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CHARACTERIZATIONS OF NORMAL QUINTIC K-3 SURFACES

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ABSTRACT. If a normal quintic surface is birational to a K-3 surface then it must contain from one to three triple points as its only essential singularities. A K-3 surface is the minimal model of a normal quintic surface with only one triple point if and only if it contains a nonsingular curve of genus two and a nonsingular rational curve crossing each other transversally. The minimal models of normal quintic K-3 surfaces with several triple points can also be characterized by the existence of some special divisors.

0. INTRODUCTION

Let $\mathbf{C}$ be the complex number field. A complete surface S over $\mathbf{C}$ is a K-3 surface if the canonical divisor of S is zero and $H^1(S) = 0$. One of the simplest examples is a smooth quartic surface in $\mathbf{P}^3$. It was shown in [YJG] that some singular quintic surfaces are birational to K-3 surfaces. The aim of this paper is to find necessary and sufficient conditions for a K-3 surface to be birational to a normal quintic surface. The main results are

Theorem 1. *A normal quintic surface in $\mathbf{P}^3$ is birational to a K-3 surface only if all its essential singularities are triple points.*

Theorem 2. *A K-3 surface S is the minimal model of a normal quintic surface with one triple point as its only essential singularity if and only if there are two nonsingular curves D and B on S with genus 2 and 0 respectively such that $DB = 1$.*

Theorem 3. *A K-3 surface S is the minimal model of a normal quintic surface with two triple points as its only essential singularities if and only if S has one of the divisors listed in Figure 1.*

(The solid dots are nonsingular elliptic curves. The hollow dots are nonsingular rational curves.)

Theorem 4. *A K-3 surface S is the minimal model of a normal quintic surface with more than two triple points if and only if there are three nonsingular elliptic curves C_1, C_2 and C_3 on S with $C_iC_j = 2$ for $1 \le i < j \le 3$.*

A generic line passing through a triple point of a quintic surface meets the quintic surface at two other points besides the triple point. So it is natural to

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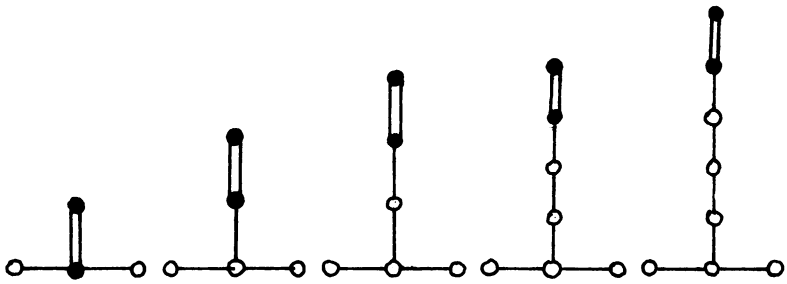

FIGURE 1

study the double cover of a normal quintic K-3 surface over a plane. In the last section some descriptions of the branch loci of such double coverings are given.

1. PRELIMINARIES

In this section we briefly mention some standard notions concerning isolated singularities of surfaces. For details see [Art1, Art2, Lauf and Yau].

Let p be an isolated singularity on a surface V and let $\pi: M \to V$ be the minimal resolution of p. The number $h = \dim_C H^0(V, R^1\pi_*(O_M))$ is the *geometric genus* of p. It is well known that

$$\chi(V) = \chi(M) + h$$

where $\chi(V)$ denotes the holomorphic Euler characteristic of V.

The set $A = \pi^{-1}(p)$ is called the *exceptional set* of p. Let $A = \bigcup A_i$, $1 \leq i \leq n$, be the decomposition of A into irreducible components.

(*Remark.* If p is a smooth point on a surface V and let $f: X \to V$ be a birational morphism, then $f^{-1}(p)$ is also called the exceptional set of p on X.)

A *cycle* D on A is an integral combination of the A_i's. There is a natural partial ordering, denoted by $<$, among cycles. For any closed subvariety B of pure dimension 1 of A, there is a unique cycle Z_B satisfying

(i) $\text{Supp}(Z_B) = B$;
(ii) $A_i Z_B \leq 0$ for all $A_i \leq B$;
(iii) Z_B is minimal with respect to these two properties.

Such a cycle is called a *fundamental cycle.* In particular, Z_A is the fundamental cycle of the singularity p, denoted by Z.

If $\chi(Z) = 0$ then p is called a *weakly elliptic* point. For any weakly elliptic point p, there is a unique cycle $E \leq Z$ such that $\chi(E) = 0$ and $\chi(D) > 0$ for all $0 < D < E$. This E is called the *minimally elliptic cycle* of p. If the fundamental cycle Z itself is the minimally elliptic cycle then p is called a *minimally elliptic* point. A singularity is called *essential* if it is not a rational double point.

2. Normal quintic K-3 surfaces with one triple point

Throughout this paper a quintic K-3 surface will mean either a singular quintic surface in $\mathbf{P}^3$ which is birational to a K-3 surface or its birational model.

Let S_0 be a normal quintic surface and let S be its minimal resolution. Since the divisor $S_0 + K_{\mathbf{P}^3}$ in $\mathbf{P}^3$ is linearly equivalent to a hyperplane, an effective canonical divisor of S, if exists, is cut out by a hyperplane H_0 passing through all essential singularities of S_0. Let C_0 be the intersection of S_0 and H. If S is birational to a K-3 surface, then the canonical divisor of S is a collection of exceptional divisors of first kind. Hence all components of the proper transform of C_0 in S must be exceptional curves of first kind. This indicates that there are not many quintic K-3 surfaces. In particular, if S is already a minimal surface then S cannot be a K-3 surface.

Lemma 2.1. *A normal quintic surface with essential singularities, among which one is a double point, cannot be* K-3.

Proof. Let S_0 be a normal quintic surface and let p be an essential double point on S_0. Let $\varphi\colon T \to \mathbf{P}^3$ be the blowing-up of $\mathbf{P}^3$ at the point p and let E be the exceptional plane. Let S be the proper transform of S_0. The canonical divisor K_T of T is $\varphi^*(K_{\mathbf{P}^3}) + 2E$ and the divisor S is linearly equivalent to $\varphi^*(S_0) - 2E$. Thus $K_T + S$ is linearly equivalent to $\varphi^*(H)$ where H is a hyperplane in $\mathbf{P}^3$.

Suppose that S is birational to a K-3 surface. Then the canonical divisor of the minimal resolution of S_0 is cut out by a hyperplane H_0 passing through the point p. On T the divisor $\varphi^*(H_0)$ is the union of E and the proper transform of H_0. Let S' be the minimal resolution of S. Since S has at most double points or double curves on E, the canonical divisor of S' contains the exceptional set A of the double point p. Since S' is birational to a K-3 surface, the divisor A is part of the exceptional set of a smooth point, which contradicts the assumption that p is an essential singularity. Therefore S_0 cannot be K-3. Q.E.D.

Proof of Theorem 1. Let S_0 be a normal quintic surface. If S_0 has a 5-tuple point, then S_0 is a cone which is birational to a ruled surface. If S_0 has a 4-tuple point, then the projection from the 4-tuple point gives a birational map from S_0 to a rational surface. Lemma 2.1 says that S_0 is not K-3 if S_0 has essential double point. The conclusion follows immediately. Q.E.D.

Let S_0 be a quintic surface with a triple point p. We may assume that the equation of S_0 is

$$f_3(x, y, z) + f_3(x, y, z) + f_5(x, y, z) = 0 \tag{1}$$

where $f_i(x, y, z)$ is a homogeneous polynomial of degree i.

Let C be the plane cubic curve defined by the equation $f_3(x, y, z) = 0$. Then the triple point p has the following types in terms of the cubic curve C:

(i) C is reduced with at most ordinary double points (i.e., the rational double points of type A_1);
(ii) C is the union of a line and a conic tangent to each other;
(iii) C is the union of three concurrent lines;
(iv) C is the union of a line and a double line;
(v) C is a triple line.

For details, see [YJG, §4].

Lemma 2.2. *An isolated triple point of type* (i) *on a quintic surface is a minimally elliptic singularity.*

Proof. See [YJG, pp. 445–446].

Lemma 2.3. *Let p be an isolated triple point of type* (ii) *on a quintic surface, then either p is minimally elliptic or p has an infinitely near essential double point.*

Proof. [YJG, p. 446(v)].

Lemma 2.4. *Let S_0 be a normal quintic surface with a triple point p of type* (ii) *which is not minimally elliptic. Then S_0 is not K-3.*

Proof. Let $\pi: T \to \mathbf{P}^3$ be the blowing-up of $\mathbf{P}^3$ at the point p and let E be the exceptional plane. Let S be the proper transform of S_0. The canonical divisor K_T of T is $\pi^*(K_{\mathbf{P}^3}) + 2E$ and the divisor S is linearly equivalent to $\pi^*(S_0) - 3E$. Thus $K_T + S$ is linearly equivalent to $\pi^*(H_0) - E$ where H_0 is a hyperplane in $\mathbf{P}^3$. So $K_T + S$ is linearly equivalent to the proper transform H of H_0 in T. Let S' be the minimal resolution on S. Then the canonical divisor of S' is cut out by the plane H_0 in $\mathbf{P}^3$ whose proper transform H passes through the essential double point of S. Then following the same argument as in the proof of Lemma 2.1 one sees that S cannot be birational to a K-3 surface. Q.E.D.

Lemma 2.5. *Let S_0 be a quintic surface with a triple point as its only essential singularity. If S_0 is K-3 then the triple point must have type* (iv) *or* (v). *Furthermore the blowing-up of S_0 at the triple point is not a normal surface.*

Proof. If the triple point is a minimally elliptic point, then S is birational to a surface of general type by computing the invariants. Hence Lemmas 2.2–2.4 imply that the triple point cannot have type (i) or (ii). If p is of type (iii) then it was shown in [YJG, p. 446] that S_0 is either of general type or an elliptic surface with Kodaira dimension 1.

Hence the triple point must have type (iv) or (v). Let S be the blowing-up of S_0 at the triple point. If S is normal, then the Kodaira dimension of S is either 1 or 2 [YJG, pp. 446–447]. Q.E.D.

Lemma 2.6. *Let S be the minimal resolution of a quintic K-3 surface S_0. If there are five disjoint exceptional curves on first kind of S then S_0 is normal.*

Proof. The canonical divisor of S is cut out by a hyperplane H in $\mathbf{P}^3$. Suppose S_0 were not normal. Then $H \cap S_0$ would not be reduced, whence it would have less than five irreducible components. This implies that S would have less than five disjoint exceptional divisors of first kind. Q.E.D.

Proof of Theorem 2. Assume that S_0 is birational to a K-3 surface. Let T be the blowing-up of $\mathbf{P}^3$ and let E be the exceptional plane. Let S be the proper transform of S_0 in T. Because of Lemma 2.5 we may assume that S_0 has the equation

$$y^2 z + y f(x, y, z) + g(x, y, z) = 0 \tag{2}$$

or

$$y^3 + y f(x, y, z) + g(x, y, z) = 0, \tag{3}$$

where $f(x, y, z)$ and $g(x, y, z)$ are homogeneous polynomials in x, y, z with degrees 3 and 5 respectively. Let H_0 be a generic plane in $\mathbf{P}^3$ passing through the triple point p. Bertini's Theorem implies that the intersection C_0 of H_0 and S_0 is an irreducible quintic curve with p as its only singularity. The equations (2) and (3) imply that C_0 has a triple point with an infinitely near double point at p. Therefore C_0 has geometric genus 2.

Assume that the equation for S_0 is (2). Then $E \cap S$ is the union of a line L_1 and a double line L_2. Let H be the proper transform of H_0 in T. Since H_0 is in general position, H meets L_1 and L_2 at two distinct points s_1 and s_2 respectively. Let C be the proper transform of C_0 in S. Then C is smooth at s_1 and C has a double point at s_2. Note that S is singular along L_2. The blowing-up of T along L_2 will normalize S and C at the same time. Let S' be the minimal resolution of S. Then the proper transform C' of C in S' is a nonsingular curve of genus 2 and the proper transform L_1' of L_1 in S' intersects C' transversally, because S is smooth at the point s_1 thanks to the general position of H. On the other hand the canonical divisor, which is a collection of exceptional divisors of first kind, is cut out by the plane $y = 0$ in $\mathbf{P}^3$. So the exceptional divisors of first kind on S' do not meet C'. Let D and B be the image of C' and L_1' in the minimal model of S'. Then $DB = 1$ and $D^2 = 2$, $B^2 = -2$ by the adjunction formula.

Next we assume that the equation of S_0 in (3). Then $E \cap S$ is a triple line L. Let H be the proper transform of H_0. Let C be the proper transform of C_0 in S. Then C has a double point at $C \cap L$. Let T^* be blowing-up of T along L and let F be the exceptional divisor. Let S^* be the proper transform of S. The equation (3) reveals that the intersection of F and the proper transform of E in T^* is a rational curve L^*, which lies in S^*. The proper transform C^* of C is a nonsingular curve meeting L^* transversally. Since H_0 is in general position, S^* is smooth at $C^* \cap L^*$ and there is no exceptional divisor of first kind passing through $C^* \cap L^*$. Let D and B be the image of C^* and L^* in the minimal model of S^* respectively. Then $DB = 1$, $D^2 = 2$ and $B^2 = -2$.

Conversely let S be a K-3 surface such that there are two nonsingular curves D and B with genera 2 and 0 respectively on S such that $DB = 1$. We want to show that S is the minimal model of a quintic K-3 surface.

The adjunction formula implies that $D^2 = 2$ and $B^2 = -2$. Let k be the canonical divisor of the curve D. Then $\deg(k) = 2$. Let p be the intersection point of D and B. Then $h^0(D, O(2k+p)) = h^0(D, O(2k)) + 1 = 4$ by the Riemann-Roch theorem. Hence a general member of the linear system $|2k+p|$ consists of five distinct points p_1, p_2, p_3, p_4, p_5 of which none is the point p.

Lemma 2.7. *Every pair of points among p_1, p_2, p_3, p_4, p_5 is not linearly equivalent to the canonical divisor k.*

Proof. If $p_1 + p_2$ were linearly equivalent to k, then $p_3 + p_4 + p_5$ would be linearly equivalent to $k + p$. Since $h^0(D, O(k)) = h^0(D, O(k+p)) = 2$, one of p_3, p_4, p_5 would be p. This would contradict our choice of $p_1, \dots, p_5$. Q.E.D.

Let S' be the blowing-up of S at these five points and let $E_1, \dots, E_5$ be the exceptional divisors. Let D' and B' be the proper transforms of D and B respectively. Since $K_S = 0, h^1(D, O_D(D)) = h^0(D, O_D) = 1$ by the adjunction formula. The short exact sequence

$$0 \longrightarrow O_S \longrightarrow O_S(D) \longrightarrow O_D(D) \longrightarrow 0$$

implies that

$$h^o(S, O(D)) = 3 \quad \text{and} \quad h^1(S, O(D)) = 0.$$

Hence $h^0(S', O(D' + E_1 + \cdots + E_5)) = 3$ and $h^1(S', O(D' + E_1 + \cdots + E_5)) = 0$. The short exact sequence

$$0 \to O_{S'}(D' + E_1 + \cdots + E_5) \to O_{S'}(D' + B' + E_1 + \cdots + E_5) \to O_{B'}(-1) \to 0$$

implies that

$$h^0(S', O(D' + B' + E_1 + \cdots + E_5)) = 3, \quad h^1(S', O(D' + B' + E_1 + \cdots + E_5)) = 0.$$

Let $H = 2D' + B' + E_1 + \cdots + E_5$. Since $P_1 + P_2 + P_3 + P_4 + P_5$ is linearly equivalent to $2k + p$ on D, the restriction of the divisor H on D' is linearly equivalent to 0 on D'. Hence the short exact sequence

$$0 \to O_{S'}(D' + B' + E_1 + \cdots + E_5) \to O_{S'}(H) \to O_{D'} \to 0$$

implies that $h^0(S', O(H)) = 4$. Next we want to show that this linear system has neither fixed components nor base points. Since $h^0(S, O(H - D')) = 3$, D' is not a fixed component of $|H|$. Since $HD' = 0$, there are no base points on D'. Let H_1 be a member of $|H|$ which does not contain D'. Since $HD' = 0$, H_1 must not contain B' or any of E_i. Therefore $|H|$ has no fixed components. A result of Saint-Donat says that on a K-3 surface any linear system without fixed components has no base points [Sai]. Thus the linear system $|D|$ on S has no base points. Hence there is an effective divisor D_1 on S' which is linearly equivalent to $D'+E_1+\cdots+E_5$ and does not meet E_1. The divisor $D'+D_1+B'$ is linearly equivalent to H which meets E_1 at $E_1 \cap D'$. Since H_1 does not meet D', H_1 and $D'+D_1+B'$ have no common points on E_1. Hence the linear system $|H|$ has no base points on E_1. For the same reason it has no base points on all E_i. Therefore the linear system $|H|$ is base point free. The linear system $|H|$ defines a morphism φ from S' to $\mathbf{P}^3$. Since $HE_i = 1$, the images of $E_1, \ldots, E_5$ are lines. Lemma 2.7 implies that for every pair $1 \leq i < j \leq 5$ there is a member H^* in $|H|$ which contains E_i but not E_j. Hence the images of $E_1, \ldots, E_5$ are distinct. Since $H^2 = 5$, the image of S' under φ is a quintic surface. Hence φ is a birational morphism. Since the images of $E_1, \ldots, E_5$ are lines, the minimal resolution of the image of S' has five disjoint exceptional curves of first kind. By Lemma 2.6, the image of S' is normal. Suppose that F is a curve on S' disjoint from H whose image in $\mathbf{P}^3$ is a point. Then the algebraic index theorem implies that $F^2 < 0$. Since $FK_{S'} = 0$, the adjunction formula implies that $\chi(F) > 0$. Hence the image of F is a rational double point. Therefore the birational image of S' in $\mathbf{P}^3$ is a normal quintic surface with a triple point as its only essential singularity. Q.E.D.

3. Normal quintic surfaces with several triple points

In this section we discuss the normal quintic surfaces with more than one triple points.

Lemma 3.1. *Let S_0 be a normal quintic surface with more than one triple points. Assume that one triple point p has type* (iv) *or* (v) *and the blowing-up of S_0 at p is not a normal surface. Then S_0 is not K-3.*

Proof. We may assume that p has the equation (2) or (3). The canonical divisor of the minimal resolution of S_0 is cut out by the plane $y = 0$. Let q be another triple point on S_0. It suffices to show that q is not on the plane $y = 0$, because then the canonical divisor of the minimal model will be $-D$ where D is the union of anticanonical divisors of all triple points other than p.

Suppose that q were on the plane $y = 0$. With a suitable linear transformation, we may assume that $q = (\infty, 0, 0)$. That would imply that the exponent of x in each term of (2) or (3) is less than or equal to 2, whence the surface S_0 is singular along the line $y = 0$, $z = 0$. This would contradict the assumption that S_0 is normal. Q.E.D.

Lemma 3.2. *Let p be a minimally elliptic triple point on a normal surface S_0 in $\mathbf{P}^3$ and let H_0 be a plane passing through p. Let C_1, C_2 and C_3 be three curves on the plane H_0 such that* (i) *all C_i pass through p*; (ii) *all C_i are smooth at p and* (iii) *C_i and C_j intersect at p transversally at p for $i \neq j$. Let S' be the* *minimal resolution of* S_0. *Let* Z *be the fundamental cycle of* p. *Let* C_1', C_2' *and* C_3' *be the proper transforms of* C_1, C_2 *and* C_3 *in* S' *respectively. Then* $C_i'Z = 1$ *for* $i = 1, 2, 3$.

Proof. Let T be the blowing up of $\mathbf{P}^3$ at p and let E be the exceptional plane. Let S be the proper transform of S_0. Then the curve $C = E \cap S$ is a plane cubic curve. Thus the intersection of the proper transform of H_0 and C consist of three points a, b, c. Since the tangent directions of C_1, C_2 and C_3 at p are distinct, the three points a, b, c on E must be distinct and C must be smooth at these three points. Hence the proper transforms of C_1, C_2 and C_3 meet C at a, b and c transversally. Since p is a minimally elliptic point, there are at most rational double points for S on C and none of a, b, c is a rational double point. The result follows immediately. Q.E.D.

Lemma 3.3. *Let* S_0 *be a normal quintic* K-3 *surface with more than* 2 *triple points. Then* S_0 *has exactly* 3 *minimally elliptic triple points which are not collinear. The minimal model of* S_0 *contains three nonsingular elliptic curves* D_1, D_2 *and* D_3 *with* $D_iD_j = 2$ *for all* $i \neq j$.

Proof. Since each triple point has a positive geometric genus. The sum of the geometric genera of the triple points of S_0 must be 3. This implies that S_0 has exactly three triple points p, q, r and all of them are minimally elliptic. Let L_{pq} be the line passing through p and q. Then L_{pq} must be on S_0, otherwise the intersection number of L_{pq} and S_0 would be greater than 5, which is impossible. Let H be a generic plane passing through L_{pq}. The intersection of H and S_0 is the union of L_{pq} and a quartic curve Q. Since p and q are triple points of the plane curve $L_{pq} \cup Q, L_{pq}$ meets Q at p and q only. This implies that the triple point r is not on L_{pq}. Let L_{pr} and L_{qr} be the lines passing through p, r and q, r respectively and let H_{pqr} be the plane passing through p, q and r. Then $H_{pqr} \cap S_0$ is the union of L_{pq}, L_{pr}, L_{qr} and a conic C which passes through p, q and r.

Let S' be the minimal resolution of S_0. Let Z_p, Z_q and Z_r be the fundamental cycles of p, q and r and S' respectively. Let L_{pq}', L_{pr}', L_{qr}' and C' be the proper transforms of L_{pq}, L_{pr}, L_{qr} and C on S' respectively. By Lemma 3.2 $L_{pq}'Z_p = L_{pr}'Z_p = C'Z_p = 1$ and etc. Let $S' \to S$ be the blowing-down of L_{pq}', L_{pr}', L_{qr}' and C'. Then S is a K-3 surface. Let B_1, B_2 and B_3 be the direct images of Z_p, Z_q and Z_r in S respectively. They are all minimally elliptic cycles. The Riemann-Roch theorem implies that the linear system $|B_i|$ has dimension 1 for each i. Since B_i is minimally elliptic, $|B_i|$ has no fixed components. Take a general member D_i from each $|B_i|$. Then D_1, D_2 and D_3 are nonsingular elliptic curves on S with $D_iD_j = 2$ for all $i \neq j$. Q.E.D.

Lemma 3.4. *Let S_0 be a normal quintic K-3 surface with two triple points p and q as its only essential singularities. Then the minimal model of S_0 contains one of the divisors in Figure* 1.

Proof. We may assume that the geometric genera of p and q are 2 and 1 respectively. By Lemmas 2.4 and 3.1 p is a triple point of type (iii), (iv) or (v) with an infinitely near triple point.

We may assume that the equation of S_0 has the form

$$f_3(x,y,z) + f_4(x,y,z) + f_5(x,y,z) = 0 \tag{4}$$

with $p = (0,0,0)$ and $q = (0,0,\infty)$. Here $f_i(x,y,z)$ are homogeneous polynomials of degree i for $i = 3,4,5$. Since q is a triple point, the exponent of z in each term of (4) is less than three. So $f_3(x,y,z)$ does not contain the monomial z^3. Either xz^2 or yz^2 must appear in $f_3(x,y,z)$, otherwise S_0 would be singular along the line $x = 0$, $y = 0$. Immediately we see that p cannot have type (v). Without loss of generality we may assume that $f_3(x,y,z)$ is yz^2 or $yz(y-z)$. Let L be the line $y = 0$, $z = 0$. This is the line whose proper transform passes through the infinitely near triple point of p. Since S_0 is assumed to have an infinitely near triple point, neither x^4 nor x^5 appears in the equation (4). Hence the line L is on S_0.

Let H be a generic plane passing through the line L. Then $H \cap S_0$ is the union of L and an irreducible quartic curve Q with a double point plus an infinitely near double point at p. Thus Q has geometric genus 1. The proper transform D of Q in the minimal model S^* of S_0 is a nonsingular elliptic curve.

Let T be the blowing-up of $\mathbf{P}^3$ at the point p and let E be the exceptional plane. Let S be the proper transform of S_0. The intersection $E \cap S$ is the union of three lines L_1, L_2 and L_3. One of them, say L_1, is on the proper transform of the plane H_0 in $\mathbf{P}^3$ passing through L and q. The intersection point s of L_1, L_2 and L_3 is a minimally elliptic triple point of S and the proper transform of Q meets E at the point s twice. Let $\pi\colon S' \to S$ be the minimal resolution of S. The fundamental cycle of the triple point s in S' is a minimally elliptic cycle Z', which meets the proper transform of Q twice.

If the triple point p is of type (iii), then $L_2 \neq L_3$. Evidently the proper transforms of Q, Z' (which can be replaced by a generic member in its linear system), L_2 and L_3 in the minimal model of S' have the configuration (a) in Figure 1.

If the triple point p has type (iv), then $L_2 = L_3$. There are following cases:

(A) S has two ordinary double points on L_2 away from s. Then the minimal model of S' contains a divisor (b) in Figure 1.

(B) S has one double point t on L_2 away from s and S has an infinitely near double point over the point t. That double point t can be represented by one of the following three equations:

$$z^2 + x^2 + xy^2 = 0, \qquad z^2 + x^3 + xy^2 = 0, \qquad z^2 + x^4 + xy^2 = 0.$$

Hence the minimal model of S' contains a divisor (c), (d) or (e) in Figure 1.

(C) S has only one ordinary double point t on L_2 away from s. Then S has an infinitely near rational double point over the point s. Thus the divisor Z' contains a rational component A_i intersecting the proper transform of L_2 transversally. Let D be a general member of the linear system $|Z'|$. Let L' be the rational exceptional curve of the double point t and let M be the proper transform of L_2 in S'. Then D, M, L' and A_i have the configuration in Figure 2, because $Z'A_i = 0$. Therefore the minimal model of S' contains a divisor (b) in Figure 1.

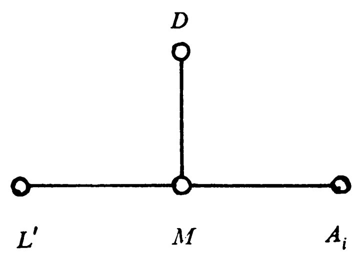

FIGURE 2

(D) s is the only singularity of S along the curve L_2. Then the cycle Z' contains a subcycle of type A_3, D_4 or D_5. following the same argument as in Case (C), one can see that the minimal model of S' contains a cycle (c), (d) or (e) in Figure 1. Q.E.D.

Proof of Theorem 4. The only part is a consequence of Lemma 3.3.

Suppose S is a K-3 surface with three nonsingular elliptic curves D_1, D_2 and D_3 with $D_iD_j = 2$ for $i \neq j$. We will show that S is birational to a quintic surface with three triple points.

Obviously these three elliptic curves are in distinct linear systems. By proper choosing the representatives in these linear systems, we may assume that D_1, D_2 and D_3 have the configuration in Figure 3.

We obtain a divisor $H = L_1 + L_2 + L_3 + Q + E_1 + E_2 + E_3$ on a surface S' as shown in Figure 4 by blowing-up the four intersection points in Figure 3.

Here E_1, E_2, E_3 are the proper transforms of D_1, D_2 and D_3 respectively and L_1, L_2, L_3 and Q are the exceptional curves. The self-intersections are $L_i^2 = Q^2 = -1$ and $E_i^2 = -2$ for $i = 1,2,3$. One can show that $h^0(S', 0(H)) = 4$ and that $|H|$ has neither fixed components nor base points. Since $H^2 = 5$, the complete linear system defines a birational morphism from S' to a quintic surface in $\mathbf{P}^3$. Let D_1' be a divisor on S which is linearly equivalent to but not equal to D_1. Let E_1' be the pull-back of D_1' in S'. Then

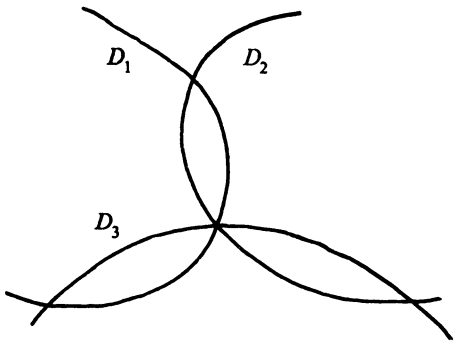

FIGURE 3

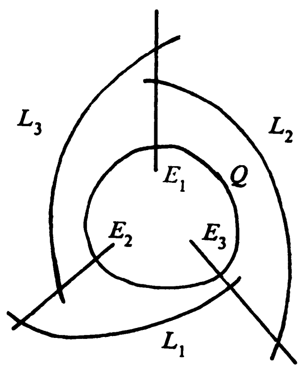

FIGURE 4

the divisor $E_1' + E_2 + E_3 + L_1$ is linearly equivalent to H. Hence the image of L_1 in S' is different from those of L_2, L_3 and Q. Hence the image of the divisor H is a reduced quintic curve, which consists of three lines and a conic. Therefore the quintic surface must be normal. Since $E_iH = 0$ for $i = 1, 2, 3$. The images of E_1, E_2 and E_3 are three isolated essential singularities. By Lemma 2.1, these must be triple points.

Proof of Theorem 3. The only part is a consequence of Lemma 3.4.

Assume that S is a K-3 surface containing a divisor in Figure 1. One can use the same method to blow up some points to get a surface S' with a connected divisor H satisfying $H^2 = 5$, $h^0(S', 0(H)) = 4$ and $|H|$ has neither fixed components nor base points. For instance in the case (a) of Figure 1, we may assume that the divisor is $D_1 + D_2 + L + M$ as in Figure 5.

Choose a general point s on D_2. Blow up S at s and at the two intersection points of D_1 and D_2 to get a divisor in Figure 6.

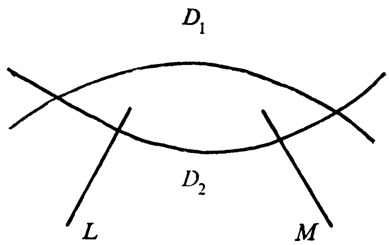

FIGURE 5

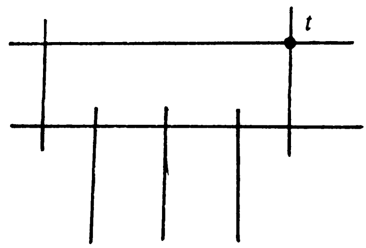

FIGURE 6

Then blow up the surface at the point t to get a surface S' with a divisor $H = C_1 + 2C_2 + E_1 + 2E_2 + 2E_3 + E_4 + L_1 + L_2$ as in Figure 7. Then one can check that the divisor H satisfies all the conditions. It can be verified that $|H|$ defines a birational morphism from S' onto a normal quintic surface in $\mathbf{P}^3$. We leave the verifications of the other cases of Figure 1 to the readers. Q.E.D.

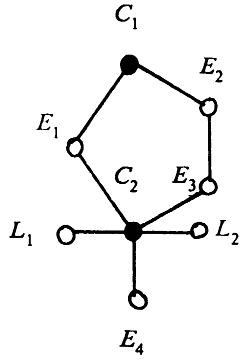

FIGURE 7

4. Characterizations by double planes

Let B be a reduced sextic curve on the plane P^2 without quadruple points and infinitely near triple points or worse singularities. Let S be the double cover of P^2 with B as the branch locus. Then S is a K-3 surface (with possibly some rational double points). The following theorem identifies those sextic curves that will give rise to normal quintic K-3 surfaces with one triple point.

Theorem 4.1. *A K-3 surface S is the minimal model of a normal quintic K-3 surface with one triple point as its only essential singularity if and only if it is a double plane branched over a sextic curve B without quadruple or infinitely near triple points and B either has a tritangent line or contains a line.*

Remark. Here a tritangent line is a line L on P^2 such that all the intersection numbers $(L, B)_p$ are even for every point p.

Proof. According to Theorem 2, if S is the minimal model of a normal quintic K-3 surface with one triple point, then there are nonsingular curves D and C with genera 2 and 0 respectively such that $DC = 1$. The linear system $|D|$ is base point-free. It defines a double cover over $\mathbf{P}^2$ branched over a sextic curve B. Since $DC = 1$, the image of C is a line L and the line L either splits or is the branch locus. If L splits, then L is a tritangent line of B.

Conversely, if S is a double cover branched over a sextic curve B and if L is a tritangent line to B, let H be a line in general position. The inverse image of H under the double cover is a nonsingular curve D of genus 2. Let L split into C and C'. Then C is a rational curve and $DC = 1$. Since H is in general position, the surface S is smooth at the point $D \cap C$. Hence Theorem 2 implies that the minimal model of S is the minimal model of a normal quintic K-3 surface with one triple point as its only essential singularity. If the sextic curve B contains a line L, then the inverse image of L is a rational curve C with $DC = 1$. Once again the surface S is the minimal model of a normal quintic K-3 surface with one triple point by Theorem 2. Q.E.D.

Some normal quintic K-3 surfaces with several triple points are also birational to sextic double planes, as can be seen by the following examples:

Examples. (1) Let B be a plane sextic curve with three ordinary double points p_1, p_2 and p_3 as its only singularities. Let S be the canonical resolution of the double cover of $\mathbf{P}^2$ branched over B. Let L_1, L_2 and L_3 be three generic lines on $\mathbf{P}^2$ passing through p_1, p_2 and p_3 respectively. Then L_i meets B at four other points besides p_i and the intersections at these four points are transversal for each i. Hence the proper transforms of L_1, L_2 and L_3 in S are three nonsingular elliptic curves with mutual intersection number 2. Thus S is birational to a normal quintic K-3 surface with three triple points by Theorem 4.

(2) Let B be a plane sextic curve with an ordinary double point p_1 and a double point p_2 which has an infinitely near double point. Let S be the canonical resolution of the double cover of $\mathbf{P}^2$ branched over B. Let L_1 and L_2 be two generic lines on $\mathbf{P}^2$ passing through p_1 and p_2 respectively. Then the proper transforms C_1, C_2 of L_1, L_2 on S are nonsingular elliptic curves with $C_1C_2 = 2$. Since B has an infinitely near double point over p_2, there are two disjoint nonsingular rational curves E_1 and E_2 on S meeting C_2 transversally. Thus S has a divisor of (a) in Figure 1. Hence S is birational to a normal quintic K-3 surface with two triple points.

Proposition 4.2. *Any normal quintic K-3 surface with more than one triple points is birational to a double cover of $\mathbf{P}^2$ branched over an octic curve with two ordinary quadruple points such that the line passing through these two quadruple points is not a component of the branch locus.*

Proof. Let S be the minimal model of a normal quintic K-3 surface with more than one triple points. Then there are two nonsingular elliptic curves C_1 and C_2 with $C_1C_2 = 2$. We may assume that C_1 and C_2 intersect at two distinct points p and q without loss of generality. Let S' be the blowing up of S at p and q. Let D_1 and D_2 be the proper transforms of C_1 and C_2 respectively and let E and F be the two exceptional curves of first kind on S'. Let $H = D_1 + D_2 + E + F$. It's easy to see that $h^0(S', 0(H)) = 3$ and the linear system $|H|$ has neither fixed components nor base points. Since $H^2 = 2$ the linear system defines a double cover over $\mathbf{P}^2$. Since $HE = HF = 1$ and $HD_1 = HD_2 = 0$, the images of E and F are the same line L on $\mathbf{P}^2$. Since L splits in the double cover, L must not be a component of the branch locus. The image of D_1 and D_2 are two points on L. Since both D_1 and D_2 are nonsingular elliptic curves, their images are ordinary quadruple points on the branch locus. Q.E.D.

Finally we give some examples of double octic planes which are birational to normal quintic K-3 surfaces.

Examples. (3) Let C be a septic curve on $\mathbf{P}^2$ with an ordinary triple point p and an ordinary quadruple point q so that the line passing through p and q is not a component of C. Let L be a generic line on $\mathbf{P}^2$ passing through the point p. Let S be the canonical resolution of the double cover of $\mathbf{P}^2$ branched over the octic curve $B = C + L$. Then the line passing through p and q splits into two exceptional curves of first kind E_1 and E_2 on S. Let D_1 and D_2 be the inverse image of p and q in S. They are nonsingular elliptic curves. After blowing down E_1 and E_2, we get two nonsingular elliptic curves intersecting at two points. Since the proper transform of L in S is a nonsingular rational curve which connects to D_1 and four other nonsingular rational curves. In particular, S contains a divisor (b) in Figure 1. Hence S is birational to a normal quintic K-3 surface with two triple points.

(4) Let B be a plane octic curve with two ordinary quadruple points p, q and two ordinary double points r, s as its only singularities. Assume that p, r and s are collinear and the line passing through p and q is not a component of B. Let L be the line passing through p and q and let M be the line passing through p, r and s. The line M is not a component of B, for otherwise B would have more singularities. Then the line M splits in the double cover. The minimal model of the double cover of $\mathbf{P}^2$ branched over B contains a divisor (a) in Figure 1. Hence this surface is birational to a normal quintic K-3 surface with two triple points.

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