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BOUNDARY LAYERS IN APPROXIMATE SOLUTIONS

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ABSTRACT. In this paper we study the development of boundary layers in parabolic approximate solutions of the initial boundary value problem for linear strictly hyperbolic systems of equations in one space variable. We also analyse the boundary layer behaviour.

1. INTRODUCTION

Let $D = \{(x,t): 0 \le x \le 1, t > 0\}$ and let $A(x,t)$ be a smooth $m \times m$ matrix which has m real distinct eigenvalues, k of them negative and $(m-k)$ of them positive:

$$\lambda_1(x,t) < \lambda_2(x,t) < \cdots < \lambda_k(x,t) < 0 < \lambda_{k+1}(x,t) < \cdots < \lambda_m(x,t).$$

Let $r_j(x,t)$ be a right eigenvector and $l_j(x,t)$ a left eigenvector corresponding to $\lambda_j(x,t)$.

It is well known (see Courant-Hilbert [3]) that a well-posed problem in D for the strictly hyperbolic system

$$u_t + A(x,t)u_x = 0, \tag{1.1}$$

where $u \in R^m$, is to prescribe $u(x,0) = u_0(x)$, $l_j(0,t)u(0,t)$, $j = k+1, \ldots, m$, and $l_j(1,t)u(1,t)$, $j = 1, 2, \ldots, k$.

The corresponding parabolic problem is

$$u_t^\varepsilon + A(x,t)u_x^\varepsilon = \varepsilon u_{xx}^\varepsilon, \tag{1.2}$$

$$u^\varepsilon(x,0) = u_0(x), \tag{1.3}$$

$$u^\varepsilon(0,t) = u_1(t), \tag{1.4}$$

$$u^\varepsilon(1,t) = u_2(t), \tag{1.5}$$

where the data are compatible at the corners. Existence, uniqueness and smoothness of u^ε are known (see A. Friedman [4]).

In §2 we study the limit as ε tends to zero in (1.2). Since more boundary conditions are prescribed for parabolic equations than for hyperbolic equations, some boundary conditions get lost in the passage $\varepsilon \to 0$. This loss manifests

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After this work was completed, the author learned from Professor Daniel Michelson's research announcement that he also considered a similar problem for several space variables.

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itself in the region of rapid change near $x = 0$ and near $x = 1$, called the boundary layer. Such problems, usually called in the literature singular perturbation problems, have been described by Lyusternik and Viski [6] for linear ode's and linear scalar elliptic equations, and by Bardos and Rauch [2] and Bardos, Brezis, Brezis [1] for maximal positive symmetric operators. In [2] Bardos and Rauch remark that one can prove H^s convergence, $s < \frac{1}{2}$ for parabolic systems, even in several space variables using the same ideas but with some extra arguments.

Here we consider the case of one space variable and get results in the maximum norm. Our approach is different from Bardos and Rauch; we construct boundary layer functions near the boundaries and derive careful energy estimates to get uniform convergence. The construction of the boundary layer function is classical, at least in the scalar elliptic case and ode's (see Lyusternik and Visik [6]).

Before taking up the general case, we give two typical and simple examples illustrating possible loss of boundary conditions.

Example 1. Consider the following problem in $x > 0$, $t > 0$.

$$u_t^\varepsilon + u_x^\varepsilon = \varepsilon u_{xx}, \quad u^\varepsilon(x, 0) = 0, \quad u^\varepsilon(0, t) = u_b(t),$$

$u_b(t)$ is smooth and compatible: $u_b(0) = 0$. When $\varepsilon = 0$, the problem is

$$u_t^0 + u_x^0 = 0, \quad u^0(x, 0) = 0, \quad u^0(0, t) = u_b(t).$$

Set $e^\varepsilon(x, t) = u^\varepsilon(x, t) - u^0(x, t)$, then $e^\varepsilon(x, t)$ solves the following:

$$\begin{gathered} e_t^\varepsilon + e_x^\varepsilon = \varepsilon e_{xx} + \varepsilon u_{xx}^0, \\ e^\varepsilon(x, 0) = 0, \quad e^\varepsilon(0, t) = 0. \end{gathered}$$

Let

$$k = \sup_{x \geq 0\ t \geq 0} |u_{xx}^0(x, t)|.$$

From the maximum principle for parabolic equations we conclude that

$$|e^\varepsilon(x, t)| \leq \varepsilon k T \quad \text{for } 0 \leq t \leq T,$$

with shows that

$$e^\varepsilon(x, t) = u^\varepsilon(x, t) - u^0(x, t) \to 0,$$

uniformly in $x \geq 0$, $0 \leq t \leq T$.

Example 2. Now we consider the following problem in $x > 0$, $t > 0$.

$$(*) \qquad u_t^\varepsilon - u_x^\varepsilon = \varepsilon u_{xx}, \qquad u^\varepsilon(x, 0) = 0, \quad u^\varepsilon(0, t) = u_b(t),$$

$u_b(t)$ is smooth and compatible: $u_b(0) = 0$. When $\varepsilon = 0$, the problem is

$$u_t^0 - u_x^0 = 0, \quad u^0(x, 0) = 0,$$

and no boundary conditions are needed. In fact $u^0(x, t) = 0$, as is seen by the method of characteristics.

For $(*)$ we can get a closed form solution; set

$$u^{\varepsilon} = e^{-x/2\varepsilon - t/4\varepsilon} v^{\varepsilon}(x,t),$$

then from $(*)$ we obtain $v_t = \varepsilon v_{xx}$, $v(x,0) = 0$, $v(0,t) = u_b(t)e^{t/4\varepsilon}$, whose solution is

$$v(x,t) = -\frac{2}{\sqrt{\pi}} \int_0^t \partial_s \left[\int_{x/2\sqrt{\varepsilon(t-s)}}^{\infty} e^{-y^2}\, dy \right] (e^{s/4\varepsilon} u_b(s))\, ds.$$

Hence

$$u^{\varepsilon}(x,t) = \frac{2}{\sqrt{\pi}} e^{-x/2\varepsilon - t/4\varepsilon} \left[u_b(t) \int_{x/2\sqrt{\varepsilon t}}^{\infty} e^{-y^2}\, dy \right.$$
$$\left. + \int_0^t \int_{x/2\sqrt{\varepsilon(t-s)}}^{\infty} e^{-y^2}\, dy \partial_s (e^{s/4\varepsilon} u_b(s))\, ds \right],$$

from which it is clear that

$$\lim_{\varepsilon \to 0} \sup_{\substack{\delta \le x < \infty \\ 0 \le t \le T}} |u^{\varepsilon}(x,t)| = 0,$$

for each $\delta > 0$.

Notice that in the first example $u^{\varepsilon}(x,t)$ preserves the boundary condition and in the second example $u^{\varepsilon}(x,t)$ loses the boundary condition as $\varepsilon \to 0$. Now we consider the general case.

2. Result for linear systems

Let $u = (u_1, \dots, u_m) \in R^m$. We use the following notations:
$(u)_k = u_k$,
$\|u\| = (\int_0^1 |u|^2\, dx)^{1/2}$.

Let $u^{\varepsilon}(x,t)$ be the solution of (1.2), (1.3), (1.4) and (1.5), and let $u^0(x,t)$ be the solution of (1.1) with initial and boundary conditions

$$u^0(x,t) = u_0(x), \tag{1.3$'$}$$

$$l_j(0,t)u^0(0,t) = l_j(0,t)u_1(t), \qquad j = k+1, \dots, m, \tag{1.4$'$}$$

$$l_j(1,t)u^0(1,t) = l_j(1,t)u_2(t), \qquad j = 1, 2, \dots, k. \tag{1.5$'$}$$

Let $B = B(x,t,\varepsilon)$ be a function with the following properties,

$$B(0,t,\varepsilon) = u_1(t) - u^0(0,t), \tag{2.1}$$

$$B(1,t,\varepsilon) = u_2(t) - u^0(1,t), \tag{2.2}$$

$$B(x,t,\varepsilon) = T(x,t) \left\{ \left[p_0\left(\frac{x}{\varepsilon}, t\right) + \varepsilon p_1\left(\frac{x}{\varepsilon}, t\right) \right] \phi_1(x) \right. \tag{2.3}$$
$$\left. + \left[q_0\left(\frac{x-1}{\varepsilon}, t\right) + \varepsilon q_1\left(\frac{x-1}{\varepsilon}, t\right) \right] \phi_2(x) \right\}$$

where $T(x,t) = (r_1(x,t), \dots, r_m(x,t))$, the matrix whose jth column is the right eigenvector of A corresponding to λ_j.

$$(2.4)_0 \qquad p_0(y,t) = M_0(t)\begin{vmatrix} e^{\lambda_1(0,t)y} \\ \vdots \\ e^{\lambda_k(0,t)y} \\ 0 \\ \vdots \\ 0 \end{vmatrix},$$

$$(2.4)_1 \qquad p_1(y,t) = c_1(t) + M_1(t)\begin{vmatrix} e^{\lambda_1(0,t)y} \\ \vdots \\ e^{\lambda_k(0,t)y} \\ 0 \\ \vdots \\ 0 \end{vmatrix} + yM_2(t)\begin{vmatrix} e^{\lambda_1(0,t)y} \\ \vdots \\ e^{\lambda_k(0,t)y} \\ 0 \\ \vdots \\ 0 \end{vmatrix} + y^2M_3(t)\begin{vmatrix} e^{\lambda_1(0,t)y} \\ \vdots \\ e^{\lambda_k(0,t)y} \\ 0 \\ \vdots \\ 0 \end{vmatrix},$$

$$(2.5)_0 \qquad q_0(y,t) = N_0(t)\begin{vmatrix} 0 \\ \vdots \\ 0 \\ e^{\lambda_{k+1}(1,t)y} \\ \vdots \\ e^{\lambda_m(1,t)y} \end{vmatrix},$$

$$(2.5)_1 \qquad q_1(y,t) = d_1(t) + N_1(t)\begin{vmatrix} 0 \\ \vdots \\ 0 \\ e^{\lambda_{k+1}(1,t)y} \\ \vdots \\ e^{\lambda_m(1,t)y} \end{vmatrix} + yN_2(t)\begin{vmatrix} 0 \\ \vdots \\ 0 \\ e^{\lambda_{k+1}(1,t)y} \\ \vdots \\ e^{\lambda_m(1,t)y} \end{vmatrix} + y^2N_3(t)\begin{vmatrix} 0 \\ \vdots \\ 0 \\ e^{\lambda_{k+1}(1,t)y} \\ \vdots \\ e^{\lambda_m(1,t)y} \end{vmatrix}.$$

$c_1(t)$ and $d_1(t)$ are smooth vectors, and $M_i(t)$ and $N_i(t)$ are smooth matrices which depend only on t and are constructed in the proof of Lemma 1. $\phi_1(x)$

and $\phi_2(x)$ are C^∞ cut off functions in $[0,1]$:

$$\phi_1(x) = \begin{cases} 1 & \text{in } 0 \le x \le \delta/2, \\ 0 & \text{in } \delta \le x \le 1, \end{cases} \tag{2.5$_3$}$$
$$\phi_2(x) = \begin{cases} 0 & \text{in } 0 \le x \le 1-\delta, \\ 1 & \text{in } 1-\delta/2 \le x \le 1. \end{cases}$$

We prove the following

Theorem.
$$\sup_{\substack{0\le x\le 1\\ 0\le t\le T}} |u^\varepsilon(x,t) - u^0(x,t) - B(x,t,\varepsilon)| \le c\varepsilon^{1/4},$$
where c depends only on T.

Remark. From the above theorem it follows that $u^\varepsilon(x,t) \to u^0(x,t)$ uniformly in $D_{\delta',T} = \{(x,t\colon \delta' \le x \le 1-\delta',\ 0 \le t \le T\}$ for each $\delta' > 0$, $T > 0$, and in the limit $u^\varepsilon(x,t)$ preserves the boundary condition at $x = 0$, $P^+u_1(t)$ and at $x = 1$, $P^-u_2(t)$.

The proof of this theorem follows from two lemmas. Denote by
$$T(x,t) = (r_1(x,t), \dots, r_m(x,t))$$
the matrix whose jth column is $r_j(x,t)$, the right eigenvalue of A. Then
$$A(x,t)T(x,t) = T(x,t)D(x,t)$$
where
$$D(x,t) = \operatorname{diag}(\lambda_1(x,t), \dots, \lambda_m(x,t))$$
is the diagonal matrix whose jth diagonal entry is $\lambda_j(x,t)$. Set

$$u = Tv, \quad u_t = T_t v + T v_t, \quad u_x = T_x v + T v_x, \tag{2.6}$$
$$u_{xx} = T_{xx} v + 2T_x v_x + T v_{xx},$$

so that from (1.2) we get $v_t = L_\varepsilon v$, where

$$\begin{aligned} L_\varepsilon v &= \varepsilon v_{xx} + (\varepsilon P - D)v_x + (R + \varepsilon Q)v, \\ P &= P(x,t) = 2T^{-1}T_x, \\ Q &= Q(x,t) = T^{-1}T_{xx}, \\ R &= R(x,t) = -T^{-1}AT_x - T^{-1}T_t. \end{aligned} \tag{2.7}$$

Denote
$$v_1(t) = T^{-1}(0,t)u_1(t), \quad v_2(t) = T^{-1}(1,t)u_2(t), \quad v_0(x) = T^{-1}(x,0)u_0(x).$$
Define $v^\varepsilon = T^{-1}u^\varepsilon$; then for $v = v^\varepsilon$ the problem becomes

$$v_t^\varepsilon = L_\varepsilon v^\varepsilon, \tag{2.8}$$
$$v^\varepsilon(x,0) = v_0(x), \tag{2.9}$$
$$v^\varepsilon(0,t) = v_1(t), \tag{2.10}$$
$$v^\varepsilon(1,t) = v_2(t). \tag{2.11}$$

The problem for u^0, in v coordinates, becomes

(2.8)′ $$v^0_t = L_0 v^0,$$

(2.9′ $$v^0(x,0) = v_0(x),$$

(2.10)′ $$(v^0(0,t))_j = (v_1(t))_j, \qquad j = k+1, \dots, m,$$

(2.11)′ $$(v^0(1,t))_j = (v_2(t))_j, \qquad j = 1, 2, \dots, k.$$

We first prove the following

Lemma 1. *There exist functions $b^\varepsilon(x,t)$, $g(x,t,\varepsilon)$ and $k(x,\varepsilon)$ such that*

(2.12) $$b^\varepsilon_t = L_\varepsilon b^\varepsilon + \varepsilon g(x,t,\varepsilon).$$

$b^\varepsilon(x,t)$ is of the form

$$\begin{aligned} b^\varepsilon(x,t) = {}& [p_0(x/\varepsilon,t) + \varepsilon p_1(x/\varepsilon,t)]\phi_1(x) \\ & + [q_0((x-1)/\varepsilon,t) + \varepsilon q_1((x-1)/\varepsilon,t)]\phi_2(x), \end{aligned}$$

where p_0, p_1, ε_0, q, ϕ_1 and ϕ_2 are as in $(2.4)_0$, $(2.4)_1$, $(2.5)_0$, (2.5), and $(2.5)_3$ respectively.

(2.13) $$\begin{aligned} b^\varepsilon(0,t) &= v^\varepsilon(0,t) - v^0(0,t), \\ b^\varepsilon(1,t) &= v^\varepsilon(1,t) - v^0(1,t), \\ b^\varepsilon(x,0) &= \varepsilon k(x,\varepsilon). \end{aligned}$$

With c depending only on T,

(2.14) $$\sup_{0\le t\le T} \|g(x,t,\varepsilon)\| \le c,$$

(2.15) $$\sup_{0\le t\le T} \|g_t(x,t,\varepsilon)\| \le c,$$

(2.16) $$\begin{aligned} &\|\partial_x^k k(x,\varepsilon)\|^2 \le c\varepsilon^{1-2k}, \qquad k = 1, 2, \\ &\|k(x,\varepsilon)\|^2 \le c. \end{aligned}$$

Proof. Expanding the coefficient of the differential operator L_ε in powers of x near $x = 0$, we get

(2.17) $$\begin{aligned} P(x,t) &= P(0,t) + xP_1(x,t), \\ R(x,t) &= R(0,t) + xR_1(x,t), \\ \lambda_i(x,t) &= \lambda_i(0,t) + x\lambda_i'(0,t) + x^2\lambda_{i1}(x,t), \end{aligned}$$

where $P_1(x,t)$ and $R_1(x,t)$ are smooth matrices, and λ_{i1} is a smooth function. Set

(2.18) $$\frac{x}{\varepsilon} = y, \quad \partial_x = \frac{1}{\varepsilon}\partial_y.$$

We seek $b^\varepsilon(x,t)$ near $x = 0$ in the form

$$b^\varepsilon(x,t) = p_0(x/\varepsilon,t) + \varepsilon p_1(x/\varepsilon,t) = p(x/\varepsilon,t).$$

In the y variable we get from (2.2), (2.6) and (2.7),

$$
\begin{aligned}
(L_\varepsilon - \partial_t)p(y,t) = {} & \frac{\varepsilon}{\varepsilon^2}p_{yy} + \frac{\varepsilon}{\varepsilon}[P(0,t) + \varepsilon y P_1(\varepsilon y, t)]p_y \\
& - \frac{1}{\varepsilon}[D(0,t) + \varepsilon y D'(0,t) + \varepsilon^2 y^2 D_1(\varepsilon y, t)]p_y \\
& + \varepsilon Q(\varepsilon y, t)p + [R(0,t) + \varepsilon y R_1(\varepsilon y, t)]p - p_t
\end{aligned} \tag{2.19}
$$

where

$$D'(0,t) = \operatorname{diag}[\partial_x\lambda_1(x,t), \dots, \partial_x\lambda_m(x,t)] \quad \text{at } x = 0,$$

and $D_1(z,t)$ is a smooth diagonal matrix. Rewriting (2.19), we get

$$(L_\varepsilon - \partial_t)p = \left(\frac{1}{\varepsilon}L_1 + L_2 + \varepsilon L_3\right)p$$

where

$$
\begin{aligned}
L_1 p &= p_{yy} - D(0,t)p_y, \\
L_2 p &= [P(0,t) - yD'(0,t)]p_y + R(0,t)p - \partial_t p, \\
L_3 p &= y p_1(\varepsilon y, t)p_y - y^2 D_1(\varepsilon y, t)p_y + [Q(\varepsilon y, t) + yR_1(\varepsilon y, t)]p,
\end{aligned} \tag{2.20}
$$

$$
\begin{aligned}
(L_\varepsilon - \partial_t)p = {} & \left(\frac{1}{\varepsilon}L_1 + L_2 + \varepsilon L_3\right)\{p_0(y,t) + \varepsilon p_1(y,t)\} \\
= {} & \frac{1}{\varepsilon}L_1 p_0(y,t) + L_1 p_1(y,t) + L_2 p_0(y,t) \\
& + \varepsilon[L_2 p_1 + L_3 p_0] + \varepsilon^2 L_3 p_1;
\end{aligned} \tag{2.21}
$$

we choose p_0 such that

$$L_1 p_0 = 0, \quad p_0(0,t) = (v_1(0,t) - v^0(0,t)), \quad p_0(y,t) \to 0 \quad \text{as } y \to \infty. \tag{2.22}$$

Notice that by (2.10) and (2.10)$'$

$$(p_0(0,t))_j = 0 \quad \text{for } j = k+1, \dots, m.$$

Since $(p_0(y,t))_j$ has to satisfy (2.22) we have

$$\partial_y^2 (p_0(y,t))_j = \lambda_j(0,t)\partial_y(p_0(y,t))_j,$$

$$(p_0(0,t))_j = \begin{cases} 0, & \text{for } j = k+1, \dots, m, \\ (v_1(0,t) - v^0(0,t))_j, & \text{for } j = 0, 1, \dots, k. \end{cases}$$

So we take

$$
\begin{aligned}
(p_0(y,t)) = ((v_1(0,t) - v^0(0,t))e^{\lambda_1(0,t)y}, \dots, \\
(v_1(0,t) - v^0(0,t))_k e^{\lambda_k(0,t)y}, 0, \dots, 0)^t.
\end{aligned}
$$

Now choose p_1 such that

$$L_1 p_1 = -L_2 p_0, \quad p_1(0,t) = 0, \quad p(y,t) \to 0 \quad \text{as } y \to \infty, \tag{2.24}$$

i.e.

$$(2.4)_1 \qquad \partial_y^2 (p_1)_j = \lambda_j(0,t)\partial_y(p_1)_j - (L_2 p_0)_j ,$$

$$(2.4)_2 \qquad (p_1)_j(0,t) = 0, \quad (p)_j \to 0 \quad \text{as } y \to \infty.$$

By variation of the constant formula, the solution of $(2.24)_1$ and $(2.24)_2$ is

$$(2.25) \qquad (p_1)_j(y,t) = c_j(t)[1 - e^{\lambda_j(0,t)y}] - \frac{1}{\lambda_j(0,t)} \int_0^y (e_1^{\lambda_j(0,t)(y-y')})(L_2 p_0)_j \, dy ;$$

$c_j(t)$ is to be chosen properly so that $(p_1)_j(y,t) \to 0$ as $y \to \infty$.

Now

$$(2.26) \qquad L_2 p_0 = \alpha(t)(e^{\lambda_1(0,t)y}, \dots, e^{\lambda_k(0,t)y}, 0 \dots, 0) + y\beta(t)(e^{\lambda_1(0,t)y}, \dots, e^{\lambda_k(0,t)y}, 0, \dots, 0)$$

where

$$(2.26)' \qquad \begin{aligned} \alpha(t) &= [P(0,t)D(0,t) + R(0,t)] \operatorname{diag}[p_0(0,t)] - \operatorname{diag}[\partial_t p_0(0,t)], \\ \beta(t) &= -D'(0,t)[I + D(0,t)] \operatorname{diag}[p_0(0,t)]. \end{aligned}$$

Let us use the notation $\alpha(t) = (\alpha_{ij}(t))$ and $\beta(t) = (\beta_{ij}(t))$. From (22.6),

$$\begin{aligned} &\int_0^y (e^{\lambda_j(0,t)(y-y')} - 1)(L_2 p_0)_j \, dy' \\ &= \int_0^y (e^{\lambda_j(0,t)(y-y')} - 1) \sum_{i=1}^k \alpha_{ij}(t) e^{\lambda_i(0,t)y'} \, dy' \\ &\quad + \int_0^y (e^{\lambda_j(0,t)(y-y')} - 1) \sum_{i=1}^k \beta_{ij}(t) e^{\lambda_i(0,t)y'} y' \, dy' \\ &= I_1^j + I_2^j. \end{aligned}$$

We first consider

$$I_1^j = e^{\lambda_j(0,t)y} \sum_{i=1}^k \alpha_{ij}(t) \int_0^y e^{(\lambda_i - \lambda_j)y'} \, dy' - \sum_{i=1}^k \alpha_{ij}(t) \int_0^y e^{\lambda_i(0,t)y'} \, dy'.$$

Rewriting this, we obtain

$$(2.27) \qquad I_1^j = e^{\lambda_j(0,t)y} \sum_{\substack{i=1 \\ j \neq i}}^k \alpha_{ij}(t) \int_0^y e^{(\lambda_i - \lambda_j)y'} \, dy' + e^{\lambda_j y} \alpha_{jj}(t) y - \sum_{i=1}^k \frac{\alpha_{ij}(t)}{\lambda_i(0,t)} (e^{\lambda_i(0,t)y} - 1).$$

For $i \neq j$,

$$\int_0^y e^{(\lambda_i-\lambda_j)y'}\,dy' = \frac{1}{\lambda_i-\lambda_j}(e^{(\lambda_i-\lambda_j)y}-1).$$

Substituting this in (2.27) we get

$$I_1^j = \sum_{\substack{i=1\\ j\neq i}}^{k} \frac{\alpha_{ij}(t)}{(\lambda_i(0,t)-\lambda_j(0,t))}(e^{\lambda_i(0,t)y}-e^{\lambda_j(0,t)y}) - \sum_{i=1}^{k}\frac{\alpha_{ij}(t)}{\lambda_i(0,t)}(e^{\lambda_i(0,t)y}-1)+\alpha_{jj}(t)ye^{\lambda_j(0,t)y}. \tag{2.28}$$

Now we consider I_2^j, where

$$I_2^j = e^{\lambda_j(0,t)y}\sum_{i=1}^{k}\beta_{ij}(t)\int_0^y e^{(\lambda_i(0,t)-\lambda_j(0,t))y'}y'\,dy' - \sum_{i=1}^{k}\beta_{ij}(t)\int_0^y y'e^{\lambda_i(0,t)y'}\,dy'.$$

Rewriting this we get

$$I_2^j = e^{\lambda_j(0,t)y}\sum_{\substack{i=1\\ i\neq j}}^{k}\beta_{ij}(t)\int_0^y e^{(\lambda_i(0,t)-\lambda_j(0,t))y'}y'\,dy' + e^{\lambda_j(0,t)y}\beta_{jj}(t)\frac{y^2}{2} - \sum_{i=1}^{k}\beta_{ij}(t)\int_0^y y'e^{\lambda_i(0,t)y'}\,dy'. \tag{2.29}$$

For $i \neq j$,

$$\begin{aligned}\int_0^y & y'e^{(\lambda_i(0,t)-\lambda_j(0,t))y'}\,dy' \\ &= \frac{y'e^{(\lambda_i(0,t)-\lambda_j(0,t))y'}}{(\lambda_i(0,t)-\lambda_j(0,t))}\bigg|_0^y - \int_0^y \frac{e^{(\lambda_i(0,t)-\lambda_j(0,t))y'}}{(\lambda_i(0,t)-\lambda_j(0,t))}\,dy' \\ &= \frac{ye^{(\lambda_i(0,t)-\lambda_j(0,t))y}}{(\lambda_i(0,t)-\lambda_j(0,t))} - \frac{(e^{(\lambda_i(0,t)-\lambda_j(0,t))y}-1)}{(\lambda_i(0,t)-\lambda_j(0,t))^2},\end{aligned}$$

$$\begin{aligned}\int_0^y y'e^{\lambda_i(0,t)y'}\,dy' &= \frac{y'}{\lambda_i(0,t)}e^{\lambda_i(0,t)y'}\bigg|_0^y - \frac{1}{\lambda_i(0,t)}\int_0^y e^{\lambda_i(0,t)y'}\,dy' \\ &= \frac{y}{\lambda_i(0,t)}e^{\lambda_i(0,t)y} - \frac{1}{\lambda_i(0,t)^2}(e^{\lambda_i(0,t)y}-1).\end{aligned}$$

Substituting these in (2.29), we obtain

$$(2.30)\quad I_2^j = \frac{y^2}{2}\beta_{jj}(t)e^{\lambda_j y} + y\left[\sum_{\substack{i=1\\ i\neq j}} \frac{\beta_{ij}(t)}{(\lambda_i(0,t)-\lambda_j(0,t))} - \sum_{i=1}^{k}\frac{\beta_{ij}(t)}{\lambda_i(0,t)}\right]e^{\lambda_i(0,t)y}$$
$$+\sum_{i=1}^{k}\frac{\beta_{ij}(t)e^{\lambda_i y}}{\lambda_i(0,t)} - \sum_{\substack{i=1\\ i\neq j}}^{k}\frac{\beta_{ij}(t)e^{\lambda_i y}}{(\lambda_i-\lambda_j)^2}$$
$$+\left(\sum_{\substack{i=1\\ i\neq j}}^{k}\frac{\beta_{ij}}{(\lambda_i-\lambda_j)^2}\right)e^{\lambda_j y} - \sum_{i=1}^{k}\frac{\beta_{ij}(t)}{\lambda_i^2(0,t)}.$$

From (2.28), (2.30) and (2.25) we get, for $j = 1, 2, \dots, k$,

(2.31)
$$(p_1)_j = c_j(t)[1-e^{\lambda_j(0,t)y}]$$
$$-\frac{1}{\lambda_j(0,t)}\left[\sum_{i=1}^{k}\frac{\alpha_{ij}(t)(e^{\lambda_i(0,t)y}-e^{\lambda_j(0,t)y})}{(\lambda_i(0,t)-\lambda_j(0,t))}\right.$$
$$\left.+\alpha_{jj}(t)ye^{\lambda_j(0,t)y} - \sum_{i=1}^{k}\frac{\alpha_{ij}(t)}{\lambda_i(0,t)}(e^{\lambda_i(0,t)y}-1)\right]$$
$$-\frac{1}{\lambda_j(0,t)}\left[\frac{y^2}{2}\beta_{jj}(t)e^{\lambda_j(0,t)y}\right.$$
$$+y\left(\sum_{\substack{i=1\\ i\neq j}}^{k}\frac{\beta_{ij}(t)e^{\lambda_i(0,t)y}}{(\lambda_i(0,t)-\lambda_j(0,t))} - \sum_{i=1}^{k}\frac{\beta_{ij}(t)e^{\lambda_i(0,t)y}}{\lambda_i(0,t)}\right)$$
$$+\sum_{i=1}^{k}\frac{\beta_{ij}(t)e^{\lambda_i y}}{\lambda_i^2(0,t)} - \sum_{\substack{i=1\\ i\neq j}}^{k}\frac{\beta_{ij}(t)e^{\lambda_i y}}{(\lambda_i(0,t)-\lambda_j(0,t))^2}$$
$$\left.+\left(\sum_{\substack{i=1\\ i\neq j}}^{k}\frac{\beta_{ij}(t)}{\lambda_i(0,t)-\lambda_j(0,t)}\right)e^{\lambda_j(0,t)y} - \sum_{i=1}^{k}\frac{\beta_{ij}(t)}{\lambda_i^2(0,t)}\right].$$

The choice of $c_j(t)$.

Case $j = 1, 2, \ldots, k$.

Since $p_0(0,0) = 0$ from equation $(2.26)'$ we have $\beta_{ij}(0) = 0$ and $\alpha_{ij}(0) = \operatorname{diag}(\partial_t(p(0,0)))$. From (2.31), we get

$$(2.32)\quad (p_1)_j(y,0) = c_j(0)[1 - e^{\lambda_j(0,0)y}] - \frac{1}{\lambda_j(0,0)}\left[\sum_{\substack{i=1\\ i\neq j}}^{k} \frac{\alpha_{ij}(0)}{\lambda_i(0,0)-\lambda_j(0,0)}(e^{\lambda_i(0,0)y} - e^{\lambda_j(0,0)y}) + \alpha_{jj}(0)ye^{\lambda_j(0,0)y} - \sum_{i=1}^{k}\frac{\alpha_{ij}(0)}{\lambda_i(0,0)}(e^{\lambda_i(0,0)y}-1)\right].$$

Take $c_j(t) = 1$.

Case $j = 1+1, \ldots, m$.

In this case in the formulas for I_1^j and I_2^j the summation in i is only from 1 to k, and for $i = 1, 2, \ldots, k$, $i \neq j$,

$$\begin{aligned}(2.33)\quad (p_1)_j = {} & c_j(t)[1 - e^{\lambda_j(0,t)y}] \\ & - \frac{1}{\lambda_j(0,t)}\left[\sum_{i=1}^{k}\frac{\alpha_{ij}(t)(e^{\lambda_i(0,t)y} - e^{\lambda_j(0,t)y})}{(\lambda_i(0,t)-\lambda_j(0,t))} - \sum_{i=1}^{k}\frac{\alpha_{ij}(t)}{\lambda_i(0,t)}(e^{\lambda_i(0,t)y}-1)\right] \\ & - \frac{y}{\lambda_j(0,t)}\left[\sum_{i=1}^{k}\frac{\beta_{ij}(t)e^{\lambda_i(0,t)y}}{(\lambda_i(0,t)-\lambda_j(0,t))} - \sum_{i=1}^{k}\frac{\beta_{ij}(t)e^{\lambda_i(0,t)y}}{\lambda_i(0,t)}\right] \\ & - \frac{1}{\lambda_j(0,t)}\left[\sum_{i=1}^{k}\frac{\beta_{ij}(t)e^{\lambda_i(0,t)y}}{\lambda_i^2(0,t)} - \sum_{i=1}^{k}\frac{\beta_{ij}(t)e^{\lambda_i(0,t)y}}{(\lambda_i(0,t)-\lambda_j(0,t))^2} + \left(\sum_{i=1}^{k}\frac{\beta_{ij}(t)}{\lambda_i(0,t)-\lambda_j(0,t)}\right)e^{\lambda_j(0,t)y} - \sum_{i=1}^{k}\frac{\beta_{ij}(t)}{\lambda_i^2(0,t)}\right].\end{aligned}$$

Since $\lambda_j(0,t) > 0$, $j = k+1, \ldots, m$, the $e^{\lambda_j(0,t)y}$ are bad terms, so we choose $c_j(t)$ such that

$$(2.34)\qquad -c_j(t) + \frac{1}{\lambda_j(0,t)}\left[\sum_{i=1}^{k}\frac{\alpha_{ij}(t)}{\lambda_i(0,t)-\lambda_j(0,t)} - \sum_{i=1}^{k}\frac{\beta_{ij}(t)}{(\lambda_i(0,t)-\lambda_j(0,t))}\right] = 0.$$

We get from (2.33) and (2.34), for $j = k+1, \dots, m$,

$$(2.35)\quad (p_1)_j(y,t) = c_j(t) - \frac{1}{\lambda_j(0,t)}\left[\sum_{i=1}^{k} \frac{\alpha_{ij}(t)e^{\lambda_i(0,t)y}}{(\lambda_i(0,t)-\lambda_j(0,t))} - \sum_{i=1}^{k} \frac{\alpha_{ij}(t)}{\lambda_i(0,t)}(e^{\lambda_i(0,t)y}-1)\right]$$
$$- \frac{y}{\lambda_j(0,t)}\left[\sum_{i=1}^{k} \frac{\beta_{ij}(t)e^{\lambda_i(0,t)y}}{(\lambda_i(0,t)-\lambda_j(0,t))} - \sum_{i=1}^{k} \frac{\beta_{ij}(t)e^{\lambda_i(0,t)y}}{\lambda_i(0,t)}\right]$$
$$- \frac{1}{\lambda_j(0,t)}\left[\sum_{i=1}^{k} \frac{\beta_{ij}(t)e^{\lambda_i(0,t)y}}{\lambda_i^2(0,t)} - \sum_{i=1}^{k} \frac{\beta_{ij}(t)e^{\lambda_i(0,t)y}}{(\lambda_i(0,t)-\lambda_j(0,t))} - \sum_{i=1}^{k} \frac{\beta_{ij}(t)}{\lambda_i^2(0,t)}\right],$$

where $c_j(t)$ is given by (2.34). For $j = k+1, \dots, m$,

$$(2.36)\quad (p_1)_j(y,0) = c_j(0) - \frac{1}{\lambda_j(0,0)}\left[\sum_{i=1}^{k} \frac{\alpha_{ij}(0)e^{\lambda_i(0,0)y}}{(\lambda_i(0,0)-\lambda_j(0,0))} - \sum_{i=1}^{k} \frac{\alpha_{ij}(0,0)}{\lambda_i(0,0)}(e^{\lambda_i(0,0)y}-1)\right].$$

Now let $\phi_1(x)$ be a C^∞ function in $[0,1]$ such that

$$\phi_1(x) = 1 \quad \text{in } 0 \le x \le \delta/2,$$
$$\phi_1(x) = 0 \quad \text{in } \delta \le x \le 1.$$

Set

$$b_1^\varepsilon(x,t) = [p_0(x/\varepsilon,t) + \varepsilon p_1(x/\varepsilon,t)]\phi_1(x).$$

$b_1^\varepsilon(x,t)$ is the boundary layer function near $x = 0$ with the following important properties.

(2.37)

(a) $(p_0(y))_j$ is a linear combination of $e^{\lambda_1(0,t)y}, \dots, e^{\lambda_k(0,t)y}$ whose coefficients depend smoothly on t and on nothing else.

(b) $(p_1(y))_j$ is a linear combination of $e^{\lambda_i(0,t)}$, $ye^{\lambda_i(0,t)}$ and $y^2e^{\lambda_i(0,t)}$, $i = 1,2,\dots,k$, whose coefficients depend smoothly on t and on nothing else.

(c) $p_0(0,t) = v_1(0,t) - v^0(0,t)$, $p_1(0,t) = 0$.

(d) $p_0(y,0) = 0$, $p_1(y,0)$ is a linear combination of 1, $e^{\lambda_i(0,0)y}, ye^{\lambda_i(1,0)y}$, $i = 1,2,\dots,k$, whose coefficients are constants.

By a similar procedure, expanding the coefficients of L_ε in powers of $(x-1)$, near $x = 1$ we construct a boundary layer function $b_2^\varepsilon(x, t)$.

The function $b^\varepsilon(x, t)$ we are seeking is given by

$$b^\varepsilon(x, t) = b_1^\varepsilon(x, t) + b_2^\varepsilon(x, t).$$

Define $k(x, \varepsilon)$ by

$$b^\varepsilon(x, 0) = \varepsilon\phi_1(x/\varepsilon, 0) = \varepsilon k(x, \varepsilon),$$

and $g(x, t, \varepsilon)$ by

$$\partial_t b^\varepsilon - L_\varepsilon b^\varepsilon = \varepsilon g(x, t, \varepsilon).$$

Notice that from (2.37) (a), (b), (c) and (d), for $b_1^\varepsilon(x, t)$ and similar properties of $b_2^\varepsilon(x, t)$, $b^\varepsilon(x, t)$ is exponentially decreasing to 0 as $\varepsilon \to 0$ in the interval $\delta/2 \le x \le 1 - \delta/2$ and so properties (2.14), (2.15), and (2.16) are verified there.

We now verify these properties in $0 \le x \le \delta/2$ and in $1-\delta/2 \le x \le 1$. We do our analysis in $0 \le x \le \delta/2$; the other case is similar.

By (2.32) and (2.36), $(k(x, \varepsilon))_j$ is a linear combination of 1, $e^{\lambda_i(0,0)x/\varepsilon}$, $(x/\varepsilon)e^{\lambda_i(0,0)x/\varepsilon}$, so that

$$\text{(2.38)} \qquad \int_0^\delta |k(x, \varepsilon)|^2 \le c, \quad \int_0^\delta |k_x(x, \varepsilon)|^2 \le \frac{c}{\varepsilon}, \quad \int_0^\delta |k_{xx}(x, \varepsilon)|^2 \le \frac{c}{\varepsilon^3}.$$

We also obtain

$$\text{(2.39)} \qquad \int_\delta^{1-\delta} |\partial_k^k k(x, t)| \le e^{-M/\varepsilon}, \qquad M > 0.$$

The same estimate as in (2.38) holds in the interval $[1-\delta, 1]$, so that we get

$$\|\partial_x^k k(x, \varepsilon)\|^2 \le c\varepsilon^{1-2k}, \qquad k = 1, 2,$$

and

$$\|k(x, \varepsilon)\|^2 \le c.$$

Now verify properties (2.14) and (2.15) of $g(x, t, \varepsilon)$. Here again we need to argue only in $0 \le x \le \delta/2$. In this interval $\phi_1(x) = 1$, so that, by (2.21)

$$\partial_t b^\varepsilon - L_\varepsilon b^\varepsilon = \varepsilon[L_2 p_1 + L_3 p_0 + \varepsilon L_3 p_1]y = x/\varepsilon.$$

The terms in $g(x, t, \varepsilon)$ and in $g_t(x, t, \varepsilon)$ are linear combinations of

$$1, (\tfrac{x}{\varepsilon})^n e^{\lambda_i(0,t)x/\varepsilon}, \qquad i = 1, 2, \dots, k,$$

with the coefficient independent of ε. It is clear that

$$\sup_{0 \le t \le T} \|g(x, t, \varepsilon)\|^2_{L^2[0,\delta]} \quad \text{and} \quad \sup_{0 \le t \le T} \|g_t(x, t, \varepsilon)\|_{L^2[0,\delta]}$$

are bounded by c.

We use a similar argument for the intervals $[1-\delta/2, 1]$ and $[\delta/2, 1-\delta/2]$, where the terms in $g(x, t, \varepsilon)$ and $g_t(x, t, \varepsilon)$ are bounded. Hence

$$\sup_{0\le t\le T} \|g(x, t, \varepsilon) \le c, \qquad \sup_{0\le t\le T} \|g_t(x, t, \varepsilon)\| \le c,$$

The proof of Lemma 1 is complete.

Define

$$z^\varepsilon = v^\varepsilon - v^0 - b^\varepsilon. \tag{2.40}$$

All the constants depend only on T.

Lemma 2.

$$\sup_{\substack{0\le x\le 1\\ 0\le t\le T}} |z^\varepsilon(x, t)| \le c\varepsilon^{1/4}, \tag{2.41}$$

where c *depends only on* T.

Proof. Clearly, by Lemma 1, (2.13),

$$z(0, t) = 0 = z(1, t), \tag{2.42}$$

$$z(x, 0) = \varepsilon k(x, \varepsilon), \tag{2.43}$$

$$z_t = \varepsilon z_{xx} + [-D(x, t) + \varepsilon P]z_x + [R + \varepsilon Q]z - \varepsilon h, \tag{2.44}$$

where $h = h(x, t, \varepsilon) = g(x, t, \varepsilon) + v^0_{xx} + Pv^0_x + Qv^0$. By using Lemma 1, we have

$$\sup_{0\le t\le T} \|h(x, t, \varepsilon)\| \le c, \qquad \sup_{0\le t\le T} \|h_t(x, t, \varepsilon)\| \le c. \tag{2.45}$$

Multiply (2.44) by z and integrate by parts with respect to x. Then using (2.42) we get

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}\|z\|^2 \le & -\varepsilon\|z_x\|^2 + c\|z\|^2 + \varepsilon c\|z\|\,\|z_x\| \\ & + c\|z\|^2 + \varepsilon c\|z\|^2 + \varepsilon\|z\|\,\|h\|.\end{aligned} \tag{2.46}$$

Using $2ab \le \delta a^2 + b^2/\delta$ we get

$$\frac{d}{dt}\|z\|^2 \le c\|z\|^2 + \varepsilon^2\|h\|.$$

By Gronwall's inequality we get

$$\|z(t)\|^2 \le e^{ct}\|z(0)\|^2 + \varepsilon^2 \int_0^t \|h\|(s)e^{c(t-s)}\,ds. \tag{2.47}$$

By Lemma 1, $z(x, 0) = \varepsilon k_\varepsilon(x)$ and $\|k_\varepsilon(x)\|^2 \le c$, so that, from (2.47) we get

$$\sup_{0\le t\le T} \|z(t)\|^2 \le c\varepsilon^2\left[1 + \sup_{0\le t\le T} \|h\|^2(s)\right].$$

Using (2.45), we get

$$\sup_{0\le t\le T} \|z(t)\|^2 \le c\varepsilon^2. \tag{2.48}$$

Next we show

$$\sup_{0\le t\le T} \|\partial_t z(x,t)\|^2 \le c\varepsilon.$$

Differentiating (2.44) for z with respect to t, we get

$$(z_t)_t = \varepsilon(z_t)_{xx} + (-D+\varepsilon P)(z_t)_x + (-D_t + \varepsilon P_t)z_x + [R+\varepsilon Q]z_t + (\varepsilon Q_t + R_t)z - \varepsilon h_t(x,t,\varepsilon). \tag{2.49}$$

Multiply (2.49) by z_t and integrate with respect to x. We get for $0 \le t \le T$,

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}\|z_t\|^2 \le {} & -\varepsilon\|z_{tx}\|^2 + c\|z_t\|^2 + \varepsilon c\|z_t\|\,\|z_{tx}\| \\ & + \int_0^1 \langle D_t z_x, z_t\rangle\,dx + \varepsilon\int_0^1 \langle p_t z_x, z_t\rangle\,dx \\ & + c\|z_t\|^2 + c\|z_t\|\,\|z\| + \varepsilon\|z_t\|\,\|h_t\|,\end{aligned} \tag{2.50}$$

$$\int_0^1 \langle D_t z_x, z_t\rangle\,dx = \sum_{i=1}^m \int_0^1 (\lambda_i)_t (z_i)_x (z_i)_t\,dx, \tag{2.51}$$

$$\begin{aligned}\int_0^1 (\lambda_i)_t(z_i)(z_i)_t\,dx &= \int_0^1 (\lambda_i)_t(z_i)_t(z_i)_x\,dx \\ &= -\int_0^1 [(\lambda_i)_t(z_i)_t]_x z_i\,dx \\ &= -\int_0^1 (\lambda_i)_{tx}(z_i)_t z_i - \int_0^1 (\lambda_i)_t(z_i)_{tx} z_i\,dx \\ &\le c[\|z\|\,\|z_t\| + \|z\|\,\|z_{tx}\|].\end{aligned} \tag{2.52}$$

We get from (2.51) and (2.52)

$$\int_0^1 \langle D_t z_x, z_t\rangle\,dx \le c[\|z\|\,\|z_t\| + \|z\|\,\|z_{tx}\|].$$

Since $\|z\| \le c\varepsilon$, by (2.48), this is

$$\le c[\varepsilon\|z_t\| + \varepsilon\|z_{tx}\|] \tag{2.53}$$

$$\int_0^1 \langle p_t z_x, z_t\rangle\,dx = \int_0^t \langle z_x, p^t z_t\rangle\,dx.$$

Integrating by parts, we get this is

$$\begin{aligned}&= \int_0^t \langle z, (p^t z_t)_x\rangle\,dx \\ &\le c\|z\|[\|z_{tx}\| + \|z_t\|].\end{aligned}$$

Since $\|z\| \le c\varepsilon$ by (2.48) this is

$$\begin{aligned}&\le c\varepsilon[\|z_{tx}\| + \|z_t\|] \\ &\le \tfrac{1}{10}\|z_{tx}\|^2 + c\varepsilon^2 + c\varepsilon\|z_t\|\end{aligned} \tag{2.54}$$

where we used $2ab \leq a^2/\delta + \delta b^2$. Using (2.50), (2.53) and (2.54), we obtain

$$\begin{aligned}\tfrac{d}{dt}\|z_t\|^2 \leq & -\varepsilon\|z_t x\|^2 + c\|z_t\|^2 + \varepsilon c\|z_t\|\,\|z_{tx}\| \\ & + c[\varepsilon\|z_t\| + \varepsilon\|z_{tx}\|] + \tfrac{\varepsilon}{10}\|z_{tx}\|^2 + c\varepsilon^3 + c\varepsilon^2\|z_t\| \\ & + c\|z_t\|^2 + c\|z_t\|\,\|z\| + \varepsilon\|z_t\|\,\|h_t\|.\end{aligned} \tag{2.55}$$

Using (2.55) and $\|z\| \leq c\varepsilon$, we get

$$\begin{aligned}\tfrac{d}{dt}\|z_t\|^2 \leq & -\varepsilon\|z_{tx}\|^2 + c\|z_t\|^2 \\ & + c\varepsilon + \varepsilon^2\|h_t\|^2 + \varepsilon\|\|z_{tx}\|^2.\end{aligned}$$

Using (2.45), we get

$$\tfrac{d}{dt}\|z_t\|^2 \leq c[\|z_t\|^2 + \varepsilon].$$

Using Gronwall's inequality we get

$$\|z_t(t)\|^2 \leq c[\|z_t(0)\| + \varepsilon]. \tag{2.56}$$

Using the partial differential equation (2.44) we can express $z_t(x, 0)$ as

$$z_t(x, 0) = L_\varepsilon z(x, t)|_{t=0} - \varepsilon h(x, 0, \varepsilon).$$

Since $z(x, 0) = \varepsilon k(x, \varepsilon)$, by (2.43)

$$\begin{aligned}z_t(x, 0) = & \varepsilon^2\partial_x^2 k(x, \varepsilon) + \varepsilon[-D(x, 0) + \varepsilon p(x, 0)]\partial_x k(x, \varepsilon) \\ & + \varepsilon[R(x, 0) + \varepsilon Q(x, 0)]k(x, \varepsilon) - \varepsilon h(x, 0, \varepsilon).\end{aligned} \tag{2.57}$$

By Lemma 1, (2.16) we have

$$\begin{gathered}\|\partial_x^k k(x, \varepsilon)\|^2 \leq \varepsilon^{1-2k}, \qquad k = 1, 2, \\ \|k(x, \varepsilon)\|^2 \leq c.\end{gathered}$$

We get from (2.57)

$$\|z_t(x, 0)\|^2 \leq c\varepsilon,$$

so that from (2.56) we get

$$\sup_{0\leq t\leq T} \|z_t(x)\|^2 \leq c\varepsilon. \tag{2.58}$$

Next we show

$$\sup_{0\leq t\leq T} \|z_x(t)\|^2 \leq c\varepsilon.$$

Multiply (2.44) by z and integrate with respect to x and integrate by parts. We get

$$\begin{aligned}\varepsilon\|z_x\|^2 \leq & \|z\|\,\|z_t\| + c\|z\|^2 + \varepsilon c\|z\|\,\|z_x\| + c\|z\|^2 \\ & + \varepsilon c\|z\|^2 + \varepsilon\|z\|\,\|h\|.\end{aligned}$$

Using previous estimates (2.48) for $\|z\|^2$, (2.58) for $\|z_t\|$ and (2.45) for $\|h\|$, we get

$$\varepsilon\|z_x\|^2 \leq c\varepsilon^3/2 + c\varepsilon^2 + c\varepsilon^2\|z_x\| + c\varepsilon^2 + c\varepsilon^3 + c\varepsilon^2.$$

Using $c\varepsilon^2\|z_x\| \le \varepsilon^2[c^2 + \|z_x\|^2]$ we get for small ε,

$$(\varepsilon - \varepsilon^2)\|z_x\|^2 \le c\varepsilon^{3/2},$$

from which we get $\|z_x\|^2 \le c\varepsilon^{1/2}$ so that

$$\sup_{0\le t\le T} \|z_x(t)\|^2 \le c\varepsilon^{1/2}. \tag{2.59}$$

Since $z(0,t) = 0$,

$$z(x,t) = \int_0^x z_y(y,t)\,dy.$$

By Schwarz's inequality, $|z(x,t)| \le \|z_x\|(t) \le c\varepsilon^{1/4}$, i.e.

$$\sup_{\substack{0\le x\le 1\\ 0\le t\le T}} |z(x,t)| \le c\varepsilon^{1/4}.$$

This completes the proof of Lemma 2.

Now we shall prove the theorem.

Proof of theorem. Let $u^\varepsilon(x,t)$ be the solution of (1.2), (1.3), (1.4) and (1.5) and $u^0(x,t)$ be the solution of (1.1), $(1.3)'$, $(1.4)'$ and $(1.5)'$. Since $u = Tv$ by (2.6), we get

$$u^\varepsilon(x,t) = T(x,t)v^\varepsilon, \quad u^0(x,t) = T(x,t)v^0(x,t).$$

Denote $B(x,t,\varepsilon) = T(x,t)b^\varepsilon(x,t)$, where $b^\varepsilon(x,t)$ is given by Lemma 1.

Consider

$$\begin{aligned} &u^\varepsilon(x,t) - u^0(x,t) - B(x,t,\varepsilon)\\ &\quad = T(x,t)[v^\varepsilon(x,t) + v^0(x,t) - b^\varepsilon(x,t)] = T(x,t)z^\varepsilon \end{aligned}$$

by (2.40). Then by Lemma 2, (2.41), we get

$$\sup_{\substack{0\le x\le 1\\ 0\le t\le T}} |u^\varepsilon(x,t) - u^0(x,t) - B(x,t,\varepsilon)| \le c\varepsilon^{1/4}.$$

The proof of the theorem is complete.

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