V_1 = \alpha X + \beta Y$  and  $V_2 = \gamma X + \delta Y$  be 2 vectors in  $P$ . Then by the note,  $\omega(V_1, V_2) = 0$ . Thus  $\omega$  vanishes on  $P$ . Let  $\rho: \mathbb{C}^4 - \{0\} \rightarrow \mathbf{P}^3$ . Since  $\xi$  is tangent to the fibers of  $\rho$  and  $\Omega|_L = \|z\|^{-2}(\xi \lrcorner \omega)|_P$ , we see that  $\Omega$  vanishes on  $L$ . Thus  $L$  is horizontal with respect to  $p$ .  $\square$

**3. The contact map.** Let  $X$  denote the blow up of  $\mathbf{P}^3$  along  $L_1$  and  $L_2$ , i.e.  $X := \{([z_0, z_1, z_2, z_3], [y_0, y_1], [y_2, y_3]) \mid z_0 y_1 = z_1 y_0, z_2 y_3 = z_3 y_2\}$ . Note that  $X$  is a  $\mathbf{P}^1$ -bundle over  $\mathbf{P}^1 \times \mathbf{P}^1$ :  $\tilde{\pi}: X \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$  where

$$\tilde{\pi}([z_0, z_1, z_2, z_3], [y_0, y_1], [y_2, y_3]) = ([y_0, y_1], [y_2, y_3]).$$

For ease of notation, let  $Y$  denote  $\text{PT}^*(\mathbf{P}^1 \times \mathbf{P}^1) \cong \text{PT}(\mathbf{P}^1 \times \mathbf{P}^1)$ . Let  $\psi: X \rightarrow Y$  be defined by

$$\begin{aligned} \psi([z_0, z_1, z_2, z_3], [y_0, y_1], [y_2, y_3]) \\ = ([y_0, y_1], [y_2, y_3], [z_0 dy_1 - z_1 dy_0, z_2 dy_3 - z_3 dy_2]). \end{aligned}$$

We have the following diagram:

$$\begin{array}{ccccc}
 \mathbf{P}^3 & \xleftarrow{\beta} & X & \xrightarrow{\psi} & Y \\
 p \downarrow & & \tilde{\pi} \downarrow & & \downarrow \pi \\
 S^4 & & \mathbf{P}^1 \times \mathbf{P}^1 & \xlongequal{\quad} & \mathbf{P}^1 \times \mathbf{P}^1
 \end{array}$$

Observe that  $\mathcal{H}$  extends to all of  $X$ , and for  $x \in X$ ,  $\tilde{\pi}_*(\mathcal{H}_x)$  is a tangent line in  $T_{\tilde{\pi}(x)}(\mathbf{P}^1 \times \mathbf{P}^1)$ , i.e.  $\tilde{\pi}_*(\mathcal{H}_x) \in \mathbf{P}T_{\tilde{\pi}(x)}(\mathbf{P}^1 \times \mathbf{P}^1)$ . Furthermore,  $\tilde{\pi} = \pi \circ \psi$  where  $\pi$  is the projection to  $\mathbf{P}^1 \times \mathbf{P}^1$ . Now let  $l := \tilde{\pi}_*(\mathcal{H}_x)$ . Then  $\pi_*^{-1}(l)$  is the contact plane at  $l \in Y$ . Now  $l = \tilde{\pi}_*(\mathcal{H}_x) = (\pi \circ \psi)_*(\mathcal{H}_x) = \pi_* \circ \psi_*(\mathcal{H}_x)$ . Thus,  $\pi_*^{-1}(l) = \psi_*(\mathcal{H}_x)$ . We thus have

**Lemma 1.3.**  $\psi$  is a contact map, i.e.  $\psi_*$  sends the horizontal plane field  $\mathcal{H}$  in  $X$  to the contact plane field  $\mathcal{H}$  in  $Y$ .

The blow ups,  $\sigma_1$  and  $\sigma_2$ , of the two distinguished skew lines  $L_1, L_2 \in \mathbf{P}^3$  are given by

$$\sigma_1 := \left\{ ([0, 0, z_2, z_3], [y_0, y_1], [z_2, z_3]) \mid [y_0, y_1] \in \mathbf{P}^1 \text{ and } [z_2, z_3] \in \mathbf{P}^1 \right\}$$

and

$$\sigma_2 := \left\{ ([z_0, z_1, 0, 0], [z_0, z_1], [y_2, y_3]) \mid [z_0, z_1] \in \mathbf{P}^1 \text{ and } [y_2, y_3] \in \mathbf{P}^1 \right\}.$$

We observe that

$$\psi(\sigma_1) = \left\{ ([y_0, y_1], [z_2, z_3], [1, 0]) \mid [y_0, y_1] \in \mathbf{P}^1 \text{ and } [z_2, z_3] \in \mathbf{P}^1 \right\}$$

and

$$\psi(\sigma_2) = \left\{ ([z_0, z_1], [y_2, y_3], [0, 1]) \mid [z_0, z_1] \in \mathbf{P}^1 \text{ and } [y_2, y_3] \in \mathbf{P}^1 \right\}.$$

**Proposition 1.4.**  $\psi$  is a branched 2-fold covering map. It is branched precisely along  $\sigma_1$  and  $\sigma_2$ .

This proposition will be proved in the next subsection.

**4. The involutions on  $X$  and  $S^4$ .** We first define an involution  $\alpha: X \rightarrow X$  by  $\alpha([z_0, z_1, z_2, z_3], [y_0, y_1], [y_2, y_3]) = ([z_0, z_1, -z_2, -z_3], [y_0, y_1], [y_2, y_3])$ . (Actually,  $\alpha$  is an involution on  $\mathbf{P}^3$  which is extended to  $X$  in a trivial manner.)

*Note.*

- (1)  $\alpha|_{\sigma_1} = \text{Id}$ ,  $\alpha|_{\sigma_2} = \text{Id}$  and  $\alpha^* \Omega = \Omega$ .
- (2) By Note 1,  $\alpha_*$  maps the horizontal plane  $\mathcal{H}_x$  at  $x \in X$  to the horizontal plane  $\mathcal{H}_{\alpha(x)}$  at  $\alpha(x)$ .
- (3) Let  $u \in L_1$  and  $v \in L_2$ . Denote by  $l_{uv}$  the line in  $\mathbf{P}^3$  uniquely defined by  $u$  and  $v$ . Since  $\alpha(u) = u$  and  $\alpha(v) = v$ , we have  $\alpha(l_{uv}) = l_{uv}$ .

Consequently,  $\tilde{\pi} \circ \alpha = \tilde{\pi}$ . (This actually follows immediately from the definition of  $\alpha$  and  $\tilde{\pi}$ .)

(1) Since  $\tilde{\pi}_*(\mathcal{H}_x) = \pi_* \circ \psi_*(\mathcal{H}_x) = \psi(x)$ , we have

$$\begin{aligned} \psi(\alpha(x)) &= \tilde{\pi}_*(\mathcal{H}_{\alpha(x)}) = \tilde{\pi}_*(\alpha_*\mathcal{H}_x) && \text{by Note 2} \\ &= (\tilde{\pi} \circ \alpha)_*(\mathcal{H}_x) \\ &= \tilde{\pi}_*(\mathcal{H}_x) && \text{by Note 3} \\ &= \psi(x). \end{aligned}$$

Thus  $\psi \circ \alpha = \psi$ , i.e.  $\psi$  is  $\alpha$ -invariant.

Notes 1–4 imply that  $\psi$  is at least 2 to 1 except along  $\sigma_1$  and  $\sigma_2$ . From the definition of  $\psi$ , it is clear that  $\psi$  is 1-to-1 on  $\sigma_1$  and  $\sigma_2$ . Let us now examine the map  $\psi$  explicitly in local coordinates. Assume that  $x \notin \sigma_1 \cup \sigma_2$ . We can then set  $z_i = y_i$  for  $i = 0, 1, 2, 3$ . Without loss of generality, we can suppose that  $z_0 = y_0 = 1$  and  $z_2 \neq 0$ . Set  $s = y_1$  and  $t = y_3/y_2$ . Then  $ds = dy_1$  and  $dt = z^{-2}(z_2 dy_3 - z_3 dy_2)$ . Thus,  $z_2^2 dt = z_2 dy_3 - z_3 dy_2$ . Hence,  $\psi([1, z_1, z_2, z_3], s, t) = (s, t, [ds, z_2^2 dt])$ . We also have

$$\psi([1, z_1, -z_2, -z_3], s, t) = (s, t, [ds, z_2^2 dt]).$$

From the above local coordinate expression for  $\psi$ , it is clear that  $\psi$  is 2-to-1 away from  $\sigma_1$  and  $\sigma_2$ . Now,  $\psi$  is a holomorphic map with finite fibers between compact complex 3-folds. Thus, it is a branched covering map of degree 2. This proves Proposition 1.4.

Let us now examine the inverse image of  $\psi$  locally. Choose a point  $y \in Y - (S_1 \cup S_2)$  where  $S_1$  and  $S_2$  are the images under  $\psi$  of  $\sigma_1$  and  $\sigma_2$  respectively. Locally,  $y$  has coordinates  $(s, t, a)$ . Recall that  $\psi([1, z_1, z_2, z_3], s, t) = (s, t, [ds, z_2^2 dt])$  where  $s = z_1$  and  $t = z_3/z_2$ . Then

$$\psi^{-1}(y) = \psi^{-1}(s, t, a) = ([1, s, \sqrt{a}, \sqrt{at}], s, t).$$

The involution  $\alpha$  on  $X$  corresponds to a permutation of the roots. Thus,

**Proposition 1.5.**  $\psi: X \rightarrow Y$  is equivalent to the projection map  $p: X \rightarrow X/\mathbb{Z}_2$  where the  $\mathbb{Z}_2$ -action on  $X$  is given by the involution  $\alpha$ .

The involution on  $\mathbb{P}^3$  descends to an involution on  $S^4$ . Identifying  $S^4$  with  $\mathbb{HP}^1$ , the stereographic projections to  $\mathbb{R}^4 = \mathbb{H}^1$  from the south and north poles are respectively given by  $\varphi_1([q_1, q_2]) = q_1^{-1}q_2$  and  $\varphi_2([q_1, q_2]) = q_2^{-1}q_1$ , with transition functions  $q \mapsto q^{-1}\|q\|^{-2}\bar{q}$ . Now  $p([z_0, z_1, z_2, z_3]) = [z_0 + z_1j, z_2 + z_3j] \in \mathbb{HP}^1$ , where  $[z_0, z_1, z_2, z_3] \in \mathbb{CP}^3$ . Thus,

$$p(\alpha[z_0, z_1, z_2, z_3]) = p([z_0, z_1, -z_2, -z_3]) = [z_0 + z_1j, -(z_2 + z_3j)].$$

The involution  $\alpha$  thus descends to an involution on  $S^4 = \mathbb{HP}^1$  as follows:  $\alpha([q_1, q_2]) = [q_1, -q_2]$  for all  $[q_1, q_2] \in \mathbb{HP}^1$ . (We will let  $\alpha$  denote the involution on both  $X$  as well as  $S^4$ .)

Now,  $\varphi_1 \circ \alpha([q_1, q_2]) = \varphi_1([q_1, -q_2]) = -q_1^{-1}q_2$  and  $\varphi_2 \circ \alpha([q_1, q_2]) = \varphi_2([q_1, -q_2]) = -q_2^{-1}q_1$ . Hence the action of  $\alpha$  on a point  $x \in S^4$  is just the antipodal map on the  $S^3 \subset S^4$  obtained by the intersection of the horizontal 4-plane through  $x$  with  $S^4$ . (This  $S^3$  is the “latitudinal  $S^3$ ”.) Thus, the geodesic 3-sphere in  $S^4$  passing through the north and south poles is invariant under  $\alpha$ .

**5. Some degree computations.** We now compute the degree of the total preimage in  $\mathbf{P}^3$  of a holomorphic curve in  $Y$ . Recall the diagram:

$$\begin{array}{ccccc} \mathbf{P}^3 & \xleftarrow{\beta} & X & \xrightarrow{\psi} & Y \\ p \downarrow & & \tilde{\pi} \downarrow & & \downarrow \pi \\ S^4 & & \mathbf{P}^1 \times \mathbf{P}^1 & \xlongequal{\quad} & \mathbf{P}^1 \times \mathbf{P}^1 \end{array}$$

Let  $l_1$  and  $l_2$  (resp.  $l'_1$  and  $l'_2$ ) denote the preimages in  $X$  (resp.  $Y$ ) of the first and second factors of  $\mathbf{P}^1 \times \mathbf{P}^1$  respectively under the map  $\tilde{\pi}: X \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$  (resp.  $\pi: Y \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$ ). Let  $S_1$  and  $S_2$  denote the 2 distinguished sections of  $Y$  corresponding to lines tangent to the second and first factors of  $\mathbf{P}^1 \times \mathbf{P}^1$  respectively. Recall that  $\psi_*(\sigma_1) = S_1$  and  $\psi_*(\sigma_2) = S_2$ . Note that  $\psi_*(l'_i) = 2l'_i$ ,  $i = 1, 2$ . Let  $H$  be a hyperplane in  $\mathbf{P}^3$ . Then  $\beta^*H = \sigma_1 + l_1 = \sigma_2 + l_2$ . Thus  $\sigma_1 - \sigma_2 = l_2 - l_1$ . Also,  $S_1 - S_2 = \psi_*(\sigma_1 - \sigma_2) = \psi_*(l_2 - l_1) = 2(l'_2 - l'_1)$ . Hence, the Picard group of  $X$  and  $Y$  are given by

$$\text{Pic}(X) = \mathbf{Z}\{l_1, l_2, \sigma_1, \sigma_2\} / \langle \sigma_1 - \sigma_2 = l_2 - l_1 \rangle$$

and

$$\text{Pic}(Y) = \mathbf{Z}\{l'_1, l'_2, S_1, S_2\} / \langle S_1 - S_2 = 2(l'_2 - l'_1) \rangle.$$

Let  $\Sigma$  be a compact Riemann surface of genus  $g$ . Let  $\phi: \Sigma \rightarrow \mathbf{P}^1$  be a holomorphic map of degree  $d$ . A point  $x \in \Sigma$  is a *ramification point* of  $\phi$  if  $d\phi(x) = 0$ , and its image  $\phi(x) \in \mathbf{P}^1$  is called a *branch point* of  $\phi$ . By the Riemann-Hurwitz Theorem the number of branch points of  $\phi$  (counting multiplicities) is  $2g + 2d - 2$ . The *ramification divisor* of  $\phi$  is the formal sum  $\sum a_i p_i$  where  $p_i$  is a ramification point of  $\phi$  with multiplicity  $a_i$ , and where the sum is taken over all ramification points of  $\phi$ . We will let  $\text{Ram}(\phi)$  denote the ramification divisor of  $\phi$ .

Let  $F = (f_1, f_2): \Sigma \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$  be a holomorphic map of bidegree  $(n, m)$ . Then the curve  $C = F(\Sigma)$  is of class  $(m, n)$ . Let  $\tilde{F}$  denote the canonical lift (i.e. Gauss lift) of  $F$  to  $Y$  and let  $C' := \tilde{F}(\Sigma)$ . (The lift of a point  $x \in C$  is the tangent line to  $C$  at  $x$ .) If we assume that  $C$  is *nonsingular*, then

$$\deg \tilde{F}^*(l'_1) = m, \quad \deg \tilde{F}^*(l'_2) = n,$$

$$\deg \tilde{F}^*(S_1) = \# \text{ branch points of } f_1 = 2g - 2 + 2n \text{ and}$$

$$\deg \tilde{F}^*(S_2) = \# \text{ branch points of } f_2 = 2g - 2 + 2m$$

where ‘deg’ refers to the intersection number of  $\tilde{F}(\Sigma)$  with the relevant generators. Let  $\tilde{C} := \psi^{-1}(C') \subset X$  and  $\gamma := \beta_*(\tilde{C}) \subset \mathbf{P}^3$ . Then for a generic hyperplane  $H$  in  $\mathbf{P}^3$ , we have

$$\begin{aligned} \deg \gamma &= H \cdot \beta_*(\tilde{C}) = \beta^* H \cdot \tilde{C} = (\sigma_1 + l_1) \cdot (\psi^{-1} C') \\ &= \psi_*(\sigma_1 + l_1) \cdot C' = (S_1 + 2l'_1) \cdot \tilde{F}_*(\Sigma) \\ &= \deg \tilde{F}^*(S_1 + 2l'_1) = 2g - 2 + 2n + 2m. \end{aligned}$$

Suppose  $\deg f_1 = \deg f_2 = d$  and  $\text{Ram}(f_1) = \text{Ram}(f_2)$ . Then the curve  $C = F(\Sigma)$  has singular points with the property that  $\deg \tilde{F}^*(S_1) = \deg \tilde{F}^*(S_2) = 0$ . Consequently,  $\deg \gamma = 2d$ .

**6. Conjugate branched superminimal surfaces.** Let us suppose that  $f: \Sigma \looparrowright S^4$  is a branched superminimal immersion of a compact Riemann surface in  $S^4$ . Generically,  $f(\Sigma)$  misses a pair of antipodal points in  $S^4$  (say the north and south poles). Also, generically,  $\alpha(f(\Sigma)) \neq f(\Sigma)$ , i.e.  $f(\Sigma)$  is not  $\alpha$ -invariant. Let  $\tilde{f}: \Sigma \rightarrow \mathbf{P}^3$  be the holomorphic horizontal lift of  $f$  to  $\mathbf{P}^3$ .

**Proposition 1.6.** *A generic branched superminimal surface  $f(\Sigma)$  in  $S^4$  has the property that its lift  $\tilde{f}(\Sigma)$  in  $\mathbf{P}^3$  is not  $\alpha$ -invariant.*

*Proof.* The proposition follows immediately from the definition of the involution  $\alpha$  and the fact that  $\alpha$ -invariance in  $\mathbf{P}^3$  descends to  $\alpha$ -invariance in  $S^4$ .  $\square$

*Note.* The converse is not necessarily true. For example, the totally geodesic  $S^2$  of area  $4\pi$  contained in the equator of  $S^4$  is obviously  $\alpha$ -invariant. However, its lift in  $\mathbf{P}^3$  is a curve  $\gamma$  of degree 1 (and hence  $\gamma \cong \mathbf{P}^1$ ) which avoids  $L_1$  and  $L_2$ , and thus is not  $\alpha$ -invariant. Observe that  $\alpha(\gamma)$  projects down to the same geodesic  $S^2$  (but with the opposite orientation).

**Corollary 1.7.** *Given a generic branched superminimal surface  $f(\Sigma)$  in  $S^4$ , we obtain a conjugate branched superminimal surface,  $\alpha \circ f(\Sigma)$ , in  $S^4$ .*

*Proof.* Since  $f(\Sigma)$  is generic, it avoids the poles and hence its lift  $\tilde{f}(\Sigma)$  avoids  $L_1$  and  $L_2$ . Thus,  $\tilde{f}(\Sigma)$  is diffeomorphic to its image  $\tilde{f}'(\Sigma)$  in  $X$  under the blow up of  $\mathbf{P}^3$  along  $L_1$  and  $L_2$ . Now by notes 1–4 in §I.4, we have  $\tilde{\pi} \circ \tilde{f}'(\Sigma) = \tilde{\pi} \circ (\alpha \circ \tilde{f}'(\Sigma))$  and that  $\alpha \circ \tilde{f}(\Sigma)$  is holomorphic and horizontal in  $\mathbf{P}^3$  and thus projects to a branched superminimal surface in  $S^4$ , i.e. we obtain conjugate branched superminimal surfaces for free!  $\square$

**7. Bidegrees and ramification divisors.** Let  $f(\Sigma)$  be a generic branched superminimal surface in  $S^4$ . Its lift  $\tilde{f}(\Sigma)$  is a holomorphic horizontal curve  $\gamma$  in  $\mathbf{P}^3$ . The homology degree of  $\gamma \subset \mathbf{P}^3$  is the fundamental class  $[\gamma] \in H_2(\mathbf{P}^3; \mathbb{Z}) \cong \mathbb{Z}$ . This degree is also the intersection number of  $\gamma$  with a generic  $\mathbf{P}^2$  in  $\mathbf{P}^3$  (i.e. homology degree = algebraic degree). Let  $\tilde{\pi} = (\tilde{\pi}_1, \tilde{\pi}_2)$  denote the projection map of  $\mathbf{P}^3 - (L_1 \cup L_2)$  to  $\mathbf{P}^1 \times \mathbf{P}^1$ . Define  $f_1, f_2: \Sigma \rightarrow \mathbf{P}^1$  by  $f_1 := \tilde{\pi}_1 \circ \tilde{f}$  and  $f_2 := \tilde{\pi}_2 \circ \tilde{f}$ .

**Proposition 1.8.** *Suppose that  $\deg(\gamma) = d$ . Then the holomorphic curve  $C = \tilde{\pi} \circ \tilde{f}(\Sigma)$  in  $\mathbf{P}^1 \times \mathbf{P}^1$  has bidegree  $(d, d)$ , i.e.  $\deg f_1 = \deg f_2 = d$ . Furthermore,  $\text{Ram}(f_1) = \text{Ram}(f_2)$ .*

*Proof.* Let  $x_1 \in L_1$ . The fiber  $\tilde{\pi}_1^{-1}(x_1) \subset \mathbf{P}^3$  is the plane  $P_1 = \text{span}(x_1, L_2)$ . Since  $\deg \gamma = d$ ,  $P_1$  has  $d$  intersection points with  $\gamma$ . Similarly, for  $x_2 \in L_2$ , the plane  $P_2 = \tilde{\pi}_2^{-1}(x_2)$  has  $d$  intersection points with  $\gamma$ . Thus  $C = \tilde{\pi}(\gamma)$  has bidegree  $(d, d)$ .

Let  $z_0$  be a ramification point of  $f_1$ . Let  $p \in \gamma$  denote the point  $\tilde{f}(z_0)$ . Then the point  $x := \tilde{\pi}_1(p)$  is a branch point of  $f_1$ . Let  $y := \tilde{\pi}_2(p)$  and let  $L_{xy}$  denote the line in  $\mathbf{P}^3$  through  $x$  and  $y$ . Finally, let  $H_x$  denote the plane  $\{v \in T_p \mathbf{P}^3 \mid \tilde{\pi}_{1*}(v) = 0\}$ . Now  $x$  is a branch point of  $f_1$  and  $\gamma$  is an integral curve of  $\mathcal{H}_p$ , so the tangent line to the curve  $\gamma$  at  $p$  must be  $L_{xy}$ —the intersection of  $\mathcal{H}_p$  and  $H_x$ . We thus have  $\tilde{\pi}_{1*}(L_{xy}) = \tilde{\pi}_{2*}(L_{xy}) = 0$ . Hence,  $y$  is a branch point of  $f_2$  and so  $z_0$  is in the ramification locus of both  $f_1$  and  $f_2$ . By genericity,  $\text{Ram}(f_1) = \text{Ram}(f_2)$ .  $\square$

**Lemma 1.9.** *A holomorphic map  $F = (f_1, f_2): \Sigma \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$  has a canonical Gauss lift  $\tilde{F}$  to  $Y = \mathbf{PT}(\mathbf{P}^1 \times \mathbf{P}^1)$ .*

*Proof.* First suppose  $(df_1(z), df_2(z)) \neq (0, 0)$ . Then the lift is given by  $\tilde{F}(z) = (f_1(z), f_2(z), [f_1'(z), f_2'(z)])$ . We are thus left with a finite set of singular points. Without loss of generality, suppose 0 is a singular point. Then  $f_1'(z) = z^p g_1(z)$  and  $f_2'(z) = z^q g_2(z)$  for some  $p, q$  and where  $g_1(0) \neq 0$  and  $g_2(0) \neq 0$ . We may assume that  $1 \leq p \leq q$ . So

$$\tilde{F}(z) = (f_1(z), f_2(z), [g_1(z), z^{q-p} g_2(z)])$$

for  $z$  in a neighborhood of 0.  $\square$

**Proposition 1.10.** *Suppose  $f: \Sigma \looparrowright S^4$  is a generic superminimal immersion. Let  $\tilde{f}: \Sigma \rightarrow \mathbf{P}^3$  be the holomorphic horizontal lift of  $f$ , and let  $f_1 := \tilde{\pi}_1 \circ \tilde{f}$  and  $f_2 := \tilde{\pi}_2 \circ \tilde{f}$ . Suppose that  $\deg f_1 = \deg f_2 = d \geq 2$ . Then  $f_2 \neq A \circ f_1$  for any  $A \in \text{PSL}(2, \mathbb{C})$ .*

*Proof.* Suppose  $f_2 = A \circ f_1$  for some  $A \in \text{PSL}(2, \mathbb{C})$ . Then  $F = (f_1, f_2) = (f_1, A \circ f_1): \Sigma \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$  factors through  $\mathbf{P}^1$  as follows:

$$\Sigma \xrightarrow{f_1} \mathbf{P}^1 \xrightarrow{G=(\text{Id}, A)} \mathbf{P}^1 \times \mathbf{P}^1.$$

Since  $G$  has bidegree  $(1, 1)$ , it is nonsingular and its canonical lift  $\tilde{G}$  to  $Y$  avoids the two sections  $S_1$  and  $S_2$ . The map  $f_1$  is necessarily branched since  $\deg f_1 \geq 2$ . Hence, the canonical lift  $\tilde{F}$  of  $F$  is a branched covering map of  $\Sigma$  into  $\tilde{G}(\mathbf{P}^1) \cong \mathbf{P}^1$ , i.e.  $\tilde{F}(\Sigma)$  is branched. Consequently, its lift to  $\mathbf{P}^3$ ,  $\tilde{\tilde{F}}(\Sigma)$ , is branched and hence projects to a branched superminimal surface in  $S^4$ . This contradicts the assumption that  $f(\Sigma) \subset S^4$  is unbranched.  $\square$

Note that for  $d = 1$ ,  $\Sigma$  must have genus zero and so  $f(\Sigma)$  is totally geodesic in  $S^4$ .

We thus have

**Theorem A.** *Every superminimal immersion  $f: \Sigma \looparrowright S^4$  arises from a pair of meromorphic functions  $f_1, f_2$  on  $\Sigma$  such that*

- (1)  $\deg f_1 = \deg f_2 = d$  for some integer  $d \geq 1$ .
- (2)  $\text{Ram}(f_1) = \text{Ram}(f_2)$
- (3) For  $d \geq 2$ ,  $f_1 \neq A \circ f_2$  for any  $A \in \text{PSL}(2, \mathbb{C})$ .

We would like to generate superminimal surfaces in  $S^4$  by considering pairs of meromorphic functions on  $\Sigma$  which satisfy the three conditions in Theorem A. Suppose  $F = (f_1, f_2)$  is such a pair. Let  $\tilde{C} = \tilde{F}(\Sigma) \subset Y$ . Our degree computations in §I.5 show that the total preimage curve  $\gamma = \beta \circ \psi^{-1}(\tilde{C})$  in  $\mathbb{P}^3$  has degree  $2d$ . Suppose  $\gamma$  consists of 2 connected (or irreducible) components  $\gamma_1$  and  $\gamma_2$ . Then  $\alpha(\gamma_1) = \gamma_2$  and consequently  $\deg \gamma_1 = \deg \gamma_2 = d$ . Under suitable conditions (to be discussed later),  $\gamma_1$  and  $\gamma_2$  will project to a conjugate pair of superminimal surfaces in  $S^4$ .

## II. GENUS ZERO

**1. Meromorphic functions, Grassmannians and resultants.** Let  $f: \mathbb{P}^1 \rightarrow \mathbb{P}^1$  be a holomorphic map of degree  $d$  (i.e.  $f$  is a meromorphic function of degree  $d$ ). Then  $f$  can be expressed as a rational function of the form  $P(z)/Q(z)$  where  $P(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0$  and  $Q(z) = b_d z^d + b_{d-1} z^{d-1} + \cdots + b_1 z + b_0$ ,  $a_i, b_i \in \mathbb{C}$ . Note that the map  $f$  is of degree  $d$  if  $\min\{\deg P(z), \deg Q(z)\} = d$  and if the resultant of the 2 polynomials does not vanish. Let  $P = (a_d, a_{d-1}, \dots, a_1, a_0)$  and  $Q = (b_d, b_{d-1}, \dots, b_1, b_0)$  denote the coefficient vectors of  $P(z)$  and  $Q(z)$  respectively. Then the resultant  $\mathcal{R}(P, Q)$  of  $P(z)$  and  $Q(z)$  is the determinant of the matrix

$$M = \begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix}$$

where

$$A_1 = \begin{pmatrix} a_d & a_{d-1} & \cdots & a_1 \\ 0 & a_d & \cdots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_d \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{d-1} & a_{d-2} & \cdots & a_0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} b_d & b_{d-1} & \cdots & b_1 \\ 0 & b_d & \cdots & b_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_d \end{pmatrix}, \quad B_2 = \begin{pmatrix} b_0 & 0 & \cdots & 0 \\ b_1 & b_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{d-1} & b_{d-2} & \cdots & b_0 \end{pmatrix}.$$

The resultant is a homogeneous polynomial of bidegree  $(d, d)$  in the  $a_i$  and the  $b_j$ . Furthermore,  $\mathcal{R}(P, Q)$  is irreducible over any arbitrary field (cf. [18]). We thus require that  $(P, Q) \in \mathbb{C}^{d+1} \times \mathbb{C}^{d+1} - \mathcal{R}$ , where  $\mathcal{R}$  is the irreducible resultant divisor. Observe that  $(\lambda P, \lambda Q)$  describes the same function as  $(P, Q)$  for any  $\lambda \in \mathbb{C}^*$ . Thus the space of meromorphic functions of degree  $d$  is

$$M_d := \mathbf{P}(\mathbb{C}^{d+1} \times \mathbb{C}^{d+1} - \mathcal{R}) \subset \mathbf{P}^{2d+1}.$$

We next define an action of  $GL(2, \mathbb{C})$  on  $\mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$  as follows:

$$g \cdot (P, Q) := (\alpha P + \beta Q, \gamma P + \delta Q) \quad \text{for } g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2, \mathbb{C}).$$

Let  $N_d := \mathbb{C}^{d+1} \times \mathbb{C}^{d+1} - \Delta$  where  $\Delta = \{(P, Q) \mid P \wedge Q = 0\}$ . Observe that for  $(P, Q) \in N_d$ ,  $g \cdot (P, Q) = (\alpha P + \beta Q, \gamma P + \delta Q) = (P_1, Q_1)$ , and  $P_1 \wedge Q_1 = (\alpha P + \beta Q) \wedge (\gamma P + \delta Q) = (\alpha\delta - \beta\gamma)P \wedge Q \neq 0$ . Thus,  $GL(2, \mathbb{C})$  acts on  $N_d$ . In fact, we have a free action on  $N_d$ :  $g \cdot (P, Q) = (\alpha P + \beta Q, \gamma P + \delta Q) = (P, Q)$  implies that  $g = I$  since  $P \wedge Q \neq 0$ . Note that we can identify  $N_d$  with the Stiefel manifold of 2-frames in  $\mathbb{C}^{d+1}$ . For  $(P, Q) \in N_d$ , let  $[P \wedge Q]$  denote the 2-plane in  $\mathbb{C}^{d+1}$  spanned by  $P$  and  $Q$ . Let  $P_1, Q_1 \in [P \wedge Q]$ . Then  $P_1 = \alpha P + \beta Q$  and  $Q_1 = \gamma P + \delta Q$  for some  $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ . If  $P_1 \wedge Q_1 \neq 0$ , then  $0 \neq P_1 \wedge Q_1 = (\alpha\delta - \beta\gamma)P \wedge Q$ , i.e.  $(\alpha\delta - \beta\gamma) \neq 0$ . Thus,  $GL(2, \mathbb{C})$  acts transitively on pairs of noncollinear vectors in  $[P \wedge Q]$ . It follows that  $N_d/GL(2, \mathbb{C}) = G(2, d+1)$  and  $\pi: N_d \rightarrow G(2, d+1)$  is a principal  $GL(2, \mathbb{C})$ -bundle (where  $\pi(P, Q) = [P \wedge Q]$ ).

**Lemma 2.1.**  $\mathcal{R}(g \cdot (P, Q)) = (\det g)^d \mathcal{R}(P, Q)$ .

*Proof.* Let  $(\tilde{P}, \tilde{Q})$  denote  $g \cdot (P, Q)$ . The resultant of  $(\tilde{P}, \tilde{Q})$  is given by the determinant of the matrix

$$\tilde{M} = \begin{pmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{B}_1 & \tilde{B}_2 \end{pmatrix}.$$

Since  $(\tilde{P}, \tilde{Q}) = (\alpha P + \beta Q, \gamma P + \delta Q)$ , we observe that

$$\begin{aligned} \tilde{A}_1 &= \alpha A_1 + \beta B_1, & \tilde{A}_2 &= \alpha A_2 + \beta B_2, \\ \tilde{B}_1 &= \gamma A_1 + \delta B_1, & \tilde{B}_2 &= \gamma A_2 + \delta B_2, \end{aligned}$$

i.e.

$$\begin{pmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{B}_1 & \tilde{B}_2 \end{pmatrix} = \begin{pmatrix} \alpha I & \beta I \\ \gamma I & \delta I \end{pmatrix} \cdot \begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix}$$

where  $I \in GL(d, \mathbb{C})$  is the identity matrix. It is straightforward to verify that

$$\det \begin{pmatrix} \alpha I & \beta I \\ \gamma I & \delta I \end{pmatrix} = (\alpha\delta - \beta\gamma)^d = (\det g)^d.$$

Thus,  $\det \tilde{M} = (\det g)^d \cdot \det M$ , i.e.  $\mathcal{R}(g \cdot (P, Q)) = (\det g)^d \cdot \mathcal{R}(P, Q)$ .  $\square$

It follows that  $\mathcal{R} \subset \mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$  is fixed under the action of  $GL(2, \mathbb{C})$ . Let  $\text{Reg}(\mathcal{R})$  denote the regular part of  $\mathcal{R}$ . Since  $\mathcal{R}$  is irreducible,  $\text{Reg}(\mathcal{R})$

is connected. Note that  $\Delta = \{(P, Q) \mid P \wedge Q = 0\} \subset \mathcal{R}$  and that  $\Delta$  has codimension  $d$  in  $\mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$ . So  $\Delta$  cannot disconnect  $\text{Reg}(\mathcal{R})$  (which has dimension  $2d + 1$ ). Consequently,  $(\text{Reg}(\mathcal{R})) \cap N_d$  is connected, i.e.  $\mathcal{R} \cap N_d$  is irreducible. For ease of notation, we shall let  $\mathcal{R}$  to denote  $\mathcal{R} \cap N_d$  also. By Lemma 2.1,  $\dim(\mathcal{R}/GL(2, \mathbb{C})) = \dim(\pi(\mathcal{R})) = 2d - 3$ . Furthermore, since  $\text{Reg}(\mathcal{R})$  is connected and  $\pi: N_d \rightarrow G(2, d+1)$  is a principal  $GL(2, \mathbb{C})$ -bundle,  $\pi(\text{Reg}(\mathcal{R})) = \text{Reg}(\pi(\mathcal{R}))$  is connected. Thus,  $\pi(\mathcal{R})$  is an irreducible divisor in  $G(2, d+1)$ .

Observe that the space of meromorphic functions of degree  $d$  is  $M_d = \mathbf{P}(N_d - \mathcal{R})$ . We thus have a free action of  $PSL(2, \mathbb{C})$  on  $M_d$ . Furthermore,  $M_d/PSL(2, \mathbb{C}) \subset G(2, d+1)$ , the Grassmannian of 2-planes in  $\mathbb{C}^{d+1}$ .

**2. The ramification divisor.** Let  $f: \mathbf{P}^1 \rightarrow \mathbf{P}^1$  be a holomorphic map of degree  $d$ . Recall that  $z_0 \in \mathbf{P}^1$  is a ramification point of  $f$  if  $f_*(v) = 0$  for all  $v \in T_{z_0} \mathbf{P}^1$ . Expressing  $f$  as a rational function  $P(z)/Q(z)$ , we have  $f'(z) = (Q(z)P'(z) - P(z)Q'(z))/(Q(z))^2$ . Then the ramification points of  $f$  are given by the zero locus of  $Q(z)P'(z) - P(z)Q'(z)$ , a polynomial of degree  $2d - 2$ . Observe that if  $\deg(Q(z)P'(z) - P(z)Q'(z)) = k < 2d - 2$ , then  $\infty$  is a ramification point of order  $2d - 2 - k$ .

Define a map  $\Psi^d: M_d = \mathbf{P}(N_d - \mathcal{R}) \rightarrow \mathbf{P}^{2d-2}$  by

$$[(P, Q)] \mapsto [\text{coeff}\{Q(z)P'(z) - P(z)Q'(z)\}],$$

where  $\text{coeff}\{R(z)\}$  denotes the coefficient vector of the polynomial  $R(z)$ . The ramification map  $\Psi^d$  is well defined since

$$\begin{aligned} (\lambda P, \lambda Q) &\mapsto [\lambda^2 \cdot \text{coeff}\{Q(z)P'(z) - P(z)Q'(z)\}] \\ &= [\text{coeff}\{Q(z)P'(z) - P(z)Q'(z)\}], \end{aligned}$$

and if  $Q(z)P'(z) - P(z)Q'(z) \equiv 0$ , we have

$$\frac{P'(z)}{P(z)} = \frac{Q'(z)}{Q(z)}, \quad \text{i.e.} \quad \log P(z) = \log Q(z) + C = \log(\tilde{C}Q(z)).$$

Thus  $P(z) = \tilde{C}Q(z)$  and so  $[(P, Q)] \notin M_d$ .

**Lemma 2.2.**  $PSL(2, \mathbb{C})$  preserves the fibers of  $\Psi^d$

*Proof.* Let  $g \in PSL(2, \mathbb{C})$ . Let  $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$  be a representative of  $g$ . Then

$$\begin{aligned} \Psi^d(g \cdot [(P, Q)]) &= \Psi^d([\alpha P(z) + \beta Q(z), \gamma P(z) + \delta Q(z)]) \\ &= [\text{coeff}\{(\gamma P(z) + \delta Q(z))(\alpha P'(z) + \beta Q'(z)) \\ &\quad - (\alpha P(z) + \beta Q(z))(\gamma P'(z) + \delta Q'(z))\}] \\ &= [\text{coeff}\{(\alpha\delta - \beta\gamma)(Q(z)P'(z) - P(z)Q'(z))\}] \\ &= [\text{coeff}\{Q(z)P'(z) - P(z)Q'(z)\}] \\ &= \Psi^d([(P, Q)]). \quad \square \end{aligned}$$

**Corollary 2.3.**  $PSL(2, \mathbb{C})$  acts freely on the fibers of  $\Psi^d$ .

*Proof.*  $PSL(2, \mathbb{C})$  acts freely on  $M_d = \mathbb{P}(\mathbb{C}^{d+1} \times \mathbb{C}^{d+1} - \mathcal{R})$ , and by Lemma 2.2, it preserves fibers.  $\square$

We thus have an induced map  $\Psi_d: G(2, d+1) \rightarrow \mathbb{P}^{2d-2}$  where

$$[P \wedge Q] \mapsto [\text{coeff}\{Q(z)P'(z) - P(z)Q'(z)\}].$$

This map is well defined.

Note that for  $d = 2$ ,  $G(2, 3) \cong G(1, 3) = \mathbb{P}^2$ .

**Proposition 2.4.**  $\Psi_2: G(2, 3) \cong \mathbb{P}^2 \rightarrow \mathbb{P}^2$  is a biholomorphism.

*Proof.* Let  $[P \wedge Q] \in G(2, 3)$ . Then  $[P \wedge Q]$  can be represented by one of the following matrices:

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \end{pmatrix}, \quad \begin{pmatrix} 1 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where  $P$  and  $Q$  correspond to the rows of the matrices. For the first matrix,  $P(z) = z^2 + a$ , and  $Q(z) = z + b$ . Then

$$\begin{aligned} \Psi_2([P \wedge Q]) &= [\text{coeff}\{Q(z)P'(z) - P(z)Q'(z)\}] \\ &= [\text{coeff}\{(z+b)(2z) - (z^2+a)\}] = [1, 2b, -a] \end{aligned}$$

i.e.

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \end{pmatrix} \mapsto [1, 2b, -a].$$

Similarly, we have

$$\begin{pmatrix} 1 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto [0, 2, a] \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto [0, 0, 1].$$

Note that in the second case,  $\infty$  is a ramification point and that the third case is a degenerate case since  $(P, Q) \in \mathcal{R}$ . From the explicit computations, it is clear that  $\Psi_2$  is one-to-one, nonsingular and is hence a biholomorphism.  $\square$

A consequence of the proposition is that  $\Psi^2: M_2 \rightarrow \mathbb{P}^2$  has connected fibers. Thus,

**Corollary 2.5.** Let  $f$  be a meromorphic function of degree 2. Let  $g$  be any other meromorphic function of degree 2 with the property that  $\text{Ram}(f) = \text{Ram}(g)$ . Then  $g = A \circ f$  for some  $A \in PSL(2, \mathbb{C})$ .

**Corollary 2.6.** There is no superminimal surface in  $S^4$  whose lifting to  $\mathbb{P}^3$  is a curve of degree 2.

*Proof.* The genus 0 case follows immediately from Proposition 1.10 and Corollary 2.5. The following argument proves the general case. Let  $\gamma$  be a holomorphic horizontal curve in  $\mathbb{P}^3$  of degree 2. Suppose  $\gamma$  is not a projective line. Pick any 3 noncollinear points  $A, B, C$  on  $\gamma$ . Let  $L_{AB}$  and  $L_{AC}$  denote the

lines through  $A&B$  and  $A&C$  respectively. Let  $P$  denote the plane spanned by these two lines. Since  $\deg(\gamma) = 2$  and  $P$  contains the points  $A$ ,  $B$  and  $C$ , necessarily,  $\gamma \subset P$ , i.e.  $\gamma$  is planar. Since there are no horizontal planes in  $\mathbf{P}^3$  (otherwise, that horizontal  $\mathbf{P}^2$  would be diffeomorphic to  $S^4$ !),  $P$  (and hence  $\gamma$ ) is in fact a projective line. Since  $\deg(\gamma) = 2$ ,  $\gamma$  is necessarily branched. (Nevertheless,  $\gamma$  projects to a totally geodesic surface in  $S^4$ .)  $\square$

**3. The orbits in the fibers of  $\Psi^d$ .** Let  $N = \frac{1}{2}(d+2)(d-1) = \binom{d+1}{2} - 1 = \dim(\mathbf{P}(\wedge^2 \mathbb{C}^{d+1}))$ . Let  $P = (a_d, \dots, a_0)$  and  $Q = (b_d, \dots, b_0)$  be two vectors in  $\mathbb{C}^{d+1}$  which span a plane,  $\binom{P}{Q}$ , in  $\mathbb{C}^{d+1}$ . Then the Plücker embedding  $G(2, d+1) \hookrightarrow \mathbf{P}^N = \mathbf{P}(\wedge^2 \mathbb{C}^{d+1})$  is given by  $\binom{P}{Q} \mapsto [P \wedge Q]$ . Choose Plücker coordinates  $x_{ij}$  on  $\mathbf{P}^N$  where  $i > j$ ,  $i = 1, \dots, d$ ,  $j = 0, \dots, d-1$ . Let  $P(z) = a_d z^d + \dots + a_1 z + a_0$  and  $Q(z) = b_d z^d + \dots + b_0$ . Then

$$Q(z)P'(z) - P(z)Q'(z) = \alpha_{2d-2} z^{2d-2} + \dots + \alpha_n z^n + \dots + \alpha_1 z + \alpha_0$$

where

$$\alpha_n = \sum_{\substack{i+j=n+1 \\ i>j}} (i-j)x_{ij}, \quad n = 0, \dots, 2d-2.$$

Consider the linear map  $L: \mathbb{C}^{N+1} \rightarrow \mathbb{C}^{2d-1}$  given by

$$(x_{ij}) \mapsto (\alpha_{2d-2}, \dots, \alpha_n, \dots, \alpha_0).$$

Observe that since  $\alpha_n$  contains only the  $x_{ij}$ 's which satisfy the condition  $i + j = n + 1$ ,  $L$  has maximal rank. Let  $K$  denote the kernel of  $L$ . Then  $\dim K = \frac{1}{2}(d^2 + d) - 2d + 1 = \frac{1}{2}(d-2)(d-1)$ . Let  $\kappa := \mathbf{P}K$ , a projective  $\frac{1}{2}d(d-3)$ -plane in  $\mathbf{P}^N$ . Note that the image of  $G(2, d+1)$  in  $\mathbf{P}^N$ ,  $G^{2d-2}$ , does not intersect  $\kappa$  by construction. Thus the map  $\Psi_d$  can be given in Plücker coordinates by

$$\Psi_d([P \wedge Q]) = [\alpha_{2d-2}, \dots, \alpha_n, \dots, \alpha_0].$$

So  $\Psi_d$  can be thought of as the restriction to  $G^{2d-2}$  of a “map” from  $\mathbf{P}^N$  to  $\mathbf{P}^{2d-2}$ . We can extend  $\Psi_d$  to a map from  $\mathbf{P}^N - \kappa$  to  $\mathbf{P}^{2d-2}$ . Let  $\tilde{\mathbf{P}}^N$  denote the blow-up of  $\mathbf{P}^N$  along  $\kappa$ . Let  $q \in \mathbf{P}^{2d-2}$ . Let  $\tilde{\Psi}_d$  denote the map induced on  $\tilde{\mathbf{P}}^N$ . Then  $\Lambda_q = (\tilde{\Psi}_d^{-1})(q)$  is a projective  $\frac{1}{2}(d-2)(d-1)$ -plane in  $\mathbf{P}^N$ , i.e. a plane of dimension complementary to that of  $G^{2d-2}$ . Therefore the number of points of intersection of  $\Lambda_q$  with  $G^{2d-2}$  is the degree of  $G^{2d-2}$  in  $\mathbf{P}^N$ , which is  $(2d-2)!/(d-1)!d!$ . As a consequence, there are generically  $(2d-2)!/(d-1)!d!$  distinct  $PSL(2, \mathbb{C})$ -orbits of holomorphic maps of degree  $d$  from  $\mathbf{P}^1$  to  $\mathbf{P}^1$  which have the same ramification divisor. We thus have

**Theorem B.** *Let  $f$  be a generic meromorphic function of degree  $d \geq 2$ . Then, under the action of  $PSL(2, \mathbb{C})$ , there are  $(2d-2)!/(d-1)!d!$  distinct orbits of meromorphic functions of degree  $d$  with ramification divisor  $\text{Ram}(f)$ .*

Note that when  $d = 2$  we have only 1 orbit. This is consistent with our previous result (Corollary 2.5).

**4. The space  $\mathfrak{H}_d$ .** Let  $F = (f_1, f_2): \mathbf{P}^1 \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$  be a holomorphic map of bidegree  $(d, d)$  such that  $\text{Ram}(f_1) = \text{Ram}(f_2)$ . By our previous results, the curve  $\tilde{F}(\mathbf{P}^1) \subset Y = \mathbf{PT}(\mathbf{P}^1 \times \mathbf{P}^1)$  avoids the 2 distinguished sections,  $S_1$  and  $S_2$  of  $Y$ . Since  $\psi: \tilde{\mathbf{P}}^3 - (\sigma_1 \cup \sigma_2) \rightarrow Y - (S_1 \cup S_2)$  is a covering map of degree 2 and since  $\pi_1(\mathbf{P}^1) = 0$ , the map  $\tilde{F}$  lifts to a map  $\tilde{\tilde{F}}: \mathbf{P}^1 \rightarrow \tilde{\mathbf{P}}^3 - (\sigma_1 \cup \sigma_2)$ . Let  $\gamma_1 := \beta \circ \tilde{\tilde{F}}(\mathbf{P}^1)$  and  $\gamma_2 := \beta \circ \alpha \circ \tilde{\tilde{F}}(\mathbf{P}^1) = \alpha(\gamma_1)$ . Then  $\gamma_1$  and  $\gamma_2$  project to a conjugate pair of branched superminimal surfaces,  $\Sigma_1$  and  $\Sigma_2$ , in  $S^4$ . If  $\tilde{F}$  is an immersion, then the pair of surfaces are unbranched. We also showed that for  $d \geq 2$ , a necessary condition for  $\Sigma_1$  and  $\Sigma_2$  to be unbranched is that  $f_1$  and  $f_2$  belong to different orbits of  $PSL(2, \mathbb{C})$ . Our search for unbranched superminimal surfaces is thus aided by the following immediate consequence of Theorem B:

**Theorem C.** *For each  $d \geq 3$ , there is a branched superminimal surface of genus 0 in  $S^4$  which arises from a pair of meromorphic functions  $(f_1, f_2)$ , each of degree  $d$  such that  $\text{Ram}(f_1) = \text{Ram}(f_2)$  and that  $f_1$  and  $f_2$  belong to distinct  $PSL(2, \mathbb{C})$ -orbits.*

*Proof.* By Theorem B, there are  $(2d - 2)!/(d - 1)!d!$  distinct orbits for each generic ramification divisor.  $\square$

Recall that a branched superminimal immersion of  $S^2$  into  $S^4$  is just a harmonic map. Also, a (branched) superminimal surface of degree  $d$  in  $S^4$  is a surface of area  $4\pi d$  whose lifting to  $\mathbf{P}^3$  is a holomorphic, horizontal curve of degree  $d$ . We say that a harmonic map  $f: S^2 \rightarrow S^4$  has *harmonic degree*  $d$  if  $f(S^2)$  has area  $4\pi d$ . Let  $\mathfrak{H}_d$  denote the space of harmonic maps of  $S^2$  into  $S^4$  of harmonic degree  $d$ .

**Theorem D.** *For each  $d \geq 1$ ,  $\mathfrak{H}_d$  is parametrized by a space of complex dimension  $2d + 4$ .*

*Proof.* A meromorphic function of degree  $d$  is determined by  $2d + 1$  complex parameters. The theorem follows immediately from the fact that the fibers of  $\Psi^d$  are 3-dimensional.  $\square$

*Note.* Theorem D is in agreement with the results of Verdier [17]. Verdier in fact shows that  $\mathfrak{H}_d$  is naturally equipped with the structure of a complex algebraic variety of pure dimension  $2d + 4$ , and for  $d \geq 3$ ,  $\mathfrak{H}_d$  possesses three irreducible components. We will show that  $\mathfrak{H}_d$  is connected.

**5. Connectivity of  $\mathfrak{H}_d$ .** Recall that a meromorphic function of degree  $d$  can be considered as an element of  $M_d = \mathbf{P}(N_d) - \mathcal{R}$  where  $N_d = \mathbb{C}^{d+1} \times \mathbb{C}^{d+1} - \{(P, Q) \mid P \wedge Q = 0\}$  and where  $\mathcal{R}$  is the resultant divisor. We have a ramification map  $\Psi^d: M_d \rightarrow \mathbf{P}^{2d-2}$ . The action of  $PSL(2, \mathbb{C})$  on  $M_d$  induces a

map  $\Psi_d: G(2, d+1) - \pi(\mathcal{R}) \rightarrow \mathbf{P}^{2d-2}$ , where  $\pi(\mathcal{R}) = \mathcal{R}/PSL(2, \mathbb{C})$  is an irreducible variety of codimension 1. For ease of notation, we will let  $\mathcal{R}$  and  $\mathcal{R}'$  denote  $\pi(\mathcal{R})$  and  $\Psi_d(\pi(\mathcal{R}))$  respectively for the rest of this section. Now,  $\Psi_d: G(2, d+1) \rightarrow \mathbf{P}^{2d-2}$  is a branched covering map. Let  $\mathfrak{R}$  and  $\mathfrak{B}$  denote the ramification locus of  $\Psi_d$  and the branch locus of  $\Psi_d$  respectively. Then

$$\Psi_d: G(2, d+1) - \mathfrak{R} - \mathcal{R} \rightarrow \mathbf{P}^{2d-2} - \mathfrak{B} - \mathcal{R}'$$

is a covering map. Now consider the diagonal map

$$\delta: \mathbf{P}^{2d-2} \rightarrow \mathbf{P}^{2d-2} \times \mathbf{P}^{2d-2}.$$

Let  $\mathcal{M}_d := G(2, d+1) - \mathcal{R}$ . From the diagram

$$\begin{array}{ccc} \delta^*(\mathcal{M}_d \times \mathcal{M}_d) & & \mathcal{M}_d \times \mathcal{M}_d \\ \downarrow & & \downarrow \Psi_d \times \Psi_d \\ \mathbf{P}^{2d-2} & \xrightarrow{\delta} & \mathbf{P}^{2d-2} \times \mathbf{P}^{2d-2} \end{array}$$

we see that modulo the action of  $PSL(2, \mathbb{C})$ , an element of  $\delta^*(\mathcal{M}_d \times \mathcal{M}_d)$  is a pair of meromorphic functions of degree  $d$  with the same ramification divisor. We will show that the space  $\delta^*(\mathcal{M}_d \times \mathcal{M}_d)$  is connected and as a consequence  $\mathfrak{H}_d$ , the space of pairs of meromorphic functions of degree  $d$  with the same ramification divisor, is connected.

**Lemma 2.7.**  $\mathcal{R}$  is not a component of  $\mathfrak{R}$ . Thus,  $\dim(\mathfrak{R} \cap \mathcal{R}) \leq 2d - 4$ .

*Proof.* In §II.1, we showed that  $\mathcal{R}$  is irreducible. Thus, it suffices to show that there exists an  $x \in \mathcal{R}$  such that  $x \notin \mathfrak{R}$ . Now in ambient coordinates,

$$\Psi^d(P, Q) = \Psi^d(a_d, \dots, a_0, b_d, \dots, b_0) = (c_{2d-2}, \dots, c_0)$$

where

$$\begin{aligned} c_m &= \sum_{j=0}^{m+1} (2j - m - 1) a_j b_{m-j+1} \\ &= \sum_{k=0}^{m+1} (m - 2k + 1) a_{m-k+1} b_k, \quad m = 0, \dots, 2d - 2. \end{aligned}$$

Thus,

$$\frac{\partial c_m}{\partial a_j} = \begin{cases} (2j - m - 1) b_{m-j+1}, & \text{for } j = 0, \dots, m+1; m-j+1 \leq d, \\ 0, & \text{for } j > m+1, \end{cases}$$

and

$$\frac{\partial c_m}{\partial b_k} = \begin{cases} (m - 2k + 1) a_{m-k+1}, & \text{for } k = 0, \dots, m+1; m-k+1 \leq d, \\ 0, & \text{for } k > m+1. \end{cases}$$

Let  $P(z) = z^d + z^2$ ,  $Q(z) = z$ . Certainly  $[P \wedge Q] \in \mathcal{R} \subset G(2, d+1)$ . Then

$$\left. \frac{\partial c_m}{\partial a_j} \right|_{(P,Q)} \neq 0, \quad \text{if } j = m = 0, 2, 3, \dots, d.$$

Also,

$$\left. \frac{\partial c_m}{\partial b_k} \right|_{(P,Q)} \neq 0, \quad \text{if } m = d + k - 1, \text{ or } m = k + 1$$

i.e. this derivative does not vanish for  $k = 0, m = 1$ ;  $k = 0, m = d - 1$ ;  $k = 1, m = d$ ;  $\dots$ ;  $k = d - 1, m = 2d - 2$ . Consequently,  $d\Psi^d|_{(P,Q)}$  has maximal rank. Thus,  $[P \wedge Q] \notin \mathfrak{R}$ .  $\square$

Recall that an element of  $\delta^*(\mathcal{M}_d \times \mathcal{M}_d)$  is (up to a Möbius transformation) a pair of meromorphic functions of degree  $d$  with the same ramification divisor. Thus, if  $q \in \mathcal{M}_d$ , the diagonal pair  $(q, q)$  is obviously in  $\delta^*(\mathcal{M}_d \times \mathcal{M}_d)$ . Since  $\mathcal{M}_d$  is connected, it is clear that a diagonal point  $(q, q) \in \delta^*(\mathcal{M}_d \times \mathcal{M}_d)$  is path connected to any other diagonal point  $(q', q') \in \delta^*(\mathcal{M}_d \times \mathcal{M}_d)$ . Thus, to show that  $\delta^*(\mathcal{M}_d \times \mathcal{M}_d)$  is path connected, it suffices to show that any point  $(x, y) \in \delta^*(\mathcal{M}_d \times \mathcal{M}_d)$  is path connected to the point  $(y, y)$ .

Now let  $(x, y) \in \delta^*(\mathcal{M}_d \times \mathcal{M}_d)$ . Let  $\Psi_d(x) = \Psi_d(y) = \star \in \mathbf{P}^{2d-2} - \mathcal{R}'$ . Without loss of generality,  $\star \in \mathbf{P}^{2d-2} - \mathfrak{B} - \mathcal{R}'$ , and so,  $x, y \notin \mathfrak{R}$ . (If  $\star \in \mathfrak{B}$ , we can find a path  $C$  in  $\mathbf{P}^{2d-2} - \mathcal{R}'$  so that  $C(0) = \star$  and  $C(1) = \star' \notin \mathfrak{B}$ ). Since  $G(2, d+2) - \mathcal{R} - \mathfrak{R}$  is connected, there is a path  $\tilde{\gamma} \subset G(2, d+1) - \mathcal{R} - \mathfrak{R}$  so that  $\tilde{\gamma}(0) = x$ ,  $\tilde{\gamma}(1) = y$ . Then  $\gamma := \Psi_d(\tilde{\gamma})$  is a based loop in  $\mathbf{P}^{2d-2} - \mathfrak{B} - \mathcal{R}'$ , i.e.  $[\gamma] \in \pi_1(\mathbf{P}^{2d-2} - \mathfrak{B} - \mathcal{R}', \star)$ . Thus  $\gamma: S^1 \rightarrow \mathbf{P}^{2d-2} - \mathfrak{B} - \mathcal{R}' \subset \mathbf{P}^{2d-2}$ . Since  $\mathbf{P}^{2d-2}$  is simply connected, we can extend  $\gamma$  to a map  $\gamma': D^2 \rightarrow \mathbf{P}^{2d-2}$ . By Thom transversality and Lemma 2.7, we can make  $\gamma'$  transversal to  $\text{Reg}(\mathfrak{B})$ ,  $\text{Reg}(\mathcal{R}')$  and  $\Psi_d(\mathfrak{R} \cap \mathcal{R}) = \mathfrak{B} \cap \mathcal{R}'$ , i.e.

$$\gamma'(D^2) \cap \{\text{Sing}(\mathfrak{B}) \cup \text{Sing}(\mathcal{R}') \cup \{\mathfrak{B} \cap \mathcal{R}'\}\} = \emptyset.$$

Then  $\gamma'(D^2)$  intersects  $\text{Reg}(\mathfrak{B})$  and  $\text{Reg}(\mathcal{R}')$  in a finite number of points, say  $\gamma'(D^2) \cap \text{Reg}(\mathfrak{B}) = \{z_1, \dots, z_n\}$  and  $\gamma'(D^2) \cap \text{Reg}(\mathcal{R}') = \{\zeta_1, \dots, \zeta_m\}$  where  $z_i \neq \zeta_j$  for any  $i, j$ . Let  $\sigma_i$  and  $\tau_j$  be tiny based loops around  $z_i$  and  $\zeta_j$  respectively. Then  $\gamma$  is homotopic to a composition of the  $\sigma_i$ 's and the  $\tau_j$ 's. Observe that the  $\tau_j$ 's act trivially on  $F = \Psi_d^{-1}(\star)$ . Let  $x_1 := x$  and  $x_{n+1} := y$ . Since  $[\gamma](x) = y$ , we have  $[\sigma_1](x_1) = x_2$ ,  $[\sigma_2](x_2) = x_3$ ,  $\dots$ ,  $[\sigma_n](x_n) = x_{n+1} = y$  for some  $x_2, \dots, x_n \in F$ . Let  $\tilde{\sigma}_i$  be the lifting of  $\sigma_i$  so that  $\tilde{\sigma}_i(0) = x_i$  and  $\tilde{\sigma}_i(1) = x_{i+1}$ . As  $\sigma_i$  traces along the boundary of a tiny disc  $D_i$  around the branch point  $z_i$ ,  $\tilde{\sigma}_i$  traces a path around some ramification point  $y_i \in \Psi_d^{-1}(z_i)$ . Let  $\tilde{D}_i$  denote the contractible disc in  $G(2, d+1) - \mathcal{R}$  around  $y_i$  which projects to  $D_i$ . Suppose  $\sigma_i(t)$  traces  $\partial D_i$  for  $t \in [t_{\alpha_i}, t_{\beta_i}]$ . Let  $u_i = \tilde{\sigma}_i(t_{\alpha_i})$  and  $v_i = \tilde{\sigma}_i(t_{\beta_i})$ . Let  $\tilde{\alpha}_i$  be a path from  $u_i$  to  $y_i$  and let  $\tilde{\beta}_i$  be

a path from  $y_i$  to  $v_i$ . Say  $\tilde{\alpha}_i(t_{\alpha_i}) = u_i$ ,  $\tilde{\beta}_i(t_{\beta_i}) = v_i$  and  $\tilde{\alpha}_i(t_{\varepsilon_i}) = \tilde{\beta}_i(t_{\varepsilon_i}) = y_i$  for some  $t_{\varepsilon_i} \in (t_{\alpha_i}, t_{\beta_i})$ . Consider the modified path  $\tilde{\sigma}'_i$  defined as follows:

$$\tilde{\sigma}'_i(t) = \begin{cases} \tilde{\sigma}_i(t), & \text{for } t \in [0, t_{\alpha_i}], \\ \tilde{\alpha}_i(t), & \text{for } t \in [t_{\alpha_i}, t_{\varepsilon_i}], \\ \tilde{\beta}_i(t), & \text{for } t \in [t_{\varepsilon_i}, t_{\beta_i}], \\ \tilde{\sigma}_i(t), & \text{for } t \in [t_{\beta_i}, 1]. \end{cases}$$

Let  $\sigma'_i := \Psi_d(\tilde{\sigma}'_i)$ . Observe that  $\sigma'_i$  is a homotopically trivial loop in  $\mathbf{P}^{2d-2} - \mathcal{R}'$ . Let  $\tilde{\sigma}''_i$  denote the lifting of  $\sigma'_i$  so that  $\tilde{\sigma}''_i(0) = \tilde{\sigma}''_i(1) = y$ . Let  $\gamma_i$  denote the path  $(\tilde{\sigma}'_i, \tilde{\sigma}''_i)$  in  $\delta^*(\mathcal{M}_d \times \mathcal{M}_d)$  from  $(x_i, y)$  to  $(x_{i+1}, y)$ . We have thus constructed a path  $\gamma_n \circ \gamma_{n-1} \circ \cdots \circ \gamma_1$  in  $\delta^*(\mathcal{M}_d \times \mathcal{M}_d)$  from  $(x, y)$  to  $(y, y)$ . Thus,

**Theorem E.** For each  $d \geq 1$ ,  $\mathfrak{H}_d$  is connected.

**6. Examples.** Consider the map  $F_d = (f_1, f_2): \mathbf{P}^1 \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$  ( $d > 2$ ) where

$$f_1(z) = \frac{P_1(z)}{Q_1(z)} = \frac{z^d + dz + 1}{z^{d-1} + z + (d-2)} \quad \text{and}$$

$$f_2(z) = \frac{P_2(z)}{Q_2(z)} = \frac{z^d - dz + 1}{z^{d-1} + z - (d-2)}.$$

We will show that for  $d > 2$ ,  $F_d$  gives rise to a conjugate pair of unbranched superminimal surfaces in  $S^4$ .

Observe that  $f_1$  and  $f_2$  belong to different  $PSL(2, \mathbb{C})$ -orbits.

**Lemma 2.8.** For  $d > 2$ ,  $F_d$  has bidegree  $(d, d)$ . Furthermore,  $\text{Ram}(f_1) = \text{Ram}(f_2)$ .

*Proof.* We must first show that  $P_i(z)$  and  $Q_i(z)$  have no common zeroes ( $i = 1, 2$ ).

Suppose  $\zeta$  is a common zero of  $P_1(z)$  and  $Q_1(z)$ . Certainly  $\zeta$  must be a zero of  $P(z) = zQ_1(z) - P_1(z) = z^2 - 2z - 1$ . But  $P(z)$  has roots  $1 \pm \sqrt{2}$  which are certainly not roots of  $P_1(z)$  or  $Q_1(z)$ . Thus,  $\deg(f_1) = d$ . A similar argument shows that  $\deg(f_2) = d$ . Now

$$f'_1(z) = \frac{R(z)}{Q_1^2(z)} = \frac{z^{2d-2} + (d-1)z^d - (d-1)z^{d-2} + d(d-2) - 1}{[z^{d-1} + z + (d-2)]^2}$$

and

$$f'_2(z) = \frac{R(z)}{Q_2^2(z)} = \frac{z^{2d-2} + (d-1)z^d - (d-1)z^{d-2} + d(d-2) - 1}{[z^{d-1} + z - (d-2)]^2}.$$

Thus,  $\text{Ram}(f_1) = \text{Ram}(f_2)$ .  $\square$

**Proposition 2.9.** *The map  $F_d$  is generically one-to-one onto its image. Hence, it is not a branched covering map.*

*Proof.*

$$F_d(0) = \left( \frac{1}{d-2}, \frac{-1}{d-2} \right).$$

Note that 0 is not a ramification point of either  $f_1$  or  $f_2$ . We shall compute

$$F_d^{-1} \left( \frac{1}{d-2}, \frac{-1}{d-2} \right).$$

This amounts to solving the simultaneous equations

$$\frac{z^d + dz + 1}{z^{d-1} + z + (d-2)} = \frac{1}{d-2} \quad \text{and} \quad \frac{z^d - dz + 1}{z^{d-1} + z - (d-2)} = \frac{-1}{d-2}.$$

We obtain

$$\begin{aligned} (d-2)(z^d + dz + 1) - (z^{d-1} + z + (d-2)) &= 0 \quad \text{and} \\ (d-2)(z^d - dz + 1) - (z^{d-1} + z - (d-2)) &= 0. \end{aligned}$$

Thus, we have to solve the simultaneous equations

$$\begin{aligned} g_1(z) &= (d-2)z^d - z^{d-1} + (d(d-2) - 1)z = 0 \quad \text{and} \\ g_2(z) &= (d-2)z^d + z^{d-1} - (d(d-2) - 1)z = 0. \end{aligned}$$

Observe that if  $\zeta$  is a common zero of  $g_1$  and  $g_2$ , then certainly it is a zero of  $(g_1 + g_2)(z) = 2(d-2)z^d$  ( $d > 2$ ). But  $g_1 + g_2$  has 0 as its only zero. Thus

$$F_d^{-1} \left( \frac{1}{d-2}, \frac{-1}{d-2} \right) = \{0\},$$

i.e.  $F_d$  is generically one to one onto its image.  $\square$

**Proposition 2.10.** *The map  $\tilde{F}_d: \mathbf{P}^1 \rightarrow \mathbf{P}T(\mathbf{P}^1 \times \mathbf{P}^1)$  is nonsingular.*

*Proof.* It suffices to show that  $\tilde{F}_{d*}$  does not vanish at the ramification points. We will consider three cases.

*Case 1.* Assume that the zeroes of  $Q_1(z)$  and  $Q_2(z)$  are not ramification points. Then  $\tilde{F}_d$  can be described locally by

$$\tilde{F}_d(z) = (f_1(z), f_2(z), G(z))$$

where

$$G(z) = \frac{f'_1(z)}{f'_2(z)} = \left( \frac{z^{d-1} + z - (d-2)}{z^{d-1} + z + (d-2)} \right)^2.$$

It suffices to show that  $G'$  does not vanish at the ramification points. Now

$$G'(z) = 2 \left( \frac{z^{d-1} + z - (d-2)}{(z^{d-1} + z + (d-2))^3} \right) \cdot 2(d-2)h(z)$$

where  $h(z) = (d-1)z^{d-2} + 1$ . Observe that  $h(z)$  vanishes when  $z^{d-2} = -1/(d-1)$ . Let  $\zeta$  be a  $(d-2)$ th root of  $-1/(d-1)$ . Then

$$\begin{aligned} R(\zeta) &= \zeta^{2d-2} + (d-1)\zeta^d - (d-1)\zeta^{d-2} + d(d-2) - 1 \\ &= \zeta^2(\zeta^{2(d-2)} + (d-1)\zeta^{d-2}) - (d-1)\zeta^{d-2} + d(d-2) - 1 \\ &= \zeta^2\left(\left(\frac{1}{d-1}\right)^2 - 1\right) + d(d-2) \neq 0. \end{aligned}$$

Thus, the zeroes of  $G'$  do not coincide with the ramification points, i.e.  $\tilde{F}_d$  is nonsingular.

*Case 2.* Suppose  $\zeta$  is a common zero of  $R(z)$  and  $Q_1(z)$ . Let  $\tilde{f}_1(z) = Q_1(z)/P_1(z)$ . Then locally,

$$\tilde{F}_d(z) = (\tilde{f}_1(z), f_2(z), G(z)) \quad \text{where } G(z) = \frac{\tilde{f}_1'(z)}{\tilde{f}_2'(z)} = -\left(\frac{Q_2(z)}{P_1(z)}\right)^2.$$

Then  $G'(z) = -2[Q_2(z)/P_1^3(z)] \cdot \Delta(z)$  where

$$\begin{aligned} \Delta(z) &= P_1(z)Q_2'(z) - Q_2(z)P_1'(z) \\ &= -z^{2d-2} + (1-d)z^d + d(2d-4)z^{d-1} + (d-1)z^{d-2} + d + d(d-2) + 1. \end{aligned}$$

Let  $S(z) = R(z) + \Delta(z) = d(2d-4)z^{d-1} + 2d(d-2)$ . First observe that  $Q_1(z)$  and  $Q_2(z)$  have no common zeroes since  $Q_1(z) + Q_2(z) = 2(d-2) \neq 0$  for  $d > 2$ . Thus  $G'(\zeta) = 0$  if and only if  $\Delta(\zeta) = 0$ . Suppose that  $\zeta$  is a common zero of  $\Delta$  and  $R$ . Then  $\zeta$  must be a zero of  $S$ . But  $S(z)$  vanishes when  $z^{d-1} = -2d(d-2)/d(2d-4) = -1$ . Then  $\zeta$  must be a  $(d-1)$ th root of  $-1$ . But  $Q_1(\zeta) = -1 + \zeta + (d-2) = \zeta + d - 3 \neq 0$  for  $d > 2$ , contradicting our assumption that  $\zeta$  was a zero of  $Q_1(z)$ . Thus,  $G'(\zeta) \neq 0$ .

*Case 3.* Suppose  $\zeta$  is a common zero of  $R(z)$  and  $Q_2(z)$ . Let  $\tilde{f}_2(z) = Q_2(z)/P_2(z)$ . Then locally,

$$\tilde{F}_d(z) = (f_1(z), \tilde{f}_2(z), G(z)) \quad \text{where } G(z) = \frac{f_1'(z)}{\tilde{f}_2'(z)} = -\left(\frac{P_2(z)}{Q_1(z)}\right)^2.$$

Then  $G'(z) = -2[P_2(z)/Q_1^3(z)] \cdot \Delta(z)$  where

$$\begin{aligned} \Delta(z) &= Q_1(z)P_2'(z) - P_2(z)Q_1'(z) \\ &= z^{2d-2} + (d-1)z^d + d(2d-4)z^{d-1} - (d-1)z^{d-2} - d(d-2) - 1. \end{aligned}$$

Let  $S(z) = R(z) - \Delta(z) = -d(2d-4)z^{d-1} + 2d(d-2)$ . If  $\zeta$  is a common zero of  $\Delta$  and  $R$ , certainly it is a zero of  $S$ . But  $S(z)$  vanishes when  $z^{d-1} = 2d(d-2)/d(2d-4) = 1$ , i.e.  $\zeta$  is a  $(d-1)$ th root of 1. But  $Q_2(\zeta) = \zeta - (d+3) \neq 0$  for  $d > 2$ , a contradiction. Thus,  $G'(\zeta) \neq 0$ .  $\square$

Thus the total preimage  $\beta \circ \psi^{-1}(\tilde{F}_d(\mathbf{P}^1))$  is a conjugate pair of nonsingular holomorphic, horizontal curves in  $\mathbf{P}^3$  which project to a conjugate pair of superminimal surfaces, each of area  $4\pi d$ , in  $S^4$  ( $d \geq 3$ ).

### III. HIGHER GENUS

We now consider branched superminimal immersions of a compact Riemann surface  $\Sigma$  of genus  $g > 0$  into  $S^4$ .

Let  $f: \Sigma \rightarrow S^4$  be a branched superminimal immersion such that  $f(\Sigma)$  has area  $4\pi d$ . Recall that generically,  $f(\Sigma)$  misses a pair of antipodal points on  $S^4$ , say the north and south poles. We have shown that  $f$  arises from a pair of meromorphic functions  $(f_1, f_2)$  of bidegree  $(d, d)$  such that  $\text{Ram}(f_1) = \text{Ram}(f_2)$ . Moreover,  $f$  is linearly full (i.e.  $f(\Sigma)$  is not contained in any strict linear subspace of  $\mathbf{R}^5$ ) provided  $d \geq 3$  and  $f_2 \neq A \circ f_1$  for any  $A \in \text{PSL}(2, \mathbb{C})$ . For each  $d \geq 3$ , we wish to construct linearly full branched superminimal immersions from such pairs of functions. Let  $F = (f_1, f_2)$  be such a pair of functions. Let  $\tilde{C}$  denote the curve  $\tilde{F}(\Sigma)$ . We require that  $\psi^{-1}(\tilde{C})$  consist of two connected components,  $\gamma_1$  and  $\gamma_2$ , such that  $\alpha(\gamma_1) = \gamma_2$  and  $\psi(\gamma_1) = \psi(\gamma_2) = \tilde{C}$ . If this is the case, then the curves  $\gamma_1$  and  $\gamma_2$  project to a conjugate pair of (branched) superminimal surfaces in  $S^4$ .

Let  $X := \tilde{\mathbf{P}}^3 - (\sigma_1 \cup \sigma_2) \cong \mathbf{P}^3 - (L_1 \cup L_2)$  and  $Y := \mathbf{PT}(\mathbf{P}^1 \times \mathbf{P}^1) - (S_1 \cup S_2)$ . Note that  $\pi_1 X = 0$  and  $\psi: X \rightarrow Y$  is a covering map of degree 2. The maps that we are considering,  $F = (f_1, f_2): \Sigma \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$ , are such that  $\tilde{F}(\Sigma) \subset Y$ . Observe that  $\tilde{F}$  lifts to a map  $\tilde{\tilde{F}}: \Sigma \rightarrow X$  if and only if  $\tilde{F}_*(\pi_1 \Sigma) = 0$ . If  $\tilde{\tilde{F}}_*(\pi_1 \Sigma) = 0$ , then we have 2 maps,  $\tilde{\tilde{F}}$  and  $\alpha \circ \tilde{\tilde{F}}$ , from  $\Sigma$  to  $X$ . Thus

**Theorem F.** *Suppose  $F = (f_1, f_2): \Sigma \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$  is a holomorphic map of bidegree  $(d, d)$  of a compact Riemann surface of genus  $g$  to  $\mathbf{P}^1 \times \mathbf{P}^1$  such that  $\text{Ram}(f_1) = \text{Ram}(f_2)$  and  $f_2 \neq A \circ f_1$  for any  $A \in \text{PSL}(2, \mathbb{C})$ . Let  $\tilde{\tilde{F}}: \Sigma \rightarrow \mathbf{PT}(\mathbf{P}^1 \times \mathbf{P}^1) - (S_1 \cup S_2)$  be the canonical Gauss lift of  $F$ . Then  $F$  gives rise to a conjugate pair of linearly full branched superminimal surfaces of genus  $g$  in  $S^4$  provided  $\tilde{\tilde{F}}_*(\pi_1 \Sigma) = 0$ .*

*Note.* The condition  $\tilde{\tilde{F}}_*(\pi_1 \Sigma) = 0$  is automatically satisfied if  $\Sigma$  has genus 0. However, if  $\tilde{\tilde{F}}_*(\pi_1 \Sigma) \neq 0$ , then we do not have a lift of  $\Sigma$  to  $X$ . Nevertheless, there is a two-fold cover  $\tilde{\Sigma}$  of  $\Sigma$  which lifts to  $X$  (where  $\text{genus}(\tilde{\Sigma}) = 2g - 1$ ). We then obtain a branched superminimal surface in  $S^4$  of genus  $2g - 1$ .

An easy way to satisfy the lifting criterion is by factoring through  $\mathbf{P}^1$ :

$$F = (F_1, F_2): \Sigma \xrightarrow{\varphi} \mathbf{P}^1 \xrightarrow{(f_1, f_2)} \mathbf{P}^1 \times \mathbf{P}^1$$

where  $\varphi$  is a holomorphic map of degree  $d_1$  and  $(f_1, f_2)$  is a holomorphic map of bidegree  $(d_2, d_2)$  which gives rise to a linearly full branched superminimal immersion of  $\mathbf{P}^1$  into  $S^4$ . Note that  $F$  has bidegree  $(d_1 d_2, d_1 d_2)$ . Certainly,

$\text{Ram}(F_1) = \text{Ram}(F_2)$  and  $F_2 \neq A \circ F_1$  for any  $A \in \text{PSL}(2, \mathbb{C})$  (since  $(f_1, f_2)$  is linearly full). Let  $\tilde{F}: \Sigma \rightarrow Y$  be the canonical Gauss lift of  $F$ . Then  $\tilde{F}_*(\pi_1 \Sigma) = 0$  and by Theorem F,  $\tilde{F}$  lifts to a holomorphic horizontal map,  $\tilde{\tilde{F}}$ , to  $\mathbb{P}^3$ . Note however that  $\tilde{\tilde{F}}(\Sigma)$  is necessarily branched. Nevertheless, it projects to a branched superminimal surface in  $S^4$  of area  $4\pi d_1 d_2$ . We thus have lots of branched superminimal immersions of  $\Sigma$  into  $S^4$ .

The construction in the previous paragraph gives us superminimal surfaces of genus  $g > 0$  in  $S^4$  which were necessarily branched. It would be interesting if the map  $F$  can be deformed (in the space of branched superminimal immersions of  $\Sigma$  into  $S^4$  of degree  $d_1 d_2$ ) to a map  $F'$  so that  $F'$  gives rise to an *unbranched* superminimal surface in  $S^4$ .

It has come to the author's attention that Verdier has obtained a result similar to Theorem E (which was his conjecture in [17]).

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