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SIMILARITY, QUASISIMILARITY AND OPERATOR FACTORIZATIONS

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ABSTRACT. We introduce and illustrate an operator factorization technique to study similarity and quasisimilarity of Hilbert space operators. The technique allows one to generate, in a systematic way, families of "test" operators, and to check for similarity and quasisimilarity with a given model. In the case of the unilateral shift U_+, we obtain a one-parameter family of nonhyponormal, noncontractive, shift-like operators in the similarity orbit of U_+. We also obtain new characterizations of quasisimilarity and similarity in terms of invariant operator ranges, and conditions for spectral and essential spectral inclusions.

1. INTRODUCTION

The purpose of this note is to introduce and illustrate an operator factorization technique for studying similarity and quasisimilarity of Hilbert space operators. Typically, the existing literature concerns conditions for a *test* operator to be similar or quasisimilar to a *given* operator, but it is difficult to decide in practice which operators are candidates for test operators. The factorization technique allows one to generate, in a systematic way, families of test operators and also to check for quasisimilarity or similarity. In the case of the unilateral shift U_+, we apply the technique to special factorizations of quadratic functions of U_+, and we obtain as a result a rather interesting one-parameter family of shift-like operators in the similarity orbit of U_+. These operators turn out to be nonhyponormal, so their similarity to U_+ does not follow from existing work [Cla1]. We also obtain new characterizations of quasisimilarity and similarity in terms of invariant operator ranges, and we study conditions for spectral and essential spectral inclusions. To describe our results, we shall first need some notation and preliminaries.

Let $\mathscr{H}$ and $\mathscr{H}_1$ denote complex Hilbert spaces and let $\mathscr{L}(\mathscr{H})$, $\mathscr{L}(\mathscr{H}_1)$ be their respective algebras of bounded operators. Operators $S \in \mathscr{L}(\mathscr{H})$ and $T \in \mathscr{L}(\mathscr{H}_1)$ are *similar* $(S \sim T)$ if there exists a bounded invertible operator

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$X: \mathscr{H}_1 \to \mathscr{H}$ such that $SX = XT$. Similar operators have isomorphic lattices of invariant and hyperinvariant subspaces; moreover, similarity preserves compactness, cyclicity, algebraicity, and the *spectral picture* (i.e., the spectrum, essential spectrum, and index function).

In the finite-dimensional case, each operator is similar to an essentially unique *Jordan model* of the form

$$\sum_{i=1}^{n} \bigoplus \left(\lambda_i + \sum_{j=1}^{m_i} \bigoplus q_{k_{ij}}^{(\alpha_{ij})} \right),$$

where $1 \le n$, m_i, $\alpha_{ij} < \infty$ and q_k denotes the Jordan nilpotent k-cell on $\mathbf{C}^{(k)}$. In the infinite-dimensional case we permit Jordan models with infinite α_{ij}'s; these operators have relatively well-understood structures but, unfortunately, they do not model each algebraic operator up to similarity:

Theorem 1.1 (C. Apostol [Ap2]). *For* $T \in \mathscr{L}(\mathscr{H})$ *the following are equivalent*:

(1) T *is similar to a Jordan model*;
(2) T *is algebraic and* $R(p(T))$ *is closed for every polynomial* p *dividing the minimal polynomial of* T (*see* [AFHV, *p.* 330]);
(3) *The inner derivation* $X \to TX - XT$ $(X \in \mathscr{L}(\mathscr{H}))$ *has closed range in* $\mathscr{L}(\mathscr{H})$.

With a view to keeping the model class as simple as possible in the infinite-dimensional case, Sz. Nagy and Foiaş [SzNF] introduced a weakened version of similarity called *quasisimilarity.* Operators $S \in \mathscr{L}(\mathscr{H})$ and $T \in \mathscr{L}(\mathscr{H}_1)$ are quasisimilar ($S \sim_{\text{qs}} T$) if there exist *quasiaffinities* (i.e., operators that are each 1–1 with dense range), $X : \mathscr{H}_1 \to \mathscr{H}$ and $Y : \mathscr{H} \to \mathscr{H}_1$, such that $SX = XT$ and $TY = YS$. In [SzNF] Sz. Nagy and Foiaş provided quasisimilarity models for certain classes of contractions as a means of studying their invariant subspace structures. Subsequently Apostol, Douglas, and Foiaş [ADF] and L. Williams [W2] proved that every algebraic operator is quasisimilar to an essentially unique Jordan model.

The ultimate value of quasisimilarity will depend on the extent to which it respects structural and spectral properties of operators, and in this regard there are several interesting open questions. In [Hoo] Hoover proved that quasisimilarity preserves the existence of nontrivial hyperinvariant subspaces. Subsequently, Fialkow [Fia2] proved that quasisimilarity preserves (up to isomorphism) a certain (possibly trivial!) sublattice of the hyperlattice. (This sublattice includes the lattice of Riesz spectral subspaces, and for a normal operator it includes the lattice of spectral subspaces.) Nevertheless, Herrero [Her2] has shown that quasisimilarity does not preserve the full hyperlattice.

Question 1.2. Does quasisimilarity preserve the existence of nontrivial invariant subspaces? Is every operator quasisimilar to an operator with a nontrivial invariant subspace?

In [Her1] Herrero proved that if $T \sim_{qs} S$, then every component of $\sigma(T)$ (the spectrum of T) intersects $\sigma(S)$ (cf. [Fia1]). In [SzNF] it is proved that every C_{11} contraction is quasisimilar to a unitary [SzNF, II.3 Proposition 3.5], and an example is given of a C_{11} contraction with spectrum the whole unit disk $\overline{\mathbf{D}}$ [SzNF, VI.4.2]. A simple example of two quasisimilar operators with different spectra was given in [Hoo]: Since $q_k \sim (1/k)q_k$, then

$$S := \sum_{k=1}^{\infty} \bigoplus q_k \sim_{qs} T := \sum_{k=1}^{\infty} \bigoplus (1/k) q_k ;$$

now $\sigma(S) = \overline{\mathbf{D}}$, while T is compact and quasinilpotent. An extension of the technique of this example depends on the following concept due to Apostol [Ap1]: A sequence of closed subspaces, $\{\mathscr{M}_n\}_{n=1}^m$, is *basic* if $\mathscr{M}_j$ and $\bigvee_{k \neq j} \mathscr{M}_k$ are complementary for every j, and

$$\bigcap_{i=1}^{m} \bigvee_{k \neq i} \mathscr{M}_k = \{0\}$$

if $m = \infty$. In [Her5] Herrero showed that in many cases, an operator S with a denumerable basic sequence of invariant subspaces is quasisimilar to an operator T with $\sigma(T) \neq \sigma(S)$; the basic sequence constructions of [Ap1, BK, Fia2, Fia3, Her5] provide the only known examples of quasisimilar operators with different spectra.

Conjecture 1.3 [Her5]. *If* Lat S *contains no denumerable basic sequence, then* $\sigma(T) = \sigma(S)$ *for every* $T \sim_{qs} S$.

Concerning Fredholm behavior, Fialkow [Fia1] and L. Williams [W1] showed that quasisimilar operators have intersecting essential spectra (cf. [St]).

Question 1.4 [St, Fia4]. If $T \sim_{qs} S$, does each component of $\sigma_e(T)$ (the essential spectrum of T) intersect $\sigma_e(S)$?

To study the preceding questions it is instructive to consider the case when $S = U_+$, the unilateral shift. A theorem of Sz. Nagy [SzN] implies that an operator T is similar to U_+ if and only if $\{T^n\}_{n=1}^{\infty}$ is uniformly bounded above and below, T is cyclic, and $T^{*n} \to_s 0$. An interesting and apparently difficult problem is to characterize the operators quasisimilar to the shift. The shift has no complementary invariant subspaces and thus falls within the scope of Conjecture 1.3.

Question 1.5 [St]. If $T \sim_{qs} U_+$, does $\sigma(T) = \sigma(U_+)$?

Since U_+ is cyclic and quasisimilarity does preserve cyclicity, it follows from a theorem of Herrero [Her4] that if $T \sim_{qs} U_+$, then $\sigma_e(T)$ is connected and contains $\sigma_e(U_+)$ (see [Fia4]).

Question 1.6. If $T \sim_{qs} U_+$, does $\sigma_e(T) = \sigma_e(U_+)$?

The literature contains few examples of operators quasisimilar to the shift, and these concern operators within restrictive classes. In [Cla1] Clary characterized the *subnormal* operators similar or quasisimilar to the shift (cf. [Con1]). Such an operator T necessarily satisfies $\sigma(T) = \sigma(U_+)$ because Clary [Cla2] proved that quasisimilar hyponormal operators have equal spectra. Raphael [R] subsequently proved that if T and S are quasisimilar cyclic subnormal operators, then $\sigma_e(T) = \sigma_e(S)$ and $\sigma_\pi(T) = \sigma_\pi(S)$ (σ_π is the approximate point spectrum) (cf. [Con2]). By using Raphael's results and the powerful Similarity Orbit Theorem [AHV], Fialkow [Fia4] proved that if T and S are quasisimilar cyclic subnormal operators, then $\overline{\mathcal{S}(T)} = \overline{\mathcal{S}(S)}$ ($\mathcal{S}(T)$ is the similarity orbit of T), so that, in particular, T and S have equal spectral pictures. In a similar vein, Fialkow [Fia4] proved that if $T \sim_{\text{qs}} U_+$, then $T \in \overline{\mathcal{S}(U_+)}$. In a different direction, Wu [Wu] characterized the contractions with finite defect indices that are quasisimilar to the shift.

Unfortunately, none of the above results seems to shed much light on Question 1.5 or Question 1.6. In this paper we explore a quite general factorization technique for generating operators similar or quasisimilar to a given operator. Starting with an operator S, a quasiaffinity Z commuting with S, and a factorization

$$(*) \qquad Z = XY \qquad (X \text{ injective}),$$

we define the linear transformation $T_Z : Y\mathcal{H} \to Y\mathcal{H}$ by $T_Z Y = YS$. T_Z is either equal to S, similar to S, quasisimilar to S, or unbounded, depending on the "extent" to which X commutes with S. We have, therefore, a way of producing elements in the quasisimilarity and similarity orbits of S; also, by varying Z in the commutant of S, we have potentially a procedure to exhibit concretely operators quasisimilar to S which are not similar to S.

After developing general properties of the factorization technique in §2 (including an illustration for the case of nilpotents of order 2), we apply the technique to $S = U_+$ in §3. Let $\{e_n\}_{n=1}^{\infty}$ denote an orthonormal basis of $\mathcal{H}$; the unilateral shift defined by $U_+ e_n = e_{n+1}$ is also uniquely determined by the relation $U_+(e_n + e_{n+1}) = e_{n+1} + e_{n+2}$. In §3 we use the factorization technique to construct a class of shift-like operators of the form $T_a h_n = h_{n+1}$, where $h_n = e_n + a_n e_{n+1}$ for special sequences $a = \{a_n\}$; this class includes, for example, the operator T defined by

$$T(e_n + ((n+1)/n)e_{n+1}) = e_{n+1} + ((n+2)/(n+1))e_{n+2}.$$

The family $\{T_a\}$ has some intriguing properties:

(1) T_a is similar to U_+;
(2) T_a is unitarily equivalent to a compact perturbation of U_+;
(3) T_a is not unitarily equivalent to any weighted shift;
(4) $\|T\| > 1$;
(5) T is not hyponormal.

To establish (1), we use our factorization technique by explicitly finding an invertible operator $J : l^2(\mathbf{Z}_+) \to \overline{R(Y)}$ such that $XJ = I + U_+$ (here X and Y are quasiaffinities in $\mathscr{L}(l^2(\mathbf{Z}_+))$, $XY = (I + U_+)^2$, and $TY = YU_+$). For (2) we use the Brown-Douglas-Fillmore theorem; for (3) we employ the characterization of weighted shifts given in [CuS]. The fact that T_a is not a weighted shift shows that the similarity with U_+ cannot be obtained from Kelley's criterion for similarity of weighted shifts [Sh, Theorem 2(b), p. 54]. Finally, we utilize a computer-aided argument to conclude that $\|T\| > 1$; the delicacy of this calculation suggests that it would be very difficult to check the similarity of T with U_+ using Sz. Nagy's theorem. Since T is similar to U_+, we see that T can't be hyponormal (for otherwise $1 = r(T) = \|T\| > 1$); thus T does not fall within the scope of Clary's theorem. The last three properties show that the family $\{T_a\}$ is a genuinely new collection of operators in the similarity orbit of U_+. The factorizations we use for $(I + U_+)^2$ are derived from the obvious commutative factorization $(I + U_+)(I + U_+)$ by perturbing each factor by a subdiagonal piece. Although our calculations in §3 may appear ad hoc or unmotivated, a little work with such factorizations will convince the reader of the naturalness of the approach; that such computations are involved is simply a reflection of the complexity of the quasisimilarity orbit of U_+.

Although Questions 1.5 and 1.6 remain open, the factorization technique permits us to reformulate these and similar questions in terms of invariant operator ranges. We show in Proposition 2.7 that with X as in $(*)$, T_Z is quasisimilar to S if and only if $SR(X_1) \subseteq R(X_1)$ (i.e., the range of X_1 is invariant under S), where $X_1 = X \mid \overline{R(Y)} : \overline{R(Y)} \to \mathscr{H}$. This result rests on a theorem of Douglas [D] characterizing the existence of a bounded solution C to the operator equation $A = BC$. In [FW], P. Fillmore and J. Williams characterize the case when C can be taken to be invertible. We use this result in Proposition 4.2 to show that if S is injective, then T_Z is invertible if and only if $SR(X_1) = R(X_1)$. Motivated by these results, in Theorem 4.6 we characterize the case when there exists a *Fredholm* operator C satisfying $A = BC$. We then show in Proposition 4.7 that if the nullity of S, $\text{nul}(S)$, is finite, then T_Z is Fredholm if and only if $R(X_1)/SR(X_1)$ is finite dimensional. We actually establish a more general result concerning the containments $\sigma(S) \subseteq \sigma(T_Z)$, $\sigma_e(S) \subseteq \sigma_e(T_Z)$, $\sigma(T_Z) \subseteq \sigma(S)$, and $\sigma_e(T_Z) \subseteq \sigma_e(S)$ (Proposition 4.9), which sheds light on Questions 1.5 and 1.6.

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2. Quasisimilarity, factorization and operator ranges

Let $S \in \mathscr{L}(\mathscr{H})$ and suppose that $T \in \mathscr{L}(\mathscr{H}_1)$ is quasisimilar to S (in symbols, $T \sim_{qs} S$); i.e., there exist quasiaffinities $X_1 : \mathscr{H}_1 \to \mathscr{H}$ and $Y_1 : \mathscr{H} \to \mathscr{H}_1$ such that $TY_1 = Y_1 S$ and $X_1 T = SX_1$. Then $S(X_1 Y_1) = X_1 T Y_1 = (X_1 Y_1) S$, so that $X_1 Y_1$ is a quasiaffinity in $(S)'$, the commutant of S. Our basic observation identifies a class of factorizations of quasiaffinities in $(S)'$ with elements in $(S)_{qs}$, the quasisimilarity orbit of S.

Definition 2.1. For $S \in \mathscr{L}(\mathscr{H})$ let

$$\mathscr{Z}_0(S) := \{(X, Y) \in \mathscr{L}(\mathscr{H})^{(2)} : X \text{ is injective}, \\ XY \text{ is a quasiaffinity, and } XY \in (S)'\}.$$

For $Z = (X, Y) \in \mathscr{Z}_0(S)$, let $\mathscr{H}_1 = \overline{R(Y)}$. Let $Y_1 : \mathscr{H} \to \mathscr{H}_1$ be given by $Y_1 x := Yx$ $(x \in \mathscr{H})$ and let $X_1 : \mathscr{H}_1 \to \mathscr{H}$ be given by $X_1 := X|_{\mathscr{H}_1}$. Clearly Y_1 is a quasiaffinity and X_1 is injective; moreover $X_1 Y_1 = XY$, so $R(X_1)$ ($\supseteq R(XY)$) is dense and thus X_1 is also a quasiaffinity. Define $\widetilde{T}_Z : R(Y) \to R(Y)$ by $\widetilde{T}_Z(Yx) = YSx$, $x \in \mathscr{H}$. $\widetilde{T}_Z$ is well defined and linear (by the injectivity of Y), although possibly unbounded. Since $\widetilde{T}_Z Y = YS$ then $X\widetilde{T}_Z Y = XYS = SXY$ and thus $X|_{R(Y)} \widetilde{T}_Z = SX|_{R(Y)}$. If $\widetilde{T}_Z$ is bounded, we denote its unique extension to $\mathscr{H}_1$ by T_Z or T. Let

$$\mathscr{Z}(S) := \{Z \in \mathscr{Z}_0(S) : \widetilde{T}_Z \text{ is bounded}\}.$$

Proposition 2.2. *If $Z = (X, Y) \in \mathscr{Z}(S)$, then T_Z is quasisimilar to S via the quasiaffinities $X_1 : \mathscr{H}_1 \to \mathscr{H}$ and $Y_1 : \mathscr{H} \to \mathscr{H}_1$.*

Proof. Since $\widetilde{T}_Z Y = YS$, then $TY_1 = Y_1 S$ and thus also $X_1 T Y_1 = X_1 Y_1 S = XYS = SXY = SX_1 Y_1$. Since Y_1 has dense range, then $X_1 T = SX_1$ and the result follows. □

Thus, each element $Z \in \mathscr{Z}(S)$ determines an operator $T_Z \in (S)_{qs}$ and each operator quasisimilar to S is unitarily equivalent to some $T_Z \in \mathscr{L}(\mathscr{H})$. Note that $Z = (X, Y) \in \mathscr{Z}(S)$ if and only if

(1) X is injective;
(2) XY is a quasiaffinity in $(S)'$; and
(3) There exists a constant $C = C(S, X, Y) > 0$ such that $\|YSx\| \le C\|Yx\|$ $(x \in \mathscr{H})$.

We next describe the cases $T_Z = S$ and $T_Z \sim S$ ($\sim$ stands for similarity).

Proposition 2.3. *Let $Z = (X, Y) \in \mathscr{Z}(S)$ and let $S_1 := S|_{\mathscr{H}_1} : \mathscr{H}_1 \to \mathscr{H}$. The following are equivalent:*

(i) $T_Z = S_1$;
(ii) $XS_1 = SX_1$;
(iii) $Y \in (S)'$.

Proof. (i) $\Rightarrow$ (iii): $YS = T_Z Y = S_1 Y = SY$.

(iii) $\Rightarrow$ (ii): $(XS_1 - SX_1)Y = XS_1 Y - SX_1 Y = XSY - SXY = XYS - SXY = 0$, so $XS_1 = SX_1$.

(ii) $\Rightarrow$ (i): $XT_Z = X_1 T_Z = SX_1 = XS_1$, and therefore $T_Z = S_1$, by the injectivity of X. □

Proposition 2.4. *Let* $Z = (X, Y) \in \mathscr{Z}(S)$. *Then* $T_Z \sim S$ *if and only if there exists an invertible operator* $J : \mathscr{H} \to \mathscr{H}_1$ *such that* $XJ \in (S)'$.

Proof. If T_Z is similar to S, let $J : \mathscr{H} \to \mathscr{H}_1$ be an invertible operator such that $T_Z J = JS$; then $SX_1J = X_1T_ZJ = X_1JS$, which gives $XJ = X_1J \in (S)'$. Conversely, if $J : \mathscr{H} \to \mathscr{H}_1$ is invertible and $XJ \in (S)'$, then $X_1T_ZJ = SX_1J = X_1JS$, so that $T_ZJ = JS$ and thus $T_Z \sim S$. □

Example 2.5. In [Her3] Herrero proved that if $S \in \mathscr{L}(\mathscr{H})$ and $(S)''$ is an algebra of finite strict multiplicity, then every operator T quasisimilar to S is similar to S. We can place this result in the present context as follows. If $T \in (S)_{\text{qs}}$, then $TY_1 = Y_1S$, $X_1T = SX_1$, and $X_1Y_1 \in (S)'$ (as above), so that $A(X_1Y_1) = (X_1Y_1)A$ for every $A \in (S)''$. Since $R(X_1Y_1)$ is dense and invariant for $(S)''$, [Her3] implies that X_1Y_1 is onto. In particular, X_1 is invertible and $J := X_1^{-1}$ satisfies the requirements of Propositions 2.4.

There is an alternate description of $\mathscr{Z}(S)$ in terms of invariant operator ranges. The main ingredient is the following basic result on majorization, factorization, and range inclusion.

Theorem 2.6 (Douglas [D]). *Let* $A, B \in \mathscr{L}(\mathscr{H})$. *The following are equivalent*:

(1) $A = BC$ *for some* $C \in \mathscr{L}(\mathscr{H})$;
(2) $R(A) \subseteq R(B)$;
(3) $AA^* \leq \lambda^2 BB^*$ *for some* $\lambda > 0$.

Proposition 2.7. *For* $Z = (X, Y) \in \mathscr{Z}_0(S)$ *the following are equivalent*:

(i) $Z \in \mathscr{Z}(S)$, *i.e.*, $T_Z \in (S)_{\text{qs}}$;
(ii) $S^*R(Y^*) \subseteq R(Y^*)$; *and*
(iii) $SR(X_1) \subseteq R(X_1)$.

Moreover, if $SR(X) \subseteq R(X)$, *then* $Z \in \mathscr{Z}(S)$.

Proof. (i) ⇒ (ii): $TY = YS$ implies $S^*Y^* = Y^*T^*$, so (ii) follows.

(ii) ⇒ (i): Since $R(S^*Y^*) \subseteq R(Y^*)$, Theorem 2.6 implies that $S^*Y^* = Y^*C$, for some $C \in \mathscr{L}(\mathscr{H})$. Then $YS = C^*Y$, so $T_Z := C^*|_{\mathscr{H}_1}$ extends $\widetilde{T}_Z$.

(i) ⇒ (iii): Since $X_1T = SX_1$ we get $SR(X_1) \subseteq R(X_1)$.

(iii) ⇒ (i): There exists $D \in \mathscr{L}(\mathscr{H}_1)$ such that $SX_1 = X_1D$. Then $X_1YS = XYS = SXY = SX_1Y = X_1DY$, whence $YS = DY$, and therefore $T_Z := D|_{\mathscr{H}_1}$ extends $\widetilde{T}_Z$.

A similar argument also proves the last assertion. □

We note that the proof of Proposition 2.7 did not use the full strength of the hypothesis that XY is a quasiaffinity; under the weaker hypotheses "X injective" and "$XY \in (S)'$" one can show that (ii), (iii), and (i′): "$Yx \mapsto YSx$ extends boundedly to $R(Y)$" are all equivalent.

Corollary 2.8. *For* $Z = (X, Y) \in \mathscr{Z}_0(S)$, *the following are equivalent*:

(i) $Z \in \mathscr{Z}(S)$ *and* $T_Z \sim S$;

(ii) *there exists an invertible operator* $J: \mathscr{H} \to \mathscr{H}_1$ *such that* $X_1 J \in (S)'$.

Moreover, if $(S)'$ *is abelian and* (i) *or* (ii) *holds, then* $X_1 Y_1 \sim Y_1 X_1$.

Proof. (i) $\Rightarrow$ (ii) follows from Proposition 2.4. Conversely, suppose (ii) holds and note that since J maps onto $\mathscr{H}_1$ and $SX_1 J = X_1 JS$, then $SR(X_1) \subseteq R(X_1)$. Proposition 2.7 (iii) thus implies $Z \in \mathscr{Z}(S)$, so (i) now follows from Proposition 2.4. If (i) or (ii) holds, let $J: \mathscr{H} \to \mathscr{H}_1$ be an invertible operator with $X_1 JS = SX_1 J$. If, moreover, $(S)'$ is abelian, then $(X_1 J)(X_1 Y_1) = (X_1 Y_1)(X_1 J)$, whence $J(X_1 Y_1) = (Y_1 X_1)J$ and the result follows. □

Corollary 2.9. *Let* $Z = (X, Y) \in \mathscr{Z}_0(S)$, *let* $R := XY$, *and suppose* $S \in W(R)$, *the weakly closed unital algebra generated by* R. *If* $X_1 Y_1 \sim Y_1 X_1$, *then* $Z \in \mathscr{Z}(S)$ *and* $T_Z \sim S$.

Proof. Let $J: \mathscr{H} \to \mathscr{H}_1$ be an invertible operator such that $JX_1 Y_1 = Y_1 X_1 J$. Then $X_1 JR = X_1 JXY = X_1 JX_1 Y_1' = X_1 Y_1 X_1 J = XYX_1 J = RX_1 J$, whence $X_1 Jp(R) = p(R) X_1 J$ for each polynomial p. Since $S \in W(R)$, it follows that $X_1 JS = SX_1 J$ and the result follows from Corollary 2.8. □

We can summarize the preceding results as follows.

Theorem 2.10. *Let* $Z = (X, Y) \in \mathscr{Z}_0(S)$. *Then*

(i) $Z \in \mathscr{Z}(S) \Leftrightarrow SXY = XEY$ *for some* $E \in \mathscr{L}(\mathscr{H}_1)$;

(ii) $Z \in \mathscr{Z}(S)$ *and* $T_Z \sim S \Leftrightarrow SXY = XEY$ *for some* $E \in \mathscr{L}(\mathscr{H}_1)$ *with* $E \sim S$;

(iii) $Z \in \mathscr{Z}(S)$ *and* $T_Z \sim S \Leftrightarrow SXJ = XJS$ *for some invertible* $J: \mathscr{H} \to \mathscr{H}_1$;

(iv) $Z \in \mathscr{Z}(S)$ *and* $T_Z = S_1 \Leftrightarrow SX_1 = XS_1$;

(v) $Z \in \mathscr{Z}(S)$ *and* $T_Z = S_1 \Leftrightarrow SY = YS$.

The operator equations in Theorem 2.10 can perhaps be thought of as measures of commutativity between S and X. In the sequel we illustrate these cases with nilpotent operators of order 2.

Let $S \in \mathscr{L}(\mathscr{H})$ be a nilpotent operator of order 2. Relative to the decomposition $\mathscr{H} = N(S) \oplus N(S)^{\perp}$ (where $N(S)$ denotes the kernel of S), the operator matrix of S is of the form

$$\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix},$$

where A is injective. For simplicity, we shall restrict attention to quasiaffinities $XY \in (S)'$, where

$$X = \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix}, \qquad Y = \begin{pmatrix} Y_{11} & 0 \\ 0 & Y_{22} \end{pmatrix},$$

and both X and Y are quasiaffinities. Note that

$$(S)' = \left\{ \begin{pmatrix} Q_{11} & Q_{12} \\ 0 & Q_{22} \end{pmatrix} : AQ_{22} = Q_{11} A \right\};$$

then X and Y (as above) must satisfy $AX_{22}Y_{22} = X_{11}Y_{11}A$ and X_{11}, X_{22}, Y_{11}, Y_{22} are all quasiaffinities. We shall let $\mathscr{Z}'(S)$ denote the subset of $\mathscr{Z}_0(S)$ consisting of all such pairs (X, Y).

Let $\binom{x_1}{x_2} \in \mathscr{H}$ and observe that the map $\widetilde{T}_Z$ can be written as

$$\begin{pmatrix} Y_{11}x_1 \\ Y_{22}x_2 \end{pmatrix} \mapsto \begin{pmatrix} Y_{11}Ax_2 \\ 0 \end{pmatrix}.$$

Since Y has dense range, if $\widetilde{T}_Z$ has a bounded extension to $\mathscr{H}_1 (= \mathscr{H})$, then T_Z is of the form

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

and a calculation of $T_Z Y = YS$ shows that $B_{11} = 0$, $B_{21} = 0$, $B_{22} = 0$, and $B_{12}Y_{22} = Y_{11}A$; conversely, if $B \in \mathscr{L}(N(S)^\perp, N(S))$ satisfies $Y_{11}A = BY_{22}$, then $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$ defines a bounded extension of $\widetilde{T}_Z$ to $\mathscr{H}$.

Proposition 2.11. *Let $S \in \mathscr{L}(\mathscr{H})$ be a nilpotent of order 2. For $Z = (X, Y) \in \mathscr{Z}'(S)$ the following statements are equivalent*:

(1) *$\widetilde{T}_Z$ has a bounded extension to $T_Z \in (S)_{qs}$ (i.e., $Z \in \mathscr{Z}(S)$).*
(2) *There exists $B \in \mathscr{L}(N(S)^\perp, N(S))$ such that $BY_{22} = Y_{11}A$.*
(3) *$A^*R(Y_{11}^*) \subseteq R(Y_{22}^*)$.*
(4) *$AR(X_{22}) \subseteq R(X_{11})$.*

Proof. The equivalence of (1) and (2) is given above. That (2) $\Rightarrow$ (3) is obvious from $A^*Y_{11}^* = Y_{22}^*B^*$, and (3) $\Rightarrow$ (2) follows from Theorem 2.6. Given (2), then $AX_{22}Y_{22} = X_{11}Y_{11}A = X_{11}BY_{22}$, so $AX_{22} = X_{11}B$ and thus (4) holds. Conversely, (4) and Theorem 2.6 imply $AX_{22} = X_{11}B$ for some $B \in \mathscr{L}(N(S)^\perp, N(S))$, whence $X_{11}BY_{22} = AX_{22}Y_{22} = X_{11}Y_{11}A$; since X_{11} is injective, $BY_{22} = Y_{11}A$, so (2) holds. □

For $Z \in \mathscr{Z}'(S) \cap \mathscr{Z}(S)$, recall that $T_Z \in (S)_{qs}$ is of the form

$$T_Z = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix},$$

where $BY_{22} = Y_{11}A$. Thus also $X_{11}BY_{22} = X_{11}Y_{11}A = AX_{22}Y_{22}$, whence $X_{11}B = AX_{22}$.

Proposition 2.12. *Let $Z \in \mathscr{Z}'(S) \cap \mathscr{Z}(S)$. Then T_Z is similar to S if and only if A and B are equivalent* [FW], *i.e., there exist invertible operators $V_{11} \in \mathscr{L}(N(S))$ and $V_{22} \in \mathscr{L}(N(S)^\perp)$ such that $BV_{22} = V_{11}A$.*

Proof. If T_Z is similar to S, then from Proposition 2.4 there exists an invertible operator $V \in \mathscr{L}(\mathscr{H})$ such that $XV \in (S)'$; thus V has the form

$$\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$$

with $X_{22}V_{21} = 0$ and $X_{11}V_{11}A = AX_{22}V_{22}$. Then $V_{21} = 0$; moreover, since $Y_{11}A = BY_{22}$, it follows that $X_{11}BY_{22} = X_{11}Y_{11}A = AX_{22}Y_{22}$, whence $X_{11}B = AX_{22}$. Now $X_{11}BV_{22} = AX_{22}V_{22} = X_{11}V_{11}A$, so $BV_{22} = V_{11}A$. It remains to show that V_{11} and V_{22} are invertible. Since V is invertible and $V_{21} = 0$, it follows at once that V_{11} is left invertible and V_{22} is onto. Since A is injective and $V_{11}A = BV_{22}$, we see that V_{22} must be injective and hence invertible. If

$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$$

is the inverse of V, we have $V_{22}W_{21} = 0$ and $V_{11}W_{11} + V_{12}W_{21} = I$, and therefore $W_{21} = 0$ and V_{11} is onto (hence invertible), completing the "only if" part of the proof.

For the "if" part, let

$$V = \begin{pmatrix} V_{11} & 0 \\ 0 & V_{22} \end{pmatrix}.$$

A straightforward calculation shows that $XV \in (S)'$, so that $T_Z \sim S$ by Proposition 2.4. □

Remark 2.13. We would like to mention here that the results of this section and those in [NRRR] provide the following program for computing quasisimilarity orbits:

Step 1. Choose $Z \in \mathscr{Z}(S)$.

Step 2. Compute $\mathscr{H}_1$, X_1, Y_1, $\widetilde{T}_Z$.

Step 3. Polar decompose $X_1 : \mathscr{H}_1 \to \mathscr{H}$ as $X_1 = U|X_1|$, U an isometric isomorphism, $|X_1|$ a quasiaffinity.

Step 4. Assume X_1 (hence $|X_1|$) is *not* invertible, for otherwise $T_Z \sim S$ by Proposition 2.4 (take $J = X_1^{-1}$).

Step 5. To show that $\widetilde{T}_Z$ extends boundedly to $\mathscr{H}_1$ it suffices to prove that $SR(X_1) \subseteq R(X_1)$ (Proposition 2.7(iii)), or, equivalently, $U^*SUR(|X_1|) \subseteq R(|X_1|)$. In principle, this step can be checked by using the structure theorem for

$$\mathscr{A}(|X_1|) := \{T \in \mathscr{L}(\mathscr{H}) : TR(|X_1|) \subseteq R(|X_1|)\}$$

[NRRR], which is expressed in terms of the spectral measure of $|X_1|$. Unfortunately, computing the spectral measure for $|X_1|$ may be very difficult.

3. Noncommutative factorizations of quadratic functions of the shift

In this section we shall apply the results of §2 to carefully study certain noncommutative factorizations of a general quadratic function of $S = U_+$, where U_+ denotes the unilateral shift acting on $l^2(\mathbf{Z}_+)$. In particular, we shall explicitly exhibit a family of operators similar to the shift. Given $Z = (\alpha + S)(\beta + S)$ $(|\alpha| = |\beta| = 1)$, we shall obtain factorizations of Z by considering the following subdiagonal perturbations of the commutative factors $\alpha + S$ and $\beta + S$:

$$X := \alpha + S + S\Delta,$$
$$Y := \beta + S + S\Sigma,$$

where Δ and Σ are operators on $l^2(\mathbf{Z}_+)$ which are diagonal with respect to the basis that S shifts. To get $XY = (\alpha + S)(\beta + S)$, Δ and Σ must satisfy the equations

$$\begin{cases} \alpha\Sigma + \beta\Delta = 0, \\ S^2\Sigma + S\Delta S + S\Delta S\Sigma = 0. \end{cases}$$

In terms of the canonical basis $\{e_n\}_{n\geq 0}$, we must have

$$\begin{cases} \alpha\sigma_n + \beta\delta_n = 0 \quad \text{and} \\ \sigma_n + \delta_{n+1} + \sigma_n\delta_{n+1} = 0 \qquad (n \geq 0) \end{cases}$$

(here $\Delta = \operatorname{diag}(\delta_n)$ and $\Sigma = \operatorname{diag}(\sigma_n)$). Then

$$\alpha\delta_{n+1} - \beta\delta_n = \beta\delta_n\delta_{n+1},$$

or

$$\lambda\delta_{n+1} - \delta_n = \delta_n\delta_{n+1} \qquad (\lambda := \alpha/\beta). \tag{3.1}$$

Case 1. $\lambda = 1$.

Without loss of generality, assume $\alpha = \beta = 1$. Then

$$\delta_n = \delta/(1 - n\delta) \qquad (n \geq 0), \tag{3.2}$$

where $\delta := \delta_0$. (Note that from (3.1) it follows at once that $\delta \neq 1/n$ for all $n = 1, 2, \ldots$; also, to avoid the trivial case, we'll always assume that $\delta \neq 0$.) Therefore $\widetilde{T}$ is given by

$$\widetilde{T}(e_n + (1 - \delta_n)e_{n+1}) = e_{n+1} + (1 - \delta_{n+1})e_{n+2} \qquad (n \geq 0). \tag{3.3}$$

We shall prove first that $\widetilde{T}$ is bounded. If we multiply equality (3.3) by $(-1)^n \prod_{i=0}^{n-1}(1 - \delta_i)$ and then add from $n = 0$ to $n = N$ we get

$$\begin{aligned} \widetilde{T}\left[e_0 + (-1)^N \prod_{i=0}^{N}(1 - \delta_i)e_{N+1}\right] \\ = \sum_{n=0}^{N}(-1)^n \left[\prod_{i=0}^{n-1}(1 - \delta_i)e_{n+1} + (1 - \delta_{n+1})\prod_{i=0}^{n-1}(1 - \delta_i)e_{n+2}\right]. \end{aligned} \tag{3.4}$$

From (3.1) and (3.2) we can derive

$$\prod_{i=0}^{n-1}(1 - \delta_i) = \frac{\delta_0}{\delta_n} = 1 - n\delta, \tag{3.5}$$

so that (3.4) becomes

$$\tag{3.6} \begin{aligned} \widetilde{T}[e_0 + (-1)^N (1-(N+1)\delta) e_{N+1}] &= \sum_{n=0}^{N} (-1)^n [(1-n\delta)e_{n+1} + (1-\delta_{n+1})(1-n\delta)e_{n+2}] \\ &= e_1 + \sum_{n=0}^{N-1} (-1)^n [(1-\delta_{n+1})(1-n\delta) - (1-(n+1)\delta)] e_{n+2} \\ &\quad + (-1)^N (1-\delta_{N+1})(1-N\delta) e_{N+2}. \end{aligned}$$

Since

$$1-(n+1)\delta = 1 - n\delta - \delta = (1-n\delta)(1-\delta_n),$$

(3.6) becomes

$$\tag{3.7} \begin{aligned} \widetilde{T}[e_0 + (-1)^N (1-(N+1)\delta) e_{N+1}] &= e_1 + \sum_{n=0}^{N-1} (-1)^{n+1} \frac{\delta^2}{1-(n+1)\delta} e_{n+2} \\ &\quad + (-1)^N \frac{(1-(N+2)\delta)(1-N\delta)}{1-(N+1)\delta} e_{N+2}. \end{aligned}$$

Therefore,

$$\tag{3.8} \begin{aligned} \widetilde{T}\left(e_{N+1} + \frac{(-1)^N}{1-(N+1)\delta} e_0\right) &= \frac{(-1)^N}{1-(N+1)\delta} \left[e_1 + \sum_{n=0}^{N-1} (-1)^{n+1} \frac{\delta^2}{1-(n+1)\delta} e_{n+2}\right] \\ &\quad + \frac{(1-(N+2)\delta)(1-N\delta)}{[1-(N+1)\delta]^2} e_{N+2}. \end{aligned}$$

If we define $W : l^2(\mathbf{Z}_+) \to l^2(\mathbf{Z}_+)$ by

$$We_0 := 0,$$

and

$$\begin{aligned} We_{N+1} := \frac{(-1)^N}{1-(N+1)\delta} &\left[e_1 + \sum_{n=0}^{N-1} (-1)^{n+1} \frac{\delta^2}{1-(n+1)\delta} e_{n+2}\right] \\ &+ \frac{(1-(N+2)\delta)(1-N\delta)}{[1-(N+1)\delta]^2} e_{N+2} \qquad (N \geq 0), \end{aligned}$$

we see immediately that

(i) W is bounded,
(ii) $W - S$ is Hilbert-Schmidt,
(iii) $WY = YS$.

Thus, $\widetilde{T}$ extends boundedly to $T_\delta := W|_{\overline{R(Y)}}$. (We incorporate δ into the notation to remind ourselves of the dependence on the parameter δ.) For the reader's convenience we give below the matrix form of W when $\delta = -1$:

$$W = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & \frac{1}{2} & -\frac{1}{3} & \frac{1}{4} & -\frac{1}{5} & \cdot \\ 0 & \frac{3}{4} & \frac{1}{6} & -\frac{1}{8} & \frac{1}{10} & \cdot \\ 0 & 0 & \frac{8}{9} & \frac{1}{12} & -\frac{1}{15} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \end{pmatrix}.$$

The main difficulty in proving the boundedness of $\widetilde{T}$ directly from (3.8) lies in the fact that the collection $\{e_{N+1} + [(-1)^N/(1-(N+1)\delta)]e_0\}_{N\geq 0}$ is not an orthonormal basis for $R(Y)$, but rather a so-called Schauder basis (see [K, V.2.5], where such collections are called simply *bases*).

Five main questions now arise:

(1) Is T_δ similar to S?
(2) Is T_δ a weighted shift?
(3) Is T_δ unitarily equivalent to a compact perturbation of S?
(4) Is T_δ a contraction?
(5) Is T_δ subnormal? (Subnormal operators similar to S have been characterized in [Cla1].)

We can answer (3) easily by using the Brown-Douglas-Fillmore theory [BDF]. For, relative to $l^2(\mathbf{Z}_+) = \overline{R(Y)} \oplus N(Y^*)$, W has the matrix $\left(\begin{smallmatrix} T & * \\ 0 & * \end{smallmatrix}\right)$. Since W differs from S by a Hilbert-Schmidt operator, to prove that $T_\delta \cong S + K$, $K \in K(\mathscr{H})$, using [BDF], it suffices to prove that T_δ is essentially normal, that $\sigma_e(T_\delta) = \sigma_e(W)$, and that $\operatorname{ind}(T_\delta - \lambda) = \operatorname{ind}(W - \lambda)$ for all $\lambda \notin \sigma_e(T_\delta)$. All of these conclusions will follow immediately from the matrix form of W once we establish that $N(Y^*)$ is finite dimensional. We shall prove that $N(Y^*)$ is actually one-dimensional:

Let $x \in l^2(\mathbf{Z}_+)$ and assume that $Y^*x = 0$, i.e.,

$$\begin{pmatrix} 1 & 1-\overline{\delta}_0 & & & \\ 0 & 1 & 1-\overline{\delta}_1 & & \\ 0 & 0 & 1 & 1-\overline{\delta}_2 & \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix},$$

so that

$$\begin{cases} x_0 + (1-\overline{\delta}_0)x_1 & = 0, \\ \qquad x_1 + (1-\overline{\delta}_1)x_2 & = 0, \\ \qquad\qquad x_2 + (1-\overline{\delta}_1)x_3 & = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots & , \end{cases}$$

and therefore $x_0 + (-1)^n \prod_{i=0}^n (1-\overline{\delta}_i)x_{n+1} = 0$; we thus get

$$x_{n+1} = \frac{(-1)^{n+1}}{1-(n+1)\overline{\delta}} x_0$$

(by 3.5) $(n \geq 0)$, showing that the vector

$$\tilde{x} := \begin{pmatrix} 1 \\ -\frac{1}{1-\overline{\delta}} \\ \frac{1}{1-2\overline{\delta}} \\ \vdots \end{pmatrix},$$

generates $N(Y^*)$. As a consequence of this fact, we obtain $W^*\tilde{x} = 0$. For, $Y^*W^*\tilde{x} = S^*Y^*\tilde{x} = 0$, so that $W^*\tilde{x} = \alpha\tilde{x}$ for some $\alpha \in \mathbf{C}$. Since $\langle \tilde{x}, e_0\rangle \neq 0$ and $\langle W^*\tilde{x}, e_0\rangle = \langle \tilde{x}, We_0\rangle = 0$, we must have $\alpha = 0$.

We shall now proceed to answer (1) in the affirmative, by exhibiting an invertible operator $J: l^2(\mathbf{Z}_+) \to \overline{R(Y)}$ such that $XJ = I + S$. Since X is injective, such a J must be unique, if it exists. A matrix calculation shows that such a J must admit the matrix form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \\ -\delta_0 & 1 & 0 & 0 & \\ \delta_1 & -\delta_1 & 1 & 0 & \\ -\delta_2 & \delta_2 & -\delta_2 & 1 & \\ \delta_3 & -\delta_3 & \delta_3 & -\delta_3 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{3.9}$$

so our task is to show that (3.9) does define a bounded operator J and that J is one-to-one and $R(J) = \overline{R(Y)}$.

Boundedness. Consider the Cesàro-like map C given by

$$(Cx)_n = \delta_n \sum_{i=0}^n x_i \qquad (n \geq 0).$$

Then

$$\begin{aligned}
\|Cx\|_2^2 &= \sum_{n=0}^{\infty}\left|\sum_{i=0}^{n}\delta_n x_i\right|^2 \\
&\le \sum_{n=0}^{\infty}\left[\sum_{i=0}^{n}(|\delta_n|^{1/2}|\delta_i|^{1/4})\left(\frac{|\delta_n|^{1/2}}{|\delta_i|^{1/4}}|x_i|\right)\right]^2 \\
&\le \sum_{n=0}^{\infty}\sum_{i=0}^{n}|\delta_n|\,|\delta_i|^{1/2}\sum_{i=0}^{n}\frac{|\delta_n|}{|\delta_i|^{1/2}}|x_i|^2 \quad \text{(by Cauchy-Schwarz)} \\
&\le c\sum_{n=0}^{\infty}|\delta_n|^{1/2}\sum_{i=0}^{n}\frac{|\delta_n|}{|\delta_i|^{1/2}}|x_i|^2 \quad \text{(for some } c>0) \\
&= c\sum_{i=0}^{\infty}\frac{|x_i|^2}{|\delta_i|^{1/2}}\sum_{n=i}^{\infty}|\delta_n|^{3/2} \\
&\le c'\sum_{i=0}^{\infty}\frac{|x_i|^2}{|\delta_i|^{1/2}}\cdot|\delta_i|^{1/2} = c'\|x\|_2^2 \quad \text{(for some } c'>0).
\end{aligned}$$

(c and c' above appear after we apply the integral test for partial sums.) Thus, C is bounded. If we now let

$$(Ux)_n = (-1)^{n+1}x_n \qquad (n \ge 0),$$

one easily checks that (3.9) corresponds to $-SUCU + I$, so that (3.9) defines a bounded operator J.

Injectivity. Clear from (3.9).

Fredholmness. We shall establish that J is Fredholm by showing that $j := \pi(J)$ is right invertible in the Calkin algebra (here $\pi\colon \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the quotient map; also, we shall use lowercase letters for elements of the Calkin algebra). Consider

$$\begin{aligned}
jj^* &= (-sucu+1)(-uc^*us^*+1) \\
&= su(-cu+us^*)(-uc^*+su)us^* \\
&= su(cc^* - us^*uc^* - cusu + 1)us^*.
\end{aligned}$$

Now, a matrix calculation shows that

$$\begin{cases} usu = -s, \\ cs = c, \\ c + c^* = cc^*. \end{cases}$$

(Actually, $USU = -S$, $CS - C \in \mathscr{C}_p$ $(p > 1)$, and $C + C^* - CC^* \in \mathscr{C}_q$ $(q > 2)$. To show, for instance, the last assertion, observe that

$$C + C^* - CC^* = \begin{pmatrix} \delta_0 + \overline{\delta}_0 - |\delta_0|^2 & \overline{\delta}_1(1-\delta_0) & \overline{\delta}_2(1-\delta_0) & \cdots \\ \delta_1(1-\overline{\delta}_0) & \delta_1 + \overline{\delta}_1 - 2|\delta_1|^2 & \overline{\delta}_2(1-2\delta_1) & \cdots \\ \delta_2(1-\overline{\delta}_0) & \delta_2(1-2\overline{\delta}_1) & \delta_2 + \overline{\delta}_2 - 3|\delta_2|^2 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

The diagonal part has a general entry given by

$$\begin{aligned} \delta_k + \overline{\delta}_k - (k+1)|\delta_k|^2 &= \frac{\delta}{1-k\delta} + \frac{\overline{\delta}}{1-k\overline{\delta}} - \frac{(k+1)|\delta|^2}{|1-k\delta|^2} \\ &= \frac{\delta + \overline{\delta} - |\delta|^2(3k+1)}{|1-k\delta|^2} \sim \frac{1}{k}, \end{aligned}$$

and it therefore belongs to $\mathscr{C}_p$ $(p > 1)$. The lower-triangular portion L has a jth column whose q-norm is

$$|1-(j+1)\overline{\delta}_j| \left(\sum_{k=j+1}^{\infty} |\delta_k|^q \right)^{1/q},$$

and therefore

$$\begin{aligned} \|L\|_q^q &= \sum_{j=0}^{\infty} |1-(j+1)\overline{\delta}_j|^q \sum_{k=j+1}^{\infty} |\delta_k|^q \\ &= \sum_{k=1}^{\infty} |\delta_k|^q \sum_{j=0}^{k-1} \left| 1 - (j+1)\frac{\overline{\delta}}{1-j\overline{\delta}} \right|^q \\ &= \sum_{k=1}^{\infty} |\delta_k|^q \sum_{j=0}^{k-1} \left| \frac{1-(2j+1)\overline{\delta}}{1-j\overline{\delta}} \right|^q \\ &\le C \sum_{k=1}^{\infty} k|\delta_k|^q \\ &= C \sum_{k=1}^{\infty} k \left| \frac{\delta}{1-k\delta} \right|^q. \end{aligned}$$

Thus, $L \in \mathscr{C}_q$ $(q > 2)$.) Hence,

$$\begin{aligned} jj^* &= su(cc^* + s^*c^* + cs + 1)us^* \\ &= su(cc^* + c^* + c + 1)us^* \\ &= su(2cc^* + 1)us^*. \end{aligned}$$

Therefore j is right invertible so that J is Fredholm.

Ontoness. To see that $R(J) = \overline{R(Y)}$, we shall check that $N(J^*) = N(Y^*)$. If $x \in l^2(\mathbf{Z}_+)$ and $J^*x = 0$, then

$$x_n + \sum_{i=n}^{\infty}(-1)^{n+1}\overline{\delta}_i x_{i+1} = 0, \qquad (n \geq 0),$$

from which it follows that $x_n + (1-\overline{\delta}_n)x_{n+1} = 0 \ (n \geq 0)$, so that $N(J^*)$ is at most one-dimensional; moreover, a straightforward calculation shows that $N(Y^*) \subseteq N(J^*)$, and therefore $N(J^*) = N(Y^*)$.

Since $T_\delta \sim S$, it follows that $R(T_\delta)$ is closed and $\overline{R(Y)} \ominus R(T_\delta)$ is one-dimensional. Moreover, since $R(W) \supseteq R(T_\delta)$ and $\mathscr{H} \ominus \overline{R(Y)}$ is one-dimensional, then $\dim N(W^*) \leq 2$. Also, since $We_0 = 0$, then $\dim N(W) \geq 1$. Since $W - S$ is Hilbert-Schmidt, it follows that $-1 = \operatorname{ind}(S) = \operatorname{ind}(W) = \dim N(W) - \dim N(W^*)$, whence $\dim N(W) = 1$ and $\dim N(W^*) = 2$. From this we conclude that $R(W) = R(T_\delta)$. Also, both e_0 and $\tilde{x}$ belong to $N(W^*)$, so that $R(W) = \{\alpha e_0 + \beta\tilde{x} : \alpha, \beta \in \mathbf{C}\}^\perp$ and the range of T_δ can be characterized in terms of e_0 and $\tilde{x}$. Observe that merely from the definition of T_δ (or from that of W), it appears to be extremely difficult to prove (or even guess!) the above statements.

Next, we use the theory of generalized Bergman kernels [CuS] to prove that T_δ is *not* a weighted shift. Since $T_\delta \sim S$, one knows that $T_\delta^* \in B_1(\mathbf{D})$, the Cowen-Douglas class of the unit disk. Therefore we may compute a *section* of the bundle associated with T_δ^*, i.e., an analytic $\mathscr{H}_1$-valued function $x(\cdot)$ such that for each $\lambda \in \mathbf{D}$, $x(\lambda)$ generates $N(T_\delta^* - \lambda)$. The normalized *kernel function* $k := k_{T_\delta^*}$ may be computed from $x(\cdot)$ as follows: $k(\lambda, \mu) = \langle x(\lambda), x(\mu)\rangle / \langle x(0), x(\mu)\rangle$ $(\lambda, \mu \in \mathbf{D})$. k is *diagonal* if it admits a power series expansion $k(\lambda, \mu) = \sum_{n=0}^{\infty} c_n \lambda^n \overline{\mu}^n$ near the origin. If k is diagonal, then clearly $\partial k(0,0) := \partial k(\lambda, \mu)/\partial\lambda|_{(\lambda,\mu)=(0,0)} = 0$; using this observation, we shall show that $k_{T_\delta^*}$ is *not* diagonal and thereby conclude from [CuS, Theorem 5.4] that T_δ is *not* a weighted shift.

To compute a section $x(\cdot)$ for T_δ^*, let $y(\lambda)$ be an analytic $\mathscr{H}$-valued function such that

$$Y^* y(\lambda) = \begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \end{pmatrix} \qquad (\lambda \in \mathbf{D}). \tag{3.10}$$

Assuming such a $y(\cdot)$ has been found, then

$$\begin{aligned} Y^*(T_\delta^* - \lambda)P_{\overline{R(Y)}}y(\lambda) &= (S^* - \lambda)Y^* P_{\overline{R(Y)}}y(\lambda) \\ &= (S^* - \lambda)Y^* y(\lambda) = 0, \end{aligned}$$

so that $(T_\delta^* - \lambda)P_{\overline{R(Y)}}y(\lambda) = 0$ (recall that Y^* is one-to-one on $\mathscr{H}_1$), and thus $P_{R(Y)}y(\lambda)$ is a good candidate for $x(\lambda)$. Such a $y(\cdot)$ must satisfy the system of equations

$$(3.11)\qquad \begin{cases} y_0(\lambda) + (1-\overline{\delta}_0)y_1(\lambda) = 1, \\ y_1(\lambda) + (1-\overline{\delta}_1)y_2(\lambda) = \lambda, \\ \cdots\cdots ; \end{cases}$$

we can take $y_0 \equiv 1$, $y_1 \equiv 0$, and

$$(3.12)\quad y_{n+1}(\lambda) = \frac{(-1)^n(1-\overline{\delta})}{1-(n+1)\overline{\delta}} \sum_{i=1}^{n} (-1)^i[1-(i-1)\overline{\delta}]\lambda^i, \qquad (n \geq 1,\ \lambda \in \mathbf{D}).$$

Before we let $x(\lambda) := P_{\overline{R(Y)}}y(\lambda)$, we must make sure that $y(\lambda)$ is never orthogonal to $\overline{R(Y)}$, i.e., $Y^*y(\lambda) \neq 0$ $(\lambda \in \mathbf{D})$, which is true in view of (3.10). Since $P_{\overline{R(Y)}}y(\lambda) = y(\lambda) - \langle y(\lambda), \tilde{x}\rangle \tilde{x}/\|\tilde{x}\|_2^2$ $(\lambda \in \mathbf{D})$, where $\tilde{x}$ is the generator of $N(Y^*)$, we see that

$$k(\lambda,\mu) = \frac{\langle \|\tilde{x}\|^2 y(\lambda) - \langle y(\lambda),\tilde{x}\rangle\tilde{x}, \|\tilde{x}\|^2 y(\mu) - \langle y(\mu),\tilde{x}\rangle\tilde{x}\rangle}{\langle \|\tilde{x}\|^2 y(0) - \langle y(0),\tilde{x}\rangle\tilde{x}, \|\tilde{x}\|^2 y(\mu) - \langle y(\mu),\tilde{x}\rangle\tilde{x}\rangle} \qquad (\lambda,\mu \in \mathbf{D}).$$

To see that k is not diagonal, it suffices to prove that $\partial k(0,0) \neq 0$. Let $\phi(\lambda) := \langle y(\lambda), \tilde{x}\rangle$. Since

$$\begin{aligned} k(\lambda,\mu) &= \frac{\langle \|\tilde{x}\|^2 y(\lambda) - \langle y(\lambda),\tilde{x}\rangle\tilde{x}, \|\tilde{x}\|^2 y(\mu) - \langle y(\mu),\tilde{x}\rangle\tilde{x}\rangle}{\langle \|\tilde{x}\|^2 y(0) - \langle y(0),\tilde{x}\rangle\tilde{x}, \|\tilde{x}\|^2 y(\mu) - \langle y(\mu),\tilde{x}\rangle\tilde{x}\rangle} \\ &= \frac{\|\tilde{x}\|^4\langle y(\lambda), y(\mu)\rangle - \|\tilde{x}\|^2\overline{\varphi(\mu)}\varphi(\lambda)}{\langle \|\tilde{x}\|^2 y(0) - \varphi(0)\tilde{x}, \|\tilde{x}\|^2 y(\mu) - \varphi(\mu)\tilde{x}\rangle}, \end{aligned}$$

we obtain

$$\partial k(0,0) = \frac{\|\tilde{x}\|^4\langle y'(0), y(0)\rangle - \|\tilde{x}\|^2\overline{\varphi(0)}\varphi'(0)}{\langle \|\tilde{x}\|^2 y(0) - \varphi(0)\tilde{x}, \|\tilde{x}\|^2 y(0) - \varphi(0)\tilde{x}\rangle}.$$

Now $\langle y'(0), y(0)\rangle = 0$ and it therefore suffices to show that $\varphi'(0) \neq 0$ (observe that $\varphi(0) = 1 \neq 0$). From (3.12) we get

$$y'_{n+1}(0) = (-1)^{n+1}(1-\overline{\delta})/(1-(n+1)\overline{\delta}) \qquad (n \geq 1)$$

and therefore

$$\begin{aligned} \varphi'(0) = \langle y'(0), \tilde{x}\rangle &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(1-\overline{\delta})}{1-(n+1)\overline{\delta}} \cdot \frac{(-1)^{n+1}}{1-(n+1)\overline{\delta}} \\ &= (1-\overline{\delta})\sum_{n=1}^{\infty} \frac{1}{[1-(n+1)\overline{\delta}]^2} \neq 0. \end{aligned}$$

This completes the proof that T_δ is not a weighted shift.

We now answer questions (4) and (5) in the negative. We shall establish that $\|T_{-1}\| > 1$. Using the fact that T_{-1} is similar to the unilateral shift, this readily implies that T_{-1} cannot be *hyponormal* (for otherwise, $\|T_{-1}\| = r(T_{-1}) = r(S) = 1$ where $r(\cdot)$ denotes spectral radius).

T_{-1} is not a contraction.

Let $a_n = 1 - \delta_n = 1 + 1/(n+1) = (n+2)/(n+1)$ $(n \geq 0)$. Then

$$T_{-1}(e_n + a_n e_{n+1}) = e_{n+1} + a_{n+1}e_{n+2} \qquad (n \geq 0).$$

Let $x := \sum_{n=0}^{N} x_n e_n$ and consider $T_{-1}Yx$. Since $T_{-1}Y = YS = (I+S-S\Delta)S = S(I+S-\Delta S)$, and since S is isometric, we see that

$$\|T_{-1}Yx\|^2 = \|(I+S-\Delta S)x\|^2 = \left\|\sum_{n=0}^{N} x_n e_n + \sum_{n=0}^{N} x_n a_{n+1} e_{n+1}\right\|^2$$
$$= |x_0|^2 + \sum_{n=0}^{N-1} |x_{n+1} + a_{n+1}x_n|^2 + |a_{N+1}x_N|^2.$$

On the other hand,

$$\|Yx\|^2 = \|(I+S-S\Delta)x\|^2 = \left\|\sum_{n=0}^{N} x_n e_n + \sum_{n=0}^{N} x_n a_n e_{n+1}\right\|^2$$
$$= |x_0|^2 + \sum_{n=0}^{N-1} |x_{n+1} + a_n x_n|^2 + |a_N x_N|^2.$$

Consider A_N, B_N: $\mathbf{C}^{N+1} \to \mathbf{C}^{N+2}$ given by

$$A_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_0 & 1 & 0 & \cdots & 0 \\ 0 & a_1 & 1 & \cdots & 0 \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots a_{N-1} & 1 \\ 0 & 0 & 0 & \cdots 0 & a_N \end{pmatrix}, \quad B_N = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & \cdots & 0 \\ 0 & a_2 & 1 & \cdots & 0 \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots a_N & 1 \\ 0 & 0 & 0 & \cdots 0 & a_{N+1} \end{pmatrix}.$$

To show that $\|T_{-1}\| > 1$ it suffices to establish that $A_N^* A_N - B_N^* B_N$ is not positive for some $N \geq 0$. A direct calculation shows that

$$A_N^* A_N - B_N^* B_N = \begin{pmatrix} a_0^2 - a_1^2 & a_0 - a_1 & 0 & \cdots & 0 \\ a_0 - a_1 & a_1^2 - a_2^2 & a_1 - a_2 & \cdots & 0 \\ 0 & a_1 - a_2 & a_2^2 - a_3^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & a_N^2 - a_{N+1}^2 \end{pmatrix}.$$

If we let

$$D_k := a_k^2 - a_{k+1}^2 = (2k^2 + 8k + 7)/(k+1)^2(k+2)^2$$

and $d_k := a_k - a_{k+1} = 1/(k+1)(k+2)$, then

$$A_N^* A_N - B_N^* B_N = \begin{pmatrix} D_0 & d_0 & 0 & \cdots & 0 \\ d_0 & D_1 & d_1 & \cdots & 0 \\ 0 & d_1 & D_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & D_N \end{pmatrix}.$$

Using Choleski's algorithm [Atk, §8.3], $A_N^* A_N - B_N^* B_N = LU$, where L is lower triangular and U is upper triangular; moreover, if $A_N^* A_N - B_N^* B_N$ is positive, $U = L^t$, the transpose of L, and the diagonal entries of L are

$$\sqrt{D_0}, \ \sqrt{D_1 - d_0^2/D_0}, \ \sqrt{D_2 - \frac{d_1^2}{D_1 - d_0^2/D_0}},$$

etc., where each of the arguments of $\sqrt{\ }$ is nonnegative. To show that $A_N^* A_N - B_N^* B_N$ is not positive for some $N \geq 0$, it thus suffices to find N such that

$$D_N - \frac{d_{N-1}^2}{D_{N-1} - d_{N-2}^2/D_{N-2} - \cdots}$$

is negative. Let $E_0 = D_0 = \frac{7}{4}$ and $E_k = D_k - d_{k-1}^2/E_{k-1}$, $k \geq 1$. By using the special form of D_k and d_k, we can write a double-precision floating-point arithmetic computer program to calculate E_k for various values of $k \geq 0$. The program output shows $E_k > 0$ for $0 \leq k < 35$ and $E_{35} = -6.30584386 \times 10^{-4} < 0$. However, this calculation of E_k involves error-generating computations, and a priori we cannot guarantee that the negativity of E_{35} is not due to error buildup. We have, therefore, designed a rational arithmetic Pascal program subject to the following recursive formulae:

$$\begin{aligned} E_k &= p_k/q_k, \qquad p_k, q_k \in \mathbf{Z}, \\ p_0 &= 7, \qquad q_0 = 4, \\ p_{k+1} &= (2k^2 + 12k + 17)(k+1)^2 p_k - (k+3)^2 q_k, \\ q_{k+1} &= (k+1)^2(k+2)^2(k+3)^2 p_k. \end{aligned}$$

(Although p_k and q_k are integers now, the direct use of the computer's fixed-point arithmetic is ruled out by the fact that the sizes of p_k and q_k are quite large; for instance, p_{10}, q_{22}, and p_{31} have 32, 93, and 142 decimal digits, respectively. Therefore, an "infinite-arithmetic" routine to handle 200-digit integers was written.) Surprisingly enough, p_{35} was the first negative value detected, and when the first significant digits were used, the value of E_{35} agreed with the originally calculated value (using the double-precision floating-point arithmetic) to six significant places. We thus concluded that $E_{35} < 0$ and, moreover, that the original routine was fairly accurate. We also mention that in the original routine, $E_k > 0$ for $0 \leq k \leq 2000$, $k \neq 35$, $k \neq 1410$.

Case 2. $\lambda \neq 1$.

We shall see that $\tilde{T}$ does *not* extend boundedly to $\overline{R(Y)}$, so we do not even reach the *quasisimilarity* orbit of S. From (3.1) we get

$$\delta_{n+1} = \delta_n/(\lambda - \delta_n) \tag{3.12}$$

and therefore

$$\delta_n = \frac{\delta}{\lambda^n - \delta(\lambda^n - 1)/(\lambda - 1)} \qquad (n \geq 0) \tag{3.13}$$

(where $\delta := \delta_0$ again). To get Δ bounded, we must exclude certain values of δ as follows:

(a) If λ is a root of unity, say $\lambda^K = 1$ (K minimal), then

$$\delta \notin \{\lambda^n(\lambda - 1)/(\lambda^n - 1) \colon n = 1, \dots, K-1\}.$$

(b) If λ is not a root of unity, so that $\{\lambda^n \colon n \geq 0\}$ is dense in the unit circle $\mathbf{T}$, then $\delta \notin \{(\lambda-1)\gamma/(\gamma-1) \colon |\gamma| = 1, \gamma \neq 1\}$ (the latter set being the straight line through $\frac{1}{2}(\lambda-1)$ and $\frac{1}{2}(\lambda-1)(1-i)$).

Since $|\lambda| = 1$, we can find a sequence $n_1 < n_2 < \cdots$ of positive integers such that $\lambda^{n_k} \to 1$ ($k \to \infty$); for this sequence,

$$\delta_{n_k} \to \delta \qquad (k \to \infty). \tag{3.14}$$

A look at (3.3) and the definition of Y reveals that in this case we must multiply (3.3) by $((-1)^n/\beta^n)\prod_{i=0}^{n-1}(1-\delta_i/\lambda)$ and then add from $n = 0$ to $n = N$ to get

$$\tilde{T}\left[\beta e_0 + \frac{(-1)^N}{\beta^N}\prod_{i=0}^{N}\left(1 - \frac{\delta_i}{\lambda}\right)e_{N+1}\right] = \sum_{n=0}^{N}\frac{(-1)^n}{\beta^n}\left[\prod_{i=0}^{n-1}\left(1-\frac{\delta_i}{\lambda}\right)e_{n+1} + (1-\delta_{n+1})\prod_{i=0}^{n-1}(1-\delta_i)e_{n+2}\right]. \tag{3.15}$$

In this case, (3.5) gets replaced by

$$\prod_{i=0}^{n-1}(\lambda - \delta_i) = \frac{\delta}{\delta_n}, \tag{3.16}$$

so that (3.15) becomes

$$\begin{aligned}\tilde{T}\left[\beta e_0 + \frac{(-1)^N}{\beta^N}\frac{\delta}{\delta_{N+1}\lambda^{N+1}}e_{N+1}\right] &= e_1 + \sum_{n=0}^{N-1}\frac{(-1)^n}{\beta^n}\left[(1-\delta_{n+1})\frac{\delta}{\delta_n\lambda^n} - \frac{\delta}{\delta_{n+1}\lambda^{n+1}}\right]e_{n+2} \\ &\quad + \frac{(-1)^N}{\beta^N}\frac{\delta}{\delta_N\lambda^N}e_{N+2}.\end{aligned} \tag{3.17}$$

Observe that

$$|\delta_{N+1}|^{-1} \leq \frac{|\lambda^{N+1}| + |\delta|(|\lambda^{N+1}|+1)/|\lambda-1|}{|\delta|} = \frac{1 + 2|\delta|/|\lambda-1|}{|\delta|},$$

so that the collection $\{\beta e_0 + ((-1)^N/\beta^N)(\delta/\delta_{N+1}\lambda^{N+1})e_{N+1}\}_{N\geq 0}$ is bounded. The coefficient of e_{n_k+2} in the right-hand-side sum is close to

$$\frac{(-1)^{n_k+1}}{\beta^{n_k+1}}\left[\left(1 - \frac{\delta}{\lambda-\delta}\right)\frac{\delta}{\delta\lambda^{n_k}} - \frac{\delta}{(\delta/(\lambda-\delta))\cdot\lambda^{n_k+1}}\right]$$

(by (3.14) and (3.12)), so that it approximately contributes $|\delta^2/(\lambda-\delta)|^2$ to the norm of the image vector (whenever $N \gg n_k$). It follows that the collection

$$\left\{\tilde{T}\left[\beta e_0 + \frac{(-1)^N}{\beta^N}\frac{\delta}{\delta_{N+1}\lambda^{N+1}}e_{N+1}\right]\right\}_{N\geq 0}$$

is *not* bounded, so that $\tilde{T}$ is unbounded.

We shall summarize the results in this section as follows.

Theorem 3.1. *Let S be the unilateral shift acting on $l^2(\mathbf{Z}_+)$ and let X and Y be the quasiaffinities given by*

$$X := \alpha + S + S\Delta,$$
$$Y := \beta + S + S\Sigma,$$

with $XY = (\alpha+S)(\beta+S)$ ($|\alpha| = |\beta| = 1$). Let $\delta := \delta_0$, the 0th diagonal entry of Δ, and let $\tilde{T}_\delta$ be the linear transformation on $R(Y)$ associated to the pair $(X, Y) \in \mathscr{Z}_0(S)$ (i.e., $\tilde{T}_\delta Yx = YSx$, $x \in l^2(\mathbf{Z}_+)$). Then

(a) *If $\lambda := \alpha/\beta = 1$ we have*
  (i) *$(X, Y) \in \mathscr{Z}(S)$ and the extension T_δ is similar to S.*
  (ii) *T_δ is unitarily equivalent to a compact perturbation of S.*
  (iii) *T_δ is not a weighted shift.*
  (iv) *$\|T_{-1}\| > 1$.*
  (v) *T_{-1} is not hyponormal.*

(b) *If $\lambda \neq 1$, then $\tilde{T}_\delta$ does not have a bounded extension to $\overline{R(Y)}$.*

Remark 3.2. By working with explicit factorizations of quadratic functions of the shift, we have illustrated how difficult it is to fully analyze the similarity and quasisimilarity orbits of a given operator. The calculations here also suggest that it may be possible to exhibit an operator quasisimilar, but not similar, to the shift, by looking at noncommutative factorizations of $R := I + U_+$. Other candidates for R may be operators such that $U_+ \in W(R)$ (recall Corollaries 2.8 and 2.9 and the fact that $(U_+)'$ is abelian [Sh]).

4. Factorization, spectra and essential spectra

In this section we address the variation of spectra and essential spectra relative to the factorization technique. Throughout, for $S \in \mathscr{L}(\mathscr{H})$ and $Z \in \mathscr{Z}(S)$ we let X, Y, X_1, Y_1, $\mathscr{H}_1$, and T_Z be as in §2. Thus, $R(X_1)$ is S-invariant and $T = T_Z$ is the *unique* solution of the operator equation $SX_1 = X_1T$ ($T \in \mathscr{L}(\mathscr{H}_1)$).

Several results in the literature assert that if S is in a prescribed class, then every $T \sim_{qs} S$ satisfies $\sigma(S) \subseteq \sigma(T)$; such is the case if S is decomposable [Fia1] or hyponormal [Cla2] (cf. [Con1]). We wish to give a criterion which insures that $\sigma(S) \supseteq \sigma(T)$ for every $T \in (S)_{qs}$. We require the following refinement of Theorem 2.6.

Theorem 4.1 [FW]. *Let $A, B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$. There exists an* <u>*invertible*</u> *operator $C \in \mathcal{L}(\mathcal{H}_1)$ such that $A = BC$ if and only if A and B have the same range and nullity.* (*For completeness, we recall from* [FW, *Theorem* 3.4] *that the condition that A and B have the same nullity and* <u>*similar*</u> *ranges corresponds to the* <u>*equivalence*</u> *of A and B.*)

Proposition 4.2. *Let $S \in \mathcal{L}(\mathcal{H})$, $T \in \mathcal{L}(\mathcal{H}_1)$, and suppose that $X \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ is an injective operator such that $SX = XT$. Then*

$$\operatorname{nul}(S) = 0 \Rightarrow (T \textit{ invertible } \Leftrightarrow SR(X) = R(X)).$$

Proof. Recall that T is the unique solution of the operator equation $SX = XC$ $(C \in \mathcal{L}(\mathcal{H}_1))$. Now, $\operatorname{nul}(SX) = \operatorname{nul}(X) = 0$, so by Theorem 4.1, T is invertible if and only if $R(SX) = R(X)$, or $SR(X) = R(X)$. □

In the sequel we will have several occasions to use the following elementary fact: If $T \in \mathcal{L}(\mathcal{H})$ is invertible and $\mathcal{M} \subseteq \mathcal{H}$ is a linear manifold that is invariant for T and T^{-1}, then $T\mathcal{M} = \mathcal{M}$.

Examples 4.3. (a) Suppose S is an algebraic operator with minimal polynomial p_S, and let X be an injective operator such that $SX = XT$, for some T. Clearly $p(S)X = Xp(T)$ for every polynomial p, so T is also algebraic and p_T divides p_S. Since the spectrum of an algebraic operator consists of the roots of its minimal polynomial [Her6], we have $\sigma(T) \subseteq \sigma(S)$. Another way to see this is to use Proposition 4.2: Since S is algebraic, if $\lambda \notin \sigma(S)$ then $(S-\lambda)^{-1} = p(S)$ for some polynomial p, and thus $(S-\lambda)R(X) = R(X)$. Thus $T - \lambda$ is invertible by Proposition 4.2.

(b) We wish to further illustrate spectral inclusion via Proposition 4.2. Let $0 < r < 1$ and let X denote the diagonal operator defined by $Xe_n = r^n e_n$ $(n \geq 0)$, where $\{e_n\}_{n \geq 0}$ is an orthonormal basis for $\mathcal{H}$. Let S be an operator whose matrix relative to this basis is upper triangular, i.e., $(Se_j, e_i) = 0$ whenever $i > j$. Theorem 3 in [NRRR] implies that $R(X)$ is S-invariant, so by Douglas's theorem, $SX = XT$ for a unique $T \in \mathcal{L}(\mathcal{H})$. We claim that $\sigma(T) \subseteq \sigma(S)$. For, if $\lambda \notin \sigma(S)$, then both $S - \lambda$ and $(S-\lambda)^{-1}$ are upper triangular and thus leave $R(X)$ invariant by [NRRR]. Therefore $(S-\lambda)R(X) = R(X)$ and Proposition 4.2 now implies that $T - \lambda$ is invertible; thus $\sigma(T) \subseteq \sigma(S)$. This inclusion can be proper: if $r = \frac{1}{2}$ and $S = U_+^*$, then $T = \frac{1}{2}U_+^*$.

Corollary 4.4. *Let $S \in \mathcal{L}(\mathcal{H})$, let $Z = (X, Y) \in \mathcal{Z}(S)$, and assume that S is injective. Then T_Z is invertible if and only if $SR(X_1) = R(X_1)$.*

Consider the following property for an operator $S \in \mathcal{L}(\mathcal{H})$:

(*) For every $Z = (X, Y) \in Z(S)$ and every $\lambda \notin \sigma(S)$,
$(S-\lambda)R(X_1) = R(X_1)$.

Corollary 4.5. *If $S \in \mathcal{L}(\mathcal{H})$ possesses (*) and $T \sim_{\mathrm{qs}} S$ then $\sigma(T) \subseteq \sigma(S)$.*

Condition (*) is satisfied (trivially) by the operators of Example 2.5. We have also just seen that algebraic operators satisfy (*). Thus quasisimilar algebraic operators have equal spectra, though they need not be similar (Theorem 1.1). Question 1.5 may be reformulated as follows: Does U_+ satisfy condition $(*)$?

We next consider the inclusion $\sigma_e(T) \subseteq \sigma_e(S)$ for T quasisimilar to S. By analogy with Douglas's theorem (Theorem 2.6) and the Fillmore-Williams result (Theorem 4.1), we prove the following characterization of the existence of Fredholm factors.

Theorem 4.6. *Let $A, B \in \mathscr{L}(\mathscr{H}_1, \mathscr{H})$. There exists a Fredholm operator $F \in \mathscr{L}(\mathscr{H}_1)$ such that $A = BF$ if and only if the following conditions hold*:

(1) $R(A) \subseteq R(B)$;

(2) $\dim(R(B)/R(A)) < \infty$; *and*

(3) $\mathrm{nul}(A) < \infty \Leftrightarrow \mathrm{nul}(B) < \infty$.

Proof. Assume that $A = BF$ for some Fredholm operator F. Since $FN(A) = N(B) \cap R(F)$, it follows at once that $\mathrm{nul}(A) < \infty \Leftrightarrow \mathrm{nul}(B) < \infty$. Also, $R(A) \subseteq R(B)$, so we must finally verify that $R(B)/R(A)$ is finite dimensional. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \to & R(F) & \to & \mathscr{H} & \to & \mathscr{H}/R(F) & \to & 0 \\ & & \downarrow B & & \downarrow B & & \downarrow [B] & & \\ 0 & \to & R(A) & \to & R(B) & \to & R(B)/R(A) & \to & 0; \end{array}$$

since B maps $\mathscr{H}$ onto $R(B)$, it follows that $[B] : \mathscr{H}/R(F) \to R(B)/R(A)$ is surjective, so that $\dim(R(B)/R(A)) \le \dim(\mathscr{H}/R(F)) < \infty$, as desired.

Conversely, assume (1), (2), and (3) hold. Let $A_1 : A|_{\mathrm{init}(A)}$ and $B_1 := B|_{\mathrm{init}(B)}$. (For an operator X, $\mathrm{init}(X)$ denotes the initial space of X, i.e., $N(X)^\perp$.) Since $R(A_1) = R(A) \subseteq R(B) = R(B_1)$, an obvious variant of the proof of Theorem 2.6 gives a bounded operator F_1: $\mathrm{init}(A) \to \mathrm{init}(B)$ such that $A_1 = B_1F_1$. Clearly F_1 is injective, and from the diagram

$$\begin{array}{ccccccccc} 0 & \to & R(F_1) & \to & \mathrm{init}(B) & \to & \mathrm{init}(B)/R(F_1) & \to & 0 \\ & & \downarrow B_1 & & \downarrow B_1 & & \downarrow [B_1] & & \\ 0 & \to & R(A) & \to & R(B) & \to & R(B)/R(A) & \to & 0 \end{array}$$

it follows at once that $[B_1]$ establishes a linear isomorphism between $\mathrm{init}(B)/R(F_1)$ and $R(B)/R(A)$, and F_1 is therefore Fredholm. Moreover, (3) implies that there exists a Fredholm operator F_2: $N(A) \to N(B)$ (F_2 may be finite rank). Therefore, $F := F_1 \oplus F_2$ (relative to the decompositions $\mathscr{H}_1 = \mathrm{init}(A) \oplus N(A)$ and $\mathscr{H}_1 = \mathrm{init}(B) \oplus N(B)$) is Fredholm and satisfies $A = BF$, completing the proof. □

As an application of Theorem 4.6 we get the following result.

Proposition 4.7. *Let $S \in \mathscr{L}(\mathscr{H})$, $T \in \mathscr{L}(\mathscr{H}_1)$, and suppose $X \in \mathscr{L}(\mathscr{H}_1, \mathscr{H})$ satisfies $SX = XT$ and* $\mathrm{nul}(X) < \infty$. *If* $\mathrm{nul}(S) < \infty$ *then T is Fredholm if and only if* $\dim(R(X)/SR(X)) < \infty$.

Proof. The necessity clearly follows from Theorem 4.6. For the sufficiency part, observe that $R(SX) \subseteq R(X)$ and that $\mathrm{nul}(SX) \le \mathrm{nul}(S) + \mathrm{nul}(X) < \infty$.

Thus, if $\dim(R(X)/SR(X)) < \infty$, an application of Theorem 4.6 shows that $SX = XF$ for some Fredholm operator F. Then $X(F - T) = 0$ implies that T is a finite rank perturbation of F, so T is Fredholm. □

Example 4.8. If S is algebraic, $\operatorname{nul}(X) < \infty$, and $SX = XT$ then $\sigma_e(T) \subseteq \sigma_e(S)$. One proof of this fact would rely on standard results about algebraic operators and Riesz projections. We sketch an alternate proof illustrating the use of Proposition 4.7. Assume S is Fredholm. Then T will be Fredholm if we can verify that $R(X)/SR(X)$ is finite dimensional. Let L be such that $SL = I + K_1$, where K_1 is finite rank. Since S is algebraic, there exist a polynomial q and a finite rank operator K_2 such that $L = q(S) + K_2$; thus $SL = Sq(S) + K_3$ (K_3 of finite rank). Now $SXq(T) = XTq(T) = Sq(S)X = (SL - K_3)X = (I + K_1 - K_3)X$, so that $X = SXq(T) + K_4$ (K_4 of finite rank), which clearly shows that $R(X)/SR(X)$ is indeed finite dimensional. As an application of this example, we see that quasisimilar algebraic operators have identical essential spectra.

We shall now explain the obstruction to "$\sigma_e(T) \subseteq \sigma_e(S)$" for $T \sim_{\mathrm{qs}} S$. Although part of the proof of the following proposition can be derived from Theorem 4.6, we shall present the argument in its entirety so the reader can visualize the simplicity of the obstruction.

Proposition 4.9. *Let* $S \in \mathscr{L}(\mathscr{H})$ *and let* $Z = (X, Y) \in \mathscr{Z}(S)$. *Then*

(i) $\operatorname{nul}(S) \le \operatorname{nul}(\widetilde{T}_Z) = \operatorname{nul}(T_Z)$;

(ii) S *invertible* $\Rightarrow T_Z$ *invertible* $\Leftrightarrow$ ($\widetilde{T}_Z$ *bijective* $\Rightarrow T_Z$ *invertible*);

(iii) T_Z *invertible* $\Rightarrow S$ *invertible* $\Leftrightarrow$ (T_Z *invertible* $\Rightarrow \widetilde{T}_Z$ *bijective*);

(iv) S *Fredholm* $\Rightarrow T_Z$ *Fredholm* $\Leftrightarrow$ ($\widetilde{T}_Z$ *algebraically Fredholm* $\Rightarrow T_Z$ *Fredholm*);

(v) T_Z *Fredholm* $\Rightarrow S$ *Fredholm* $\Leftrightarrow$ (T_Z *Fredholm* $\Rightarrow \widetilde{T}_Z$ *algebraically Fredholm*).

(*A linear transformation* $A: \mathscr{V} \to \mathscr{W}$ *between two vector spaces is algebraically Fredholm if* $N(A)$ *and* $\mathscr{W}/A(\mathscr{V})$ *are both finite dimensional.*)

Proof. Consider the following diagram:

$$\begin{array}{ccccccccccc}
0 & \to & N(S) & \to & \mathscr{H} & \xrightarrow{S} & \mathscr{H} & \to & \mathscr{H}/R(S) & \to & 0 \\
 & & \downarrow Y & & \downarrow Y & & \downarrow Y & & \vdots\, [Y] & & \\
0 & \to & N(\widetilde{T}_Z) & \to & R(Y) & \xrightarrow{\widetilde{T}_Z} & R(Y) & \to & R(Y)/\widetilde{T}_Z R(Y) & \to & 0 \\
 & & \downarrow i & & \downarrow i & & \downarrow i & & \vdots\, [i] & & \\
0 & \to & N(T_Z) & \to & \mathscr{H}_1 & \xrightarrow{T_Z} & \mathscr{H}_1 & \to & \mathscr{H}_1/R(T_Z) & \to & 0 \\
 & & \downarrow X_1 & & \downarrow X_1 & & \downarrow X_1 & & \vdots\, [X_1] & & \\
0 & \to & N(S) \cap R(X_1) & \to & R(X_1) & \xrightarrow{S} & R(X_1) & \to & R(X_1)/SR(X_1) & \to & 0 \\
 & & \downarrow i & & \downarrow i & & \downarrow i & & \vdots\, [i] & & \\
0 & \to & N(S) & \to & \mathscr{H} & \xrightarrow{S} & \mathscr{H} & \to & \mathscr{H}/R(S) & \to & 0
\end{array}$$

(i denotes inclusion).

(i) Looking at the first column of maps,

$$\operatorname{nul}(S) = \operatorname{nul}(\widetilde{T}_Z) \leq \operatorname{nul}(T_Z) = \operatorname{nul}(S|_{R(X_1)}) \leq \operatorname{nul}(S),$$

and therefore $\operatorname{nul}(S) = \operatorname{nul}(\widetilde{T}_Z) = \operatorname{nul}(T_Z)$.

(ii) ($\Rightarrow$) If $\widetilde{T}_Z$ is bijective, then $N(\widetilde{T}_Z) = 0$ and $R(Y)/\widetilde{T}_Z R(Y) = 0$, so $N(S) = 0$ and $\mathscr{H}/R(S) = 0$, whence S is invertible. By hypothesis, it follows that T_Z is invertible. ($\Leftarrow$) If S is invertible, the diagram shows that $\widetilde{T}_Z$ is bijective, and by hypothesis this implies that T_Z is invertible.

(iv) Imitate the proof of (ii).

(iii) and (v) Imitate the proof of (ii) using rows 3 and 4 of the diagram.

Remark 4.10. (i) By Proposition 4.9, spectral inclusions between T_Z and S can be checked by comparing corresponding spectral properties for T_Z and $\widetilde{T}_Z$. For instance, $\sigma(T_Z) \subseteq \sigma(S) \Leftrightarrow [\widetilde{T}_Z - \lambda$ bijective $\Rightarrow T_Z - \lambda$ invertible] (all $\lambda \in \mathbf{C}$) (observe that $\widetilde{T}_Z - \lambda = \widetilde{T}_W$ for some $W \in \mathscr{Z}(S - \lambda)$).

(ii) As we showed in Proposition 4.9, the obstruction to "$\sigma_e(T) \subseteq \sigma_e(S)$" is measured by the implication "$\widetilde{T}_Z$ algebraically Fredholm $\Rightarrow T_Z$ Fredholm." Since $\operatorname{nul}(\widetilde{T}_Z) = \operatorname{nul}(T_Z)$, the distinction between the Fredholmness of S and that of T is actually given by whether $\dim(R(Y)/T_Z R(Y)) < \infty$ implies $\dim(\overline{R(Y)}/T_Z\overline{R(Y)}) < \infty$. This delicate point (Fredholmness preserved under closure of the domain) is at the heart of many of our calculations in §3 and underlines the reason why spectral inclusion results of the above type are so hard to prove.

Consider now the following property for $S \in \mathscr{L}(\mathscr{H})$:

(**) For every $Z = (X, Y) \in \mathscr{Z}(S)$ and every $\lambda \notin \sigma_e(S)$, $\dim(R(X_1)/(S-\lambda)R(X_1)) < \infty$.

Corollary 4.11. *Let $S \in \mathscr{L}(\mathscr{H})$ possess property* (**). *Then $\sigma_e(T) \subseteq \sigma_e(S)$ for every $T \sim_{\text{qs}} S$.*

Proof. From the diagram used in the proof of Proposition 4.9 we get at once that $\dim(\mathscr{H}_1/R(T-\lambda)) = \dim(R(X_1)/(S-\lambda)R(X_1)) < \infty$, for every $\lambda \notin \sigma_e(S)$. Also, $\operatorname{nul}(T-\lambda) = \operatorname{nul}(S-\lambda) < \infty$ for $\lambda \notin \sigma_e(S)$. Therefore, $T-\lambda$ is Fredholm for every $\lambda \notin \sigma_e(S)$, or $\sigma_e(T) \subseteq \sigma_e(S)$. □

Remark 4.12. Condition (**) above will be satisfied if S has the property:

(***) For each S-invariant operator range $\mathscr{M}$ and every $\lambda \notin \sigma_e(S)$, $\dim(\mathscr{M}/(S-\lambda)\mathscr{M}) < \infty$.

Example 4.13. If $S = \lambda + F$, where λ is a nonzero scalar and F is a finite rank operator, then $\dim(\mathscr{M}/S\mathscr{M}) < \infty$ for every S-invariant linear manifold $\mathscr{M}$. For, suppose instead that $\{m_k + S\mathscr{M}\}_{k=1}^{\infty}$ is a linearly independent set in $\mathscr{M}/S\mathscr{M}$, and let $\mathscr{N}$ denote the linear span of $\{m_k\}_{k=1}^{\infty}$. Then $\mathscr{N} \cap N(F) =$

$\mathscr{N} \cap (R(F^*))^\perp$ is infinite dimensional, so there exists a nonzero vector $y \in \mathscr{N} \cap N(F)$. Write $y = \sum_{i=1}^K c_i m_i$. Now

$$(\lambda + F)((1/\lambda)y) = y,$$

so that $y \in S\mathscr{M}$ and therefore $\sum_{i=1}^K c_i(m_i + S\mathscr{M}) = S\mathscr{M}$, a contradiction. As it turns out, $(S)_{qs} = \mathscr{S}(S)$ in this case: Indeed, if $T \sim_{qs} S$, then $TY_1 = Y_1(\lambda+F)$ for some quasiaffinity Y_1. If $\widetilde{G}$ is defined on $R(Y_1)$ by $\widetilde{G}Y_1 := Y_1 F$, then $\widetilde{G}$ is finite rank and hence extends to a finite rank operator G on $\mathscr{H}$. Thus $TY_1 = Y_1(\lambda + F) = \lambda Y_1 + GY_1 = (\lambda + G)Y_1$, whence $T = \lambda + G$. Therefore, $G \sim_{qs} F$ and hence $\operatorname{rank} p(G) = \operatorname{rank} p(F)$ for every polynomial p. It follows from [Her6, Corollary 2.8] that $G \sim F$, whence $T \in \mathscr{S}(S)$. We conjecture that operators of the form $\lambda + F$ ($\lambda \in \mathbf{C}$, F of finite rank) are the only operators satisfying $(***)$.

The following example was kindly supplied to us by H. Salas.

Example 4.14. The unilateral shift S does not satisfy $(***)$. for, let $Xe_n := \alpha_n e_n$ $(n \geq 0)$, where $\alpha_0 = 1$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = 1$, $\alpha_3 = \frac{1}{2}$, $\alpha_4 = \frac{1}{4}$, $\alpha_5 = 1$, $\alpha_6 = \frac{1}{2}$, $\alpha_7 = \frac{1}{4}$, $\alpha_8 = \frac{1}{8}$, $\alpha_9 = 1, \ldots$. The range of X is S-invariant because $\alpha_n/\alpha_{n+1} \leq 2$ $(n \geq 0)$. Let $A := \{k : \alpha_k/\alpha_{k+1} < 1\}$ and partition A as $\bigcup_{j=1}^\infty A_j$, where each A_j is infinite. The vector $y_j := \sum_{k\in A_j} a_k e_{k+1} \in R(X)$ because $\alpha_{k+1} = 1$ if $k \in A_j$ (so that $Xy_j = y_j$) $(j = 1, 2, \ldots)$. Moreover, the family $\{y_j + SR(X)\}_{j=1}^\infty$ is linearly independent: for, if

$$\sum_{j=1}^M c_j y_j = SX\left(\sum_{n=0}^\infty g_n e_n\right).$$

we must have

$$\sum_{j=1}^M c_j \sum_{k\in A_j} \alpha_k e_{k+1} = \sum_{j=1}^\infty \sum_{k\in A_j} \alpha_k g_k e_{k+1};$$

if $c_{j_0} \neq 0$ then

$$c_{j_0} \sum_{k\in A_j} \alpha_k e_{k+1} = \sum_{k\in A_j} \alpha_k g_k e_{k+1},$$

so that $g_k = c_{j_0}$ for all $k \in A_{j_0}$, a contradiction. Thus $\mathscr{M} := R(X)$ is S-invariant and $\dim \mathscr{M}/S\mathscr{M} = \infty$. Observe that X is a quasiaffinity but $XY \in (S)'$ only when $Y = 0$, so $(**)$ might still be true for S.

Question 4.15. Does the unilateral shift satisfy $(**)$?

Remark 4.16. It is quite easy to see that Theorem 4.6 extends to the case when $A: \mathscr{H}_1 \to \mathscr{H}_3$, $B: \mathscr{H}_2 \to \mathscr{H}_3$, and $F: \mathscr{H}_1 \to \mathscr{H}_2$. Several analogues of Douglas's theorem for operators on Banach spaces are known [B, E].

Question 4.17. What is the analogue of Theorem 4.6 for Banach space operators?

Observe that the proof of the necessity of (1)–(3) in Theorem 4.6 works well in the Banach space case. The proof of sufficiency makes use of the Hilbert space structure in two ways: when we consider $\operatorname{init}(A)$ and $\operatorname{init}(B)$ (for which, in the Banach space case, one could assume that $N(A)$ and $N(B)$ are complemented subspaces), and when we postulate the existence of a Fredholm operator from $N(A)$ to $N(B)$.

Let $\mathscr{Q}(\mathscr{H}) := \mathscr{L}(\mathscr{H})/\mathscr{K}(\mathscr{H})$ be the Calkin algebra. As usual, we let t denote the image in $Q(\mathscr{H})$ of $T \in \mathscr{L}(\mathscr{H})$. For $A, B \in \mathscr{L}(\mathscr{H})$ consider the following properties:

(1) $A + K = BC$ for some $C \in \mathscr{L}(\mathscr{H})$ and $K \in K(\mathscr{H})$;

(2) $a = bc$ for some $c \in Q(\mathscr{H})$;

(3) $aa^* \le \lambda^2 bb^*$ for some constant $\lambda > 0$.

It is clear that $(1) \Leftrightarrow (2)$. By using Douglas's theorem and a faithful representation of $\mathscr{Q}(\mathscr{H})$, one sees that $(2) \Rightarrow (3)$.

Question 4.18. Does condition (3) imply condition (2)?

It is not difficult to see that Theorem 2.6 cannot be extended to elements of commutative C^*-algebras.

Example 4.19. Let $\{e_n\}_{n=1}^{\infty}$ denote an orthonormal basis for $\mathscr{H}$. Define $A, B \in \mathscr{L}(\mathscr{H})$ as follows:

$$Be_n = \tfrac{1}{n} e_n \qquad (n \ge 1),$$
$$Ae_n = \begin{cases} \tfrac{1}{n} e_n & (n \ge 1,\ n \text{ odd}), \\ 0 & (n \text{ even}). \end{cases}$$

Let $\mathscr{A} = C^*(A, B)$, the unital C^*-subalgebra of $\mathscr{L}(\mathscr{H})$ generated by A and B. Clearly $AA^* \le BB^*$. However, the unique operator $C \in \mathscr{L}(\mathscr{H})$ such that $A = BC$ is the orthogonal projection onto the span of $\{e_n : n \text{ odd}\}$, and $C \notin \mathscr{A}$. Thus Douglas's theorem fails in $\mathscr{A}$.

Question 4.20. Is there an analogue of Theorem 4.6 for the Calkin algebra?

The second author and H. Salas have recently proved that $(3) \Rightarrow (2)$. In a forthcoming note they show more generally that $(3) \Rightarrow (2)$ holds in any AW^*-algebra and that if $(3) \Rightarrow (2)$ holds in a C^*-algebra $\mathscr{A}$, then it holds in every $*$-homomorphic image of $\mathscr{A}$ [FiS].

Added in proof. Recently, D. Herrero has answered Question 1.4 in the affirmative [Her7] and K. Takahaski [T] has shown that for T a contraction, Question 1.6 also has an affirmative answer. Finally, D. Herrero has proved in [Her8] that property $(***)$ in Remark 4.12 characterizes the algebraic operators.

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