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ON TWO-CARDINAL PROPERTIES OF IDEALS

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ABSTRACT. We investigate two-cardinal properties of ideals. These properties involve notions such as Luzin sets, special coverings, etc. We apply our results to the ideals of meagre sets and of negligible sets in the real line. In case of the negligible sets, we relate these properties to caliber and precalibers of the measure algebra.

1. INTRODUCTION

We shall use standard set theoretical notation and terminology (see e.g. [K]). We work in the ZFC set theory. By CH we denote the Continuum Hypothesis, by GCH the General Continuum Hypothesis and by MA the Martin's Axiom. If κ is a cardinal then by MA_κ we denote the Martin's Axiom restricted to c.c.c. partial orders of powers less then κ. By c we denote continuum. Letters κ, λ, μ denote cardinal numbers and $\alpha, \beta, \gamma, \delta$ denote ordinal numbers.

By $P(X)$ we denote the family of all subsets of the set X. A family $\mathfrak{I} \subseteq P(X)$ is called an ideal on the set X if it is closed under finite unions and subsets, and contains all finite subsets of the set X. For any ideal $\mathfrak{I}$ on the set X we define the following four cardinal numbers:

$$\begin{aligned}
&\mathrm{add}(\mathfrak{I}) = \min\left\{|\mathfrak{X}|\colon \mathfrak{X} \subseteq \mathfrak{I} \& \bigcup \mathfrak{X} \notin \mathfrak{I}\right\}, \\
&\mathrm{cov}(\mathfrak{I}) = \min\left\{|\mathfrak{X}|\colon \mathfrak{X} \subset \mathfrak{I} \& \bigcup \mathfrak{X} = X\right\}, \\
&\mathrm{non}(\mathfrak{I}) = \min\{|A|\colon A \subseteq X \& A \notin \mathfrak{I}\}, \\
&\mathrm{cof}(\mathfrak{I}) = \min\{|\mathfrak{X}|\colon \mathfrak{X} \subset \mathfrak{I} \& (\forall A \in \mathfrak{I})(\exists B \in \mathfrak{X})(A \subseteq B)\}.
\end{aligned}$$

By ω^ω we denote the family of all sequences of natural numbers. On the set ω^ω we define the relation $\leq^*$ by

$$f \leq^* g \Leftrightarrow (\exists n \in \omega)(\forall m > n)(f(m) \leq g(m)).$$

Let

$$\mathfrak{b} = \min\{|\mathfrak{X}|\colon \neg(\exists f \in \omega^\omega)(\forall g \in \mathfrak{X})(g \leq^* f)\}$$

and

$$\mathfrak{d} = \min\{|\mathfrak{X}|\colon (\forall f \in \omega^\omega)(\exists g \in \mathfrak{X})(f \leq^* g)\}.$$

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Note that the notations $\mathfrak{b}$ and $\mathfrak{d}$ are due to E. van Douwen. A sequence $\langle f_\alpha : \alpha < \kappa \rangle$ of functions from ω^ω is called a κ-scale if $f_\alpha \leq^* f_\beta$ for $\alpha < \beta < \kappa$ and for every function $g \in \omega^\omega$ there exists $\alpha < \kappa$ such that $g \leq^* f_\alpha$. Note that κ-scale exists for some κ if and only if $\mathfrak{b} = \mathfrak{d}$.

Let $\mathbf{R}$ denote the set of real numbers and $\mathbf{Q}$ the set of rational numbers. Let $\mathscr{L}$ denote the ideal of Lebesgue measure subsets of $\mathbf{R}$ and let $\mathscr{K}$ denote the ideal of the first category subsets of $\mathbf{R}$. The following diagram shows the relations which holds between the cardinal functions introduced above for ideals $\mathscr{K}$ and $\mathscr{L}$:

$$\begin{array}{ccccccc}
\operatorname{cov}(\mathscr{L}) & \longrightarrow & \operatorname{non}(\mathscr{K}) & \longrightarrow & \operatorname{cof}(\mathscr{K}) & \longrightarrow & \operatorname{cof}(\mathscr{L}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
| & & \mathfrak{b} & \longrightarrow & \mathfrak{d} & & | \\
| & & \uparrow & & \uparrow & & | \\
\operatorname{add}(\mathscr{L}) & \longrightarrow & \operatorname{add}(\mathscr{K}) & \longrightarrow & \operatorname{cov}(\mathscr{K}) & \longrightarrow & \operatorname{non}(\mathscr{L})
\end{array}$$

In this diagram $\kappa \to \lambda$ means ZFC $\kappa \leq \lambda$. Two additional relations hold: $\operatorname{add}(\mathscr{K}) = \min(\operatorname{cov}(\mathscr{K}), \mathfrak{b})$ and $\operatorname{cof}(\mathscr{K}) = \max(\operatorname{non}(\mathscr{K}), \mathfrak{d})$. A detailed discussion of this diagram can be found in [F].

Let $\mathscr{B}$ denote the family of all Borel subsets of the real line. The measure algebra $\mathscr{R}$ is the quotient algebra $\mathscr{B}/\mathscr{L}$. Let $\mathfrak{B}$ be a complete boolean algebra and let κ, λ be cardinal numbers. We say that the algebra $\mathfrak{B}$ has (κ, λ)-caliber if for every family $\mathfrak{X} \subseteq \mathfrak{B}$ of cardinality κ there exists a subfamily $\mathscr{Y} \subseteq \mathfrak{X}$ of cardinality λ such that $\inf(\mathscr{Y}) > 0$. We say that the algebra $\mathfrak{B}$ has (κ, λ)-precaliber if for every family $\mathfrak{X} \subseteq \mathfrak{B}$ of cardinality κ there exists a centered subfamily $\mathscr{Y} \subseteq \mathfrak{X}$ of cardinality λ. It is well known that any c.c.c. boolean algebra has $(\aleph_1, \aleph_0)$-precaliber and that the measure algebra has $(\aleph_1, \aleph_0)$-caliber (see e.g. [T]).

Let $\mathscr{LOC}$ denote the family of all functions from ω into finite subsets of ω such that for every $n \in \omega$ we have $|f(n)| \leq n$. The following three lemmas are proved in [F]:

Lemma 1.1. *There are functions* $\mathbf{F}: \mathscr{K} \to \mathscr{L}$ *and* $\mathbf{G}: \mathscr{L} \to \mathscr{K}$ *such that for every* $A \in \mathscr{K}$ *and* $B \in \mathscr{L}$ *if* $\mathbf{F}(A) \subseteq B$ *then* $A \subset \mathbf{G}(B)$.

Lemma 1.2. *There are functions* $\mathbf{F}: \omega^\omega \to \mathscr{K}$ *and* $\mathbf{G}: \mathscr{K} \to \omega^\omega$ *such that for any* $A \in \mathscr{K}$ *and* $f \in \omega^\omega$ *if* $\mathbf{F}(f) \subseteq A$ *then* $f \leq^* \mathbf{G}(A)$.

Lemma 1.3. *There are functions* $\mathbf{F}: \omega^\omega \to \mathscr{L}$ *and* $\mathbf{G}: \mathscr{L} \to \mathscr{LOC}$ *such that for any* $f \in \omega^\omega$ *and* $A \in \mathscr{L}$ *if* $\mathbf{F}(f) \subseteq A$ *then for all but finitely many* $n \in \omega$ *we have* $f(n) \in \mathbf{G}(A)(n)$.

2. Arbitrary ideals

Let $\mathfrak{I}$ be an ideal on a set X. A subset A of X is a (κ, λ)-Luzin set for the ideal $\mathfrak{I}$ if $|A| = \kappa$ and for every $B \in \mathfrak{I}$ we have $|A \cap B| < \lambda$. Recall that $(\mathfrak{c}, \aleph_1)$-Luzin sets for $\mathscr{K}$ are called Luzin sets and $(\mathfrak{c}, \aleph_1)$-Luzin sets for $\mathscr{L}$ are called Sierpiński sets. It is well known that MA implies the existence of $(\mathfrak{c}, \mathfrak{c})$-Luzin sets for $\mathscr{K}$ and $\mathscr{L}$. The construction of $(\mathfrak{c}, \aleph_1)$-Luzin sets for this ideal in the absence of CH requires the forcing techniques. Namely, if $C = \{c_\alpha : \alpha < \kappa\}$ is a sequence of independent Cohen reals over the model M of ZFC then in the model $M[C]$ the set C is a $(\kappa, \aleph_1)$-Luzin set for the ideal $\mathscr{K}$. A similar fact is true for random reals. If $C = \{r_\alpha : \alpha < \kappa\}$ is a sequence of independent random reals over M then in $M[C]$ the set is a $(\kappa, \aleph_1)$-Luzin set for the ideal $\mathscr{L}$. It was proved by A. Krawczyk that if M $(\mathfrak{b} = \mathfrak{d})$ and r is a random real over M then in the model $M[r]$ there exists a $(\mathfrak{b}, \mathfrak{b})$-Luzin set for $\mathscr{L}$ (see [P, Theorem 3.1]). If c is a Cohen real over M then there are a $(\mathfrak{b}, \mathfrak{b})$-Luzin set and a $(\mathfrak{d}, \mathfrak{d})$-Luzin set for the ideal $\mathscr{K}$ in $M[c]$ (this easily follows from results from [CP]).

Definition 2.1. $L(\mathfrak{J}, \kappa, \lambda) \Leftrightarrow$ there exists a (κ, λ)-Luzin set for the ideal $\mathfrak{J}$.

A family $\mathfrak{X} \subseteq \mathfrak{J}$ is called a (κ, λ)-Rothberger family for the ideal if $|\mathfrak{X}| = \kappa$ and for every $\mathscr{Y} \subseteq \mathfrak{X}$ if $|\mathscr{Y}| = \lambda$ then $\bigcup \mathscr{Y} = X$. The existence of $(\mathfrak{c}, \mathfrak{c})$-Rothberger families for $\mathscr{K}$ or $\mathscr{L}$ can be easily deduced from MA. One Cohen real always produces a $(\mathfrak{c}, \aleph_1)$-Rothberger family for the ideal $\mathscr{L}$ (see [CP]).

Definition 2.2. $R(\mathfrak{J}, \kappa, \lambda) \Leftrightarrow$ there exists a (κ, λ)-Rothberger family for the ideal $\mathfrak{J}$.

A family $\mathfrak{X} \subseteq \mathfrak{J}$ is called a (κ, λ)-nonadditive family for the ideal $\mathfrak{J}$ if $|\mathfrak{X}| = \kappa$ and for every $\mathscr{Y} \subseteq \mathfrak{X}$ if $|\mathscr{Y}| = \lambda$ then $\bigcup \mathscr{Y} \notin \mathfrak{J}$. Notice that every (κ, λ)-Rothberger family for $\mathfrak{J}$ is a (κ, λ)-nonadditive family for $\mathfrak{J}$.

Definition 2.3. $N(\mathfrak{J}, \kappa, \lambda) \Leftrightarrow$ there exists a (κ, λ)-nonadditive family for the ideal $\mathfrak{J}$.

A family $\mathfrak{X} \subseteq \mathfrak{J}$ is called a (κ, λ)-base of $\mathfrak{J}$ if $|\mathfrak{X}| = \kappa$ and for every $A \in \mathfrak{J}$ we have $|\{B \in \mathfrak{X} : \neg(B \supseteq A)\}| < \lambda$. Note that if $\mathrm{add}(\mathfrak{J}) = \mathrm{cof}(\mathfrak{J})$ then there exists a $(\mathrm{add}(\mathfrak{J}), \mathrm{add}(\mathfrak{J}))$-base of $\mathfrak{J}$.

Definition 2.4. $C(\mathfrak{J}, \kappa, \lambda) \Leftrightarrow$ there exists a (κ, λ)-base of $\mathfrak{J}$.

It is easy to see that $C(\mathfrak{J}, \kappa, \lambda) \Rightarrow R(\mathfrak{J}, \kappa, \lambda)$, $C(\mathfrak{J}, \kappa, \lambda) \Rightarrow L(\mathfrak{J}, \kappa, \lambda)$, $R(\mathfrak{J}, \kappa, \lambda) \Rightarrow N(\mathfrak{J}, \kappa, \lambda)$ and $L(\mathfrak{J}, \kappa, \lambda) \Rightarrow N(\mathfrak{J}, \kappa, \lambda)$ for all cardinal numbers κ and λ. Hence the property $C(\mathfrak{J}, \kappa, \lambda)$ is the strongest one among those introduced above and $N(\mathfrak{J}, \kappa, \lambda)$ is the weakest one.

Lemma 2.5. *If $\kappa > \lambda$ then $C(\mathfrak{J}, \kappa, \lambda)$ does not hold.*

Proof. Suppose that $\lambda < \kappa$ and that $\mathfrak{X}$ is a (κ, λ)-base of $\mathfrak{J}$. Let $\{T_\alpha : \alpha < \lambda\}$ be any subfamily of $\mathfrak{X}$ of different sets. For every $\alpha < \lambda$ let C_α be any set from $\mathfrak{J}$ such that $C_\alpha T_\alpha$. Let D be any element of $\{\{T \in \mathfrak{X} : C_\alpha \subseteq T\} : \alpha < \lambda\}$. Then $\{T \in \mathfrak{X} : \neg(D \subseteq T)\} \supseteq \{C_\alpha : \alpha < \lambda\}$. This contradicts the definition of a (κ, λ)-base. □

The sequence $\langle A_\alpha : \alpha < \kappa\rangle$ of elements of $\mathfrak{J}$ is called a κ-base of $\mathfrak{J}$ if for every element A of $\mathfrak{J}$ there is $\alpha < \kappa$ such that $A \subseteq A_\alpha$ and $A_\alpha \subseteq A_\beta$ if $\alpha < \beta < \kappa$.

Lemma 2.6. *If* $C(\mathfrak{J}, \kappa, \kappa)$ *then there exists a* κ*-base of the ideal* $\mathfrak{J}$.

Proof. If κ is a regular cardinal number then an easy induction may be applied. Suppose hence that $\mathrm{cf}(\kappa) < \kappa$ and let $\langle \kappa_\alpha : \alpha < \kappa\rangle$ be a monotonic and cofinal in κ sequence of cardinal numbers. Let $\mathfrak{X}$ be a (κ, κ)-base of $\mathfrak{J}$. For any $\alpha < \mathrm{cf}(\kappa)$ let $T_\alpha = \{A \in \mathfrak{X} : |\{T \in \mathfrak{X} : \neg(T \supseteq A)\}| < \kappa_\alpha\}$. Note that if $\alpha < \beta < \mathrm{cf}(\kappa)$ then $T_\alpha \subseteq T_\beta$. We claim that for every $\alpha < \mathrm{cf}(\kappa)$ we have $|T_\alpha| < \kappa$. Suppose that the claim is not true for some $\alpha < \mathrm{cf}(\kappa)$. Then T_α is an ordinary base of $\mathfrak{J}$ and it follows that it is a (κ, κ_α)-base, contradicting Lemma 2.5.

We define by induction a sequence $\langle A_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ as follows:

$$A_\alpha \in \{\{B \in \mathfrak{X} : B \supseteq C\} : C \in T_\alpha \cup \{A_\zeta : \zeta < \alpha\}\}.$$

It is easy to check that there is a $\mathrm{cf}(\kappa)$-base of the ideal $\mathfrak{J}$. □

Theorem 2.7. *If* $C(\mathfrak{J}, \kappa, \lambda)$ *then* $\kappa = \lambda$ *and* $\mathrm{cf}(\kappa) = \mathrm{add}(\mathfrak{J}) = \mathrm{cof}(\mathfrak{J})$. *Conversely, if* $\mathrm{add}(\mathfrak{J}) = \mathrm{cof}(\mathfrak{J})$ *then* $C(\mathfrak{J}, \mathrm{add}(\mathfrak{J}), \mathrm{add}(\mathfrak{J}))$. *Moreover, if* $\mathrm{cf}(\kappa) = \mathrm{add}(\mathfrak{J}) = \mathrm{cof}(\mathfrak{J})$, $\kappa \leq |X|$ *and* $(\forall A \in \mathfrak{J})(\exists B \in \mathfrak{J})(|B - A| = |X|)$ *then* $C(\mathfrak{J}, \kappa, \kappa)$.

Proof. The first part of the theorem follows from Lemmas 2.5 and 2.6. The proof of the second part is by standard transfinite induction. Note that if $\langle A_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ is a $\mathrm{cf}(\kappa)$-base of $\mathfrak{J}$ and $(\forall A \in \mathfrak{J})(\exists B \subset \mathfrak{J})(|B - A| = |X|)$ then we may assume that if $\alpha < \beta$ then $|A_\beta - A_\alpha| = |X|$. Hence it is possible to refine the sequence $\langle A_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ to length κ. □

The next two propositions show the connection between properties of ideals introduced previously and the cardinal functions.

Proposition 2.8. (1) $N(\mathfrak{J}, \kappa, \kappa) \Rightarrow (\mathrm{add}(\mathfrak{J}) \leq \kappa \,\&\, \mathrm{cf}(\kappa) \leq \mathrm{cof}(\mathfrak{J}))$,

(2) $R(\mathfrak{J}, \kappa, \kappa) \Rightarrow (\mathrm{cof}(\mathfrak{J}) \leq \kappa \,\&\, \mathrm{cf}(\kappa) \leq \mathrm{non}(\mathfrak{J}))$,

(3) $L(\mathfrak{J}, \kappa, \kappa) \Rightarrow (\mathrm{non}(\mathfrak{J}) \leq \kappa \,\&\, \mathrm{cf}(\kappa) \leq \mathrm{cov}(\mathfrak{J}))$.

Proposition 2.9. *Suppose that* $\lambda < \kappa$. *Then*

(1) $N(\mathfrak{J}, \kappa, \lambda) \Rightarrow (\mathrm{add}(\mathfrak{J}) \leq \lambda < \kappa \leq \mathrm{cof}(\mathfrak{J}))$,

(2) $R(\mathfrak{J}, \kappa, \lambda) \Rightarrow (\mathrm{cov}(\mathfrak{J}) \leq \lambda < \kappa \leq \mathrm{non}(\mathfrak{J}))$,

(3) $L(\mathfrak{J}, \kappa, \lambda) \Rightarrow (\mathrm{non}(\mathfrak{J}) \leq \lambda < \kappa \leq \mathrm{cov}(\mathfrak{J}))$.

We omit the elementary proofs of these propositions.

Suppose that $A \subseteq U \times V$, $u \in U$ and $v \in V$. Then $A_u = \{y \in V : \langle u, y\rangle \in A\}$ and $A^v = \{x \in U : \langle x, v\rangle \in A\}$.

Lemma 2.10. *The following two conditions are equivalent*:

(1) $\mathrm{cf}(\kappa) = \mathrm{cf}(\lambda)$,

(2) *there are two sets* $A, B \subseteq \kappa \times \lambda$ *such that* $A \cup B = \kappa \times \lambda$, *for every* $\alpha < \kappa$ *we have* $|A_\alpha| < \lambda$ *and for every* $\beta < \lambda$ *we have* $|B^\beta| < \kappa$.

Proof. It is clear that condition (1) implies condition (2). Hence suppose that condition (2) holds. Observe that from the symmetry of assumption it follows that it is sufficient to prove that $\mathrm{cf}(\kappa) \geq \mathrm{cf}(\lambda)$. Let us suppose that $\mathrm{cf}(\kappa) < \mathrm{cf}(\lambda)$. We will consider two cases.

Case 1. $\lambda < \kappa$. Note that $\mathrm{cf}(\kappa) < \mathrm{cf}(\lambda) \leq \lambda < \kappa$, hence κ is a singular cardinal number. Let $\langle \kappa_\alpha : \alpha < \mathrm{cf}(\kappa) \rangle$ be a monotonic and cofinal in κ sequence of cardinal numbers. For any $\beta < \lambda$ we define

$$f(\beta) = \min\{\alpha < \mathrm{cf}(\kappa) \colon |B^\beta| < \kappa_\alpha\}.$$

Then $f \colon \lambda \to \mathrm{cf}(\kappa)$. Hence there exists $\alpha_0 < \mathrm{cf}(\kappa)$ such that $|f^{-1}(\{\alpha_0\})| = \lambda$. Let $T = \bigcup\{B^\beta \colon \beta \in f^{-1}(\{\alpha_0\})\}$. Then $|T| < \kappa$. Let us take $\alpha_1 \in \kappa - T$. Then $f^{-1}(\{\alpha_0\}) \subseteq A_{\alpha_1}$ but this is impossible. Hence Case 1 is eliminated.

Case 2. $\lambda > \kappa$. We may assume that $\mathrm{cf}(\lambda) \neq \kappa$ since otherwise we would have $\mathrm{cf}(\kappa) = \mathrm{cf}(\lambda)$ and this contradicts our assumption.

Subcase 2.1. $\mathrm{cf}(\lambda) > \kappa$. For $\beta < \lambda$ we put $f(\beta) = \min(\kappa - B^\beta)$. Then $f \colon \lambda \to \kappa$ and we can find $\alpha_0 < \kappa$ such that $|f^{-1}(\{\alpha_0\})| = \lambda$. But $f^{-1}(\{\alpha_0\}) \subseteq A_{\alpha_0}$, which is impossible.

Subcase 2.2. $\mathrm{cf}(\lambda) < \kappa$. Let $\langle \kappa_\eta : \eta < \mathrm{cf}(\kappa) \rangle$ be a monotonic and cofinal in κ sequence of regular cardinal numbers. For any $\beta < \lambda$ we define

$$f(\beta) = \min\{\eta < \mathrm{cf}(\kappa) \colon (\forall \zeta \geq \eta) |B^\beta \cap \kappa_\zeta| < k_\zeta\}.$$

Then $f \colon \lambda \to \mathrm{cf}(\kappa)$. Since $\mathrm{cf}(\kappa) < \mathrm{cf}(\lambda)$ there exists $\eta_0 < \mathrm{cf}(\kappa)$ such that $|f^{-1}(\{\eta_0\})| = \lambda$. Let η_1 be such that $\eta_0 \leq \eta_1 < \mathrm{cf}(\kappa)$ and $\kappa_{\eta_1} > \mathrm{cf}(\lambda)$. Note that if $\beta \in f^{-1}(\{\eta_0\})$ then $|B^\beta \cap \kappa_{\eta_1}| < \kappa_{\eta_1}$. For $\zeta < \kappa_{\eta_1}$ we define $H_\zeta = \{\beta < \lambda \colon f(\beta) = \eta_0$ and $\kappa_{\eta_1} \cap B^\beta \subseteq \zeta\}$. Then $H_\zeta \subseteq H_\xi$ if $\zeta \leq \xi$ and $f^{-1}(\{\eta_0\}) = \bigcup\{H_\zeta \colon \zeta < \kappa_{\eta_1}\}$.

We claim that there exists $\zeta < \kappa_{\eta_1}$ such that $|H_\zeta| = \lambda$. Suppose that this is not true. Since $\mathrm{cf}(\lambda) < \lambda$ we can find a monotonic and cofinal in λ sequence $\langle \lambda_\alpha \colon \alpha < \mathrm{cf}(\lambda) \rangle$ of regular cardinal numbers. For $\zeta < \kappa_{\eta_1}$ we put $g(\zeta) = \min\{\eta < \mathrm{cf}(\lambda) \colon |H_\zeta| < \lambda_\eta\}$. Then $g \colon k_{\eta_1} \to \mathrm{cf}(\lambda)$ and g is a monotonic function. Hence g is a bounded function! Let μ be such that for every $\zeta < \kappa_{\eta_1}$ we have $|H_\zeta| < \lambda_\mu$. Then $\lambda = |f^{-1}(\{\eta_0\})| = |\bigcup\{H_\zeta \colon \zeta < \kappa_{\eta_1}\}| < \lambda_\mu * \kappa_{\eta_1} < \kappa$. Hence the claim is proved.

Thus we can find $\xi < \kappa$ such that $|H_\zeta| = \lambda$. But then if $\beta \in H_\zeta$ then $\langle \xi, \beta \rangle \notin B$ hence $\langle \zeta, \beta \rangle \in A$. Thus $H_\zeta \subseteq A_\zeta$ and hence $\lambda \leq \kappa$. □

Theorem 2.11. *If* $L(\mathfrak{J}, \kappa, \kappa_1)$ *and* $R(\mathfrak{J}, \lambda, \lambda_1)$ *then* $\kappa = \kappa_1$, $\lambda = \lambda_1$ *and* $\mathrm{cf}(\kappa) = \mathrm{cf}(\lambda)$.

Proof. Notice that if $L(\mathfrak{I},\kappa,\lambda)$ and $\kappa \geq \nu \geq \mu \geq \lambda$ then $L(\mathfrak{I},\nu,\mu)$. Similarly, if $R(\mathfrak{I},\kappa,\lambda)$ and $\kappa \geq \nu \geq \mu \geq \lambda$ then $R(\mathfrak{I},\nu,\mu)$. Hence it is sufficient to prove that from assumption $L(\mathfrak{I},\kappa,\kappa)$ and $R(\mathfrak{I},\lambda,\lambda)$ it follows that $\mathrm{cf}(\kappa) = \mathrm{cf}(\lambda)$.

Let $\{x_\alpha : \alpha < \kappa\}$ be a (κ,κ)-Luzin set for $\mathfrak{I}$ and let $\{A_\alpha : \alpha < \lambda\}$ be a (λ,λ)-Rothberger family for $\mathfrak{I}$. Let $A = \{\langle\alpha,\beta\rangle \in \kappa \times \lambda : x_\alpha \notin A_\beta\}$ and $B = \kappa \times \lambda - A$. It is easy to check that sets A and B satisfy condition (2) of Lemma 2.10. Hence $\mathrm{cf}(\kappa) = \mathrm{cf}(\lambda)$. □

If $\mathfrak{I}$ and $\mathfrak{J}$ are ideals on a set X then we say that they are orthogonal if there are two sets A and B such that $A \in \mathfrak{I}$, $B \in \mathfrak{J}$ and $A \cup B = X$.

Suppose now that $\mathfrak{J}$ is an ideal on the group $\mathscr{G}$. We say that $\mathfrak{J}$ is an invariant ideal on a group $\mathscr{G}$ if for every $A \in \mathfrak{J}$ and $a \in \mathscr{G}$ we have $a + A = \{a + x : x \in A\} \in \mathfrak{J}$.

Note that $\mathscr{K}$ and $\mathscr{L}$ are invariant and orthogonal ideals on the group $(\mathbf{R},+)$.

Theorem 2.12. *If* $\mathfrak{I}$ *and* $\mathfrak{J}$ *are invariant and orthogonal ideals on a group* $\mathscr{G}$ *and* $L(\mathfrak{I},\kappa,\lambda)$ *then* $R(\mathfrak{J},\kappa,\lambda)$.

Proof. Let $A \in \mathfrak{I}$ and $B \in \mathfrak{J}$ be such that $A \cup B = \mathscr{G}$. Let L be a (κ,λ)-Luzin set for the ideal $\mathfrak{I}$. Consider the family $\mathfrak{X} = \{B - x : x \in L\}$. We claim that this is a (κ,λ)-Rothberger family for the ideal $\mathfrak{J}$. Suppose that $T \subseteq L$ and $|T| = \lambda$ and that $\bigcup\{B - x : x \in T\} \neq \mathscr{G}$. Let $g \in \mathscr{G} - \bigcup\{B - x : x \in T\}$. Then for every $x \in T$ we have $g \notin B - x$, hence $g + x \notin B$. Thus $(g + T) \cap B = \varnothing$ and therefore $T \cap ((-g) + B) = \varnothing$. But then $T \subseteq (-g) + A$. The ideal $\mathfrak{I}$ is an invariant ideal on $\mathscr{G}$, hence $(-g) + A \in \mathfrak{I}$. Thus $T \in \mathfrak{I}$ and we get a contradiction. □

Corollary 2.13. *If* $\mathfrak{I}$ *and* $\mathfrak{J}$ *are invariant and orthogonal ideals on a group,* $L(\mathfrak{I},\kappa,\kappa)$ *and* $L(\mathfrak{J},\lambda,\lambda)$ *then* $\mathrm{cf}(\kappa) = \mathrm{cf}(\lambda)$.

Proof. From Theorem 2.12 it follows that if $\mathfrak{I}$ and $\mathfrak{J}$ satisfy assumptions of the corollary then $R(\mathfrak{J},\kappa,\kappa)$ holds. Thus Theorem 2.11 may be applied to ideals $\mathfrak{J}$ and $\mathfrak{J}$. □

Corollary 2.13 is a generalization of a result of Rothberger who showed that from the same assumptions the inequality $\mathrm{cf}(\kappa) \leq \lambda$ follows.

Corollary 2.14 (Rothberger). $((\mathscr{K},\mathfrak{c},\aleph_1)\&L(\mathscr{L},\mathfrak{c},\aleph_1) \Leftrightarrow \mathrm{CH}$.

Proof. It is clear that CH implies existence of a Luzin and Sierpiński set. Suppose now that CH is false. Recall that for every ideal $\mathfrak{I}$ if $L(\mathfrak{I},\kappa,\lambda)$ and $\kappa \geq \nu \geq \mu \geq \lambda$ then $L(\mathfrak{I},\nu,\mu)$. Hence we get $L(\mathscr{K}I,\kappa_1,\kappa_1)$ and $L(\mathscr{L},\kappa_2,\kappa_2)$. By Corollary 2.13 we get $\mathrm{cf}(\aleph_1) = \aleph_1 = \mathrm{cf}(\aleph_2) = \aleph_2$. □

3. Connections between measure and category

We shall consider connections between properties of the ideals $\mathscr{K}$ and $\mathscr{L}$ which were introduced in §2. Let $\mathscr{K}_\sigma$ be the ideal of subsets of the Baire space

ω^ω generated by the family $\{K(f)\colon f \in \omega^\omega\}$ where $K(f) = \{g \in \omega^\omega\colon g \leq^* f\}$. In other words, $\mathscr{K}_\sigma$ is the ideal generated by the σ-compact subsets of ω^ω. It is easy to see that for any cardinal numbers κ and λ we have $N(\mathscr{K}_\sigma, \kappa, \lambda) \Leftrightarrow L(\mathscr{K}_\sigma, \kappa, \lambda)$ and $C(\mathscr{K}_\sigma, \kappa, \lambda) \Leftrightarrow R(\mathscr{K}_\sigma, \kappa, \lambda) \Leftrightarrow (\kappa = \lambda \ \& \ \mathrm{cf}(\kappa) = \mathfrak{b} = \mathfrak{d})$.

Theorem 3.1. *For any two cardinal numbers κ and λ the following implication holds*:

$$\begin{array}{ccccccc}
R(\mathscr{L}, \kappa, \lambda) & \longleftarrow & L(\mathscr{K}, , \lambda) & \longleftarrow & C(\mathscr{K}, \kappa, \lambda) & \longleftarrow & C(\mathscr{L}, \kappa, \lambda) \\
\Big| & & \downarrow & & \downarrow & & \Big| \\
\Big| & & N(\mathscr{K}_\sigma, \kappa, \lambda) & \longleftarrow & C(\mathscr{K}_\sigma, \kappa, \lambda) & & \Big| \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
N(\mathscr{L}, \kappa, \lambda) & \longleftarrow & N(\mathscr{L}, \kappa\lambda) & \longleftarrow & R(\mathscr{K}, \kappa, \lambda) & \longleftarrow & N(\mathscr{L}, \kappa, \lambda).
\end{array}$$

Proof. There are only four implications in the above diagram which do not follow directly from results in §2.

(1) $N(\mathscr{K}, \kappa, \lambda) \leftarrow N(\mathscr{L}, \kappa, \lambda)$.

Let $\mathbf{F}\colon \mathscr{K} \to \mathscr{L}$ and $\mathbf{G}\colon \mathscr{L} \to \mathscr{K}$ be functions from Lemma 1.1 and let $\mathfrak{X}$ be a (κ, λ)-nonadditive family for $\mathscr{K}$. It is easy to check that the family $\mathscr{Y} = \{\mathbf{F}(A)\colon A \in \mathfrak{X}\}$ is a (κ, λ)-nonadditive family for the ideal $\mathscr{L}$.

(2) $N(\mathscr{K}_\sigma, \kappa, \lambda) \leftarrow N(\mathscr{K}, \kappa, \lambda)$.

Let $\mathbf{F}\colon \omega^\omega \to \mathscr{K}$ and $\mathbf{G}\colon \mathscr{K} \to \omega^\omega$ be functions from Lemma 1.2 and let $\mathfrak{X}$ be a (κ, λ)-nonadditive family for $\mathscr{K}_\sigma$. It is easy to check that the family $\mathscr{Y} = \{\mathbf{F}(A)\colon A \in \mathfrak{X}\}$ is a (κ, λ)-nonadditive family for the ideal $\mathscr{K}$.

(3) $R(\mathscr{K}_\sigma, \kappa, \lambda) \leftarrow R(\mathscr{K}, \kappa, \lambda)$.

Let $\mathbf{F}\colon \omega^\omega \to \mathscr{K}$ and $\mathbf{G}\colon \mathscr{K} \to \omega^\omega$ be functions from Lemma 1.2 and let $\mathfrak{X}$ be a (κ, λ)-Rothberger family for $\mathscr{K}_\sigma$. It is easy to check that the family $\mathscr{Y} = \{\mathbf{F}(A)\colon A \in \mathfrak{X}\}$ is a (κ, λ)-Rothberger family for the ideal $\mathscr{K}$.

(4) $L(\mathscr{K}, \kappa, \lambda) \leftarrow N(\mathscr{K}_\sigma, \kappa, \lambda)$.

Suppose that $N(\mathscr{K}_\sigma, \kappa, \lambda)$ is false. Then also $L(\mathscr{K}_\sigma, \kappa, \lambda)$ is false. Let $\mathfrak{J}$ be the ideal of first category subsets of the Baire space ω^ω. Then $\mathscr{K}_\sigma \subseteq \mathfrak{J}$, hence $L(\mathfrak{J}, \kappa, \lambda)$ is also false. Recall that the Baire space ω^ω and the real line are Borel isomorphic. Moreover, there exists such an isomorphism which preserves first category sets of both spaces. Hence $L(\mathscr{K}, \kappa, \lambda)$ is false.

Theorem 3.2. *If $N(\mathscr{K}, \kappa, \lambda)$ then for every μ such that $\lambda \leq \mu \leq \kappa$ we have $R(\mathscr{K}, \kappa, \mu)$ or $N(\mathscr{K}_\sigma, \mu, \lambda)$.*

Proof. Suppose that $\lambda \leq \mu \leq \kappa$ but $\neg R(\mathscr{K}, \kappa, \mu)$ and $\neg N(\mathscr{K}_\sigma, \mu, \lambda)$. We shall prove that $N(\mathscr{K}, \kappa, \lambda)$ is false. Note that it will be sufficient to show that if $\mathfrak{X}$ is a family of closed nowhere dense subsets of $\mathbf{R}$ and $|\mathfrak{X}| = \kappa$ then there exists a subfamily $\mathscr{Y} \subseteq \mathfrak{X}$ such that $|\mathscr{Y}| = \lambda$ and $\bigcup \mathscr{Y} \in \mathscr{K}$. Hence let $\mathfrak{X}$ be a family of nowhere dense subsets of $\mathbf{R}$ and $|\mathfrak{X}| = \kappa$. Let us consider the family $\mathscr{Y} = \{T + \mathbf{Q}\colon T \in \mathfrak{X}\}$, where $\mathbf{Q}$ denotes the rational numbers and $+$ the complex sum of subsets of the real line. Since there are no (κ, μ)-Rothberger families for the ideal $\mathscr{K}$ we can find a subfamily $\mathscr{A} \subseteq \mathfrak{X}$ such that $|\mathscr{A}| = \mu$ and $\bigcup\{T + \mathbf{Q}\colon T \in \mathscr{A}\}$ is a proper subset of $\mathbf{R}$.

Let $c \in \mathbf{R} - \bigcup\{T+\mathbf{Q}\colon T \in \mathscr{A}\}$. Then for every $T \in \mathscr{A}$ we have $(c+\mathbf{Q}) \cap T = \emptyset$. Let $\{q_n\colon n \in \omega\} = c + \mathbf{Q}$. For any $T \in \mathscr{A}$ let f_T be such a function that for every $n \in \omega$ we have

$$\left(q_n - \frac{1}{f_T(n)}, q_n + \frac{1}{f_T(n)}\right) \cap T = \emptyset .$$

Since $N(\mathscr{K}_\sigma, \mu, \lambda)$ is false we may find a subfamily $\mathscr{C} \subseteq \mathscr{A}$ and a function $f \in \omega^\omega$ such that $|\mathscr{C}| = \lambda$ and for every $T \in \mathscr{C}$ we have $f_T \leq^* f$. Let

$$G = \left\{x \in \mathbf{R}\colon (\forall m \in \omega)(\exists n > m) \left(x \in \left(q_n - \frac{1}{f(n)}, q_n + \frac{1}{f(n)}\right)\right)\right\} .$$

Then G is a dense G_δ subset of $\mathbf{R}$ and $\bigcup \mathscr{C} \cap G = \emptyset$. Thus $N(\mathscr{K}, \kappa, \lambda)$ is false. □

4. Luzin sets

Let us fix an ideal $\mathfrak{I}$ on a set X.

Definition 4.1. The ideal $\mathfrak{I}$ is nice if for every $A \in \mathfrak{I}$ there exists $B \in \mathfrak{I}$ such that $|B - A| = |X|$.

Note that the ideals $\mathscr{K}$ and $\mathscr{L}$ are nice. We shall deal from now on only with nice ideals.

Definition 4.2. A family $\mathscr{M} \subseteq [X]^{|X|}$ is called a (κ, λ)-Luzin family for the ideal $\mathfrak{I}$ if $|\mathscr{M}| = \kappa$ and for every $A \in \mathfrak{I}$ we have

$$|\{M \in \mathscr{M}\colon A \cap M = \emptyset\}| < \lambda .$$

Note that if $\lambda \leq \kappa$ then a (κ, λ)-Luzin family is also a (κ, κ)-Luzin family.

Proposition 4.3. *If there exists a (κ, κ)-Luzin family for $\mathfrak{I}$, $\mathrm{cf}(\kappa) = \kappa \leq \lambda \leq |X|$ and $\kappa = \mathrm{cf}(\lambda)$ then there exists a (λ, λ)-Luzin set for $\mathfrak{I}$.*

A typical transfinite induction gives the following sufficient condition for the existence of (κ, κ)-Luzin families:

Proposition 4.4. *Suppose that $\mathrm{add}(\mathfrak{I}) = \mathrm{cof}(\mathfrak{I})$. Then there exists a $(\mathrm{add}(\mathfrak{I}), \mathrm{add}(\mathfrak{I}))$-Luzin family for the ideal $\mathfrak{I}$.*

The strong assumption $\mathrm{add}(\mathfrak{I}) = \mathrm{cof}(\mathfrak{I})$ of Proposition 4.4 can be weakened if we assume slightly more about the ideal $\mathfrak{I}$.

Definition 4.5. The ideal $\mathfrak{I}$ is kind if for every family $\mathscr{M} \subseteq \mathfrak{I}$ of cardinality less than $\mathrm{cov}(\mathfrak{I})$ we have $|X - \bigcup \mathscr{M}| = |X|$.

Proposition 4.6. *If the ideal $\mathfrak{I}$ is kind and $\mathrm{cof}(\mathfrak{I}) = \mathrm{cof}(\mathfrak{I})$ then there exists a $(\mathrm{cov}(\mathfrak{I}), \mathrm{cov}(\mathfrak{I}))$-Luzin family for $\mathfrak{I}$.*

Definition 4.7. A κ-tower in the ideal $\mathfrak{I}$ is an increasing sequence of elements of $\mathfrak{I}$, the union of which is X.

Note that if there exists a κ-tower in $\mathfrak{J}$ then there exists a $\mathrm{cf}(\kappa)$-tower in $\mathfrak{J}$. If $\mathrm{non}(\mathfrak{J}) = |X|$ then there exists a $\mathrm{non}(\mathfrak{J})$-tower in $\mathfrak{J}$. If $\mathrm{add}(\mathfrak{J}) = \mathrm{cov}(\mathfrak{J})$ then there exists an $\mathrm{add}(\mathfrak{J})$-tower in $\mathfrak{J}$. Conversely, if there exists a κ-tower then $\mathrm{cof}(\mathfrak{J}) \leq \mathrm{cf}(\kappa) \leq \mathrm{non}(\mathfrak{J})$. Hence, if for some cardinal number κ there exists a κ-tower in $\mathfrak{J}$ then $\mathrm{cov}(\mathfrak{J}) \leq \mathrm{non}(\mathfrak{J})$. It is easy to show that the inequality $\mathrm{cov}(\mathfrak{J}) \leq \mathrm{non}(\mathfrak{J})$ is not a sufficient condition for existence of a tower in $\mathfrak{J}$.

Proposition 4.8. *If there exists a κ-tower in $\mathfrak{J}$ and $L(\mathfrak{J}, \lambda, \lambda)$ then* $\mathrm{cf}(\kappa) = \mathrm{cf}(\lambda)$.

Proof. We may assume that κ is a regular cardinal number. Suppose that $\langle A_\alpha : \alpha < \kappa \rangle$ is a κ-tower in $\mathfrak{J}$ and let L be a (λ, λ)-Luzin set for the ideal $\mathfrak{J}$. Note that $\langle A_\alpha \cap L : \alpha < \kappa \rangle$ is a monotonic sequence of sets from $[L]^{<\lambda}$.

Since $L = \bigcup\{A_\alpha \cap L : \alpha < \kappa\}$ we have $\mathrm{cf}(\lambda) = \mathrm{cf}(|L|) \leq \kappa$. If λ is a regular cardinal then we immediately see that $\lambda = \kappa$. Suppose hence that $\mathrm{cf}(\lambda) < \lambda$, $\mathrm{cf}(\lambda) < \kappa$ and let $\langle \lambda_\alpha : \alpha < \mathrm{cf}(\lambda) \rangle$ be a sequence of cardinals cofinal with λ. For every $\alpha < \kappa$ let ζ_α be the first ordinal number ζ such that $|A_\alpha \cap L| \leq \lambda_\zeta$. Then $\langle \zeta_\alpha : \alpha < \kappa \rangle$ is a monotonic sequence of elements from $\mathrm{cf}(\lambda)$. But $\mathrm{cf}(\lambda) < \kappa$ hence $\langle \zeta_\alpha : \alpha < \kappa \rangle$ is bounded. Let $\mu < \lambda$ be such that for every $\alpha < \kappa$ we have $|A_\alpha \cap L| < \mu$. Then $\lambda = |\bigcup\{A_\alpha \cap L : \alpha < \kappa\}| \leq \mu^+ < \lambda$. □

From the results above it follows that if $\mathfrak{J}$ is a nice ideal and $\mathrm{add}(\mathfrak{J}) = \mathrm{cof}(\mathfrak{J})$ then $L(\mathfrak{J}, \kappa, \kappa) \Leftrightarrow (\kappa \leq |X| \,\&\, \mathrm{cf}(\kappa) = \mathrm{add}(\mathfrak{J}))$. Note that MA implies that $\mathrm{add}(\mathscr{L}) = \mathrm{add}(\mathscr{K}) = \mathrm{cof}(\mathscr{L}) = \mathrm{cof}(\mathscr{K}) = \mathfrak{c}$. It is well known that the theory $\mathrm{ZFC} + \mathrm{add}(\mathscr{L}) = \mathrm{add}(\mathscr{K}) = \mathrm{cof}(\mathscr{L}) = \mathrm{cof}(\mathscr{K}) = \aleph_1 + \mathfrak{c} = \aleph_2$ is relatively consistent.

5. The idea of first category sets

We will prove that the ideal $\mathscr{K}$ is kind and that the existence of Luzin sets and for $\mathscr{K}$ implies existence of Luzin families for $\mathscr{K}$.

Lemma 5.1. *There exists a Borel set $P \subseteq \mathbf{R} \times \mathbf{R}$ such that for every $t \in \mathbf{R}$ the set $\{y \in \mathbf{R} : \langle t, y \rangle \in P\}$ is uncountable and for every $A \in \mathscr{K}$ we have $\{x \in \mathbf{R} : (\exists y \in A)(\langle x, y \rangle \in A)\} \in \mathscr{K}$.*

Proof. Let BIN be the family of all nonempty and finite functions the domain of which is contained in ω and the range in $\{0, 1\}$. Let

$$S = \{f \in \mathrm{BIN}^\omega : (\forall n \in \omega)(\max(\mathrm{dom}(f(n)) + 2) = \min(\mathrm{dom}(f(n+1))))\}.$$

We treat S as a polish space. Let

$$P = \{\langle f, x \rangle \in S \times \{0,1\}^\omega : (\forall n \in \omega)(f(n) \subseteq x)\}.$$

Then P is a Borel subset of the space $S \times \{0,1\}^\omega$ with perfect vertical sections.

Suppose that A is a first category subset of the Cantor set $\{0,1\}^\omega$. Let $\langle G_n : n \in \omega \rangle$ be a decreasing sequence of open dense subsets of $\{0,1\}^\omega$ such that $\{G_n : n < w\} \cap A = \emptyset$. Let $f \in S$ be such that $\{x \in \{0,1\}^\omega : f(n) \subseteq x\} \subseteq G_n$ for every $n \in \omega$. Let

$$T = \{h \in S : (\forall n \in \omega)(\exists m > n)(f(m) = h(m))\}.$$

Then T is a comeager subset of the space S and

$$(\forall h \in T)(\forall y \in A)(\langle h, y\rangle \notin P).$$

Hence $\{h \in S\colon (\forall y \in A)(\langle x, y\rangle \in A)\}$ is a meager subset of S.

If we use Borel isomorphisms which preserves meager subsets between S and $\mathbf{R}$ and also between $\{0,1\}^{\omega}$ and $\mathbf{R}$ then we obtain a required subset of the plane $\mathbf{R} \times \mathbf{R}$. $\square$

Lemma 5.1 is in fact a descriptive version of the well-known observation: if c is a Cohen real over a model M then in the model $M[c]$ there exists a perfect set of Cohen reals over the model M.

Theorem 5.2. *The ideal $\mathscr{K}$ is kind.*

Proof. Let $P \subseteq \mathbf{R} \times \mathbf{R}$ be such a set whose existence is proved in Lemma 5.1. Suppose that $\mathscr{A} \subseteq \mathscr{K}$ and $|\mathscr{A}| < \mathrm{cov}(\mathscr{K})$. For $T \in \mathscr{A}$ we put $T^* = \{x \in \mathbf{R}\colon (\exists y \in T)(\langle x, y\rangle \in P)\}$ and $\mathscr{A}^* = \{T^*\colon T \in \mathscr{A}\}$. Then $\mathscr{A}^* \subseteq \mathscr{K}$ and $|\mathscr{A}^*| < \mathrm{cov}(\mathscr{K})$ so $\bigcup \mathscr{A}^* \mathbf{R}$. Let $c \in \mathbf{R} - \bigcup \mathscr{A}^*$ and $S = \{y \in \mathbf{R}\colon \langle c, y\rangle \in P\}$. Then S is an uncountable Borel set and $S \cap \bigcup \mathscr{A} = \emptyset$. $\square$

Theorem 5.3. *$L(\mathscr{K}, \kappa, \lambda) \Leftrightarrow$ there exists a (κ, λ)-Luzin family for $\mathscr{K}$.*

Proof. It follows from Proposition 4.3 that if there exists a (κ, λ)-Luzin family for $\mathscr{K}$ then $L(\mathscr{K}, \kappa, \lambda)$ holds. Suppose that L is a (κ, λ)-Luzin set for $\mathscr{K}$ and let $P \subseteq \mathbf{R} \times \mathbf{R}$ be such a set whose existence is proved in Lemma 5.1. We claim that the set $\{\{y \in \mathbf{R}\colon \langle x, y\rangle \in P\}\colon x \in L\}$ is a (κ, λ)-Luzin family for the ideal $\mathscr{K}$. Suppose that $A \subseteq \mathbf{R}$ and

$$|\{x \in L\colon \{y \in \mathbf{R}\colon \langle x, y\rangle \in P\} \cap A \neq \emptyset\}| \geq \lambda.$$

Let $A^* = \{x \in \mathbf{R}\colon (\exists y \in A)(\langle x, y\rangle \in P)\}$. Then $|L \cap A^*| \geq \lambda$ hence $A \notin \mathscr{K}$. $\square$

Corollary 5.4. *If $L(\mathscr{K}, \kappa, \kappa)$, $\kappa \leq \lambda \leq c$ and $\mathrm{cf}(\kappa) = \mathrm{cf}(\kappa)$ then $L(\mathscr{K}, \lambda, \lambda)$.*

Proof. If $L(\mathscr{K}, \kappa, \kappa)$ then, by Theorem 5.3, there exists a (κ, κ)-Luzin family for $\mathscr{K}$. Hence by Proposition 4.3 we get $L(\mathscr{K}, \lambda, \lambda)$. $\square$

Let us notice that Mokobodzki [M] has proved that if P is a Σ^1_1 subset of the plane with all vertical sections uncountable then there exists a set $A \in \mathscr{L}$ such that

$$\mathbf{R} - \{x \in \mathbf{R}\colon (\exists y \in A)(\langle x, y\rangle \in P\} \in \mathscr{L}.$$

A forcing version of this result is the following: if r is a random real over a model M of ZFC then in $M[r]$ there is no perfect set of random reals over the model M.

Problems.

(1) Suppose that $L(\mathscr{L}, \kappa, \lambda)$. Does there exists a (κ, λ)-Luzin family for the ideal $\mathscr{L}$?

(2) Suppose that $L(\mathscr{L}, \kappa, \kappa)$, $\kappa \leq \lambda \leq c$ and $\mathrm{cf}(\kappa) = \mathrm{cf}(\lambda)$. Does $L(\mathfrak{L}, \lambda, \lambda)$?

6. Luzin sets in forcing extensions

Suppose that c is a Cohen real over a model M and that A is a Borel set coded in the model $M[c]$. Then there exists a Borel subset A^* of $\mathbf{R}\times\mathbf{R}$ such that $A = A^*_c$ $(=\{y\in\mathbf{R}: \langle c, y\rangle \in A^*\})$. Moreover if $A\in\mathscr{K}$ in the model $M[c]$ then A^* is a meager subset of the plane (see for example [CP]). An analogous fact is true for random reals and the ideal $\mathfrak{L}$.

Lemma 6.1. *Suppose that c is a Cohen real over the model M and that A is a Borel set coded in the model $M[c]$ and $A\in\mathscr{K}$. Then there exists a Borel set $B\in\mathscr{K}$ coded in M such that $A\cap\mathbf{R}^M\subseteq B$.*

Proof. Let C be the Borel set coded in M such that C is a meager subset of the plane and $A = C_c$. Let $B=\{y\in\mathbf{R}: C^y\notin\mathscr{K}\}$ (where $C^y=\{x\in\mathbf{R}: \langle x, y\rangle\in C\}$). Then from the Banach-Kuratowski theorem it follows that $B\in\mathscr{K}$. Suppose now that $a\in A\cap\mathbf{R}^M$. Then

$$0<[a\in A]=[\langle c, a\rangle\in C]=[C_a]_{\mathscr{K}}$$

where $[U]_{\mathscr{K}}$ is the equivalence class of the set U in the algebra $\mathscr{B}/\mathscr{K}$ and $[[\phi]]$ is the boolean value of the sentence ϕ in the algeba $\mathscr{B}/\mathscr{K}$. Hence $a\in B$. □

Corollary 6.2. *If c is Cohen real over the model M and $M\models$ (A is a (κ,λ)-Luzin set for $\mathscr{K}$) then $M[c]\models$ (A is a (κ,λ)-Luzin set for $\mathscr{K}$).*

An analogous lemma and corollary are true for the ideal $\mathfrak{L}$ and for random reals. As we mentioned in §2 if c is a Cohen real over M then $M[c]\models R(\mathscr{L},\mathfrak{c},\aleph_1)$. Hence if $M\models\neg$CH then $M[c]\models(\forall\kappa)\neg L(\mathscr{L},\kappa,\kappa)$. Let us remark that Lemma 6.1 and Corollary 6.2 are also true for any number of Cohen reals. For every ideal $\mathfrak{J}$ let $L(\mathfrak{J})=\{\kappa: L(\mathfrak{J},\kappa,\kappa)\}$.

Theorem 6.3. *The theory $ZFC+\mathfrak{c}=\aleph_3+L(\mathscr{K})=\{\aleph_1,\aleph_3\}$ is relatively consistent.*

Proof. Let M be a model of ZFC + MA + $\mathfrak{c}=\aleph_3$. Let $C=\langle c_\alpha: \alpha<\aleph_1\rangle$ be a sequence of independent Cohen reals over M. Then in the model $M[C]$, C is a $(\aleph_1,\aleph_1)$-Luzin set for $\mathscr{K}$. Moreover, since in M there exists a $(\mathfrak{c},\mathfrak{c})$-Luzin set for $\mathscr{K}$, from Lemma 6.1 we deduce that the same is true in the model $M[C]$.

Suppose now that $A\in M[C]$, $A\subseteq\mathbf{R}$ and $M[C]\models(|A|=\aleph_2)$. Then there exists $\alpha<\aleph_1$ such that $|A\cap M[\langle c_\zeta: \zeta<\alpha\rangle]|=\aleph_2$ and $A\cap M[\langle c_\zeta: \zeta<\alpha\rangle]\in M[\langle c_\zeta: <\alpha\rangle]$. Let $B=A\cap M[\langle c_\zeta: \zeta<\alpha\rangle]$. Note that $\alpha<\aleph_1$, hence the boolean algebra which adds the sequence $\langle c_\zeta: \zeta<\alpha\rangle$ is isomorphic with the algebra which adds one Cohen real. It was proved in [CP] that if one Cohen real is added to the model of ZFC then the additivity of the ideal $\mathscr{K}$ is not changed. Hence $M[\langle c_\zeta: \zeta<\alpha\rangle]\models\mathrm{add}(\mathscr{K})=\aleph_3$. Thus $B\in\mathscr{K}$ in $M[\langle c_\zeta: \zeta<\alpha\rangle]$ and hence $B\in\mathscr{K}$ in the model $M[C]$. □

Note that an analogous lemma for the ideal $\mathscr{L}$ is also true. Notice also that if $L(\mathscr{K}) = \{\aleph_1, \aleph_3\}$ then $L(\mathscr{L}) = \varnothing$ and if $L(\mathscr{L}) = \{\aleph_1, \aleph_3\}$ then $L(\mathscr{K}) = \varnothing$ (Corollary 2.13).

Theorem 6.4. *Suppose that $\langle P_\xi, Q_\xi : \xi < \alpha\rangle$ is a finite support iteration of notions of forcing such that for every $\xi < \alpha$ we have $\Vdash_\xi$ (Q_ξ satisfies c.c.c. and $|Q_\xi| \le \kappa$), $A \subseteq \mathbf{R}$, $|A| > \kappa$ and that A is a $(|A|, \kappa^+)$-Luzin set for $\mathscr{K}$. Then $\Vdash_\alpha$ (A is a $(|A|, \kappa^+)$-Luzin set for $\mathscr{K}$).*

Proof. If $M \subseteq N$ are transitive models of ZFC and $A \subseteq \mathbf{R}$, $A \in M$ then $M \vDash$ (A is a nowhere dense set) if and only if $N \vDash$ (A is a nowhere dense set).

We will prove the theorem by the induction on α. Note that if the conclusion of the theorem is not true for some α then there exists a P_α-name B such that $\Vdash_\alpha$ ($B \subseteq \bigcup\{F_n : n \in \omega\}$ and $(\forall n \in \omega)(F_n$ is nowhere dense) & $B \subseteq A$ & $|B| = \kappa^+$). Hence there exists a P_α-name B such that

$$\Vdash_\alpha (B \subseteq A \;\&\; |B| = \kappa^+ \;\&\; (B \text{ is a nowhere dense subset of } \mathbf{R}).$$

Assume that α is the least ordinal for which the theorem is false.

Case 1. $\alpha = \beta + 1$ for some β. Since $\Vdash_\beta (|Q_\beta| \le \kappa)$ and $\Vdash_\beta (Q_\beta(|B| > \kappa))$ we find a P_β-name C such that $\Vdash_\beta (|C| > \kappa \;\&\; \Vdash_{Q_\beta} (C \subseteq B))$. Hence we have

$$\Vdash_\beta (|C| > \kappa \;\&\; C \subseteq A \;\&\; C \text{ is nowhere dense}).$$

This gives a contradiction with the inductive hypothesis.

Case 2. α is a limit ordinal and $\mathrm{cf}(\alpha) > \omega$. Since $\Vdash_\alpha$ (B is a nowhere dense set) there exists a P_α-name f such that $\Vdash_\alpha$ (f is a code of a Borel nowhere dense set & $B \subseteq \#f$). (If f is a Borel code of a Borel set then $\#f$ denotes the set which is coded by f.) Then there exists $\beta < \alpha$ such that f is a P_β-name. Hence $\vDash (|\#f \cap A| > \kappa)$. This is a contradiction with the inductive hypothesis.

Case 3. α is a limit ordinal and $\mathrm{cf}(\alpha) = \omega$. Let f be a P_α-name such that $\Vdash_\alpha (f : \kappa^+ \xrightarrow{\text{1-1}} B)$. For each $\xi < \kappa^+$ let M_ξ be a maximal family of pairwise disjoint elements from P_α which decides the value of $f(\xi)$. Let $M_\xi = \{p_{n\xi} : n \in \omega\}$. For each $n \in \omega$ we fix $x_{n\xi}$ such that $p_{n\xi} \Vdash (f(\xi) = x_{n\xi})$. Let $\langle \beta_n : n \in \omega\rangle$ be a monotonic and cofinal sequence in α. For every $p \in P_\alpha$ let $\mathrm{supp}(p)$ denote the support of the element p.

Let us fix $k \in \omega$. We extend the set $\{p_{n\xi} : n \in \omega \;\&\; \max(\mathrm{supp}(p_{n\xi})) < \beta_n\}$ to a maximal antichain in P_{β_k}. Let

$$\{p_{n\xi} : n \in \omega \;\&\; \max(\mathrm{supp}(p_{n\xi})) < \beta_k\} \cup \{q_{n\xi} : n < \omega\}$$

be such an antichain. Let f_k be such a P_{β_k}-name that for every $n \in \omega$ we have

(1) $p_{n\xi} \Vdash_{\beta_k} (f_k(\xi) = x_{n\xi})$,
(2) $q_{n\xi} \Vdash_{\beta_k} (f_k(\xi)$ is not defined).

Note that $\Vdash_\alpha (\forall \xi < \kappa^+)(\forall n \in \omega)(f_n(\xi)$ is defined $\to f_n(\xi) = f(\xi))$ and $\Vdash_\alpha (\forall \xi < \kappa^+)(\exists n < \omega)(f_n(\xi) = f(\xi))$. Thus

$$\Vdash_\alpha (\exists n < \omega)(|\{\xi < \kappa^+ : f_n(\xi) \text{ is defined}\}| = \kappa^+).$$

Let $p \in P_\alpha$ and $n_0 \in \omega$ be such that

$$p \Vdash_\alpha (|\{\xi < \kappa^+ : f_{n_0}(\xi) \text{ is defined}\}| = \kappa^+).$$

Let $t \in \omega$ be such that $n_0 < t$ and $\text{supp}(p) \subseteq \beta_t$. Then

$$p \Vdash_{\beta_t} (|\{\xi < \kappa^+ : f_{n_0}(\xi) \text{ is defined}\}| = \kappa^+).$$

Let C be a P_{β_t}-name such that $p \Vdash_{\beta_t} (C = \{f_{n_0}(\xi) : \xi < \kappa^+ \;\&\; f_{n_0}(\xi) \text{ is defined}\})$. Then $p \Vdash_\alpha (C \subseteq B)$. Hence $p \Vdash_\alpha$ (C is a nowhere dense set). Therefore p_{β_t} does not force that A is a $(|A|, \kappa^+)$-Luzin set for $\mathscr{K}$. Hence we get a contradiction. □

Theorem 6.5. *Let M be a model of ZFC + GCH and let κ be a cardinal number in M such that $M \vDash (\text{cf}(\kappa) > \aleph_1)$. Then there exists a c.c.c. generic extension N of the model M such that*

$$N \vDash (\mathfrak{c} = \kappa + MA_{\aleph_2} + L(\mathscr{K}, \kappa, \aleph_2)).$$

Proof. Let $C = \langle c_\alpha : \alpha < \kappa \rangle$ be a sequence of independent Cohen reals over model M and let $N_1 = M[C]$. Then $\mathfrak{c} = \kappa$ and C is a $(\kappa, \aleph_1)$-Luzin set for $\mathscr{K}$. We use now the Solovay-Tennenbaum technique to force $\text{MA}_{\aleph_2}$. Recall that this can be done using c.c.c. finite support iteration of notions of partial orders of size less than $\aleph_2$. Hence we may use Theorem 6.4 to deduce that in the obtained model $L(\mathscr{K}, \kappa, \aleph_2)$ holds. □

If we apply the last theorem to $\kappa = \aleph_{\omega_1+1}$ then we obtain a model of ZFC+ $\text{MA}_{\aleph_2} + L(\mathscr{K}, \aleph_{\omega_1+1}, \aleph_2)$. Hence

$$L(\mathscr{K}) = \{\lambda \le \aleph_{\omega_1+1} : \lambda \ge \aleph_2\}$$

in this model. Thus the assumption $\lambda \in L(\mathscr{K})$ does not imply that $\text{cf}(\lambda) \in L(\mathscr{K})$.

Problem. Suppose that $\kappa \in L(\mathscr{L})$. Does $\text{cf}(\kappa) \in L(\mathscr{L})$?

Theorem 6.6. *The theory $ZFC + L(\mathscr{K}) = \{\aleph_1, \aleph_{\omega_1}, \aleph_{\omega_1+1}\}$ is relatively consistent.*

Proof. Let M be a model of ZFC + GCH. Let $C = \langle c_\alpha : \alpha < \aleph_{\omega_1+1} \rangle$ be a sequence of independent Cohen reals over model M and let $N_1 = M[C]$. Then $\mathfrak{c} = \aleph_{\omega_1+1}$ and C is a $(\aleph_{\omega_1+1}, \aleph_1)$-Luzin set for $\mathscr{K}$.

Note that if N is any model of ZFC and $N \vDash \mathfrak{c} = 2^{\aleph}$ then there exists a c.c.c. notion of forcing A_κ which is an iteration of c.c.c. notions of forcing of powers less than κ such that $|A_\kappa| = \mathfrak{c}$ and $\Vdash_{A_\kappa} (\text{MA}_\kappa + \mathfrak{c} = \mathfrak{c}^V)$.

We shall extend the model N_1 by c.c.c. finite support iteration of length $\aleph_1$. For every $\alpha < \aleph_1$ we put $P_{\alpha+1} = P_\alpha * A_{\aleph_{\alpha+1}}$. Let P be the direct limit of $\langle P_\alpha : \alpha < \aleph_1\rangle$ and let G be a P-generic set over model N_1. Let $N = N_1[G]$.

It is easy to check that $N \models \mathfrak{c} = \aleph_{\omega_1+1}$. Moreover, using Theorem 6.4 it is easy to prove that $N \models$ (C is a $(\aleph_{\omega_1+1}, \aleph_{\omega_1})$-Luzin set for $\mathscr{K}$). Since at each stage of the iteration a Cohen real is added an $(\aleph_1, \aleph_1)$-Luzin set exists in the model N.

Suppose that $N \models (A \subseteq \mathbf{R}\ \&\ \aleph_1 < |A| < \aleph_{\omega_1})$. Then there exists $\alpha < \aleph_{\omega_1}$ such that $A \cap N_1[G \cap P_\alpha] \in N_1[G \cap P_\alpha]$ and

$$N \models (|A| = |A \cap N_1[G \cap P_\alpha]|).$$

Then $N_1[G \cap P_{\alpha+1})] \models (A \cap N_1[G \cap P_\alpha] \in \mathscr{K})$, hence $N \models (A \cap N_1[G \cap P_\alpha] \in \mathscr{K})$. Thus $N \models$ (A is not $(|A|, |A|)$-Luzin set for $\mathscr{K}$). □

7. The measure algebra

In this part we will show some connections between combinatorial properties of the ideal $\mathscr{L}$ and the measure algebra $\mathscr{R}$. One such connection is shown in [CKP]. It is shown there that $\mathrm{cof}(\mathscr{L})$ is equal to the minimal cardinality of a dense subset of $\mathscr{R} - \{0\}$.

If A is a Lebesgue measurable subset of $\mathbf{R}$ then $\mu(A)$ will denote its Lebesgue measure.

Theorem 7.1. *Suppose that* $\mathrm{cf}(\kappa) > \omega$ *and that* $\lambda \le \kappa$. *Then the following conditions are equivalent*:

(1) *the algebra* $\mathscr{R}$ *has* (κ, λ)*-precaliber*;
(2) *for any family* $\mathscr{A}$ *of sets of positive Lebesgue measure if* $|\mathscr{A}| = \kappa$ *then there exists a subfamily* $\mathscr{C} \subseteq \mathscr{A}$ *such that* $|\mathscr{C}| = \lambda$ *and* $\bigcap \mathscr{C} \neq \emptyset$;
(3) $\neg\mathscr{R}(\mathscr{L}, \kappa, \lambda)$.

Proof. Suppose that condition (1) holds and that $\mathscr{A}$ is a family of sets of positive Lebesgue measure and $|\mathscr{A}| = \kappa$. For every $A \in \mathscr{A}$ let A^* be a compact subset of A of positive measure. We may assume that if $A, B \in \mathscr{A}$ and $A \neq B$ then $[A]_{\mathscr{L}} \neq [B]_{\mathscr{L}}$. There exists a subfamily $\mathscr{C} \subseteq \mathscr{A}$ such that $|\mathscr{C}| = \lambda$ and $\{[A]_{\mathscr{L}} : A \in \mathscr{A}\}$ is a centered subfamily of $\mathscr{R}$. Hence $\bigcap \mathscr{C} \neq \emptyset$.

Suppose now that condition (2) holds. Let $\mathscr{A} \subseteq \mathscr{L}$ and $|\mathscr{A}| = \kappa$. For any $A \in \mathscr{A}$ let A^* be a measurable set such that $\mu(A^*) > 0$ and $A \cap A^* = \emptyset$. Let $\mathscr{C} \subseteq \mathscr{A}$ be such that $|\mathscr{C}| = \lambda$ and $\bigcap\{A^* : A \in \mathscr{A}\} \neq \emptyset$. Then $\bigcup \mathscr{A} \neq \mathbf{R}$.

Suppose now that condition (3) holds. Let $\langle a_\alpha : \alpha < \kappa\rangle$ be a sequence of elements from $\mathscr{R} - \{0\}$. For every $\alpha < \kappa$ we fix a set A_α such that $[A_\alpha]_{\mathscr{L}} = a_\alpha$ and for every $x \in A_\alpha$ the density of A_α at the point x is 1.

We define two sequences $\langle b_\alpha : \alpha < \kappa\rangle$ and $\langle I_\alpha : \alpha < \kappa\rangle$ such that

(a) $\langle b_\alpha : \alpha < \kappa\rangle$ is a decreasing sequence of elements of $\mathscr{R}$;
(b) $\langle I_\alpha : \alpha < \kappa\rangle$ is a sequence of pairwise disjoint countable subsets of κ;
(c) $(\forall \alpha < \kappa)(b_\alpha = \{a_\xi : \xi \in \kappa - \{I_\beta : \beta < \alpha\} = \{a_\xi : \xi \in I_\alpha\})$.

Since $\mathrm{cf}(\kappa) > \omega$ there exists a set $T \subseteq \kappa$ such that $|T| = \kappa$ and $b \in \mathscr{R} - \{0\}$ such that $(\forall \alpha \in T)(b_\alpha = b)$. Let $b = [B]_{\mathscr{L}}$ and let

$$C_\alpha = B - \{A_\xi : \xi \in I_\alpha\}.$$

Then for every $\alpha \in T$ we have $C_\alpha \in \mathscr{L}$. Hence there exists a set $S \subseteq T$ such that $|S| = \lambda$ and $\bigcup\{C_\alpha : \alpha \in S\} \neq B$. Let $x \in B - \{C_\alpha : \alpha \in S\}$. Then $x \in \{\{A_\xi : \xi \in I_\alpha\} : \alpha \in S\}$. Let f be a function such that $\mathrm{dom}(f) = S$ and for every $\alpha \in S$ we have $f(\alpha) \in I_\alpha$ and $x \in A_{f(\alpha)}$. The family $\{A_{f(\alpha)} : \alpha \in S\} = \{a_{f(\alpha)} : \alpha \in S\}$ is centered. Hence the theorem is proved. □

A special case of Theorem 7.1 was proved in [CSW], where it is shown that the conditions (1) $\mathscr{R}$ has $(\aleph_1, \aleph_1)$-precaliber and (2) $\mathrm{cov}(\mathscr{L}) > \aleph_1$ are equivalent.

Theorem 7.2. *Suppose that* $\mathrm{cf}(\kappa) > \omega$, $\mathrm{cf}(\lambda) > \omega$ *and* $\lambda \leq \kappa$. *Then the following conditions are equivalent:*

(1) $\mathscr{R}$ *has* (κ, λ)*-caliber;*
(2) $\neg N(\mathscr{L}, \kappa, \lambda)$;
(3) *for every* $\mathfrak{X} \subseteq \omega^\omega$ *if* $|\mathfrak{X}| = \kappa$ *then there exists a family* $\mathscr{Y} \subseteq \mathfrak{X}$ *such that* $|\mathscr{Y}| = \lambda$ *and a function* $F \in \mathscr{L}\mathscr{O}\mathscr{C}$ *such that*

$$(\forall f \in \mathscr{Y})(\exists n \in \omega)(\forall m > n)(f(n) \in F(n)).$$

Proof. Suppose that condition (1) holds. Let $\{A_\alpha : \alpha < \kappa\} \subseteq \mathscr{L}$. For any $\alpha < \kappa$ let U_α be an open set of $\mathbf{R}$ such that $A_\alpha + \mathbf{Q} \subseteq U_\alpha$ and $\mu(U_\alpha) < 1$. We may assume that for every $\alpha < \kappa$ and every open set U if $U - U_\alpha \in \mathscr{L}$ then $U \subseteq U_\alpha$. We also assume that if $\alpha < \beta < \kappa$ then $\mu(U_\alpha) \neq \mu(U_\beta)$. Then $\{[U_\alpha]_{\mathscr{L}} : \alpha < \kappa\}$ is a subfamily of $\mathscr{R}$ of power κ. Hence there exists a set $T \in [\kappa]^\lambda$ and a measurable set S such that $\mu(\mathbf{R} - S) > 0$ and $\sum\{[U_\alpha] : \alpha \in T\} = [S]_{\mathscr{L}}$. We may assume that for every open set U if $U - S \in \mathscr{L}$ then $U \subseteq S$. Hence we have $\{U_\alpha : \alpha \in T\} \subseteq S$. Therefore for every $\alpha \in T$ the set A_α is contained in the set $D = \bigcap\{S - q : q \in \mathbf{Q}\}$. Notice that $D \in \mathscr{L}$. Hence $\{A_\alpha : \alpha \in T\} \in \mathscr{L}$.

The implication $(2) \to (3)$ follows immediately from Lemma 1.3.

Suppose that condition (3) holds and that $\langle a_\alpha : \alpha < \kappa \rangle$ is a sequence of different elements from $\mathscr{R}$. For every $\alpha < \kappa$ we choose a measurable subset A_α of $\mathbf{R}$ such that $a_\alpha = [A_\alpha]_{\mathscr{L}}$. Since $\mathrm{cf}(\kappa) > \omega$ we may assume that there exists $\varepsilon > 0$ such that for every $\alpha < \aleph$ we have $\mu(A_\alpha) < 1 - \varepsilon$. Let $\mathscr{G} = \{U : U$ is a finite union of intervals with rational endpoints$\}$.

For every $n \in \omega$ let $\mathscr{D}(n) = \{U \in \mathscr{G} : \mu(U) < \varepsilon/(n^* 2^{n+2})\}$. For every $\alpha < \kappa$ we fix a sequence $\langle I_{\alpha n} : n \in \omega \rangle$ such that

(a) $A_\alpha \subseteq \{I_{\alpha n} : n \in \omega\}$;
(b) $\mu(I_{\alpha 0}) < 1 - \varepsilon$;
(c) if $n > 0$ then $I_{\alpha n} \in \mathscr{D}(n)$.

We may assume that there exists $J \in \mathscr{G}$ such that for every $\alpha < \kappa$ we have $I_{\alpha 0} = J$. By the assumption there exists a function F such that $\mathrm{dom}(F) = \omega$ and $(\forall n \in \omega)(F(n) \in \mathscr{D}(n)^{\leq n})$ and a set $T \in [\kappa]^{\lambda}$ such that $(\forall \alpha \in T)(\exists n \in \omega)(\forall m > n)(I_{am} \in F(m))$. For every $n \in \omega$ let $K_n = F(n)$. Then $\mu(K_n) < \varepsilon/2^{n+2}$ for every $n > 0$. Moreover, for every $\alpha \in T$ there exists $n_\alpha \in \omega$ such that

$$A_\alpha - (J \cup \{K_n : 0 < n < \omega\}) \subseteq I_{\alpha 1} \cup \cdots \cup I_{\alpha n_\alpha}.$$

But $\mathrm{cf}(\lambda) > 0$, hence there exist a set $S \in [T]^{\lambda}$, $N \in \omega$ and $J_1, \ldots, J_n$ such that for every $\alpha \in S$ we have:

(a) $n_\alpha = N$,
(b) $J_1 = J_{\alpha 1}, \ldots, J_N = J_{\alpha N}$.

Hence for every $\alpha \in S$ we have

$$A_\alpha \subseteq J \cup \{K_n : 0 < n < \omega\} \cup J_1 \cup \cdots \cup J_N.$$

But

$$\mu(J \cup \{K_n : 0 < n < \omega\} \cup J_1 \cup \cdots \cup J_N) < 1,$$

hence $\{a_\alpha : \alpha \in S\} < 1$. □

Theorem 7.2 is a generalization of an unpublished result of A. Kamburelis. He showed that conditions (1) $\mathscr{R}$ has $(\aleph_1, \aleph_1)$-caliber and (2) $\mathrm{add}(\mathscr{L}) > \aleph_1$ are equivalent.

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