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LORENTZ SPACES THAT ARE ISOMORPHIC TO SUBSPACES OF L^1

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ABSTRACT. We show which Lorentz spaces are isomorphic to subspaces of L^1 and which are not.

The class of subspaces of L^1 is known to be quite big. Even if one considers only symmetric subspaces of L^1, i.e. subspaces that have symmetric basis or that are isomorphic to a rearrangement invariant function space, one finds that the class is quite rich.

It can be shown that those spaces are isomorphic to averages of Orlicz spaces [3, 11]. It was first believed that those spaces are in fact just Orlicz spaces [1]. Examples were given that this is not the case [3]. These examples have an abstract nature. Therefore our paper serves two purposes. On one hand it decides the natural question, which Lorentz spaces embed into L^1 and on the other hand it gives some examples of spaces that show up naturally in functional analysis and that are symmetric subspaces of L^1 though they are not Orlicz spaces.

In particular we show that the Lorentz spaces $L^{p,q}$ embed into L^1 if and only if $1 \le q \le p < 2$ or $p = q = 2$.

We would like to mention that infinite-dimensional versions of the Orlicz spaces that we use in our standard embedding were already used in [6] for some other purpose. This type of space was first studied by Rosenthal [12].

In §1 we state the theorem and derive some corollaries. Also we present some propositions from which the theorem follows. The propositions are proved in the following section.

0. PRELIMINARIES

A basis $\{e_i\}_{i=1}^n$ of a normed space is C-symmetric if we have for all signs $\varepsilon_i = \pm 1$, $x_i \in \mathbf{R}$, and all permutations π

$$\left\| \sum_{i=1}^n x_i e_i \right\| \le C \left\| \sum_{i=1}^n \varepsilon_i x_{\pi(i)} e_i \right\| .$$

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The Banach-Mazur distance of two Banach spaces E and F is $d(E,F) = \inf\{\|I\|\|I^{-1}\| | I$ is an isomorphism between E and $F\}$. For a given $x \in \mathbf{R}^n$ the sequence x_i^*, $i = 1, \dots, n$, is the decreasing rearrangement of $|x_i|$, $i = 1, \dots, n$. If f is a measurable function on $[0,1]$ we denote by f^* the decreasing rearrangement of $|f|$. Let $1 = a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. The Lorentz space $d(a,p)$ is $\mathbf{R}^n$ with norm

$$\|x\| = \left(\sum_{i=1}^{n} a_i |x_i^*|^p\right)^{1/p}$$

and, if W is a nonincreasing, continuous, nonnegative function on $(0,1]$ with $\int_0^1 W(t)\,dt = 1$, the Lorentz space $L_W^p[0,1]$ is the space of measurable functions with norm

$$\|f\| = \left(\int_0^1 |f^*(t)|^p W(t)\,dt\right)^{1/p}.$$

A Banach space has cotype 2 with constant C if we have for all sequences of vectors x_i, $i = 1, \dots, n$,

$$\left(\sum_{i=1}^{n} \|x_i\|^2\right)^{1/2} \leq C \operatorname{Ave}_{\pm} \left\|\sum_{i=1}^{n} \pm x_i\right\|$$

and a Banach lattice is 2-concave with constant C if

$$\left(\sum_{i=1}^{n} \|x_i\|^2\right)^{1/2} \leq C \left\|\left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}\right\|.$$

For a given Banach space we denote the infimum of these numbers C by $\operatorname{concave}_2(E)$.

1. The theorem

Theorem 1.1. *Let $1 \leq p < 2$ and W be a nonincreasing, positive, continuous function on $(0,1]$ with $\int_0^1 W(t)\,dt = 1$. Then the following are equivalent:*

(i) *L_W^p is isomorphic to a subspace of L^p.*

(ii) *L_W^p has cotype 2 (is 2-concave).*

(iii) *There is a $C > 0$ such that we have for all $x \in [0,1]$*

$$\int_0^x W(t) t^{-p/2}\,dt \leq C x^{-p/2} \int_0^x W(t)\,dt. \tag{1.1}$$

Proposition 1.2. *Let $1 \leq p \leq 2$ and $1 = a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ such that*

$$a_1 + \left(1 - \frac{p}{2}\right) \sum_{j=1}^{k} a_j j^{-p/2} \leq C k^{1-p/2} a_k, \qquad k = 1, \dots, n. \tag{1.2}$$

Then there is a subspace E of l^p such that $d(E, d(a,p)) \leq CD$ where D is a universal constant.

Proposition 1.2 will be proved in §3.

Proposition 1.3. *Let $1 \leq p < 2$ and $1 = a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. Then we have*

$$\sum_{i=1}^{k} a_i i^{-p/2} \leq D_p k^{-p/2} \sum_{i=1}^{k} a_i, \qquad k = 1, \ldots, n,$$

where D_p depends on p and the 2-concavity constant of $d(a, p)$ but not on n.

Proposition 1.3 follows from Lemma 4.3 and Proposition 4.4 with $s = 2/(2-p)$.

Lemma 1.4 [9, p. 306]. *Let X be a separable Banach space and $1 \leq p < \infty$ such that for every finite-dimensional subspace E of X there is a subspace $\widetilde{E}$ of l^p with $d(E, \widetilde{E}) \leq d$. Then there is a subspace Y of $L^p[0,1]$ such that $d(X, Y) \leq d$.*

Proof of Theorem 1.1. If L_W^p is isomorphic to a subspace of L^p then L_W^p has cotype 2, or, which is equivalent [4, 10], is 2-concave. We show first that L_W^p is isomorphic to a subspace of L^p if there is a constant C such that (1.1) holds. By Lemma 1.4 it is enough to show that the subspaces of L_W^p spanned by the vectors $\chi_{[(i-1)/n, i/n]}$, $i = 1, \ldots, n$, are uniformly isomorphic to subspaces of L^p. We have

$$\left\| \sum_{i=1}^{n} x_i \chi_{[(i-1)/n, i/n]} \right\|_{L_W^p} = \left(\sum_{i=1}^{n} |x_i^*|^p \int_{(i-1)/n}^{i/n} W(t)\, dt \right)^{1/p}.$$

We put

$$a_i = \int_{(i-1)/n}^{i/n} W(t)\, dt \left(\int_0^{1/n} W(t)\, dt \right)^{-1}, \qquad i = 1, \ldots, n, \tag{1.3}$$

and the space $d(a, p)$ is isometric to the subspace of L_W^p spanned by

$$\chi_{[(i-1)/n, i/n]}, \qquad i = 1, \ldots, n.$$

By our hypothesis (1.1) we have

$$\sum_{i=1}^{k} \int_{(i-1)/n}^{i/n} W(t) \left(\frac{i}{n}\right)^{-p/2} dt \leq \int_0^{k/n} W(t) t^{-p/2}\, dt$$
$$\leq C \left(\frac{k}{n}\right)^{-p/2} \int_0^{k/n} W(t)\, dt = C \left(\frac{k}{n}\right)^{-p/2} \sum_{i=1}^{k} \int_{(i-1)/n}^{i/n} W(t)\, dt.$$

Therefore we get

$$\sum_{i=1}^{k} a_i i^{-p/2} \leq C k^{-p/2} \sum_{i=1}^{k} a_i. \tag{1.4}$$

This means in particular

$$\frac{1}{2} \sum_{i \leq k/(2C)^{2/p}} a_i i^{-p/2} \leq \sum_{i \leq k/(2C)^{2/p}} a_i (i^{-p/2} - C k^{-p/2}) \leq C k^{-p/2} \sum_{i \geq k/(2C)^{2/p}}^{k} a_i.$$

Therefore we obtain

$$\frac{1}{2}\sum_{i\le k/(2C)^{2/p}} a_i i^{-p/2} \le C k^{1-p/2} a_{[k/(2C)^{2/p}]}.$$

This implies

$$\sum_{i=1}^{k} a_i i^{-p/2} \le C' k^{1-p/2} a_k, \qquad 1 \le k \le \frac{n}{(2C)^{2/p}},$$

and C' depends only on C. In case this inequality does not hold for $k \ge n(2C)^{-2/p}$ we simply change the a_k. We put

$$\tilde{a}_k = a_{[n/(2C)^{2/p}]} \quad \text{for } k > n(2C)^{-2/p}.$$

By this we do not change the norm of $d(a,p)$ significantly. This follows by triangle-inequality. On the other hand we establish inequality (1.2) so that we can apply Proposition 1.2 to $d(a,p)$.

Now we show that (1.1) holds if L_W^p has cotype 2 which is in particular true if L_W^p is isomorphic to a subspace of L^p.

By [4, 10] a Banach lattice has cotype 2 if and only if it is 2-concave. If L_W^p is 2-concave then the 2-concavity constants of the spaces $d(a,p)$ with a defined by (1.3) have to be uniformly bounded. We apply Proposition 1.3 and get

$$\sum_{i=1}^{k} a_i i^{-p/2} \le D_p k^{-p/2} \sum_{i=1}^{k} a_i, \qquad k = 1, \dots, n.$$

Therefore we get

$$\begin{aligned}\int_{1/n}^{k/n} W(t) t^{-p/2}\, dt &= \sum_{i=2}^{k} \int_{(i-1)/n}^{i/n} W(t) t^{-p/2}\, dt \\ &\le \sum_{i=2}^{k} \left(\frac{i-1}{n}\right)^{-p/2} \int_{(i-1)/n}^{i/n} W(t)\, dt.\end{aligned}$$

By (1.3)

$$= \int_0^{1/n} W(t)\, dt \sum_{i=2}^{k} \left(\frac{i-1}{n}\right)^{-p/2} a_i \le \int_0^{1/n} W(t)\, dt\, 2 \sum_{i=2}^{k} \left(\frac{i}{n}\right)^{-p/2} a_i.$$

By Proposition 1.3

$$\le \int_0^{1/n} W(t)\, dt\, C_p \left(\frac{k}{n}\right)^{-p/2} \sum_{i=1}^{k} a_i = C_p \left(\frac{k}{n}\right)^{-p/2} \int_0^{k/n} W(t)\, dt.$$

We obtain

$$\int_{1/n}^{k/n} W(t) t^{-p/2}\, dt \le C_p \left(\frac{k}{n}\right)^{-p/2} \int_0^{k/n} W(t)\, dt$$

for all $1 \le k \le n < \infty$. □

Corollary 1.5. *The Lorentz spaces* $L^{p,q}[0,1]$ *are isomorphic to subspaces of* $L^1[0,1]$ *if and only if* $1 \le q \le p < 2$ *or* $p = q = 2$.

For the proof we also require the following notion and lemma [8]. Suppose that $\{e_i\}_{i=1}^n$ is a 1-symmetric basis of E. Then there is an Orlicz function M_E such that

$$\left\|\sum_{i=1}^k e_i\right\|_E \le \left\|\sum_{i=1}^k e_i\right\|_{M_E} \le 2\left\|\sum_{i=1}^k e_i\right\|_E, \qquad k = 1, \dots, n.$$

We say that M_E is associated to E.

Lemma 1.6. *Let* $\{e_i\}_{i=1}^n$ *be a* 1*-symmetric basis of* E *and suppose that* E *is* C*-isomorphic to a subspace of* L^1. *Then we have for all* x

$$\left\|\sum_{i=1}^n x_i e_i\right\|_{M_E} \le CD\left\|\sum_{i=1}^n x_i e_i\right\|_E$$

where D *is a universal constant and* M_E *an associated Orlicz function.*

Proof of Corollary 1.5. The positive direction for $1 \le q \le p < 2$ is a consequence of Theorem 1.1. The negative direction follows from Lemma 1.6. The spaces $L^{2,q}$ do not have cotype 2 unless $q = 2$ [2]. Therefore those spaces cannot be isomorphic to subspaces of L^1. If $p > 2$ then $L^{p,q}$ does not have cotype 2 [2]. □

2. Some required lemmas

Here we collect some results that were proved in other places and that are essential for the proof of the main result.

We prove the results by using averages over permutations. For other methods see [6, 12].

Lemma 2.1. *Let* $1 \le p \le 2$ *and* $a_{ij} \ge 0$, $i = 1, \dots, n$, $j = 1, \dots, N$, *with* $\sum_{i=1}^n \sum_{j=1}^N |a_{ij}|^p = n$. *Then* $\mathbf{R}^n$ *with norm*

$$\|x\| = \left(\sum_{j=1}^N \operatorname{Ave}_\pi \left(\sum_{i=1}^n |x_{\pi(i)} a_{ij}|^2\right)^{p/2}\right)^{1/p}$$

is $\sqrt{2}$*-isomorphic to a* 1*-symmetric subspace of* l^p.

Proof. Let I be the isomorphism mapping $x \in \mathbf{R}^n$ onto an element from $l^p_{N2^n n!}$ given by $(2^n n!)^{-1/p}(\sum_{i=1}^n x_i \varepsilon_i a_{\pi(i)j})_{j,\varepsilon,\pi}$. We have

$$\|I(x)\|_p = \left(\sum_{j=1}^N \frac{1}{n!}\sum_\pi 2^{-n}\sum_\varepsilon \left|\sum_{i=1}^n x_i \varepsilon_i a_{\pi(i)j}\right|^p\right)^{1/p}.$$

By Khintchine-inequality we get

$$\|I(x)\|_p \le \left(\sum_{j=1}^{N} \mathrm{Ave}_\pi \left(\sum_{i=1}^{n} |x_{\pi(i)} a_{ij}|^2\right)^{p/2}\right)^{1/p} \le \sqrt{2}\|I(x)\|_p. \quad \square$$

Lemma 2.2. *Let* $1 \le p \le q < \infty$, *then we have*

$$\frac{1}{10}\left\{\left(\frac{1}{n}\sum_{k=1}^{n} |S(k)|^p\right)^{1/p} + \left(\frac{1}{n}\sum_{k=n+1}^{n^2} |S(k)|^q\right)^{1/q}\right\}$$

$$\le \left(\frac{1}{n!}\sum_{\pi}\left(\sum_{i=1}^{n} |a_{i\pi(i)}|^q\right)^{p/q}\right)^{1/p}$$

$$\le \left(\frac{1}{n}\sum_{k=1}^{n} |S(k)|^p\right)^{1/p} + \left(\frac{1}{n}\sum_{k=n+1}^{n^2} |S(k)|^q\right)^{1/q}$$

where $S(k)$, $k = 1, \dots, n^2$, *is the decreasing rearrangement of* a_{ij}, $i, j = 1, \dots, n$.

Proof. The result for $p = 1$ and a constant equal to $\frac{1}{5}$ is contained in [7]. To pass to the general case we consider the matrix a_{ij}^p instead of a_{ij}. Also, as second exponent we have to choose q/p which is possible since $p \le q$. $\square$

As an immediate consequence we get the following lemma.

Lemma 2.3. *Let* $1 \le p \le 2$. *Then we have for all* $x \in \mathbf{R}^n$ *and* $k = 1, \dots, n$

$$C\left(\left(\sum_{i=1}^{k} |x_i^*|^p\right)^{1/p} + k^{1/p-1/2}\left(\sum_{i=k+1}^{n} |x_i^*|^2\right)^{1/2}\right)$$

$$\le k^{1/p}\left(\frac{1}{n!}\sum_{\pi}\left(\sum_{i \le n/k} |x_{\pi(i)}|^2\right)^{p/2}\right)^{1/p}$$

$$\le \left(\sum_{i=1}^{k} |x_i^*|^p\right)^{1/p} + k^{1/p-1/2}\left(\sum_{i=k+1}^{n} |x_i^*|^2\right)^{1/2}$$

where x_i^*, $i = 1, \dots, n$, *denotes the decreasing rearrangement of* $|x_i|$, $i = 1, \dots, n$, *and* $C > 0$ *is a universal constant.*

Proof. If we apply Lemma 2.2 to the matrix

$$a_{ij} = \begin{cases} x_i & \text{if } 1 \le j \le n/k, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$\left[\frac{n}{k}\right]\sum_{i\le k}|x_i^*|^p \le \sum_{k=1}^{n}|S(k)|^p \le \frac{n}{k}\sum_{i\le k}|x_i^*|^p,$$

$$\left[\frac{n}{k}\right]\sum_{i>2k}|x_i^*|^2 \le \sum_{k=n+1}^{n^2}|S(k)|^2 \le \frac{n}{k}\sum_{i>k}|x_i^*|^2. \quad \square$$

3. The standard embedding

Proof of Proposition 1.2. We consider $\mathbf{R}^n$ with norm

(3.1)

$$\|x\| = \left\{\sum_{k=1}^{n-1}(a_k - a_{k+1})\left|\left(\sum_{i=1}^{k}|x_i^*|^p\right)^{1/p} + k^{1/p-1/2}\left(\sum_{i=k+1}^{n}|x_i^*|^2\right)^{1/2}\right|^p + a_n\sum_{i=1}^{n}|x_i|^p\right\}^{1/p}.$$

By Lemmas 2.1 and 2.3 these spaces are C-isomorphic to subspaces of L^p where C is a universal constant. We verify that this norm is equivalent to the norm of the n-dimensional Lorentz space $d(a,p)$. We have

$$\begin{aligned}\|x\| &\ge \left(\sum_{k=1}^{n-1}(a_k - a_{k+1})\sum_{i=1}^{k}|x_i^*|^p + a_n\sum_{i=k}^{n}|x_i|^p\right)^{1/p}\\ &= \left(\sum_{i=1}^{n}|x_i^*|^p\left(a_n + \sum_{k=i}^{n-1}a_k - a_{k+1}\right)\right)^{1/p}\\ &= \left(\sum_{i=1}^{n}|x_i^*|^p a_i\right)^{1/p}.\end{aligned}$$

On the other hand, by the same calculation we get

$$\|x\| \le \left(\sum_{i=1}^{n}|x_i^*|^p a_i\right)^{1/p} + \left(\sum_{k=1}^{n-1}(a_k - a_{k+1})k^{1-p/2}\left(\sum_{i=k+1}^{n}|x_i^*|^2\right)^{p/2}\right)^{1/p}.$$

It is left to estimate the second summand. Since we have

$$\left(\sum_{i=k+1}^{n}|x_i^*|^2\right)^{1/2} \le \left(|x_k^*|^p\sum_{i=1}^{k}C_i + \sum_{i=k+1}^{n}C_i|x_i^*|^p\right)^{1/p}$$

with $\sum_{i=1}^{j} C_i = j^{p/2}$ for $j = 1, \ldots, n$ we get

$$\left(\sum_{k=1}^{n-1}(a_k - a_{k+1})k^{1-p/2}\left(\sum_{i=k+1}^{n}|x_i^*|^2\right)^{p/2}\right)^{1/p}$$

$$\le \left(\sum_{k=1}^{n-1}(a_k - a_{k+1})k^{1-p/2}\left(|x_k^*|^p\sum_{i=1}^{k}C_i + \sum_{i=k+1}^{n}C_i|x_i^*|^p\right)\right)^{1/p}$$

$$\le \left(\sum_{k=1}^{n-1}(a_k - a_{k+1})k|x_k^*|^p\right)^{1/p} + \left(\sum_{k=1}^{n-1}(a_k - a_{k+1})k^{1-p/2}\sum_{i=k+1}^{n}C_i|x_i^*|^p\right)^{1/p}$$

Since

$$\sum_{k=1}^{j}k(a_k - a_{k+1}) = -ja_{j+1} + \sum_{k=1}^{j}a_k \le \sum_{k=1}^{j}a_k \quad \text{for } j = 1, \ldots, n-1$$

we can estimate the first summand by $(\sum_{k=1}^{n-1} a_k|x_k^*|^p)^{1/p}$. Again, it is left to estimate the second summand

$$\left(\sum_{k=1}^{n-1}(a_k - a_{k+1})k^{1-p/2}\sum_{i=k+1}^{n}C_i|x_i^*|^p\right)^{1/p}$$

$$\le \left(\sum_{i=1}^{n-1}|x_i^*|^pC_i\sum_{k=1}^{i}(a_k - a_{k+1})k^{1-p/2} + C_n|x_n^*|^p\sum_{k=1}^{n-1}(a_k - a_{k+1})k^{1-p/2}\right)^{1/p}.$$

We have

$$C_1 = 1, \quad C_k = k^{p/2} - (k-1)^{p/2} \le \frac{p}{2}(k-1)^{p/2-1} \quad \text{for } k = 2, \ldots, n$$

and

$$\sum_{k=1}^{i}(a_k - a_{k+1})k^{1-p/2} \le a_1 + \sum_{k=2}^{i}a_k(k^{1-p/2} - (k-1)^{1-p/2})$$

$$\le a_1 + \sum_{k=2}^{i}a_k\left(1 - \frac{p}{2}\right)(k-1)^{-p/2}$$

$$\le a_1 + 2\left(1 - \frac{p}{2}\right)\sum_{k=2}^{i}a_kk^{-p/2}.$$

By the hypothesis (1.2) we get

$$\sum_{k=1}^{i}(a_k - a_{k+1})k^{1-p/2} \le 2Ci^{1-p/2}a_i$$

Therefore we get, with a universal constant D,

$$\left(\sum_{k=1}^{n-1}(a_k - a_{k+1})k^{1-p/2}\sum_{i=k}^{n} C_i|x_i^*|^p\right)^{1/p}$$
$$\leq CD\left(\sum_{i=1}^{n-1}|x_i^*|^p a_i + a_{n-1}|x_n^*|^p\right)^{1/p}. \quad \square$$

4. 2-CONCAVITY OF LORENTZ SPACES

Proposition 4.1. *Let* $1 \leq p \leq 2$ *and* $1 = a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. *We have*

$$\text{concave}_2(d(a,p)) = \sup\left\{ \frac{\left(\sum_{i=1}^{n} a_i|b_i|^p\right)^{1/p}}{\left(\sum_{i=1}^{n} a_i|c_i|^p\right)^{1/p}} \;\middle|\; \|b\|_2 = \|c\|_2, \; \sum_{i=1}^{k} b_i^2 \geq \sum_{i=1}^{k} c_i^2, \; \begin{array}{l} b_1 \geq b_2 \geq \cdots \geq b_n \geq 0, \\ c_1 \geq c_2 \geq \cdots \geq c_n \geq 0, \end{array} k = 1, \ldots, n \right\}. \tag{4.1}$$

Compare also [13].

For the proof we require the following lemma which is essentially the same as Theorem 4.6 in [5].

Lemma 4.2. *Let* $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$ *and* $y_1 \geq y_2 \geq \cdots \geq y_n \geq 0$ *such that*

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i, \quad \sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i, \qquad k = 1, \ldots, n.$$

Then there are numbers $d_r \geq 0$ *with* $\sum_{r=1}^{N} d_r = 1$ *and permutations* π_r, $r = 1, \ldots, N$, *such that*

$$y_k = \sum_{r=1}^{N} d_r x_{\pi_r(k)}, \qquad k = 1, \ldots, n.$$

Proof of Proposition 4.1. For simplicity we refer to the right-hand expression as C_2. We show first that the left-hand expression is smaller than the right-hand one.

$$\left(\sum_{l=1}^{N}\|x_l\|^2\right)^{1/2} = \left(\sum_{l=1}^{N}\left(\sum_{i=1}^{n}|x_l(i)^*|^p a_i\right)^{2/p}\right)^{1/2}$$

$$\leq \left(\sum_{i=1}^{n}|a_i|\left(\sum_{l=1}^{N}|x_l(i)^*|^2\right)^{p/2}\right)^{1/p}$$

$$\leq C_2^{\sigma}\left(\sum_{i=1}^{n}|a_i|\left(\sum_{l=1}^{N}|x_l(\sigma(i))|^2\right)^{p/2}\right)^{1/p}.$$

The last inequality holds because

$$\sum_{i=1}^{k}\sum_{l=1}^{N}|x_l^*(i)|^2 = \sum_{l=1}^{N}\sum_{i=1}^{k}|x_l^*(i)|^2 \geq \sum_{l=1}^{N}\sum_{i=1}^{k}|x_l(\sigma(i))|^2$$

is valid for all permutations σ. Therefore we get

$$\left(\sum_{l=1}^{N}\|x_l\|^2\right)^{1/2} \leq C_2\left\|\left\{\left(\sum_{l=1}^{N}|x_l(i)|^2\right)^{1/2}\right\}_{i=1}^{n}\right\|.$$

Now we show the opposite inequality. Let $b, c \in \mathbf{R}^n$ with $\|b\|_2 = \|c\|_2$ and

$$\sum_{i=1}^{k} b_i^2 \geq \sum_{i=1}^{k} c_i^2$$

for all $k = 1, \dots, n$. By Lemma 4.2 we find numbers $d_r > 0$ and $\sum d_r^2 = 1$ and permutations π_r such that

$$c_k^2 = \sum_{r=1}^{N} d_r^2 b_{\pi_r(k)}^2, \qquad k = 1, \dots, n.$$

We have

$$\|b\| = \left(\sum_{r=1}^{N} d_r^2\|b\|^2\right)^{1/2} = \left(\sum_{r=1}^{N}\|(d_r b_{\pi_r(k)})_{k=1}^{n}\|^2\right)^{1/2}$$

$$\leq \operatorname{concave}_2(d(a,p))\left\|\left\{\left(\sum_{r=1}^{N}|d_r b_{\pi_r(k)}|^2\right)^{1/2}\right\}_{k=1}^{n}\right\|$$

$$= \operatorname{concave}_2(d(a,p))\|c\|. \quad \square$$

Lemma 4.3. *Let* $1 \leq p < 2$ *and* $1 = a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. *Then we have*

$$\left(\sum_{i=1}^{k}|a_i|^{2/(2-p)}\right)^{(2-p)/2} \leq (\operatorname{concave}_2(d(a,p)))k^{-p/2}\sum_{i=1}^{k} a_i, \qquad k = 1, \dots, n.$$

Lemma 4.3 follows immediately from Proposition 4.1.

Proposition 4.4. *Let* $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$ *and* $2 \le s < \infty$ *such that*

$$\left(\sum_{i=1}^{k} |a_i|^s\right)^{1/s} \le Ck^{1/s-1}\sum_{i=1}^{k} a_i, \qquad k = 1, \dots, n. \tag{4.2}$$

Then we have

$$\sum_{i=1}^{k} a_i i^{1/s-1} \le Dk^{1/s-1}\sum_{i=1}^{k} a_i, \qquad k = 1, \dots, n,$$

where $D = D(C, s)$ *does not depend on* n.

Lemma 4.5. *Let* $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$ *and* $2 \le s < \infty$ *such that*

$$k^{1/s-1}\sum_{i=1}^{k} a_i \le Cl^{1/s-1}\sum_{i=1}^{l} a_i, \qquad 1 \le k \le l \le n. \tag{4.3}$$

Then we have

$$k^{1/s-1}\sum_{i=1}^{k} a_i \le 2(2C)^{s/(s-1)}k^{1/s}a_k, \qquad 1 \le k \le \frac{n}{(2C)^{s/(s-1)}}.$$

Proof. Let $\alpha = [(2C)^{s/(s-1)}] + 1$.

$$k^{1/s-1}\sum_{i=1}^{k} a_i \le C(\alpha k)^{(1-s)/s}\sum_{i=1}^{\alpha k} a_i \le \frac{1}{2}k^{1/s-1}\sum_{i=1}^{k} a_i + \frac{1}{2}k^{1/s-1}\sum_{i=k+1}^{\alpha k} a_i.$$

Therefore

$$\frac{1}{2}k^{1/s-1}\sum_{i=1}^{k} a_i \le \frac{1}{2}k^{1/s-1}\sum_{i=k+1}^{\alpha k} a_i \le \frac{1}{2}k^{1/s-1}\alpha k a_k = \frac{\alpha}{2}k^{1/s}a_k. \quad \square$$

Lemma 4.6. *Let* $2 \le s < \infty$ *and* $a_1 \ge a_2 \ge a_3 \ge \cdots \ge a_n \ge 0$ *such that*

$$\left(\sum_{i=1}^{k} |a_i|^s\right)^{1/s} \le Ck^{1/s}a_k, \qquad k = 1, \dots, n.$$

Then we have $a_k \le \frac{1}{2}D^{1/s}a_{Dk}$, $1 \le k \le n/D$, *where* $D \ge 2\exp(2^s C^{2s})$ *and* $D \in \mathbf{N}$.

Proof. We have by hypothesis

$$\sum_{i=1}^{k+j} |a_i|^s \le C^s(k+j)|a_{k+j}|^s.$$

This implies

$$k|a_k|^s \le C^s(k+j)|a_{k+j}|^s$$

which implies

$$C^s Dk|a_{Dk}|^s \ge \sum_{i=1}^{Dk} |a_i|^s \ge \sum_{i=0}^{Dk-k} |a_{k+i}|^s \ge C^{-s}|a_k|^s k \sum_{i=0}^{Dk-k} \frac{1}{k+i}.$$

Since

$$\sum_{i=0}^{Dk-k} \frac{1}{k+i} \geq \ln\left(\frac{Dk}{k+1}\right) \geq \ln\frac{D}{2},$$

we get $C^{2s}D|a_{Dk}|^s \geq |a_k|^s \ln\frac{D}{2}$. Since $D \geq 2e^{2^s C^{2s}}$ we get $(D/2^s)|a_{Dk}|^s \geq |a_k|^s$. □

Lemma 4.7. *Let* $2 \leq s < \infty$, $D \in \mathbf{N}$, *and* $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ *such that*

(4.4) $$a_k \leq \tfrac{1}{2}D^{1/s}a_{Dk} \quad \textit{for } 1 \leq k \leq n/D.$$

Then we have

$$\sum_{i=1}^{k} a_i i^{1/s-1} \leq D^2 k^{1/s} a_k, \qquad 1 \leq k \leq \frac{n}{D}.$$

Proof. We show by induction

$$\sum_{i=1}^{D^k} a_i i^{1/s-1} \leq D(D^k)^{1/s} a_{D^k}, \qquad 1 \leq D^k \leq n.$$

We have

$$\begin{aligned}\sum_{i=1}^{D^{k+1}} a_i i^{1/s-1} &= \sum_{i=1}^{D^k} a_i i^{1/s-1} + \sum_{i=D^k+1}^{D^{k+1}} a_i i^{1/s-1} \\ &\leq D(D^k)^{1/s}a_{D^k} + a_{D^k}(D^k)^{1/s-1}(D^{k+1} - D^k) \\ &= a_{D^k}(D^k)^{1/s}(2D-1) \leq a_{D^k}(D^k)^{1/s}2D \leq a_{D^{k+1}}(D^{k+1})^{1/s}D. \quad \square\end{aligned}$$

Proof of Proposition 4.4. By (4.2) we have in particular

$$l^{1/s-1}\sum_{i=1}^{l} a_i \leq Ck^{1/s-1}\sum_{i=1}^{k} a_i, \qquad 1 \leq l \leq k \leq n.$$

By Lemma 4.5 we get

$$k^{1/s-1}\sum_{i=1}^{k} a_i \leq 2(2C)^{s/(s-1)}k^{1/s}a_k, \qquad 1 \leq k \leq \frac{n}{(2C)^{s/(s-1)}}.$$

This inequality together with (4.2) gives

$$\left(\sum_{i=1}^{k}|a_i|^s\right)^{1/s} \leq (2C)^{s/(s-1)+1}k^{1/s}a_k, \qquad 1 \leq k \leq \frac{n}{(2C)^{s/(s-1)}}.$$

By Lemma 4.6 we get

$$a_k \leq \frac{D^{1/s}}{2}a_{Dk}, \qquad 1 \leq k \leq \frac{n}{D(2C)^{s/(s-1)}},$$

and D is only depending on C. By Lemma 4.7 we obtain

$$\sum_{i=1}^{k} a_i i^{1/s-1} \leq D^2 k^{1/s} a_k, \qquad 1 \leq k \leq \frac{n}{D(2C)^{s/(s-1)}},$$

which gives

$$\sum_{i=1}^{k} a_i i^{1/s-1} \leq D^2 k^{1/s-1} \sum_{i=1}^{k} a_i, \qquad 1 \leq k \leq \frac{n}{D(2C)^{s/(s-1)}}.$$

For $k > n/D(2C)^{s/(s-1)}$ we get the inequality because a_i, $i = 1, \dots, n$, is a decreasing sequence. □

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