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OPTIMAL STOPPING OF TWO-PARAMETER PROCESSES ON NONSTANDARD PROBABILITY SPACES

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ABSTRACT. We prove the existence of optimal stopping points for upper semicontinuous two-parameter processes defined on filtered nonstandard (Loeb) probability spaces that satisfy a classical conditional independence hypothesis. The proof is obtained via a lifting theorem for elements of the convex set of randomized stopping points, which shows in particular that extremal elements of this set are ordinary stopping points.

1. INTRODUCTION

The optimal stopping problem for two-parameter processes has been the object of much research in recent years, starting with the fundamental paper [CG] of Cairoli and Gabriel. The discrete time version of the problem was then solved with increasing generality by Mandelbaum and Vanderbei [MV], Krengel and Sucheston [KS] and Mazziotto and Szpirglas [MS]. Several papers concerning the continuous time version of this problem have also appeared: Mazziotto [Ma] shows the existence of optimal stopping points for bi-Markov processes, and similar results are stated in [Mi] and [MM] for general two-parameter processes. However, the proofs contained in these two papers are not complete, and the question of existence of optimal stopping points for general two-parameter processes in continuous time is to be regarded as open (see Remark 7.4). However, in this paper, we shall prove the existence of optimal stopping points for upper semicontinuous two-parameter processes defined on a nonstandard (Loeb) probability space that satisfies the commutation property F4 of Cairoli and Walsh [CW].

The approach in this paper was motivated by the following considerations.

—The discrete time optimal stopping problem was well understood, but no continuous time extension had been obtained. In particular, no discretization argument seems feasible.

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—Nonstandard probability theory, as developed by Loeb [L], Anderson [A], Keisler [K] and Hoover and Perkins [HP] provides a powerful tool for extending discrete case results to continuous time.

It thus seemed natural to study the optimal stopping problem via these methods, which have so far been little used in the general theory of two-parameter processes (the only case we are aware of is [MMe]).

Our main tool in this study of the optimal stopping problem is the notion of randomization. The convex compact set of randomized stopping times was first introduced in continuous time by Baxter and Chacon [BC], and used in the context of the single-parameter optimal stopping problem by Ghoussoub [G]: the property that makes this set useful is that extremal elements of the set of randomized stopping times are exactly ordinary stopping times. Now when trying to follow a similar procedure for two-parameter processes, one is hindered by the fact that the set $\mathscr{U}$ of randomized stopping points generally contains extremal elements which are not stopping points (a simple example is provided in [MM]). This fact turns out to be a consequence of the complex combinatorial structure of two-parameter filtrations (see [DTW]), and led Millet [Mi] and Mazziotto and Millet [MM] to try different randomizations.

As a matter of fact, the set of extremal elements of $\mathscr{U}$ seems to remain the set $\mathscr{T}$ of stopping points when the two-parameter filtration satisfies certain classical conditions, such as Hypothesis CQI of Krengel and Sucheston [KS] or Hypothesis F4 of Cairoli and Walsh [CW]. This was proved on finite probability spaces in [DTW] and on arbitrary complete probability spaces but in discrete time in [D2].

The main result of this paper is that the property $\mathscr{T} = \operatorname{ext}\mathscr{U}$ is again valid in continuous time, provided the underlying probability space is a nonstandard (Loeb) space. Existence of optimal stopping points for upper semicontinuous two-parameter processes is then obtained using a generalization of the regularity result for functionals of randomized stopping points obtained in [D1].

The use of nonstandard probability theory seems particularly natural due to the fact that the discrete time proof that $\mathscr{T} = \operatorname{ext}\mathscr{U}$ contained in [D2] relies on the construction of a particular optional increasing path $(Z_n)_{n\in\mathbf{N}}$ by a step by step procedure. In continuous time, one would imagine that a path $(Z_u)_{u\in\mathbf{R}_+}$ with similar properties could be defined as the solution of a (random) differential equation of the form

$$(*) \qquad \frac{dZ_u}{du}(\omega) = f(u, (Z_v)_{v\leq u}, \omega).$$

However, no regularity is to be expected from the function $f(\cdot, \cdot, \omega)$. Now certain stochastic differential equations with insufficiently regular coefficients are known not to have any (strong) solution (see Barlow [Ba]), and so it is improbable that $(*)$ would have a solution in any useful sense. On the other hand, Keisler [K] (Theorems 5.2 and 5.5) has shown under minimal regularity assumptions that stochastic differential equations have a (strong) solution when the probability space is hyperfinite, hence the use of these spaces in this paper. We feel that nonstandard probability theory may lead to solutions of several other problems in the theory of two-parameter processes, particularly in instances where the discrete case is solved, but the continuous time extension via classical methods does not seem to succeed.

2. The set of randomized stopping points

Throughout this paper, we will primarily be concerned with stochastic processes indexed by $\mathbb{N}$, $\mathbb{D}_n$ or $\mathbb{R}_+$ (single-parameter processes) or $\mathbb{N}^2$, $\mathbb{D}_n^2$ or $\mathbb{R}_+^2$ (two-parameter processes). Here $\mathbb{D}_n$ denotes the set of dyadic real numbers of order n. In the continuous case, we will often replace $\mathbb{R}_+$ by $[0, 1]$.

The letter I (respectively I^2) will denote a single-parameter (respectively two-parameter) index set. The set I is equipped with the usual total order, denoted $\leq$, whereas on I^2 it is natural to consider the two orders $\leq$ and $\triangle$ defined by

$$s = (s_1, s_2) \leq t = (t_1, t_2) \Leftrightarrow s_1 \leq t_1 \text{ and } s_2 \leq t_2,$$

$$s = (s_1, s_2) \triangle t = (t_1, t_2) \Leftrightarrow s_1 \leq t_1 \text{ and } s_2 \geq t_2.$$

We will use the notation $s < t$ to express that $s \leq t$ and $s \neq t$, whereas $s \wedge t$ will mean $s \triangle t$ and $s \neq t$, and $s \ll t$ will mean $s_1 < t_1$ and $s_2 < t_2$. Several kinds of intervals can be defined on I^2: $[s, t] = \{u \in I^2 : s \leq u \leq t\}$, $]s, t] = \{u \in I^2 : s \ll u < t\}$ and so forth. In order to avoid introducing special symbols, we will set $]s, t] = \{u \in I^2 : s < u \leq t\}$ when $s \leq t$ but $s_1 = t_1$ or $s_2 = t_2$.

In several instances, we will use the lexicographic (total) order $\leq_1$ on I^2:

$$s \leq_1 t \Leftrightarrow (s_1 < t_1 \text{ or } (s_1 = t_1 \text{ and } s_2 \leq t_2)).$$

The notation $s <_1 t$ will mean $s \leq_1 t$ and $s \neq t$.

We will often add to I or I^2 an extra element, denoted in both cases ∞, and will set $\overline{I} = I \cup \{\infty\}$, $\overline{I}^2 = I^2 \cup \{\infty\}$. These sets will be equipped with their usual metric topologies, making them compact. We will also suppose that $t \leq \infty$, for all t in either I or I^2. The notations $\mathscr{B}(I)$, $\mathscr{B}(\overline{I})$, $\mathscr{B}(I^2)$, $\mathscr{B}(\overline{I}^2)$ will denote in each case the Borel σ-algebra of the index set.

Let $(\Omega, \mathscr{F}, P)$ be a (complete) probability space. A *two-parameter filtration* is a family $(\mathscr{F}_t)_{t \in I^2}$ of sub-σ-algebras of $\mathscr{F}$ with the following properties.

F1. $\mathscr{F}_{0,0}$ contains all P-null sets;

F2. $s \leq t \Rightarrow \mathscr{F}_s \subset \mathscr{F}_t$;

F3. When $I = [0, 1]$, $\mathscr{F}_s = \bigcap_{t \in]s,(1,1)]} \mathscr{F}_t$, $\forall s \in I^2$.

These properties are termed the "usual conditions" [DM, IV. 48].

Many results in the theory of two-parameter processes require a supplementary hypothesis on the two-parameter filtration, usually Hypothesis F4 of Cairoli and Walsh [CW].

F4. If $s, t, u \in I^2$ are such that $s \triangle t$ and $u = (s_1, t_2)$, then $\mathscr{F}_s$ is conditionally independent of $\mathscr{F}_t$ given $\mathscr{F}_u$.

This condition restricts the combinatorial complexity of the filtration (see [DTW, Theorems 3.6, 5.8 and 5.9]).

Associated with a two-parameter filtration is a set $\mathscr{T}$ of *stopping points*: a random variable $T: \Omega \to \overline{I}^2$ is a stopping point provided $\{T \leq t\} \in \mathscr{F}_t$, $\forall t \in I^2$.

Given a measurable real-valued process $X = (X_t)_{t \in \overline{I}^2}$, the *optimal stopping problem* is to determine a stopping point T_0 such that

$$E(X_{T_0}) = \sup_{T \in \mathscr{T}} E(X_T);$$

T_0 is then called *optimal*. We shall prove that optimal stopping points do exist on nonstandard filtered Loeb probability spaces that satisfy Hypothesis F4, under suitable regularity assumptions on the reward process X. This process may or may not be adapted (a process $(X_t)_{t \in I^2}$ is *adapted* to $(\mathscr{F}_t)_{t \in I^2}$ provided X_t is $\mathscr{F}_t$-measurable, for all t).

The problem of existence of optimal stopping points reduces to the following: consider the map $\phi: \mathscr{T} \to \mathbf{R}$ defined by $T \mapsto \phi(T) = E(X_T)$, and show that this map attains its maximum on $\mathscr{T}$. It is thus natural to embed $\mathscr{T}$ into some larger "randomized" set $\mathscr{U}$ with certain convexity and compactness properties and on which ϕ can be extended to a function with sufficient regularity that a maximum over $\mathscr{U}$ will exist. The choice of randomization should be such that one can then recover a maximum in $\mathscr{T}$.

The regularity question for upper semicontinuous processes will be solved by a generalization of the result of [D1]. Furthermore, a natural way to randomize is to take the convex closure of $\mathscr{T}$ in an appropriate sense. This leads to the set of randomized stopping points, introduced by Baxter and Chacon [BC] in the single-parameter setting. The presentation of this set by Meyer [Me] and Ghoussoub [G] will be the most convenient for our purposes.

A *randomized stopping point* is a random probability measure $\mu(\omega, B)$, $\omega \in \Omega$, $B \in \mathscr{B}(\overline{I}^2)$ such that $\mu(\cdot, [0, t])$ is $\mathscr{F}_t$-measurable, for all t. Each stopping point T identifies with the randomized stopping point μ_T defined by

$$\mu_T(\omega, B) = I_{\{T \in B\}}(\omega), \qquad \omega \in \Omega, \ B \in \mathscr{B}(\overline{I}^2),$$

so $\mathscr{T}$ is "contained" in $\mathscr{U}$.

Let $\mathscr{C}$ denote the set of continuous real-valued processes $(X_t)_{t \in \overline{I}^2}$ such that $E(\sup_{t \in \overline{I}^2} |X|_t) < +\infty$. $\mathscr{C}$ equipped with the norm $\|X\| = E(\sup_{t \in \overline{I}^2} |X_t|)$ is a Banach space. It is well known that $\mathscr{U}$ is a subset of the unit ball in the dual $\mathscr{C}^*$ of $\mathscr{C}$ that is compact in the weak topology $\sigma(\mathscr{C}^*, \mathscr{C})$ (see [Me, G]).

Furthermore, for $(X_t)_{t \in \overline{I}^2} \in \mathscr{C}$, the map $\Phi: \mathscr{U} \to \mathbf{R}$, defined by

$$\Phi(\mu) = E\left(\int_{\overline{I}^2} X_t(\cdot)\mu(\cdot, dt)\right),$$

is continuous on $\mathscr{U}$ and is an extension of $T \mapsto E(X_T)$. Hence, the existence of an optimal randomized stopping point is clear in this case. Now since Φ is affine and $\mathscr{U}$ is convex, Φ attains its maximum at an extremal element of $\mathscr{U}$. Thus we will have shown the existence of an optimal stopping point provided $\mathscr{T} = \text{ext}\,\mathscr{U}$. This method was in fact used by Ghoussoub [G] for continuous single-parameter processes.

Now for two-parameter processes, it is clear that $\mathscr{T} \subset \text{ext}\,\mathscr{U}$, but as mentioned in the introduction, the converse inclusion is false in general. Our purpose here is to show that the property $\mathscr{T} = \text{ext}\,\mathscr{U}$ also holds in continuous time when Ω is a nonstandard (Loeb) space and the two-parameter filtration satisfies Hypothesis F4.

To see why this extension is feasible, let us first look at the set $\mathscr{U}$ when $I = \mathbf{N}$. In this case, a randomized stopping point can be identified with a *positive weight process* $(a_t)_{t\in\overline{\mathbf{N}}^2}$ defined by $a_t(\omega) = \mu(\omega, \{t\})$ (i.e. a_t is the random weight of t for μ). This weight process satisfies the following conditions:

(2.1) $a_t \geq 0$ a.s.;

(2.2) a_t is $\mathscr{F}_t$-measurable, $\forall t \in \mathbf{N}^2$;

(2.3) $\sum_{t\in\overline{\mathbf{N}}^2} a_t = 1$ a.s.

These three properties characterize weight processes that correspond to randomized stopping points. The weights $(a_t)_{t\in\overline{\mathbf{N}}^2}$ are very convenient to work with, and this was exploited in [D2]. Now when $I = \mathbf{R}_+^2$, a randomized stopping point can only be identified with a right-continuous nonnegative adapted process $(A_t)_{t\in\overline{\mathbf{R}}_+^2}$ such that $A_\infty = 1$ a.s. and $\Delta_{]s,t]}A \geq 0$ a.s., where

$$\Delta_{]s,t]}A = \begin{cases} A_t - A_{(s_1,t_2)} - A_{(t_1,s_2)} + A_s & \text{if } s \ll t, \\ A_t - A_s & \text{if } s \leq t \text{ and } s_1 = t_1 \text{ or } s_2 = t_2. \end{cases}$$

The main idea of this paper will be to "lift" a continuous time randomized stopping point to an (internal) weight process indexed by a hyperfinite set (the terminology from nonstandard probability theory will be recalled in §3). This weight process can then be manipulated as in the discrete case. Of course this procedure can only be carried out on a Loeb space and as mentioned in the introduction, it is not clear that a discretisation on a standard space can lead to a continuous time solution to the question of equality of $\mathscr{T}$ and $\text{ext}\,\mathscr{U}$. A corollary of this study will be a proof of the existence of optimal stopping points in continuous time.

Before introducing the nonstandard framework we will be working in, we recall the discrete case result of [D2]. For this, we need the notion of optional increasing path [W].

2.1. **Definition.** A family $Z = (Z_u)_{u\in\overline{I}}$ of stopping points is an *optional increasing path* (*o.i.p.*) provided $Z_0 \equiv (0, 0)$ a.s., $u \leq v \Rightarrow Z_u \leq Z_v$ a.s., and $|Z_u| = u$ a.s., $\forall u \in \overline{I}$ (for $t = (t_1, t_2)$, $|t|$ denotes the sum $t_1 + t_2$). If $I = \mathbf{D}_n$, we impose the supplementary condition $Z_{u+2^{-n}}$ is $\mathcal{F}_{Z_u}$-measurable, $\forall u \in \mathbb{D}_n$ (these o.i.p.'s are often called *tactics*: see [MV]).

Though the theorem below was proved under the weaker hypothesis CQI of Krengel and Sucheston [KS], we only need it for filtrations that satisfy Hypothesis F4.

2.2. **Splitting Theorem.** *Let* $(\Omega, \mathcal{F}, P)$ *be a (complete) probability space, and* $(\mathcal{F}_t)_{t\in\mathbb{N}^2}$ *be a two-parameter filtration satisfying Hypothesis* F4. *Then*:

(a) *all extremal elements of the set of randomized stopping points are stopping points*;

(b) *furthermore, for any randomized stopping point* $(a_t)_{t\in\overline{\mathbb{N}}^2} \in \mathcal{U}$, *there are* $(a_t^1)_{t\in\overline{\mathbb{N}}^2}$, $(a_t^2)_{t\in\overline{\mathbb{N}}^2} \in \mathcal{U}$ *and an o.i.p.* $(Z_n)_{n\in\overline{\mathbb{N}}}$ *such that*:

(b1) $a_t = \frac{1}{2}a_t^1 + \frac{1}{2}a_t^2$ *a.s.*, $\forall t \in \overline{\mathbb{N}}^2$;

(b2) *for almost all* $\omega \in \Omega$,

$$t \wedge Z_{|t|}(\omega) \Rightarrow a_t^1(\omega) = 2a_t(\omega), \qquad a_t^2(\omega) = 0,$$

$$Z_{|t|}(\omega) \wedge t \Rightarrow a_t^1(\omega) = 0, \qquad a_t^2(\omega) = 2a_t(\omega).$$

(For a proof, see [D2, (4.22) and Theorem 4.23].)

In order to apply the Transfer Principle of Nonstandard Analysis (see (3.3)), we shall only need this result for index sets I^2 of the form $\{s \in \mathbb{N}^2 : s \leq (n, n)\}$, for some $n \in \mathbb{N}$.

3. Preliminaries from nonstandard probability theory

The nonstandard framework will be that of Keisler [K]: we work in an ω_1-saturated enlargement $V(^*S)$ of a superstructure $V(S)$, where $S \supset \mathbb{R}$. The reader interested in familiarizing himself with the basics of nonstandard analysis should consult [HL]. The nonstandard theory of single-parameter stochastic processes is contained in [SB], and we follow their notation. In the hyperfinite setting, a comprehensive presentation with applications is given in [AFHL].

(3.1) Internal functions will generally be written $\tilde{f}$, $\tilde{g}$.

(3.2) The standard part of a finite element $r \in {}^*\mathbb{R}$ is denoted $\mathrm{st}(r)$. When $x, y \in {}^*\mathbb{R}$ are such that $|x-y| < 1/n$, $\forall n \in \mathbb{N}$, we write $x \approx y$. If $s, t \in {}^*\mathbb{R}^2$, $s \approx t$ means $s_1 \approx t_1$ and $s_2 \approx t_2$.

(3.3) *Transfer Principle.* Let $S_1, \ldots, S_n \in V(S)$. Any elementary statement which is true of $S_1, \ldots, S_n$ is true of $^*S_1, \ldots, {}^*S_n$.

(3.4) *Countable Comprehension Principle.* Let X be an internal set, and $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of X. Then there exists an internal sequence $(y_n)_{n\in{}^*\mathbb{N}}$ of elements of X such that $y_n = x_n$, $\forall n \in \mathbb{N}$.

(3.5) We fix $n_0 \in {}^*\mathbb{N}\backslash\mathbb{N}$, and set $L = n_0!$, $\Delta u = 1/L$. $\mathbb{T}$ denotes the internal set $\{0, \Delta u, 2\Delta u, \ldots, 1\}$. Since L is an infinite factorial, $\mathbb{T}$ contains all rational numbers in $[0, 1]$.

(3.6) If $(\Omega, \mathscr{A}, \overline{P})$ is an internal probability space, $(\Omega, L(\mathscr{A}), P)$ denotes the corresponding *Loeb space*, that is $L(\mathscr{A})$ is the (external) σ-algebra generated by $\mathscr{A}$, and P is the unique σ-additive extension of $\mathrm{st}(\overline{P})$ to $L(\mathscr{A})$ [HP, §3].

(3.7) An *internal two-parameter filtration* will be an internal family $(\mathscr{A}_t)_{t\in\mathbf{T}^2}$ of internal $*$-sub-σ-algebras of $\mathscr{A}$, such that

$$s \le t,\ s, t \in \mathbf{T}^2 \Rightarrow \mathscr{A}_s \subset \mathscr{A}_t.$$

This filtration is *complete* provided any internal subset N of an internal set $M \in \mathscr{A}$ with $\overline{P}(M) = 0$ belongs to $\mathscr{A}_{0,0}$. The *standard part* of $(\mathscr{A}_t)_{t\in\mathbf{T}^2}$ is the (ordinary) filtration $(\mathscr{F}_t)_{t\in[0,1]^2}$ defined by

$$\mathscr{F}_t = \left(\bigcap_{\mathrm{st}(s)\gg t} \sigma(\mathscr{A}_s)\right) \vee \mathscr{N}, \qquad t \in [0, 1]^2,$$

where $\mathscr{N}$ denotes the family of null sets of P. It is easy to see that properties F1, F2 and F3 are satisfied.

(3.8) The family $(A_t)_{t\in\mathbf{T}^2}$ satisfies *Hypothesis* $\overline{\mathrm{F4}}$ provided $s, t, u \in \mathbf{T}^2$, $s \triangle t$, $u = (s_1, t_2)$, $B \in \mathscr{A}_s$, and $C \in \mathscr{A}_t$ imply

$$\overline{P}(B \cap C \mid \mathscr{A}_u) = \overline{P}(B \mid \mathscr{A}_u)\overline{P}(C \mid \mathscr{A}_u).$$

(3.9) A *lifting* of a random variable X defined on $(\Omega, L(\mathscr{A}), P)$ is an internal function $\widetilde{X}: \Omega \to {}^*\mathbf{R}$ which is $\mathscr{A}$-measurable (i.e. constant on atoms of $\mathscr{A}$), and such that $X = \mathrm{st}(\widetilde{X})$ P-a.s.

Throughout this paper, we will work on a fixed filtered Loeb space $(\Omega, \mathscr{F} = L(\mathscr{A}), P, (\mathscr{F}_t)_{t\in[0,1]^2})$, where P is the Loeb measure associated with an internal probability measure on $\mathscr{A}$, and $(\mathscr{F}_t)_{t\in[0,1]^2}$ is the standard part of an internal (complete) filtration $(\mathscr{A}_t)_{t\in\mathbf{T}^2}$.

3.1. **Lemma.** *Fix $t \in [0, 1]^2$. A random variable X is $\mathscr{F}_t$-measurable provided X has a lifting $\widetilde{X}$ which satisfies the following condition*:

$$\textit{there exists } s \in \mathbf{T}^2,\ s \approx t,\ \textit{such that } \widetilde{X} \textit{ is } \mathscr{A}_s\textit{-measurable}.$$

The proof of this lemma is omitted, as it is similar to the single-parameter case (see [HP, Theorem 3.2]).

3.2. **Lemma.** *Let X be a bounded random variable, and $\widetilde{X}$ a bounded lifting of X. Fix $t \in [0, 1]^2$. Then there is $u \approx t$, $u \in \mathbf{T}^2$ (depending on X) such that for $s \ge u$, $s \approx t$, $\overline{E}(\widetilde{X} \mid \mathscr{A}_s)$ is a lifting of $E(X \mid \mathscr{F}_t)$.*

Proof. By [HP, Lemma 3.3],

$$\mathrm{st}(\overline{E}(\widetilde{X} \mid \mathscr{A}_s)) = E(X \mid L(\mathscr{A}_s)) \quad \text{a.s.}, \forall s \in \mathbf{T}^2.$$

Hence it is only necessary to prove that for some $u \in \mathbf{T}^2$, $u \approx t$,

$$s \in \mathbf{T}^2,\ s \ge u,\ s \approx t \Rightarrow E(X \mid L(\mathscr{A}_s)) = E(X \mid \mathscr{F}_t) \quad \text{a.s.}$$

The proof of this statement is the straightforward two-parameter extension of the Remark following Lemma 8.4 in [Ke]. □

3.3. **Proposition.** *Suppose* $(A_t)_{t\in\mathbf{T}^2}$ *satisfies Hypothesis* $\overline{\text{F4}}$. *Then* $(\mathcal{F}_t)_{t\in[0,1]^2}$ *satisfies Hypothesis* F4.

Proof. Fix $s, t \in [0,1]^2$ such that $s \triangle t$, and set $u = (s_1, t_2)$. Let $B \in \mathcal{F}_s$, $C \in \mathcal{F}_t$. By Lemma 3.1, there are $\tilde{s}, \tilde{t} \in \mathbf{T}^2$, $\tilde{s} \approx s$, $\tilde{t} \approx t$, and internal sets $\widetilde{B} \in \mathcal{A}_{\tilde{s}}$, $\widetilde{C} \in \mathcal{A}_{\tilde{t}}$ such that $B = \widetilde{B}$ a.s. and $C = \widetilde{C}$ a.s. Using Lemma 3.2, we get for sufficiently large $\tilde{u} \approx u$, $\tilde{u} \in \mathbf{T}^2$:

$$\begin{aligned} P(B \cap C \mid \mathcal{F}_u) &= \operatorname{st}(\overline{P}(\widetilde{B} \cap \widetilde{C} \mid \mathcal{A}_{\tilde{u}})) \\ &= \operatorname{st}(\overline{P}(\widetilde{B} \mid \mathcal{A}_{\tilde{u}})\overline{P}(\widetilde{C} \mid \mathcal{A}_{\tilde{u}})) \\ &= P(B \mid \mathcal{F}_u)P(C \mid \mathcal{F}_u). \quad \square \end{aligned}$$

The following proposition provides a canonical example of a filtered hyperfinite probability space which satisfies properties F1–F4.

3.4. **Proposition.** *Let* $\Omega = \Omega_0^{\mathbf{T}^2}$ *be the (internal) set of all internal functions from* $\mathbf{T}^2$ *into some hyperfinite set* Ω_0, $\mathcal{A}$ *be the algebra of internal sets in* Ω, *and* $\overline{P}$ *the uniform counting measure on* $\mathcal{A}$ (*see* [Ke, §1]). *For* $t \in \mathbf{T}^2$, *let* $\mathcal{A}_t$ *be the algebra of internal sets closed under the equivalence relation* $\approx_t$ *defined by*

$$\omega \approx_t \omega' \quad \Leftrightarrow \quad \omega(s) = \omega'(s),\ \forall s \leq t,\ s \in \mathbf{T}^2.$$

Then $(\Omega, L(\mathcal{A}), P, (\mathcal{F}_t)_{t\in[0,1]^2})$ *satisfies properties* F1, F2, F3 *and* F4.

Proof. We only check Hypothesis F4. By Proposition 3.3, it is sufficient to check Hypothesis $\overline{\text{F4}}$ for $(A_t)_{t\in\mathbf{T}^2}$.

If A is an internal set, let $|A|$ denote the internal cardinality of A, and let $\rho_t(\omega)$ denote the equivalence class of ω for $\approx_t$. Since each element of $\mathcal{A}_t$ is a hyperfinite union of disjoint equivalence classes $\rho_t(\omega)$, Hypothesis $\overline{\text{F4}}$ will hold provided for $s, t, u \in \mathbf{T}^2$ such that $s \triangle t$ and $u = (s_1, t_2)$,

$$\overline{P}(\rho_s(\omega') \cap \rho_t(\omega'')|\rho_u(\omega)) = \overline{P}(\rho_s(\omega')|\rho_u(\omega))\overline{P}(\rho_t(\omega'')|\rho_u(\omega)),$$

for all $\omega, \omega', \omega'' \in \Omega$. Observe that both sides above are zero unless $\omega' \approx_u \omega \approx_u \omega''$. In this case, the above equality is equivalent to

$$|\rho_s(\omega') \cap \rho_t(\omega'')| = |\rho_s(\omega')|\,|\rho_t(\omega'')|/|\rho_u(\omega)|.$$

Since $\Omega = \Omega_0^{\mathbf{T}^2}$, the left-hand side of this equality is equal to

$$|\Omega_0|^{L^2((1-s_2)+(1-t_1)s_2+(s_2-t_2)(t_1-s_1))} = |\Omega_0|^{L^2(1-s_1s_2-t_1t_2+s_1t_2)},$$

where $L \in {}^*\mathbb{N}$ is defined in (3.5), and the right-hand side is equal to

$$|\Omega_0|^{L^2(1-s_1s_2)}|\Omega_0|^{L^2(1-t_1t_2)}/|\Omega_0|^{L^2(1-u_1u_2)}.$$

Since $u_1 u_2 = s_1 t_2$, these two quantities are equal, completing the proof. □

4. The simultaneous lifting theorem

The first step towards obtaining a lifting theorem for continuous time randomized stopping points is to obtain such a theorem on a finite index set. This is no problem in the single-parameter case, but as will become apparent, it is quite nontrivial in the presence of two parameters.

Throughout the rest of this paper, we make the following assumption.

4.1. **Assumption.** The internal filtration $(\mathscr{A}_t)_{t\in[0,1]^2}$ satisfies Hypothesis $\overline{\text{F4}}$.

4.2. **Simultaneous Lifting Theorem.** *Fix $n \in \mathbb{N}$, and set $I = \{0, 1/n, 2/n, \ldots, 1\}$. Let $(a_t)_{t\in I^2}$ be a family of real random variables such that*

(4.1) a_t *is $\mathscr{F}_t$-measurable,* $\forall t \in I^2$;

(4.2) $a_t \geq 0$ *a.s.,* $\forall t \in I^2$;

(4.3) $\sum_{t\in I^2} a_t = 1$ *a.s.*

Then there is a family $(\tilde{a}_t)_{t\in I^2}$ of internal functions from Ω into ${}^*\mathbf{R}$ such that

(4.4) $\operatorname{st}(\tilde{a}_t) = a_t$ *a.s.*;

(4.5) *for each $t \in I^2$, there is $s \in \mathbf{T}^2$, $s \approx t$ such that $\tilde{a}_t$ is $\mathscr{A}_s$-measurable*;

(4.6) $\tilde{a}_t(\omega) \geq 0$, $\forall \omega \in \Omega$, $\forall t \in I^2$;

(4.7) $\sum_{t\in I^2} \tilde{a}_t(\omega) = 1$, $\forall \omega \in \Omega$.

4.3. *Remark.* The difficult point in this theorem is to replace the (external) "a.s." relationships in (4.2) and (4.3) by the internal relations (4.6) and (4.7) valid for each $\omega \in \Omega$. Though the proof seems nontrivial already for $n = 2$, and uses the conditional supremum operator introduced in [D2], its proof would be quite straightforward in the single-parameter case, when I^2 is replaced by I. We briefly indicate how the theorem could be proved in this case.

Let $\tilde{b}_t$ be a lifting of $\sum_{s\leq t} a_s$, such that $0 \leq \tilde{b}_t(\omega) \leq 1$, $\forall \omega \in \Omega$, and for some $s^t \approx t$, $\tilde{b}_t$ is $\mathscr{A}_{s^t}$-measurable. Set $\tilde{c}_t = \sup_{s\leq t} \tilde{b}_s$, $\tilde{a}_0 = \tilde{c}_0$, and

$$\tilde{a}_t = \tilde{c}_t - \tilde{c}_{t-1/n}, \qquad t \in I\backslash\{0, 1\},$$
$$\tilde{a}_1 = 1 - \tilde{c}_{(n-1)/n}.$$

Then $(\tilde{a}_t)_{t\in I}$ has the desired properties. □

Before proving Theorem 4.2, we recall the definition and main properties of the *conditional supremum operator* $S(Y|\mathscr{G})$ introduced in [D2]: given a sub-σ-algebra $\mathscr{G}$ of $\mathscr{F}$, and a bounded random variable Y, $S(Y|\mathscr{G})$ is the $\mathscr{G}$-measurable random variable defined by $S(Y|\mathscr{G}) = \operatorname{ess\,inf} Z$, where the essential infimum is taken over all $Z \geq Y$ which are $\mathscr{G}$-measurable. $S(\cdot \mid \cdot)$ has the following properties, which we recall for ease of reference.

(4.8) $\mathscr{G}_1 \subset \mathscr{G}_2 \Rightarrow S(Y|\mathscr{G}_2) \leq S(Y|\mathscr{G}_1)$;

(4.9) If X is $\mathscr{G}$-measurable, then $S(X + Y|\mathscr{G}) = X + S(Y|\mathscr{G})$;

(4.10) If $(\mathscr{F}_t)_{t\in I^2}$ satisfies Hypothesis F4, and $s, t, u \in I^2$ are such that $s \triangle t$, $u = (s_1, t_2)$ and if Y is $\mathscr{F}_s$-measurable, then $S(Y|\mathscr{F}_t) = S(Y|\mathscr{F}_u)$.

((4.8) is clear; (4.9) follows from [D2, Lemma 4.7 (f)] and (4.10) follows from [D2, Proposition 4.12(a) and (b)].)

Proof of Theorem 4.2. For $t \in I^2$, set $R_t^- = \{s \in I^2 : s \triangle t\}$, and $A_t^- = \sum_{s \in R_t^-} a_s$. Observe that $0 \le S(A_t^- | \mathscr{F}_t) \le 1$ a.s., since $0 \le A_t^- \le 1$ a.s. Since a_t and $S(A_t^- | \mathscr{F}_t)$ are $\mathscr{F}_t$-measurable, there exist by Lemma 3.1 two internal functions $\tilde{b}_t$, $\widetilde{S}_t : \Omega \to {}^*[0, 1]$ such that

(4.11) $\mathrm{st}(\tilde{b}_t) = a_t$ a.s., $\mathrm{st}(\widetilde{S}_t) = S(A_t^- | \mathscr{F}_t)$ a.s.,

(4.12) for some $s^t \approx t$, $\tilde{b}_t$ and $\widetilde{S}_t$ are $\mathscr{A}_{s^t}$-measurable.

We can now define $\tilde{a}_t$, $t \in I^2$, by induction in increasing order for $\le_1$ (the lexicographic order on I^2). Throughout this proof, k and l will denote elements of I. Set

$$\tilde{a}_{0,0} = \min(\tilde{b}_{0,0}, \widetilde{S}_{0,0}),$$

and suppose by induction that $\tilde{a}_s$ has been defined, for $s <_1 t$. Then set

$$(4.13) \qquad \tilde{a}_t = \max\left(0, \min\left(\tilde{b}_t, \min_{\substack{0 \le k < t_1 \\ 0 \le l \le t_2}} \left(\widetilde{S}_{t_1,l} - \widetilde{S}_{k,t_2} - \sum_{\substack{u < t \\ u \in R_{t_1,l}^- \setminus R_{k,t_2}^-}} \tilde{a}_u\right)\right)\right)$$

if $t \ne (1, 1)$, and

$$\tilde{a}_{1,1} = 1 - \sum_{t \in I^2 \setminus \{(1,1)\}} \tilde{a}_t .$$

Then property (4.7) is trivially satisfied. Before showing that properties (4.4), (4.5) and (4.6) hold, we prove the following lemmas.

4.4. **Lemma.** *Fix* $t \in I^2$, *and* $0 \le k < t_1$, $0 \le l \le t_2$. *Then*

$$a_t \le S(A_{t_1,l}^- \mid \mathscr{F}_{t_1,l}) - S(A_{k,t_2}^- \mid \mathscr{F}_{k,t_2}) - \sum_{\substack{u<t \\ u \in R_{t_1,l}^- \setminus R_{k,t_2}^-}} a_u \quad a.s.$$

Proof. Since $(\mathscr{F}_t)_{t \in I^2}$ satisfies Hypothesis F4, (4.10) implies that

$$S(A_{k,t_2}^- \mid \mathscr{F}_{k,t_2}) + \sum_{\substack{u \le t \\ u \in R_{t_1,l}^- \setminus R_{k,t_2}^-}} a_u = S(A_{k,t_2}^- \mid \mathscr{F}_t) + \sum_{\substack{u \le t \\ u \in R_{t_1,l}^- \setminus R_{k,t_2}^-}} a_u ,$$

which, by (4.9), is equal to

$$S\left(A_{k,t_2}^- + \sum_{\substack{u \le t \\ u \in R_{t_1,l}^- \setminus R_{k,t_2}^-}} a_u \mid \mathscr{F}_t\right) \le S(A_{t_1,l}^- \mid \mathscr{F}_t).$$

By (4.8), this is not greater than $S(A_{t_1,l}^- \mid \mathscr{F}_{t_1,l})$. This clearly implies the statement of the lemma. □

4.5. **Lemma.** *Fix* t_1, $l \in I$, *and* $\omega \in \Omega$, *and suppose* $\tilde{a}_{t_1,t_2}(\omega) > 0$, *for some* $t_2 \geq l$ *with* $(t_1, t_2) \neq (1, 1)$. *Then*

$$\sum_{\substack{s \in R^-_{t_1,l} \\ s \neq (1,1)}} \tilde{a}_s(\omega) \leq \widetilde{S}_{t_1,l}(\omega).$$

Proof. We first show that the statement of the lemma holds when $t_1 = 0$. Suppose $\tilde{a}_{0,t_2}(\omega) > 0$ for some $t_2 \geq l$. Let $t_2 \in I$ be maximal with this property. Then

$$\sum_{s \in R^-_{0,l}} \tilde{a}_s(\omega) = \sum_{l \leq s_2 \leq t_2} \tilde{a}_{0,s_2}(\omega). \tag{4.14}$$

Now by (4.13), $\tilde{a}_{0,t_2}(\omega) > 0$ implies

$$\tilde{a}_{0,t_2}(\omega) \leq \widetilde{S}_{0,l}(\omega) - \sum_{l \leq s_2 < t_2} \tilde{a}_{0,s_2}(\omega),$$

and thus

$$\sum_{l \leq s_2 \leq t_2} \tilde{a}_{0,s_2}(\omega) \leq \widetilde{S}_{0,l}(\omega).$$

By (4.14), the lemma holds for $t_1 = 0$.

Suppose now by induction that the statement of the lemma holds for $0 \leq t_1' < t_1$, and show that it holds for t_1. Fix $l \in I$, and suppose $\tilde{a}_{t_1,t_2}(\omega) > 0$, for some $t_2 \geq l$, with $(t_1, t_2) \neq (1, 1)$. Let t_2 be maximal with this property.

Case 1. $\tilde{a}_{t_1',t_2'}(\omega) = 0$, $\forall t_1' < t_1$, $t_2' \geq t_2$. Then

$$\sum_{\substack{s \in R^-_{t_1,l} \\ s \neq (1,1)}} \tilde{a}_s(\omega) = \sum_{\substack{s \leq (t_1,t_2) \\ s \in R^-_{t_1,l} \setminus R^-_{t_1-1,t_2}}} \tilde{a}_s(\omega) = \tilde{a}_{t_1,t_2}(\omega) + \sum_{\substack{s < (t_1,t_2) \\ s \in R^-_{t_1,l} \setminus R^-_{t_1-1,t_2}}} \tilde{a}_s(\omega).$$

By (4.13), $\tilde{a}_{t_1,t_2}(\omega) > 0$ implies that the last expression above is not greater than

$$\widetilde{S}_{t_1,l}(\omega) - \widetilde{S}_{t_1-1,t_2}(\omega) \leq \widetilde{S}_{t_1,l}(\omega),$$

which implies the desired property.

Case 2. For some $k < t_1$ and $t_2' \geq t_2$, $\tilde{a}_{k,t_2'}(\omega) > 0$. Let k be maximal with this property. Then

$$\sum_{s \in R^-_{t_1,l}} \tilde{a}_s(\omega) = \sum_{s \in R^-_{k,t_2}} \tilde{a}_s(\omega) + \sum_{\substack{s < (t_1,t_2) \\ s \in R^-_{t_1,l} \setminus R^-_{k,t_2}}} \tilde{a}_s(\omega) + \tilde{a}_{t_1,t_2}(\omega). \tag{4.15}$$

Applying the induction hypothesis to the first term on the right-hand side of (4.15) and using the fact that $\tilde{a}_{t_1,t_2}(\omega) > 0$, we see by (4.13) that this last expression is not greater than

$$\widetilde{S}_{k,t_2}(\omega) + \widetilde{S}_{t_1,l}(\omega) - \widetilde{S}_{k,t_2}(\omega) = \widetilde{S}_{t_1,l}(\omega).$$

This completes the proof of the lemma. □

End of the proof of Theorem 4.2. To see (4.4), we proceed by induction in increasing order for $\leq_1$. Use Lemma 4.4 and (4.11) and (4.13) to see that

$$\begin{aligned}\mathrm{st}(\tilde{a}_t) &= \max\left(0, \min\left(\mathrm{st}(\tilde{b}_t), \min_{\substack{0\leq k<t_1 \\ 0\leq l\leq t_2}}\left(\mathrm{st}\left(\widetilde{S}_{t_1,l} - \widetilde{S}_{k,t_2} - \sum_{\substack{u<t \\ u\in R^-_{t_1,l}\setminus R^-_{k,t_2}}} \tilde{a}_u\right)\right)\right)\right) \\ &= \mathrm{st}(\tilde{b}_t) = a_t \quad \text{a.s.}\end{aligned}$$

Again proceeding by induction in increasing order for $\leq_1$, we see that (4.5) is implied by (4.12) and (4.13). Now (4.6) clearly holds for all $t \in I^2\setminus\{(1, 1)\}$ by (4.13). To see that (4.6) holds for $t = (1, 1)$, we must show that

$$\sum_{s<(1,1)} \tilde{a}_s(\omega) \leq 1, \qquad \forall\omega\in\Omega.$$

Let $t \in I^2\setminus\{(1, 1)\}$ be $\leq_1$-maximal such that $\tilde{a}_t(\omega) > 0$. Using Lemma 4.5, we see that

$$\sum_{s<(1,1)} \tilde{a}_s(\omega) = \sum_{s\in R^-_{t_1,0}\setminus\{(1,1)\}} \tilde{a}_s(\omega) \leq \widetilde{S}_{t_1,0}(\omega) \leq 1.$$

This concludes the proof of the theorem. □

5. A lifting theorem and a projection theorem for randomized stopping points

5.1. **Definition.** An *internal weight process* on $\mathbf{T}^2$ is an internal function $\delta\alpha\colon \Omega \times\mathbf{T}^2 \to {}^*[0, 1]$. Such a weight process defines a random internal additive measure $\overline{\alpha}$ on the internal algebra of internal subsets of $\mathbf{T}^2$ by the formula

$$\overline{\alpha}(\omega, B) = \sum_{t\in B} \delta\alpha(\omega, t),$$

where $\omega \in \Omega$ and B is an internal subset of $\mathbf{T}^2$. If $\overline{\alpha}$ is finite a.s., the σ-additive extension of $\mathrm{st}(\overline{\alpha}(\omega, \cdot))$ to the Borel σ-algebra on $\mathbf{T}^2$ is denoted $\alpha(\omega, \cdot)$ (the *Borel* σ-algebra is generated by the algebra of all internal subsets of $\mathbf{T}^2$).

The object of this section is to show how to lift a randomized stopping point to an internal weight process, and conversely, how to obtain a randomized stopping point from an internal weight process. Our method for lifting relies on the Simultaneous Lifting Theorem 4.1, and is quite different from the single-parameter lifting theorem of [SB, Chapter 7.1], which uses Skorohod's topology on right-continuous processes with left limits. Recall that Assumption 4.1 is in force.

5.2. **Lifting Theorem.** *Let* $(A_t)_{t\in[0,1]^2}$ *be a randomized stopping point. Then there is* $h \in {}^*\mathbf{N}\backslash\mathbf{N}$, $h \leq n_0$ (n_0 *is defined in* (3.5)), *an internal weight process* $\delta\alpha$, *and a (generally external) P-null set* $N \subset \Omega$ *such that*

(a) *for each* $t \in \mathbf{T}^2$, $\delta\alpha(\cdot, t)$ *is* $\mathscr{A}_{t+(1/h,1/h)}$*-measurable*;

(b) $\Delta_{]s,t]}A(\omega) = \alpha(\omega, {}^*]s, t] \cap \mathbf{T}^2)$, $\forall s, t \in \mathbf{D}^2$, $s < t$, $\forall \omega \in \Omega\backslash N$;

(c) $\overline{\alpha}(\omega, \mathbf{T}^2) = 1$, $\forall \omega \in \Omega$.

($\mathbf{D}$ *denotes the dyadics in* $[0, 1]$. *Throughout this section we use the following convention*: $t + (1/h, 1/h) = (\min(t_1 + 1/h, 1), \min(t_2 + 1/h, 1)))$.

Proof. Set $k = (k_1, k_2)$, $k^- = (k_1 - 1, k_2 - 1)$, $k^+ = (k_1 + 1, k_2 + 1)$. Using Theorem 4.2, we see that for each $n \in \mathbf{N}$ and $0 \leq k_1$, $k_2 \leq 2^n$, there is a P-null set $N^n_{(k_1,k_2)}$ and an internal function $\delta\alpha^n_{(k_1,k_2)} : \Omega \to {}^*[0, 1]$ such that

$$\omega \in \Omega\backslash N^n_k \Rightarrow \mathrm{st}(\delta\alpha^n_k(\omega)) = \Delta_{2^{-n}]k^-,k]}A(\omega), \tag{5.1}$$

$$\delta\alpha^n_k \text{ is } \mathscr{A}_{2^{-n}k^+}\text{-measurable}, \tag{5.2}$$

$$\sum_{0\leq k_1,k_2\leq 2^n} \delta\alpha^n_{(k_1,k_2)}(\omega) = 1, \qquad \forall \omega \in \Omega. \tag{5.3}$$

Let B denote the set of internal functions from $\Omega \times \mathbf{T}^2$ into ${}^*[0, 1]$. B is internal (see [HL, Ex. II.6.12]). For $n \in \mathbf{N}$, we define an element $\delta\alpha^n$ of B by setting

$$\delta\alpha^n(\omega, t) = \begin{cases} \delta\alpha^n_{(k_1,k_2)}(\omega) & \text{if } t = k2^{-n}, \text{ for some } 0 \leq k_1, k_2 \leq 2^n; \\ 0 & \text{otherwise.} \end{cases} \tag{5.4}$$

Observe that by (5.2),

$$\delta\alpha^n(\cdot, t) \text{ is } \mathscr{A}_{2^{-n}k^+}\text{-measurable}, \quad \forall t \in {}^*]2^{-n}k^-, 2^{-n}k] \cap \mathbf{T}^2. \tag{5.5}$$

Set $\widetilde{N} = \bigcup_{n\in\mathbf{N}} \bigcup_{0\leq k_1,k_2\leq 2^n} N^n_{(k_1,k_2)}$. Then there is a sequence $(N^n)_{n\in\mathbf{N}}$ of internal subsets of Ω such that $N^n \supset \widetilde{N}$ and

$$\overline{P}(N^n) < 1/n \quad \text{and} \quad N^m \supset N^n, \qquad \forall m \leq n, \tag{5.6}$$

$$0 \leq k_1, k_2 \leq 2^n, \qquad m \leq n, \qquad \omega \in \Omega\backslash N^n$$
$$\Rightarrow \left| \sum_{t\in{}^*]k^-2^{-m},k2^{-m}]\cap\mathbf{T}^2} \delta\alpha^m(\omega, t) - \sum_{t\in{}^*]k^-2^{-m},k2^{-m}]\cap\mathbf{T}^2} \delta\alpha^n(\omega, t) \right| < \frac{1}{n}, \tag{5.7}$$

$$\sum_{t\in\mathbf{T}^2} \delta\alpha^n(\omega, t) = 1, \qquad \forall \omega \in \Omega. \tag{5.8}$$

Using the Countable Comprehension Principle, we can extend the sequence $(\delta\alpha^n, N^n)_{n\in\mathbb{N}}$ to an internal sequence $(\delta\alpha^n, N^n)_{n\in{}^*\mathbb{N}}$. Set

$$C = \{n \in {}^*\mathbb{N}\colon (5.5), (5.6), (5.7) \text{ and } (5.8) \text{ hold, and } 2^{n+1} \le n_0\}.$$

By the Internal Definition Principle (see [HL, Theorem 6.4]), C is an internal set, which contains the (external) set $\mathbb{N}$. Hence there is $\underline{m} \in C\backslash\mathbb{N}$.

We set $\delta\alpha = \delta\alpha^{\underline{m}}$. Observe that (c) is satisfied by (5.8), and (a) holds by (5.5) with $h = 2^{\underline{m}+1}$. Set $N = \tilde{N} \cup N^{\underline{m}}$. Then $P(N) = 0$ by (5.6), and for all $m \in \mathbb{N}$ and $0 \le k_1, k_2 \le 2^m$, (5.7) implies that

$$\omega \in \Omega\backslash N \Rightarrow \left| \sum_{t\in{}^*]k^-2^{-m}, k2^{-m}]\cap\mathbb{T}^2} \delta\alpha^m(\omega, t) - \overline{\alpha}(\omega, {}^*]k^-2^{-m}, k2^{-m}]\cap\mathbb{T}^2) \right| < \frac{1}{\underline{m}}.$$

By (5.1) and (5.4), this implies that for all $m \in \mathbb{N}$ and $0 \le k_1, k_2 \le 2^m$,

$$\begin{aligned}\alpha(\omega, {}^*]k^-2^{-m}, k2^{-m}]\cap\mathbb{T}^2) &= \mathrm{st}(\overline{\alpha}(\omega, {}^*]k^-2^{-m}, k2^{-m}]\cap\mathbb{T}^2)) \\ &= \mathrm{st}\left(\sum_{t\in{}^*]k^-2^{-m}, k2^{-m}]\cap\mathbb{T}^2} \delta\alpha^m(\omega, t)\right) \\ &= \Delta_{2^{-m}]k^-, k]} A(\omega).\end{aligned}$$

This proves (b), and concludes the proof. □

5.3. **Corollary.** *Let $(A_t)_{t\in[0,1]^2}$ be a randomized stopping point, and let $\delta\alpha$ be the internal weight process and N the null set given by Theorem 5.2. For any Borel set $B \subset [0, 1]^2$,*

(a) *$\mathrm{st}^{-1}(B) \cap \mathbb{T}^2$ is a Borel subset of $\mathbb{T}^2$;*

(b) *$\int_B d_t A_t(\omega) = \alpha(\omega, \mathrm{st}^{-1}(B) \cap \mathbb{T}^2)$, $\forall \omega \in \Omega\backslash N$.*

Proof. (a) is a consequence of Theorem (2.2.6) of [SB]. Furthermore, by a classical Monotone Class argument, it is sufficient to prove (b) when B is a rectangle with dyadic edges, $B =]s, t[$, $s < t$, $s, t \in \mathbb{D}^2$. We fix $\omega \in \Omega\backslash N$, and only consider the case $s \ll t$.

Let μ_ω be the random measure on $[0, 1]^2$ whose distribution function is $t \mapsto A_t(\omega)$. By Theorem 5.2(b),

$$\mu_\omega(]s, t]) = \alpha(\omega, {}^*]s, t] \cap \mathbb{T}^2). \tag{5.9}$$

The remainder of the proof follows that of Lemma (2.3.2) of [SB]. Since

$$\mathrm{st}^{-1}(]s, t[) \subset {}^*]s, t] \subset \mathrm{st}^{-1}([s, t]),$$

we get by (5.9) that

$$\alpha(\omega, \mathrm{st}^{-1}(]s, t[) \cap \mathbb{T}^2) \le \mu_\omega(]s, t]) \le \alpha(\omega, \mathrm{st}^{-1}([s, t]) \cap \mathbb{T}^2).$$

Now

$$\begin{aligned}\alpha(\omega, \mathrm{st}^{-1}(]s, t[) \cap {}^*\mathbf{T}^2) &= \lim_{n\to\infty} \alpha(\omega, \mathrm{st}^{-1}([s + (1/n, 1/n), t - (1/n, 1/n)]) \cap \mathbf{T}^2) \\ &= \mu_\omega(]s, t[)\end{aligned}$$

since

$$\begin{aligned}\mu_\omega(]s, t[) &= \lim_{n\to\infty} \mu_\omega(]s + (1/n, 1/n), t - (1/n, 1/n)]) \\ &\le \lim_{n\to\infty} \alpha(\omega, \mathrm{st}^{-1}([s + (1/n, 1/n), t - (1/n, 1/n)]) \cap \mathbf{T}^2) \\ &\le \lim_{n\to\infty} \alpha(\omega, \mathrm{st}^{-1}(]s + (1/2n, 1/2n), t - (1/2n, 1/2n)[) \cap \mathbf{T}^2) \\ &\le \lim_{n\to\infty} \mu_\omega(]s + (1/2n, 1/2n), t - (1/2n, 1/2n)]) \\ &= \mu_\omega(]s, t[).\end{aligned}$$

This completes the proof. □

Theorem 5.2 and Corollary 5.3 provide the desired liftings of randomized stopping points. The projection theorem is simpler.

5.4. **Projection Theorem.** *Let $\delta\alpha\colon \Omega \times \mathbf{T}^2 \to {}^*[0, 1]$ be an internal weight process, such that*

$$\sum_{t\in\mathbf{T}^2} \delta\alpha(\omega, t) = 1, \qquad \forall\omega \in \Omega\backslash M,$$

where M is an (internal) $\overline{P}$-null set. Suppose that for some $h \in {}^*\mathbf{N}\backslash\mathbf{N}$, $\delta\alpha$ is adapted to the internal fibration $(\mathscr{A}_t^h)_{t\in\mathbf{T}^2}$, where $\mathscr{A}_t^h = \mathscr{A}_{t+(1/h, 1/h)}$, $\forall t \in \mathbf{T}^2$. Set

$$A_t(\omega) = \inf_{q\in\mathbf{D}^2\cap]t, (1, 1)]} \alpha(\omega, {}^*[0, q] \cap \mathbf{T}^2), \qquad t \in [0, 1]^2\backslash\{(1, 1\},$$

$$A_{(1,1)}(\omega) \equiv 1.$$

Then $A = (A_t)_{t\in[0,1]^2}$ is a randomized stopping point such that for almost all $\omega \in \Omega$,

$$\int_B d_tA_t(\omega) = \alpha(\omega, \mathrm{st}^{-1}(B) \cap \mathbf{T}^2),$$

for all Borel sets $B \subset [0, 1]^2$ (A is termed the projection of $\delta\alpha$).

Proof. The definition of A clearly implies that $A_\cdot(\omega)$ is right-continuous and has positive planar increments. Since $A_{(1,1)} \equiv 1$ a.s., A will be a randomized stopping point provided A_t is $\mathscr{F}_t$-measurable, for all $t \in [0, 1]^2$. This is the case since $(\mathscr{F}_t)$ is right-continuous and $\alpha(\cdot, {}^*[0, q] \cap \mathbf{T}^2)$ is $\mathscr{F}_q$-measurable by Lemma 3.1. As for the last statement of the theorem, it is sufficient to observe that by the definition of A, $A_q(\omega) = \alpha(\omega, \mathrm{st}^{-1}([0, q]) \cap \mathbf{T}^2)$, $\forall q \in \mathbf{D}^2$. □

6. Extremal elements of the set of randomized stopping points

The purpose of this section is to show that on any filtered Loeb probability space that satisfies properties F1 to F4, all extremal elements of the set of randomized stopping points are (ordinary) stopping points. As mentioned in §2, this will be the key step in our proof of existence of optimal stopping points.

Throughout this section, we work, under Assumption 4.1, with a fixed randomized stopping point $A = (A_t)_{t\in[0,1]^2}$. Using the Lifting Theorem 5.2, together with the Splitting Theorem 2.2 and the Transfer Principle (3.3), we shall build two randomized stopping points $A^i = (A^i_t)_{t\in[0,1]^2}$, $i = 1, 2$, and an optional increasing path Z^* such that

$$A = \tfrac{1}{2}A^1 + \tfrac{1}{2}A^2 \tag{6.1}$$

and Z^* splits $[0, 1]^2$ into two parts, one of which contains the support of the random probability measure associated with A^1, and the other, the support of the random measure associated with A^2 (of course, if A is a stopping point, the supports of A, A^1 and A^2 will be contained in the graph of Z^*).

Let $\delta\alpha$ be the internal weight process given by Theorem 5.2, together with $h \in {}^*\mathbb{N}\backslash\mathbb{N}$ and the P-null set N: $\delta\alpha$ is adapted to the internal (complete) filtration $(\mathscr{A}_t^h)_{t\in\mathbf{T}^2}$, which satisfies Hypothesis $\overline{\text{F4}}$.

Let $\mathbf{T} + \mathbf{T} = \{0, \Delta u, 2\Delta u, \ldots, 2\}$. The Transfer Principle, applied to Theorem 2.2 in the case of a finite index set, affirms the existence of an internal $\overline{P}$-null set M, an internal function $\widetilde{Z}: \Omega \times (\mathbf{T} + \mathbf{T}) \to \mathbf{T}^2$ and of two internal weight processes $\delta\alpha^1$, $\delta\alpha^2: \Omega \times \mathbf{T}^2 \to {}^*[0, 1]$ with the following properties for all $t \in \mathbf{T}^2$, $\omega \in \Omega\backslash M$, $p \in \mathbf{T} + \mathbf{T}$:

$$\delta\alpha(\omega, t) = \tfrac{1}{2}\delta\alpha^1(\omega, t) + \tfrac{1}{2}\delta\alpha^2(\omega, t)\,; \tag{6.2}$$

$$\delta\alpha^i \text{ is } \mathscr{A}_t^h\text{-measurable}, \qquad i = 1, 2\,; \tag{6.3}$$

$$\sum_{s\in\mathbf{T}^2} \delta\alpha^i(\omega, s) = 1, \qquad i = 1, 2\,; \tag{6.4}$$

$$\widetilde{Z}(\omega, p + \Delta u) \in \{\widetilde{Z}(\omega, p) + (\Delta u, 0), \widetilde{Z}(\omega, p) + (0, \Delta u)\}\,; \tag{6.5}$$

$$\{\omega \in \Omega: \widetilde{Z}(\omega, p) \le t\} \in \mathscr{A}_t^h\,; \tag{6.6}$$

$$t \curlywedge \widetilde{Z}(\omega, |t|) \Rightarrow (\delta\alpha^1(\omega, t) = 2\delta\alpha(\omega, t), \delta\alpha^2(\omega, t) = 0)\,; \tag{6.7}$$

$$\widetilde{Z}(\omega, |t|) \curlywedge t \Rightarrow (\delta\alpha^1(\omega, t) = 0,\ \delta\alpha^2(\omega, t) = 2\delta\alpha(\omega, t)). \tag{6.8}$$

Let A^i be the projection of $\delta\alpha^i$, $i = 1, 2$. It follows from the definition of A^i (see Theorem 5.4) and from (6.2) that (6.1) holds. It remains to be shown that if A is not in fact a stopping point, then $A^1 \ne A \ne A^2$.

Recall that a map $f: \mathbf{T} \to \mathbf{T}^2$ is termed S-continuous provided $u \approx v \Rightarrow f(u) \approx f(v)$, $\forall u, v \in \mathbf{T}$ (see [SB, Appendix 1.4]).

6.1. **Lemma.** (a) *For* $\omega \in \Omega \backslash M$, $p \mapsto \tilde{Z}(\omega, p)$ *is S-continuous*;

(b) *Define* $Z^* = (Z_u^*)_{u \in [0,2]}$ *by* $Z_u^*(\omega) = \operatorname{st}(\tilde{Z}(\omega, \operatorname{st}^{-1}(u)))$, $\omega \in \Omega$, $u \in [0, 2]$. *Then* Z *is an optional increasing path.*

Proof. Property (a) is a consequence of the equality

$$|\tilde{Z}(\omega, p) - \tilde{Z}(\omega, q)| = |p - q|, \qquad \forall \omega \in \Omega,\ p, q \in \mathbf{T} + \mathbf{T},$$

which follows from (6.5). As for (b), observe that Z_u^* is well defined by (a), since if ω is not in the $\overline{P}$-null set M and $u = \operatorname{st}(p) = \operatorname{st}(\tilde{p})$, then $\tilde{Z}(\omega, p) \approx \tilde{Z}(\omega, \tilde{p})$, so $\operatorname{st}(\tilde{Z}(\omega, p)) = \operatorname{st}(\tilde{Z}(\omega, \tilde{p}))$. Furthermore, $u \mapsto Z_u^*(\cdot)$ is increasing by (6.5), and if $p \in \mathbf{T} + \mathbf{T}$ is such that $\operatorname{st}(p) = u$, then

$$|Z_u^*(\omega)| = \operatorname{st}(|\tilde{Z}(\omega, p)|) = \operatorname{st}(p) = u,$$

also by (6.5). Now fix $u \in [0, 2]$ and $t \in [0, 1]^2$. We must show that

$$\{\omega \in \Omega\colon Z_u^*(\omega) \le t\} \in \mathcal{F}_t .$$

Since the filtration $(\mathcal{F}_t)$ is right-continuous, it is sufficient to show that for $t \in [0, 1]^2 \cap \mathbf{D}^2$,

$$F = \{\omega \in \Omega\colon \operatorname{st}(\tilde{Z}(\omega, p)) \ll t\} \in \mathcal{F}_t,$$

where $p \in \mathbf{T} + \mathbf{T}$ is such that $p \approx u$. Since t also belongs to $\mathbf{T}^2$,

$$F = \bigcup_{n \in \mathbf{N}} \{\omega \in \Omega\colon \tilde{Z}(\omega, p) \le t - (1/n, 1/n)\}.$$

But then (6.6) implies that $F \in \mathcal{F}_t$. This completes the proof. □

The following lemma shows that A^1, A^2 and Z^* have a property similar to that of Theorem 2.2(b2).

6.2. **Lemma.** *Fix* $\omega \in \Omega \backslash M$ *and* $s, t \in \mathbf{D}^2$ *such that* $s \le t$.

(a) *Suppose* $(t_1, s_2) \wedge Z_{t_1+s_2}^*(\omega)$. *Then* $\Delta_{]s,t]} A^2(\omega) = 0$.

(b) *Suppose* $Z_{s_1+t_2}^*(\omega) \wedge (s_1, t_2)$. *Then* $\Delta_{]s,t]} A^1(\omega) = 0$.

Proof. We only prove (a). By the hypothesis and (6.7), there is $\varepsilon > 0$, $\varepsilon \in \mathbf{R}$, such that

$$\delta\alpha^2(\omega, u) = 0, \qquad \forall u \in {}^*]s, t + (\varepsilon, \varepsilon)] \cap \mathbf{T}^2,$$

so if $p, q \in {}^*]s, t + (\varepsilon, \varepsilon)] \cap \mathbf{D}^2$, $p \le q$,

$$\sum_{p < u \le q} \delta\alpha^2(\omega, u) = 0.$$

Thus, where $\alpha^2(\omega, [a, b])$ is an abbreviation of $\alpha^2(\omega, {}^*[a, b] \cap \mathbf{T}^2)$,

$$\alpha^2(\omega, [0, q]) - \alpha^2(\omega, [0, (p_1, q_2)]) - \alpha^2(\omega, [0, (q_1, p_2)]) + \alpha^2(\omega, [0, p]) = 0.$$

Taking the limit as $q \downarrow t$, $p \downarrow s$ gives the desired result. □

If $(Z_u)_{u \in [0,2]}$ is an o.i.p. we set $\operatorname{Im} Z_\cdot(\omega) = \{Z_u(\omega)\colon 0 \le u \le 2\}$, and if ν is a measure, $\operatorname{supp} \nu$ denotes the support of ν.

6.3. **Proposition.** *Suppose* $\mu_\omega(\cdot)$ *is the random measure whose distribution function is the randomized stopping point* $(A_t)_{t\in[0,1]^2}$, *and suppose*

$$P\{\omega\in\Omega:\ \operatorname{supp}\mu_\omega(\cdot)\subset \operatorname{Im} Z_\cdot(\omega)\}<1$$

for all o.i.p.'s $(Z_u)_{u\in[0,2]}$. *Then* $A^1\neq A\neq A^2$.

Proof. Let $(Z^*_u)_{u\in[0,2]}$ be the o.i.p. defined in Lemma 6.1, and set

$$F^*=\{\omega\in\Omega:\ \operatorname{supp}\mu_\omega(\cdot)\not\subset \operatorname{Im} Z^*_\cdot(\omega)\}.$$

Since $P(F^*)>0$, we may suppose for example that $P(F)>0$, where

$$F=\{\omega\in\Omega:\ \text{there is } s,t\in \mathbb{D}^2,\ s\leq t \text{ such that } (t_1,s_2)\wedge Z^*_{t_1+s_2}(\omega) \text{ and } \Delta_{]s,t]}A(\omega)>0\}.$$

Now for each $\omega\in F\backslash M$, since $A=\frac{1}{2}A^1+\frac{1}{2}A^2$, we have by Lemma 6.2:

$$\Delta_{]s,t]}A^1(\omega)=2\Delta_{]s,t]}A(\omega)\neq\Delta_{]s,t]}A(\omega)$$

for some $s,t\in\mathbb{D}^2$ with $s\leq t$. This implies that the sample paths $t\mapsto A_t(\omega)$, $t\mapsto A^i_t(\omega)$, $i=1,2$, are distinct for $\omega\in F\backslash M$. Since $P(F\backslash M)>0$, $A^1\neq A\neq A^2$. □

The following lemma is a straightforward extension of a result for single-parameter randomized stopping points.

6.4. **Lemma.** *Let* $(\Omega,\mathscr{F},P)$ *be an arbitrary (complete) probability space, and* $(\mathscr{F}_t)_{t\in[0,1]^2}$ *an arbitrary two-parameter filtration (with or without* CQI *or* F4). *Suppose* $\mu_\omega(\cdot)$ *is a random measure whose distribution function is some randomized stopping point* $A=(A_t)_{t\in[0,1]^2}$. *If there is an optional increasing path* $(Z_u)_{u\in[0,2]}$ *such that*

$$P\{\omega\in\Omega:\ \operatorname{supp}\mu_\omega(\cdot)\subset \operatorname{Im} Z_\cdot(\omega)\}=1,$$

then A *is an extremal element of the set of randomized stopping points if and only if* A *is a stopping point.*

Proof. Set $B^1_t=\min(2A_t,1)$, $B^2_t=\max(2A_t-1,0)$. Clearly $A_t=\frac{1}{2}B^1_t+\frac{1}{2}B^2_t$, and the sample paths

$$t\mapsto A_t(\omega)\quad\text{and}\quad t\mapsto B^i_t(\omega),\qquad i=1,2,$$

are distinct if and only if $0<A_t(\omega)<1$ for some t. If $s,t\in[0,1]^2$ are such that $s\leq t$, it is easy to see that $\Delta_{]s,t]}B^i\geq 0$ a.s. by examining the relative positions of s,t and the path $u\mapsto Z_u$ (see Figure 1).

Since $B^i_{(1,1)}\equiv 1$, this implies that B^1 and B^2 are randomized stopping points. Thus if A is extremal, we must have

$$A_t\in\{0,1\}\quad\text{a.s.}$$

But then A is a stopping point. □

It is now straightforward to prove the continuous time extension of Theorem 2.2.

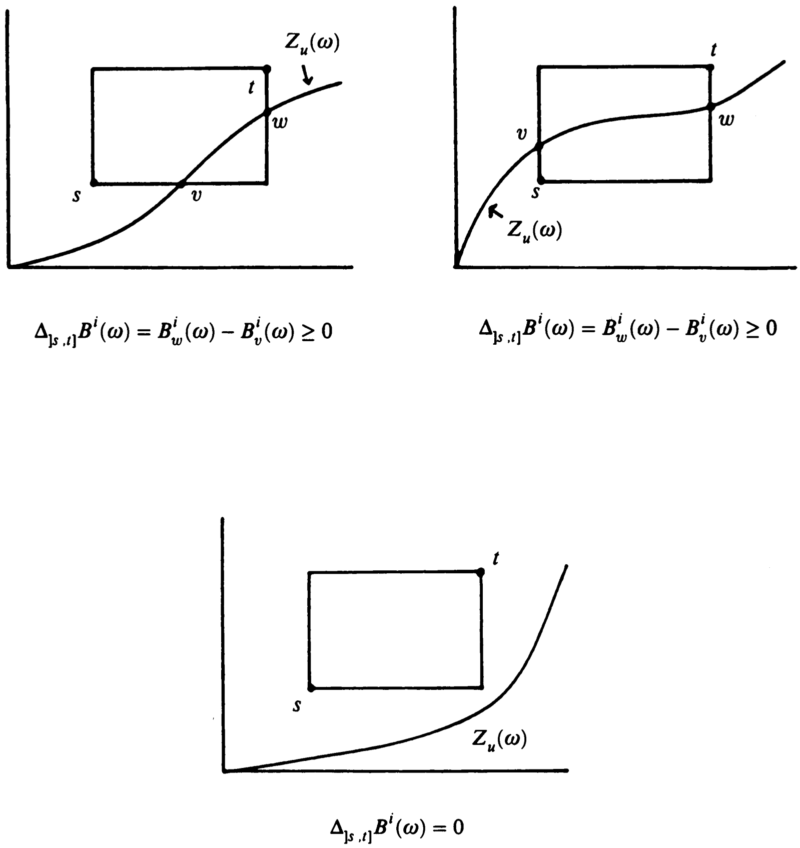

FIGURE 1

6.5. **Theorem.** *Let* $(\Omega, \mathscr{A}, \overline{P})$ *be an internal probability space,* $(\Omega, L(\mathscr{A}), P)$ *the corresponding Loeb space. Suppose* $(\mathscr{F}_t)_{t\in[0,1]^2}$ *is the standard part of an internal (complete) two-parameter filtration that satisfies Hypothesis* $\overline{\text{F4}}$. *Then all extremal elements of the set of randomized stopping points are stopping points.*

Proof. Let $A = (A_t)_{t\in[0,1]^2}$ be a randomized stopping point. Suppose

$$P\{\omega \in \Omega: \operatorname{supp} \mu_\omega(\cdot) \subset \operatorname{Im} Z_\cdot(\omega)\} < 1$$

for all optional increasing paths Z, where $\mu_\omega(\cdot)$ is the measure on $[0, 1]^2$ whose distribution function is $t \mapsto A_t(\omega)$. Then by Proposition 6.3, A is the midpoint of two distinct randomized stopping points, and thus is not extremal. This implies that any extremal randomized stopping point must satisfy

$$P\{\omega \in \Omega: \operatorname{supp} \mu_\omega(\cdot) \subset \operatorname{Im} Z_\cdot(\omega)\} = 1$$

for some optional increasing path Z. But then the statement of the theorem is a consequence of Lemma 6.4. □

6.6. *Remark.* It is not known whether the conclusion of this theorem remains valid for filtered probability spaces that satisfy Hypothesis F4 but are not Loeb spaces.

7. Application: The existence of optimal stopping points

As mentioned in §2, Theorem 6.5 leads to a proof of the existence of optimal stopping points, under integrability assumptions as weak as those in [Kl, Theorem 1]. For this we need the following proposition.

7.1. **Proposition.** *Let $(\Omega, \mathscr{F}, P)$ be an arbitrary complete probability space, and $X = (X_t)_{t \in [0,1]^2}$ a measurable process with upper semicontinuous (u.s.c.) sample paths such that $E(\sup_{t \in [0,1]^2} X_t) < +\infty$. Then the map $\Phi_X : \mathscr{U} \to \mathbb{R} \cup \{-\infty\}$ defined by*

$$\Phi_X((A_t)_{t \in [0,1]^2}) = E\left(\int_{[0,1]^2} X_t(\cdot) d_t A_t(\cdot)\right)$$

is u.s.c. (for the weak topology induced by $\sigma(\mathscr{C}^*, \mathscr{C})$; see §2).

Proof. For separable bounded processes, this was proved in [D1, Theorem 3.5]. Our proof here is more direct and gives the more general result above.

We should perhaps point out that the map $\omega \mapsto \sup_{t \in [0,1]^2} X_t$ is measurable since the process X is (see the proof of [DM, IV. 33a]), and so it makes sense to speak of sup integrability for X, and Φ_X is well-defined.

If the sample paths of the process X were continuous, then the function Φ_X would be continuous by the definition of the weak topology $\sigma(\mathscr{C}^*, \mathscr{C})$. Now suppose there were a nonincreasing sequence $(Y^k)_{k \in \mathbb{N}}$ of continuous processes in $\mathscr{C}$ such that

$$\lim_{k \to \infty} \downarrow Y_t^k(\omega) = X_t(\omega),$$

for almost all $\omega \in \Omega$. Then we would have $\Phi_{Y^k} \downarrow \Phi_X$ by monotone convergence, and so Φ_X, as the nonincreasing limit of a sequence of continuous functions, would be u.s.c. [B1, IV.6.2 Theorem 4]. Thus the proposition will be proved if we construct the sequence $(Y^k)_{k \in \mathbb{N}}$.

It is well known that an u.s.c. bounded function defined on a metric space is the nonincreasing limit of a sequence of continuous functions, so the problem here is to choose the sequence for fixed $\omega \in \Omega$ in such a way that the resulting $Y_t^k(\omega)$ are measurable functions of ω and such that $Y^k \in \mathscr{C}$. In order to do this, we need the following lemma.

7.2. **Lemma.** *Consider $F \subset \mathscr{F} \times \mathscr{B}(\bar{I}^n)$ such that for each $\omega \in \Omega$, the section $F_\omega = \{t \in \bar{I}^n : (\omega, t) \in F\}$ is closed. Then the mapping $\omega \mapsto \mathrm{dist}(t, F_\omega)$ is $\mathscr{F}$-measurable ($\mathrm{dist}(t, F_\omega)$ denotes the distance between t and the set F_ω for the usual metric on $\bar{I}^n$).*

Proof. For $r > 0$,

$$A = \{\omega \in \Omega : \mathrm{dist}(t, F_\omega) < r\} = \{\omega \in \Omega : \text{there is } s \in F_\omega,\ d(s, t) < r\},$$

so A is the projection on Ω of the $\mathscr{F}\times\mathscr{B}(\overline{I}^n)$-measurable set $F\cap(\Omega\times B(t,r))$, where $B(t,r)$ denotes the open ball centered at t with radius r. Thus A is $\mathscr{F}$-analytic by Theorem II.13 of [DM], and since $\mathscr{F}$ is complete, $F\in\mathscr{F}$ by III.33 of [DM]. This proves the lemma. □

End of the proof of Proposition 7.1. Our proof follows that of [B1, IX §2.7, Proposition 11]. Since we can always replace the process X by the process $(X_t - \sup_t X_t)_{t\in[0,1]^2}$, we may suppose without loss of generality that $X \leq 0$. Set

$$X_t^n(\omega) = -2^{-n}\sum_{k=1}^{\infty} I_{U^{k,n}}(\omega, t),$$

where

$$U^{k,n} = \{(\omega, t)\in\Omega\times\overline{I}^n : X_t(\omega) < -k2^{-n}\},$$

and observe that $(X^n)_{n\in\mathbf{N}}$ is a nonincreasing sequence which converges to X. Now since X is u.s.c., the section $U_\omega^{k,n}$ of $U^{k,n}$ is open for each $\omega\in\Omega$. Furthermore, since $\sup_t X_t < +\infty$ a.s., there is a measurable map $\omega\mapsto K_\omega$ from Ω into $\mathbf{N}$ such that $k > 2^n K_\omega \Rightarrow I_{U^{k,n}}(\omega, t) = 0$, $\forall t$, for almost all $\omega\in\Omega$.

For each fixed k, l, and n, set

$$Z_t^{k,n,l}(\omega) = \min(1, l\,\mathrm{dist}(t, \overline{I}^n\backslash U_\omega^{k,n})).$$

Then $\omega\mapsto Z^{k,n,l}(\omega)$ is a measurable map by Lemma 7.2, $t\mapsto Z_t^{k,n,l}(\omega)$ is continuous and

$$(t\in\overline{I}^n\backslash U_\omega^{k,n} \text{ or } \mathrm{dist}(t,\overline{I}^n\backslash U^{k,n}) > 1/l) \Rightarrow Z_t^{k,n,l}(\omega) = I_{U^{k,n}}(\omega, t),$$

so

$$\lim_{l\to\infty}\uparrow Z_t^{k,n,l}(\omega) = I_{U^{k,n}}(\omega, t), \qquad \forall t\in\overline{I}^n, \quad \forall\omega\in\Omega.$$

Thus if we define a bounded continuous process $X^{n,l}$ by setting

$$X_t^{n,l}(\omega) = \max\left(-2^{-n}\sum_{k=1}^{2^n K_\omega} Z_t^{n,k,l}(\omega) - l\right),$$

we have

$$\lim_{l\to\infty}\downarrow X_t^{n,l}(\omega) = X_t^n(\omega), \qquad \forall t\in\overline{I}^n, \quad \text{for almost all } \omega\in\Omega.$$

But then the sequence $(Y^k)_{k\in\mathbf{N}}$ of continuous processes defined by

$$Y_t^k(\omega) = \min_{n,l\leq k} X_t^{n,l}(\omega)$$

satisfies the required conditions. □

7.2. **Theorem.** *Let* $(\Omega, \mathscr{F}, P, (\mathscr{F}_t)_{t\in[0,1]^2})$ *be a filtered Loeb space satisfying the assumptions of Theorem* 6.5, *and let* $(X_t)_{t\in[0,1]^2}$ *be a measurable process with* *upper semicontinuous sample paths, such that* $E(\sup_{t\in[0,1]^2} X_t) < +\infty$. *Then there is a stopping point* T_0 *such that*

$$E(X_{T_0}) = \sup_{T\in\mathcal{T}} E(X_T).$$

Proof. This proof is similar to that of Ghoussoub [G, Proposition II.3]. Consider the functional $\Phi\colon \mathcal{U} \to \mathbb{R}$ defined by

$$\Phi((A_t)_{t\in[0,1]^2}) = E\left(\int_{[0,1]^2} X_t(\cdot)d_tA_t(\cdot)\right).$$

By Lemma 7.1, this functional is u.s.c. on $\mathcal{U}$. Since Φ is affine, it attains its maximum on $\mathcal{U}$ at an extremal element $A^0 \in \operatorname{ext}\mathcal{U}$ [B2, II §7, Proposition 1]. By Theorem 6.5, A^0 is in fact a stopping point, which we denote T_0. This stopping point is clearly optimal. □

7.3. *Remark.* From the point of view of applications, it does not seem too restrictive to impose that the underlying probability space be Loeb. In the single-parameter case, this would be no restriction at all due to the result of Hoover and Keisler [HK], which shows that these spaces are universal and saturated.

7.4. *Remark.* The papers [Mi and MM] claim, under certain regularity assumptions on the reward process, the existence of optimal stopping points in the two-parameter optimal stopping problem on arbitrary probability spaces (in [MM], there is even no Hypothesis F4 on the filtration). Both these papers use a "randomized" set $\mathcal{U}$ which is different from the one considered here, and both papers use the following theorem: "a separately continuous bilinear map is jointly continuous", in a situation where the hypothesis of this theorem is not satisfied [Mi, Theorem 1.5; MM, Proposition 7]. Thus the problem of existence of optimal stopping points on arbitrary probability spaces, even for continuous processes and under Hypothesis F4, is open.

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