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LIE GROUPS THAT ARE CLOSED AT INFINITY

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ABSTRACT. A noncompact Riemannian manifold M is said to be closed at infinity if no bounded volume form which is also bounded away from zero can be written as the exterior derivative of a bounded form on M. The isoperimetric constant of M is defined by $h(M) = \inf\{\mathrm{vol}(\partial S)/\mathrm{vol}(S)\}$ where S ranges over compact domains with boundary in M. It is shown that a Lie group G with left invariant metric is closed at infinity if and only if $h(G) = 0$ if and only if G is amenable and unimodular. This result relates these geometric invariants of G to the algebraic structure of G since the conditions amenable and unimodular have algebraic characterizations for Lie groups. G is amenable if and only if G is a compact extension of a solvable group and G is unimodular if and only if $\mathrm{Tr}(\mathrm{ad}\, X) = 0$ for all X in the Lie algebra of G. An application is the clarification of relationships between several conditions for the existence of transversal invariant measures for a foliation of a compact manifold by the orbits of a Lie group action.

INTRODUCTION

Let L be a complete Riemannian manifold. L has *isoperimetric constant* $h(L) = 0$ if there is a sequence of domains $\{S_i\}$ in L such that

$$\lim \mathrm{vol}(\partial S_i)/\mathrm{vol}(S_i) = 0, \quad [2].$$

L is *closed at infinity* if no bounded volume form which is bounded away from zero can be written as the derivative of a bounded form [15]. This paper characterizes these two invariants for Lie groups with a natural left invariant metric in terms of well known analytic properties, amenability and unimodularity. The conditions amenable and unimodular have algebraic characterizations for Lie groups. G is amenable if and only if G is a compact extension of a solvable group and G is unimodular if and only if $\mathrm{Tr}(\mathrm{ad}\, X) = 0$ for all X in the Lie algebra of G. Thus we are relating geometric invariants of G to its algebraic structure.

These asymptotic metric properties arise naturally in the study of the dynamical properties of a foliation $\mathscr{F}$ of a compact manifold M, [1, 9, 14, 15]. A

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foliation may be thought of simply as a partition of M into immersed submanifolds called leaves such that each point has a "flow box" neighborhood. Certain leaves are known to give rise to holonomy invariant measures in the sense of Plante [14]. Such measures may be thought of as elements of $H^k(M)$ where k is the codimension of $\mathcal{F}$. Taking the dual point of view one would say that certain leaves L gives rise to asymptotic homology classes of the same dimension as L. The similar nature of these dual points of view is apparent in the following theorem on the existence of invariant measures.

Theorem A [4, 15]. *If there is a leaf $L \subset \mathcal{F}$ such that either*
(1) $h(L) = 0$ *or*
(2) L *is closed at infinity*
then there is a holonomy invariant measure for $\mathcal{F}$ with support in the closure of L.

If $\mathcal{F}$ is given by orbits of the action of a connected Lie group G on the compact, connected manifold M there are conditions on the group which guarantee the existence of an invariant measure. Let us first recall the structure of such a foliation. Let $\varphi: G \times M \to M$ be a $C^k (k \geq 1)$ left action. For $x \in M$ let $\mathcal{O}_x$ and G_x denote respectively the orbit of x and the isotropy subgroup of x. If G_x is discrete for each $x \in M$, the action is said to be *locally free* and in this case the orbits determine the leaves of a foliation of M. $\mathcal{O}_x$ is naturally homeomorphic to G/G_x and the action of G on $\mathcal{O}_x$ corresponds to the action of G on G/G_x by left multiplication. Furthermore the map $\varphi_x : G \to \mathcal{O}_x$ defined by $\varphi_x(g) = \varphi(g, x)$ is a covering map.

Theorem B [14]. *Let φ be a locally free action of a connected Lie group G on a compact manifold M. If G is amenable and unimodular then there is an invariant measure for the orbit foliation of M.*

The main result of this paper ties together the conditions in Theorems A and B above. We show that if G is a connected Lie group with left invariant metric then the conditions $h(G) = 0$, G is closed at infinity, and G is amenable and unimodular are equivalent. The proof of this result is in §3. In §1 we review some properties of Lie group structure which are used in the final section. In §2 we give precise definitions and examples of the metric invariants discussed above.

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1. Some background on Lie groups

In this section we review the structure of a Lie group and bring together from various references [7,12,16] some results which will be useful in the last section.

In the following G will be connected Lie group with Lie algebra $\mathcal{G}$. For $g \in G$ we will use L_g and R_g to denote the automorphisms of G given by left and right multiplication by g.

When discussing Riemannian structure of Lie groups we will be referring to a left invariant metric given by specifying an inner product $\langle , \rangle$ on $\mathcal{G} \approx T_eG$. Thus choosing a basis for $\mathcal{G}$ gives a global orthonormal basis for the tangent space of G.

Recall that for any basis of $\mathcal{G}$ consisting of left invariant vector fields $\{X_1, \dots, X_n\}$, there are constants c^i_{rs} such that $[X_r, X_s] = \sum_{i=1}^n c^i_{rs}X_i$. Moreover, if $\{\omega_1, \dots, \omega_n\}$ is the basis of left invariant one forms dual to $\{X_1, \dots, X_n\}$, their exterior derivatives are given by the Maurer-Cartan equations:

$$d\omega_i = -\frac{1}{2}\sum_{r,s=1}^{n} c^i_{rs}\omega_r \wedge \omega_s = -\sum_{r<s\leq n} c^i_{rs}\omega_r \wedge \omega_s.$$

Also recall a few facts about the adjoint representation of a Lie group G, denoted by $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathcal{G})$ where $\mathrm{Aut}(\mathcal{G})$ is the group of automorphisms of $\mathcal{G}$. For $g \in G$ let $a_g\colon G \to G$ be the map $a_g(h) = ghg^{-1}$, then $\mathrm{Ad}(g) = da_g|T_eG$. So $\mathrm{Ad}(g)$ acts on $\mathcal{G}$ since $T_eG \approx \mathcal{G}$. The derivative of this representation is denoted by $\mathrm{ad}\colon \mathcal{G} \to \mathrm{End}(\mathcal{G})$ where $\mathrm{End}(\mathcal{G})$ is the group of endomorphisms of $\mathcal{G}$. For all X and Y in $\mathcal{G}$, $\mathrm{ad}(X)(Y) = [X, Y]$ and $\mathrm{Ad}(\exp(X)) = \exp(\mathrm{ad}\, X)$ where $\exp\colon \mathcal{G} \to G$ is the exponential map, see [16] for details.

Given an ordered basis $\{X_1, \dots, X_n\}$ for $\mathcal{G}$, the structure constants above give the matrix representation of $\mathrm{ad}\, X_i$, $1 \leq i \leq n$. Using this it is an easy computation with the Maurer-Cartan equations to show.

Lemma 1.1. *Let G be a Lie group with Lie algebra $\mathcal{G}$ and let $\{X_1, \dots, X_n\}$ be a basis for $\mathcal{G}$ with dual basis of forms $\{\omega_1, \dots, \omega_n\}$. Then for $1 \leq i \leq n$*

$$d(\omega_1 \wedge \cdots \wedge \omega_{i-1} \wedge \omega_{i+1} \wedge \cdots \wedge \omega_n) = (-1)^{n-1}\,\mathrm{Tr}(\mathrm{ad}\, X_i)\omega_1 \wedge \cdots \wedge \omega_n$$

where $\mathrm{Tr}(A)$ is the trace of A.

It is well known that a locally compact topological group G has a unique measure (up to constant factor) which is invariant under the action of G on itself by left multiplication. This measure is called Haar measure on G.

Definition. A locally compact topological group G is said to be *unimodular* if its Haar measure is both left and right invariant.

If G is a Lie group of dimension n one can be more specific. Let $\{\omega_1, \dots, \omega_n\}$ be a basis of left invariant one forms for G and let vol_G be the left invariant n-form $\omega_1 \wedge \cdots \wedge \omega_n$. Denote by $C_c(G)$ the Banach space of continuous functions on G with compact support with the supremum norm. Then vol_G defines a continuous linear functional in the dual of $C_c(G)$ by $f \to \int f\,\mathrm{vol}_G$. This integral is left invariant and therefore corresponds to a left invariant measure by the Riesz representation theorem. Thus Haar measure on G is just the measure corresponding to vol_G.

Proposition 1.2. *For a connected Lie group* G *of dimension* n *the following are equivalent*:

(i) G *is unimodular.*

(ii) $|\det(\mathrm{Ad}(g))| = 1$ *for all* $g \in G$.

(iii) $\mathrm{Tr}(\mathrm{ad}\, X) = 0$ *for all* $X \in \mathcal{G}$.

(iv) $d(\omega_1 \wedge \cdots \wedge \omega_{i-1} \wedge \omega_{i+1} \wedge \cdots \wedge \omega_n) = 0$ *for* $1 \leq i \leq n$.

Proof. Since vol_G is left invariant we have the formula

$$(R_g - 1)^* \mathrm{vol}_G(e) = (R_g - 1)^*(L_g^* \mathrm{vol}_G)(e) = (L_g \circ R_g - 1)^* \mathrm{vol}_G(e)$$
$$= \det(\mathrm{Ad}(g))\, \mathrm{vol}_G(e)$$

which implies the equivalence of (i) and (ii). (ii) and (iii) are equivalent by the well-known formula $\det(\mathrm{Ad}(\exp X)) = e^{\mathrm{Tr}(\mathrm{ad}\, X)}$ and Lemma 1.1 gives the equivalence of statements (iii) and (iv). □

Since condition (iii) of Proposition 1.2 is satisfied for nilpotent groups we see that all connected nilpotent groups are unimodular. From condition (ii) we see that all compact groups are unimodular, in fact any group with $\mathrm{Ad}(G)$ compact must be unimodular.

Given a finite dimensional Lie algebra $\mathcal{G}$, one defines the Killing form $B: \mathcal{G} \times \mathcal{G} \to \mathbf{R}$ by $B(X, Y) = \mathrm{Tr}(\mathrm{ad}\, X \circ \mathrm{ad}\, Y)$. The Killing form is preserved by elements of $\mathrm{Aut}(\mathcal{G})$ and satisfies the formula $B(\mathrm{ad}\, Z(X), Y) = -B(X, \mathrm{ad}\, Z(Y))$ for all X, Y, and Z in $\mathcal{G}$. $\mathcal{G}$ is said to be *semisimple* if its Killing form is nondegenerate. A Lie group is semisimple if its Lie algebra is semisimple. Note that since $\mathrm{Ad}(g)$ preserves the Killing form for all $g \in G$, Proposition 1.2 implies that every semisimple Lie group is unimodular.

Every semisimple Lie algebra $\mathcal{G}$ has a *Cartan decomposition* $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$ where $\mathcal{K}$ is a subalgebra and $\mathcal{P}$ is a vector subspace. The Cartan decomposition is characterized by the properties: the Killing form is strictly negative on $\mathcal{K}$, strictly positive on $\mathcal{P}$, and $\mathcal{K}$ and $\mathcal{P}$ satisfy the bracket relations $[\mathcal{K}, \mathcal{K}] \subseteq \mathcal{K}$, $[\mathcal{K}, \mathcal{P}] \subseteq \mathcal{P}$ and $[\mathcal{P}, \mathcal{P}] \subseteq \mathcal{K}$. In a semisimple Lie group G the subgroup $K \subset G$ corresponding to the subalgebra $\mathcal{K}$ in the Cartan decomposition of the Lie algebra of G may be assumed to be a maximal compact subgroup of G [7, p. 184].

Recall that the *radical* of a Lie algebra $\mathcal{G}$ is the largest solvable ideal of $\mathcal{G}$. We will find the following structure theorem is of great use in proving the applications in §3.

Theorem 1.3 (Levi) [12, p. 525]. *Let* G *be a Lie group with Lie algebra* $\mathcal{G}$ *and let* $\mathcal{S}$ *be the radical of* $\mathcal{G}$, *then there is a semisimple Lie subalgebra* $\mathcal{T} \subset \mathcal{G}$ *such that* $\mathcal{G} = \mathcal{S} \oplus \mathcal{T}$. *Futhermore if* S *is the subgroup of* G *corresponding to* $\mathcal{S}$ *then* S *is a connected, closed, solvable normal subgroup and the quotient group* G/S *is a semisimple Lie group with Lie algebra isomorphic to* $\mathcal{T}$.

Let G be a Lie group and as above let S be the largest solvable subgroup of G. We will say that *the radical of* G *is cocompact* if G/S is compact. From the point of view of analysis a locally compact group G is *amenable* if there is a left invariant mean on $L^\infty(G)$. The properties of amenable groups have been well studied and in particular they have nice averaging properties (i.e. the Følner Condition). The interested reader should see [5].

Theorem 1.5 [11]. *Let G be a connected Lie group. G is amenable if and only if the radical of G is cocompact.*

Since the algebraic and analytic conditions are equivalent, to be concise we will use the term amenable to describe those Lie groups that are compact extensions of solvable groups.

We finish this section with a brief discussion of the structure of simply connected solvable Lie groups. Let G and H be connected Lie groups and let $\alpha: h \to \alpha_h$ be a homomorphism from H into the group of automorphisms of G. The semidirect product of G and H under α is the Lie group $G \times_\alpha H$ which consists of all ordered pairs (g, h) with the multiplication $(g, h)(g_1, h_1) = (g\alpha_h(g_1), hh_1)$. The following proposition describes the decomposition of a simply connected group into a semidirect product.

Proposition 1.5 [12, p. 518]. *Let T be a simply connected Lie group with Lie algebra $\mathscr{T}$. Let $\mathscr{G}$ be an ideal in $\mathscr{T}$ and $\mathscr{H}$ a Lie subalgebra of $\mathscr{T}$ such that $\mathscr{T} = \mathscr{G} \oplus \mathscr{H}$. Let G and H be subgroups of T corresponding to $\mathscr{G}$ and $\mathscr{H}$. For all $g \in G$ and $h \in H$ define $\alpha_h(g) = hgh^{-1}$. The mapping $(g, h) \to gh$ is an isomorphism of the Lie groups $G \times_\alpha H$ and T. The groups G and H are closed and simply connected and $GH = T$, $G \cap H = \{e\}$.*

Proposition 1.6. *Let G be a simply connected solvable Lie group of dimension n. Since G is solvable the commutator subgroup $N = [G, G]$ is a nilpotent subgroup of G. Let $\mathscr{N}$ be the Lie algebra of N and $\mathscr{G}$ the Lie algebra of G. There is a basis $\{X_1, \dots, X_k, X_{k+1}, \dots, X_n\}$ for $\mathscr{G}$ which has the following properties*

(a) *$\{X_1, \dots, X_k\}$ is a basis for $\mathscr{N}$.*

(b) *Let $\mathscr{L}_i = \operatorname{span}\{X_1, \dots, X_i\}$, $1 \le i \le k$. Then $\mathscr{L}_i$ is a subalgebra of $\mathscr{N}$ and $\mathscr{L}_i$ is an ideal in $\mathscr{L}_{i+1}$.*

(c) *For $1 \le j \le n - k$, let $\mathscr{L}_j = \operatorname{span}\{X_1, \dots, X_k, X_{k+1}, \dots, X_{k+j}\}$. Then $\mathscr{L}_j$ is an ideal in $\mathscr{L}_{j+1}$ and moreover $\mathscr{L}_j$ is an ideal in $\mathscr{G}$.*

(d) *With the basis for $\mathscr{G}$ chosen as above the map $\varphi: \mathbf{R}^n \to G$ defined by $\varphi(t_1, \dots, t_n) = \exp(t_1X_1) \times \cdots \times \exp(t_nX_n)$ is a diffeomorphism and the restriction of φ to $\mathbf{R}^k \times \{0\}$ is a diffeomorphism onto N.*

From 1.6 we may conclude that a solvable simply connected Lie group G of dimension n with commutator subgroup N is parametrized by $\mathbf{R}^n$ and has a structure described as follows. Let $H_0 = N$, and $H_i = N\exp(t_1X_{k+1}) \times \cdots \times \exp(t_iX_{k+i})$ for $1 \le i \le n - k$; then each H_i is a closed normal subgroup of H_{i+1} and also a closed normal subgroup of G. It follows from Proposition 1.5

that H_{i+1} is isomorphic to $H_i \times_\tau \mathbf{R}$ where $\tau: \mathbf{R} \to \mathrm{Aut}(H_i)$ is given by

$$\tau(t)(h) = \exp(tX_{k+i})h[\exp(tX_{k+i})]^{-1}$$

and multiplication is defined by $(h, t)(h_1, t_1) = (h \cdot \tau(t)h_1, t + t_1)$. Notice that $d\tau(t)|e_{H_i} = \mathrm{Ad}[\exp(tX_{k+1})] = \exp(t\,\mathrm{ad}\,X_{k+1})$. In particular, if G is unimodular the determinant of this derivative will be one implying that the volume of a domain in H_i is preserved by left translation by (e, t), see §3.

2. Asymptotic metric properties

Let V be a vector space of dimension n with an inner product $\langle , \rangle$ and let β be the isomorphism from V onto its dual space V^* given by $\beta(v)[w] = \langle w, v\rangle$. $\Lambda(V) = \sum_{k=1}^n \Lambda_k(V)$ will denote the exterior algebra of V. The inner product on V is extended to $\Lambda_k(V)$ via the relation

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k\rangle = \beta(v_1) \wedge \cdots \wedge \beta(v_k)[w_1 \wedge \cdots \wedge w_k].$$

For $\alpha \in \Lambda_k(V)$ we define the Euclidean norm of α to be $|\alpha| = \sqrt{\langle \alpha, \alpha\rangle}$. A choice of an orthonormal basis $\{e_1, \dots, e_n\}$ for V gives an orthonormal basis for $\Lambda_k(V)$ which consists of the set of k-fold exterior products of elements of $\{e_1, \dots, e_n\}$. By duality we can extend the Euclidean norm to $\Lambda^*(V) = \sum_{k=1}^n \Lambda_k(V^*)$.

For the remainder of this section M^n will be a complete C^∞ Riemannian manifold of dimension n. ∇ will denote the Levi-Civita connection on M and $\langle , \rangle_p$ will be a corresponding inner product on T_pM, the tangent space to M at p. Let $\{X_1, \dots, X_n\}$ be an orthonormal frame field defined on an open set in M and let $\{\theta_1, \dots, \theta_n\}$ be the dual coframe field. Recall from the structural equations that, [7]:

$$\omega_{jk} = \sum_{r=1}^n \omega_{jk}(X_r)\theta_r \quad \text{where } \omega_{jk}(X_r) = \langle \nabla_{X_r} X_j, X_k\rangle, \tag{1}$$

$$d\theta_k = -\sum_{j=1}^n \omega_{jk} \wedge \theta_j. \tag{2}$$

A choice of metric $\langle , \rangle_p$ on M allows us to define as above the Euclidean norm $|\cdot|_p$ on $\Lambda^k(M)_p = \Lambda_k(T_pM)^*$. Using this we may define a *supremum norm*, on the smooth differential forms on M by $\|\omega\| = \sup\{|\omega|_p | p \in M\}$.

Definition. A differential form ω on a Riemannian manifold M is *bounded* if its supremum norm is finite.

Definition (Sullivan [15]). A Riemannian manifold M of dimension n is said to be *closed at infinity* if there is no nontrivial solution to the equation $d\eta = \omega$ where η is a bounded $(n-1)$-form and $\omega = f\,\mathrm{vol}_M$ where $0 < c < f(x) < C < \infty$ for all $x \in M$ and vol_M is the unit volume form ($\theta_1 \wedge \cdots \wedge \theta_n$ locally).

Note that ω is bounded if its coefficient function is bounded. This definition also requires ω to be bounded away from zero. The $(n-1)$-form η is bounded if and only if its coefficient functions are all bounded in some choice of basis.

Examples. 1. $\mathbf{R}^n$ is closed at infinity for all n. For the case $n = 1$ we are asking whether a function of one real variable which is bounded and bounded away from zero can have a bounded antiderivative. The Fundamental Theorem of Calculus tells us this is not possible. We can reduce the case $n > 1$ to the previous case by averaging over the compact group of isometries fixing the origin, that is, the orthogonal group. The compactness of $O(n)$ implies that if we have a nontrivial solution $d\eta = \omega$ as above, we can produce a nontrivial $O(n)$ invariant solution. This would give a bounded function of the radial distance which is bounded away from zero and has bounded derivative. This is not possible.

2. A slightly different example is the affine group of the line. This is the simply connected solvable two dimensional Lie group which is not $\mathbf{R}^2$. Referring to the structure of such groups discussed after Proposition 1.6, we see that this group is the semidirect product of the additive real numbers with itself via the map $\tau(y) = e^y$. So we are just considering the set of ordered pairs (x, y) with multiplication $(x, y)(u, v) = (x + e^y u, y + v)$. A basis of left invariant vector fields for this parameterization is $\{e_1 = e^y \partial/\partial x, e_2 = \partial/\partial y\}$ and there is the one bracket relation $[e_1, e_2] = -e_1$. Dual to this basis we have the 1-forms $\{\omega_1 = e^{-y}\, dx, \omega_2 = dy\}$. Give the affine group of the line a Riemannian metric by simply specifying that the vector fields $\{e_1, e_2\}$ be an orthonormal basis at each point. With this metric the affine group is not closed at infinity since ω_1 and $\omega_1 \wedge \omega_2$ are both bounded of norm one and $d\omega_1 = \omega_1 \wedge \omega_2$. It is interesting to note that the affine group of the line with this metric is isometric to the hyperbolic plane so that space is not closed at infinity either.

The proof of Proposition 2.2, the main result in this section, depends on the following technical lemma, the proof of which is deferred.

Lemma 2.1. *Let $\pi\colon M^{n+k} \to N^n$ be a Riemannian submersion. Let ω be the k-form on M that at each point x gives the unit volume form on F_x, the fiber containing x. Let $\mathscr{H}$ denote the 1-form dual to the mean curvature vector of F_x. Then*

$$d\omega = \omega \wedge \mathscr{H} + \sum_{j=1}^{k} (-1)^j v_j \wedge h_j$$

where v_j is a vertical $(k-1)$-form and h_j is a horizontal 2-form.

Proposition 2.2. *Let $\pi\colon M^{n+k} \to N^n$ be a Riemannian submersion with minimal fibers (the fibers $\pi^{-1}(p)$ are minimal submanifolds of M). If N is open at infinity then M is open at infinity.*

Proof. Assuming Lemma 2.1, we prove the proposition. Let η be a bounded form on N such that $d\eta = f\,\mathrm{vol}_N$ where f is bounded and bounded away from zero and let ω be as in Lemma 2.1. Let $\eta^* = \pi^*(\eta) \wedge \omega$. Since η is bounded so is η^* and

$$\begin{aligned} d\eta^* &= d(\pi^*(\eta)) \wedge \omega + \pi^*(\eta) \wedge d\omega \\ &= f \circ \pi \operatorname{vol}_N \wedge \omega \pm \pi^*(\eta) \wedge d\omega. \end{aligned}$$

It remains to show that the second term is zero. By the lemma

$$\pi^*(\eta) \wedge d\omega = \pi^*(\eta) \wedge \omega \wedge \mathscr{H} + \sum_{j=1}^{k} (-1)^j \pi^*(\eta) \wedge v_j \wedge h_j$$

and both of these terms vanish. The first vanishes because $\mathscr{H} = 0$ by minimality and the second because $\pi^*(\eta) \wedge h_j$ is horizontal of degree $(n-1)+2 = n+1 > n$. Hence M is open at infinity. □

Proof of Lemma 2.1. Let $\{X_1, \dots, X_k, Y_1, \dots, Y_n\}$ be an adapted frame field in an open set of M. That is, $\{X_1, \dots, X_k\}$ is a vertical frame field and $\{Y_1, \dots, Y_n\}$ is a horizontal frame field. Let $\{\theta_1, \dots, \theta_k, \eta_1, \dots, \eta_n\}$ be the dual coframe field. Then $\omega = \theta_1 \wedge \cdots \wedge \theta_k$ and

$$d\omega = \sum_{j=1}^{k} (-1)^{j-1} \theta_1 \wedge \cdots \wedge \theta_{j-1} \wedge d\theta_j \wedge \theta_{j+1} \wedge \cdots \wedge \theta_k.$$

The structural equations [formula (2) of §2] tell us that

$$d\theta_j = -\sum_{r=1}^{k} \omega_{rj} \wedge \theta_r - \sum_{s=1}^{n} \omega_{sj} \wedge \eta_s.$$

Substituting this expression into the formula for $d\omega$ and using the facts

$$\omega_{\alpha j} = \sum_{p=1}^{k} \omega_{\alpha j}(X_p)\theta_p + \sum_{q=1}^{n} \omega_{\alpha j}(Y_q)\eta_q \quad \text{and} \quad \omega_{\alpha\alpha} = 0,$$

we obtain

$$\begin{aligned} d\omega &= \sum_{j=1}^{k} (-1)^{j-1} \theta_1 \wedge \cdots \wedge \theta_{j-1} \wedge \left(-\sum_{s=1}^{n} \omega_{sj} \wedge \eta_s \right) \wedge \theta_{j+1} \wedge \cdots \wedge \theta_k \\ &= \sum_{j=1}^{k} \sum_{s=1}^{n} (-1)^j \theta_1 \wedge \cdots \wedge \theta_{j-1} \wedge \omega_{sj} \wedge \eta_s \wedge \theta_{j+1} \wedge \cdots \wedge \theta_k \\ &= \omega \wedge \left(\sum_{s=1}^{n} \sum_{j=1}^{k} \omega_{sj}(X_j)\eta_s \right) + \sum_{j=1}^{k} (-1)^j v_j \wedge h_j \end{aligned}$$

where

$$\begin{aligned} v_j &= \theta_1 \wedge \cdots \wedge \theta_{j-1} \wedge \theta_{j+1} \wedge \cdots \wedge \theta_k, \\ h_j &= \sum_{s,q=1}^{n} \omega_{sj}(Y_q)\eta_q \wedge \eta_s. \end{aligned}$$

Finally, for each s let $\mathscr{H}_s = \sum_{j=1}^{k} \omega_{sj}(X_j)\eta_s$ and let $\mathscr{H} = \sum_{s=1}^{n} \mathscr{H}_s$. Then $\mathscr{H}$ is dual to the mean curvature vector field of the fibers since $\omega_{sj}(X_j) = \langle \nabla_{X_j} Y_s, X_j \rangle = -\langle Y_s, \nabla_{X_j} X_j \rangle$. □

Closely related to the notion of closed at infinity is the isoperimetric constant of M^n, [1, 2]. To deal with this invariant it is necessary to work with geometric integration theory at some level. The work of Whitney [17] provides a level of generality sufficient for the results in this paper and the reader is referred to that very readable text for details.

As before, corresponding to a choice of a local orthonormal frame field $\{\theta_1, \dots, \theta_n\}$ in a neighborhood of each point of M we get a *unit volume* form $\Omega = \theta_1 \wedge \cdots \wedge \theta_n$ on M. For a measurable set $A \subset M$ we will write $\mathrm{vol}(A) = \int_A \Omega$. In general an n-form ω on M is of the form $f\Omega$ where f is a smooth function. If $f(p) > 0$ for all $p \in M$ then we will refer to $\omega = f\Omega$ as a *volume form.*

Let S be a compact, connected subset of M^n. We will be interested in finding an $(n-1)$-dimensional volume for the topological boundary of S, if possible. If the boundary of S smooth submanifold of M^n of dimension $n-1$ with metric restricted from M one may find a unit normal vector N to the boundary at each boundary point. Then $i_N\Omega$ [interior product] defines a volume form on the boundary which allows the calculation of the $(n-1)$-volume of sets contained in it. If the boundary is not a smooth submanifold then there are obvious difficulties in defining its volume. For our purposes the slightly more general standard domain [17] will suffice.

A point z_0 contained in the topological boundary of S is called a *regular boundary point* if there exists a coordinate neighborhood U containing z_0 and a chart map $\varphi\colon U \to \mathbf{R}^n$ such that $\varphi(U \cap S) \subset \{(x_1, \dots, x_n) \in \mathbf{R}^n | x_n \leq 0\}$ and for any point z in the topological boundary of S we have $\varphi(z) \in \{(x_1, \dots, x_n) \in \mathbf{R}^n | x_n = 0\}$. For a subset $S \subset M$ let ∂S denote the set of all regular boundary points of S and let $\partial_0 S$ denote the set of all boundary points of S which are not regular. Briefly, a *standard domain* S in M^n is a compact set whose interior is an orientable open n-submanifold of M. Its topological boundary is the union of ∂S which is a collection of orientable smooth $n-1$ submanifolds and $\partial_0 S$ which is a closed subset that is negligible in integration. In the simplest cases $\partial_0 S$ might be a finite collection of submanifolds of dimension less than $n-1$ or a subset of measure zero in an otherwise smooth boundary or a combination of these. All examples in this paper are of these types. The structure of ∂S allows the definition of a unit normal vector at each point of ∂S and hence a volume form on ∂S. So it is easy to find the volume of ∂S for a standard domain.

The nice property of standard domains is that Stokes' Theorem works [17]. That is, if S is an oriented standard domain in M and ω is a smooth $(n-1)$-form on M then $\int_{\partial S} \omega = \int_{S^0} d\omega$ where S^0 is the interior of S.

Definition. Let M be a noncompact Riemannian manifold of dimension n. The *isoperimetric constant* $h(M)$ is defined by

$$h(M) = \inf\{\text{vol}(\partial S)/\text{vol}(S) | S \text{ is a standard domain in } M\},$$

where we write $\text{vol}(S)$ for $\text{vol}(S^0)$.

Examples. Let $D(k)$ be the disk of radius K in $\mathbf{R}^n$. Then $h(\mathbf{R}^n) = 0$ since $\text{vol}(\partial D(k))/\text{vol}(D(k))$ goes to zero as $k \in \mathbf{Z}^+$ increases. Gromov [6] shows that the upper half space H^n has $h(H^n) > 0$. It is easy to see that the open disk of radius r in $\mathbf{R}^n$ with restricted metric has isoperimetric constant larger than $1/r$. We have the following easy relationship to closed at infinity which shows for example that the affine group of the line has $h > 0$.

Proposition 2.3. *If $h(M) = 0$ then M is closed at infinity.*

Proof. Suppose that $d\eta = \omega$ with η and ω bounded. Let S be any standard domain in M. Then for some $c > 0$,

$$c\,\text{vol}(S) \leq \int_{S^0} \omega = \int_{\partial S} \eta \leq \int_{\partial S} \|\eta\| = \|\eta\|\,\text{vol}(\partial S)$$

so $h(M) > 0$. □

As a further example we point out the relationship between the isoperimetric constant and the growth of volume in a Riemannian manifold [10]. Let M be a complete Riemannian manifold with volume form Ω. The disk of radius r about a point $x \in M$ is the set $D_r(x) = \{y \in M | d(x, y) \leq r\}$.

Definition. The *growth function* on M at x is the function $g_x\colon \mathbf{R}^+ \to \mathbf{R}^+$ defined by $g_x(r) = \text{vol}(D_r(x))$. M is said to have *exponential growth* if $g(n) \geq a\exp(bn)$ for some positive real numbers a and b. Otherwise, G is said to have *subexponential* growth. In particular, if $g(n) \leq P(n)$ for a polynomial P of degree d then G is said to have *polynomial growth* of degree d.

Note that the growth function of M depends on the point x but the type of growth does not depend on x. The type of growth is dependent on the Riemannian metric on M unless M is compact.

It is shown in [13] that if M is a Riemannian manifold such that the exponential map at $x \in M$ is of class C^1 then for almost all r, $D_r(x)$ is a standard domain and the derivative of the growth function is given by $g_x(r) = \text{vol}(\partial D_r(x))$. This easily yields the following result.

Proposition 2.4. *Let M be a Riemannian manifold with exponential map which is C^1. If $h(M) > 0$ then M has exponential growth.*

3. Applications to Lie groups

In this section we restrict our attention to the case of a Lie group G with Lie algebra $\mathscr{G}$ of left invariant vector fields. In the following, as in §2, we consider the Riemannian connection on G, or equivalently we give G a left invariant metric by putting an inner product $\langle\,,\rangle_e$ on $\mathscr{G}$. In this case we are simply choosing a basis $\{X_1, \dots, X_n\}$ for $\mathscr{G}$ and specifying that it be an orthonormal basis. Then the inner product $\langle\,,\rangle_g$ on T_gG is given by $\langle v, w\rangle_g = \langle DL_g - 1(v), DL_g - 1(w)\rangle_e$. A basis of left invariant one forms $\{\omega_1, \dots, \omega_n\}$ dual to $\{X_1, \dots, X_n\}$ is an orthonormal basis of one forms for G. The k-fold exterior products of this basis form a global orthonormal basis of bounded k-forms on G.

Lemma 3.1. *Let G be a Lie group. If G is closed at infinity then G is unimodular.*

Proof. This follows immediately from Lemma 1.1. □

Lemma 3.2. *Let G be a semisimple Lie group. G is closed at infinity if and only if G is compact.*

Proof. Since compact manifolds are closed at infinity we need only show that G noncompact implies that G is open at infinity. Let K be the maximal compact subgroup corresponding to the Cartan decomposition $\mathscr{G} = \mathscr{K} \oplus \mathscr{P}$. We show first that the projection $\pi\colon G \to G/K$ is a Riemannian submersion with minimal fibers. Let $\{X_1, \dots, X_k, Y_1, \dots, Y_n\}$ be a basis for $\mathscr{G}$ such that $\{X_1, \dots, X_k\}$ is a basis for $\mathscr{K}$ and $\{Y_1, \dots, Y_n\}$ is a basis for $\mathscr{P}$. Let $\{\theta_1, \dots, \theta_k, \eta_1, \dots, \eta_n\}$ be the corresponding dual basis of left invariant one forms. Using the fact that the structural constants for G are related to the connection one forms by the relation

$$c_{sj}^k = \omega_{jk}(X_s) - \omega_{sk}(X_j)$$

together with $\omega_{jj} = 0$ we see that the form $\mathscr{H}$ of Lemma 2.1 is given by

$$\mathscr{H} = \sum_{s=1}^{n}\left(\sum_{j=1}^{k} c_{sj}^{j}\right)\eta_s = \sum_{s=1}^{n} \operatorname{Tr}(\operatorname{ad} Y_s)\eta_s.$$

The second equality is true because of the Lie bracket relation $[\mathscr{P}, \mathscr{P}] \subseteq \mathscr{K}$. Now we see that $\mathscr{H} = 0$ either directly from the bracket relation $[\mathscr{K}, \mathscr{P}] \subseteq \mathscr{P}$ or from the fact that semisimple Lie groups are unimodular.

It remains to show that G/K, a symmetric space of noncompact type, is not closed at infinity. To see this let x be a point at infinity of G/K and let f be a Busemann function at x [3]. We let $X = \operatorname{grad} f$ and Ω be the unit volume form on G/K. Since X is a unit vector field the form $\eta = i_X\Omega$ is bounded. Moreover, $d\eta = d(i_X\Omega) = L_X\Omega = (\operatorname{div} X)\Omega = (\Delta f)\Omega$. Δf is constant since Δ is invariant under the action of G on G/K and the isotropy subgroup of x acts transitively on G/K. Geometrically, $f(p)$ represents the signed radial distance from p to a fixed horosphere $H = f^{-1}(0)$ at x. So $\operatorname{grad} f$ is a "radial" vector field and $\Delta f(p)$ represents the mean curvature of the horosphere through x at p. □

Corollary 3.3. *If a Lie group G is closed at infinity then G is amenable.*

Proof. Since solvable groups are amenable, assume that G is not solvable. By Levi's theorem (1.3) the Lie algebra $\mathscr{G}$ of G may be written as $\mathscr{G} = \mathscr{S} \oplus \mathscr{T}$ where $\mathscr{S}$ is solvable and $\mathscr{T}$ is semisimple. As in Lemma 3.2 let $(\theta_1, \ldots, \theta_k, \eta_1, \ldots, \eta_n\}$ be a basis of left invariant one forms for G, the first k of which are dual to $\mathscr{S}$ and the last n of which are dual to $\mathscr{T}$.

Let S be the closed normal subgroup corresponding to $\mathscr{S}$, then G/S is a semisimple Lie group with Lie algebra isomorphic to $\mathscr{T}$. Let $\pi\colon G \to G/S$ be the projection map. As in 3.2 this projection has minimal fibers. Again we find that $\mathscr{H} = \sum_{s=1}^{n} \operatorname{Tr}(\operatorname{ad} Y_s)\eta_s$. In this case the equality is true and $\mathscr{H} = 0$ because by 3.1 G closed at infinity implies that G (and hence S) is unimodular. Now Lemma 3.2 shows that G closed at infinity implies G/S is compact. Thus G is amenable. □

We now turn to the isoperimetric constant, considering first the case of simply connected solvable Lie group. Recall that from Propositions 1.5 and 1.6 we know that such groups may be decomposed into a series of groups of the form $G = H \times_\tau \mathbf{R}$ where H is a normal subgroup of G containing the commutator subgroup of G. Let $\{X_1, \ldots, X_n, T\}$ be a basis for the Lie algebra of G such that $\{X_1, \ldots, X_n\}$ is a basis for the Lie algebra $\mathscr{H}$ of H. Recall that the map $\tau\colon \mathbf{R} \to \operatorname{Aut}(H)$ is given by $\tau(t)(h) = \exp(tT)h\exp(-tT)$ and multiplication is defined by $(h, t)(h_1, t_1) = (h \cdot \tau(t)h_1, t + t_1)$.

We will assume that G has left invariant metric corresponding to orthonormal basis $\{X_1, \ldots, X_n, T\}$. Let $\{\omega_1, \ldots, \omega_n, dT\}$ be the basis of dual left invariant one forms. As usual L_g and R_g will denote left and right multiplication by $g \in G$.

Let φ_t denote the one parameter group of diffeomorphisms associated to the left invariant vector field T on G. Then $\varphi_t(h, s) = (h, s + t) = R_{(e,t)}(h, s)$. In a unimodular group we know that the volume is left and right invariant. For G as described above this is also true for the n-volume of sets in H.

Lemma 3.4. *Let G and H be as above. Let S be a standard domain in H. For $t \in \mathbf{R}$ let $S_t = \{(s, t) \in G | s \in S\}$. If G is unimodular then* $\operatorname{vol}(S_t) = \operatorname{vol}(S_0)$ *for all t.*

Proof. Let L_T denote the Lie derivative with respect to T. Then

$$L_T(\omega_1 \wedge \cdots \wedge \omega_n) = i_T d(\omega_1 \wedge \cdots \wedge \omega_n) + d(i_T\omega_1 \wedge \cdots \wedge \omega_n) = 0$$

by Lemma 1.1 (Again, this reflects the minimality of H_t in G for all t.). □

Lemma 3.5. *Let G be a simply connected solvable Lie group which is unimodular. Let H be a normal subgroup of codimension one in G and containing the commutator subgroup. If $h(H) = 0$ then $h(G) = 0$.*

Proof. Keep the notation from the discussion above. In particular let $\operatorname{vol}_G = \omega_1 \wedge \cdots \wedge \omega_n \wedge dT$ and $\operatorname{vol}_H = \omega_1 \wedge \cdots \wedge \omega_n$. Let S be a standard domain in H, let $S_{[0,a]} = \{(s, t) \in G | s \in S$ and $t \in [0, a]\}$ and let S_t be as in 3.4. The content of Lemma 3.4 is that $\operatorname{vol}(S_t) = \operatorname{vol}(S_a)$ and $\operatorname{vol}(S_{[0,a]}) = a \operatorname{vol}(S)$.

Consider the map $f\colon H\times\mathbf{R}\to H\times_\tau\mathbf{R}$ given by $f(h,t)=(h,t)=\varphi_t(h,0)$. Note that this is simply the identity map topologically but the spaces differ algebraically and metrically. We wish to calculate the $\operatorname{vol}(\partial S_{[0,a]})=2\operatorname{vol}(S)+\operatorname{vol}[f(\partial S\times[0,a])]$. With respect to the orthonormal bases in question, $Df\colon\mathscr{H}\times\mathbf{R}\to TG$ is given by

$$Df|_{(h,t)}=\left(\begin{array}{c|c} DR_{(e,t)}|_{(h,0)} & \begin{matrix}0\\ \vdots\\ 0\end{matrix}\\ \hline 0\ \cdots 0 & 1\end{array}\right)$$

so that $\|Df\|\le\|DR_{(e,t)}|_{(h,0)}\|+1$. Now if $v\in\mathscr{G}\approx T_{(e,0)}G$ then

$$|DR_{(e,t)}(v)|_{(e,t)}=|DL_{(e,-t)}DR_{(e,t)}(v)|_{(e,0)}=|\operatorname{Ad}[(e,-t)](v)|=|e^{-t\operatorname{ad}(T)}(v)|.$$

Since $R_{(e,t)}$ and $L_{(h,0)}$ commute for all h and t, and left translation is an isometry we have $\|DR_{(e,t)}|_{(h,0)}\|=\|DR_{(e,t)}|_{(e,0)}\|=\|e^{-t\operatorname{ad}(T)}\|\le e^{t\|\operatorname{ad}(T)\|}$. Thus $\|Df\|\le e^{t\|\operatorname{ad}(T)\|}+1\le 2e^{t\|\operatorname{ad}(T)\|}$ for $t\ge 0$. Let Ω be a volume form on $f(\partial S\times[0,a])$ then

$$\begin{aligned}\operatorname{vol}[f(\partial S\times[0,a])]&=\int_{\partial S\times[0,a]}f^*\Omega\le\int_{\partial S\times[0,a]}\|Df\|^n\\ &\le\|Df\|^n a\operatorname{vol}(\partial S)\le a2^ne^{tn\|\operatorname{ad}(T)\|}\operatorname{vol}(\partial S).\end{aligned}$$

We now have a constant λ so that $\operatorname{vol}(\partial S_{[0,a]})\le 2\operatorname{vol}(S)+a2^ne^{\lambda a}\operatorname{vol}(\partial S)$.

Let $\varepsilon>0$ and fix $a>0$ large enough so that $2/a<\varepsilon/2$. Since H satisfies the isoperimetric condition we may find a standard domain S in H so that $\operatorname{vol}(\partial S)/\operatorname{vol}(S)<\varepsilon/2^{n+1}e^{\lambda a}$. Then $\operatorname{vol}(\partial S_{[0,a]})/\operatorname{vol}(S_{[0,a]})<\varepsilon$ and we see that $h(G)=0$. □

Corollary 3.6. *If G is a simply connected solvable Lie group then $h(G)=0$ if and only if G is unimodular.*

Proof. $h(G)=0$ implies that G is closed at infinity. From 3.1 we know in general that closed at infinity implies unimodular. Conversely, since G is solvable its commutator subgroup N is nilpotent and so has polynomial growth, Wolf [18]. Proposition 2.4 shows that polynomial growth implies that $h(N)=0$. Propositions 1.5 and 1.6 show that G is built up by one dimensional unimodular extension of the type in Lemma 3.5 starting with $H_0=N$. So by 3.5 $h(G)=0$. □

Lemma 3.7. *Let G be a simply connected Lie group which is amenable and unimodular then $h(G)=0$.*

Proof. Once again by Levi's Theorem we may conclude that G isomorphic to a group of the form $S\times_\alpha T$ where $\alpha(t)(s)=tst^{-1}$, S is a solvable normal subgroup of G and T is semisimple. By assumption the radical of G is cocompact, so T is compact.

As in Lemma 3.4, G unimodular implies that volume in S is preserved by translation on the right by elements of the form (e, t). Let M be a standard domain in S and $M_T = \{(s, t) \in G | s \in M \text{ and } t \in T\}$, then $\text{vol}(M_T) = \text{vol}(M)\,\text{vol}(T)$.

Let $K = \sup_{t \in T}\{\|DR_{(e,t)}\|\}$. Then $\text{vol}(\partial M_T) \leq K\,\text{vol}(\partial M)\,\text{vol}(T)$. So we have $\text{vol}(\partial M_T)/\text{vol}(M_T) \leq K\,\text{vol}(\partial M)/\text{vol}(M)$. By 3.6 $h(S) = 0$ so $h(G) = 0$ also. □

Finally, we state the main result which was referred to in the introduction.

Theorem 3.8. *Let G be a simply connected Lie group. The following are equivalent:*

1. *G is amenable and unimodular;*
2. *$h(G) = 0$;*
3. *G is closed at infinity.*

Proof. This theorem simply combines the results of this section. Lemma 3.7 gives (1) implies (2), (2) implies (3) is Stokes' Theorem as in 2.3, and (3) implies (1) is given by 3.1 and 3.3.

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