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MAPPINGS OF TREES AND THE FIXED POINT PROPERTY

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ABSTRACT. We investigate weakly confluent, universal, and related mappings of trees and their relationships to the fixed point property for tree-like continua. This investigation leads to some new results, to generalizations of some known results, and to a partial solution of a question of H. Cook.

1. INTRODUCTION

In 1969, R. H. Bing [2] asked if each tree-like continuum has the fixed point property (f.p.p.). David Bellamy [1] answered Bing's question by giving an example of a tree-like continuum that admits a fixed-point-free mapping. Shortly thereafter, Oversteegen and Rogers [14] gave inverse limit descriptions of tree-like continua without the f.p.p. Viewing tree-like continua as inverse limits of trees, one might add conditions to either the bonding mappings or the projection mappings in an effort to obtain fixed point theorems. We discuss results of this nature below.

W. Holsztynski called a mapping $f: X \to Y$ *universal* if for each mapping $g: X \to Y$, there is a point x in X such that $f(x) = g(x)$. Holsztynski [7, Corollary 1] proved the following theorem.

Theorem 1.1. *If X is an inverse limit of absolute neighborhood retracts with universal bonding mappings, then X has the fixed point property.*

In the proof of this theorem, Holsztynski showed that the projection mappings must also be universal. It is then easy to show that X has the f.p.p.

In 1941, A. D. Wallace [16] showed that monotone maps from continua onto generalized trees are universal. In 1967, H. Schirmer [15] generalized Wallace's result by replacing monotone with weakly monotone. The next theorem follows from Schirmer's result.

Theorem 1.2. *If X is a tree-like continuum with weakly monotone projection mappings, then X has the f.p.p.*

Weakly monotone maps between finite trees must be confluent. C. A. Eberhart and J. B. Fugate [7] have shown that confluent maps of trees are weakly

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arc preserving which, in turn, are universal. Theorem 1.3 below follows from their result and Theorem 1.1.

Theorem 1.3. *If X is an inverse limit of trees with weakly arc preserving bonding mappings, then X has the f.p.p.*

The author [8, Theorem 1] has shown that mappings between trees which have restrictions to u-mappings are universal. It was also shown in [8] that the class of such mappings properly contains the class of weakly arc preserving mappings. Hence, we have, in the following theorem, the most general known fixed point result for inverse limits of trees where conditions are placed only on the bonding mappings.

Theorem 1.4. *If X is an inverse limit of trees and each bonding mapping has a restriction that is a u-mapping, then X has the f.p.p.*

Other fixed point results for tree-like continua can be found in [5, 6, 9–11, and 12].

It is unknown if tree-like continua with weakly confluent (or even semiconfluent) bonding mappings must have the f.p.p. H. Cook has asked two questions pertaining to inverse limits with weakly confluent bonding or projection mappings.

(1) (Problem 122, U. of Houston Problem Book) Does each (n-cell)-like continuum with weakly confluent projection mappings have the f.p.p.?

(2) (see [3]) Does each tree-like continuum with weakly confluent bonding mappings have the f.p.p.? The first of these two questions has been answered for $n = 2$ by S. B. Nadler [13].

Theorem 1.5. *If X is disk (2-cell)-like with weakly confluent projection mappings, then X has the f.p.p.*

This paper is concerned with weakly confluent and related mappings of trees. We give a partial solution to Cook's second question which was obtained independently by the author in his dissertation [12] and by Eberhart and Fugate in [3]. Specifically, we show that inverse limits on a single tree with weakly confluent bonding maps must have the f.p.p. We also prove some interesting theorems about weakly confluent mappings between trees which may help solve Cook's second question. The theorems and corollaries of §4 establish cardinality comparisons between the endpoints, the branchpoints, and the edges emanating from branchpoints in the domain and image trees. In the case of finite trees, one of these theorems generalizes a result of Eberhart, Fugate, and Gordh [4, Lemma II.5]. Finally, in §5 we look at relationships between Cook's second question and S. Young's simple folds [17]. Note that semiconfluent (thus, weakly confluent) mappings between finite trees need not be universal and hence Theorem 1.1 cannot be applied directly to Cook's question.

2. Preliminary definitions

A continuum is a compact, connected metric space. A tree is a finite, connected, simply connected graph. Each continuous function will be referred to as a *map* or *mapping*. The topological space X is said to have the *fixed point property* if for each mapping f from X to itself, there is a point x in X such that $f(x) = x$.

Suppose that X is a tree. For $x \in X$, the order of x in X, denoted $o_X(x)$, will be the cardinality of the set of components of $X - \{x\}$. We will omit the reference to X if it is clear from the context. For any set A, the cardinality of A will be denoted by $|A|$. We define the sets $E(X)$ and $B(X)$ of *endpoints* and *branchpoints* of X respectively by

$$E(X) = \{x \in X | o_X(x) = 1\} \quad \text{and} \quad B(X) = \{x \in X | o_X(x) \geq 3\}.$$

The set $V(X) = E(X) \cup B(X)$ will be referred to as the set of *vertices* of X.

For each pair of points x_1, x_2 in X, the unique arc in X linearly ordered from x_1 to x_2 will be denoted by $[x_1, x_2]$; (x_1, x_2) and $(x_1, x_2]$ will denote respectively the open segment from x_1 to x_2, and the half-open segment from x_1 to x_2. The arc $[v_1, v_2]$ in X will be called an *edge* of X only in case $[v_1, v_2] \cap V(X) = \{v_1, v_2\}$. If $[v_1, v_2]$ is an edge of X and one of v_1 or v_2 is in $E(X)$, then $[v_1, v_2]$ is *a terminal edge* of X. Otherwise, $[v_1, v_2]$ is an *interior edge* of X.

For each tree X in this paper, we will assume that we have a metric d defined on $X \times X$ so that each edge of X has length one and, for x_1, x_2 in X, $d(x_1, x_2)$ is the length of the arc $[x_1, x_2]$.

3. Combinatorial lemmas

Lemma 3.1. *Suppose that X is a tree with endpoint set E and branchpoint set B. Then $|E| = 2 + \sum_{v \in B} o(v) - 2|B|$.*

Proof. Let α_0 denote the number of vertices of X and α_1 the number of edges of X. It is well known that $\alpha_0 - \alpha_1 = 1$. Since each edge of X has exactly two vertices and, for $v \in V(X)$, $o(v)$ is the number of edges emanating from v, $\sum_{v \in E \cup B} o(v)$ adds each edge twice; i.e., $2\alpha_1 = \sum_{v \in E \cup B} o(v)$. Thus,

$$2\alpha_1 = \sum_{v \in E} o(v) + \sum_{v \in B} o(v) = |E| + \sum_{v \in B} o(v).$$

We have the following two expressions for α_1:

$$\alpha_1 = \alpha_0 - 1 = |E| + |B| - 1, \quad \alpha_1 = \frac{1}{2}\left(|E| + \sum_{v \in B} o(v)\right).$$

Equating the two, we get that $|E| = 2 + \sum_{v \in B} o(v) - 2|B|$.

Lemma 3.2. *Suppose that X and Y are trees and $f\colon V(X) \to V(Y)$ is a function such that $f(B(X))$ is a proper subset of $B(Y)$, $E(Y)$ is a subset of*

$f(E(X))$, and $f|_{B(X)}$ is one-to-one. Then there is a point $v \in B(X)$ such that $o_X(v) > o_Y(f(v))$.

Proof. Suppose the contrary. Let $H = \{f(v) | v \in B(X)\}$ and $K = B(Y) - H$. By Lemma 3.1 and supposition, we get that

$$\begin{aligned} |E(X)| &= 2 + \sum_{v \in B(X)} o_X(v) - 2|B(X)| \\ &\leq 2 + \sum_{v \in B(X)} o_Y(f(v)) - 2|B(X)|. \end{aligned}$$

Since $\sum_{w \in K} o_Y(w) - 2|K| > 0$ and $f|_{B(X)}$ is one-to-one, it follows that

$$2 + \sum_{v \in B(X)} o_Y(f(v)) - 2|B(X)| < 2 + \sum_{w \in H} o_Y(w) - 2|B(X)| + \sum_{w \in K} o_Y(w) - 2|K|.$$

Thus,

$$\begin{aligned} |E(X)| &< 2 + \sum_{w \in H} o_Y(w) - 2|B(X)| + \sum_{w \in K} o_Y(w) - 2|K| \\ &= 2 + \sum_{w \in B(Y)} o_Y(w) - 2|H| - 2|K| \\ &= 2 + \sum_{w \in B(Y)} o_Y(w) - 2|B(Y)| \\ &= |E(Y)|. \end{aligned}$$

However, $E(Y) \subseteq f(E(X))$ implies that $|E(Y)| \leq |E(X)|$, which is a contradiction.

Lemma 3.3. *Suppose that X and Y are trees and $f: V(X) \to V(Y)$ is a function such that $f(B(X)) = B(Y)$, $E(Y) \subseteq f(E(X))$, and $f|_{B(X)}$ is one-to-one. Then there is a point $v \in B(X)$ such that $o_X(v) \geq o_Y(f(v))$.*

Proof. Suppose the contrary. By Lemma 3.1 and supposition,

$$\begin{aligned} |E(X)| &= 2 + \sum_{v \in B(X)} o_X(v) - 2|B(X)| \\ &< 2 + \sum_{v \in B(X)} o_Y(f(v)) - 2|B(Y)| \\ &= |E(Y)|. \end{aligned}$$

As in Lemma 3.2, this yields a contradiction.

4. Weakly confluent maps of trees

A mapping $f: X \to Y$ is *weakly confluent* if for each subcontinuum K of Y, there is a component H of $f^{-1}(K)$ such that $f(H) = K$. Our first theorem shows that if X and Y are trees, the weak confluence of f allows us to choose a continuum H in X such that f maps H onto K in a certain manner.

Theorem 4.1. *Suppose that $f: X \to Y$ is a weakly confluent map from a tree X onto a tree Y and K is a subtree of Y such that $B(K)$ is not empty. Then there is a tree H in X such that*

(1) $f(H) = K$,
(2) $B(H)$ *is not empty,*
(3) $E(K) = f(E(H))$, *and*
(4) *either*
   a. $|B(H)| \geq |B(K)|$,
   b. $f|_{B(H)}$ *is not one-to-one, or*
   c. *there is a point* $v \in B(H)$ *such that* $o_H(v) > o_K(f(v))$.

Proof. Let $\{K_i\}_{i=1}^{\infty}$ be a sequence of trees in Y such that $\overline{\bigcup_{i=1}^{\infty} K_i} = K$, and for each integer $i \geq 1$, $K_i \subseteq K_{i+1} \subseteq K$, $E(K_{i+1}) \cap K_i$ is empty, and $B(K_i) = B(K)$. For each $i \geq 1$, let M_i be a subtree of X such that $f(M_i) = K_i$. Finally, for each $i \geq 1$, let H_i be a subtree of M_i which is minimal with respect to mapping onto K_i.

First we show that, for each $i \geq 1$, $f(E(H_i)) \subseteq E(K_i)$. Let n be a positive integer, e a point of $E(H_n)$, and suppose that $f(e)$ is in $K_n - E(K_n)$. Since $f^{-1}(E(K_n))$ is closed and $e \notin f^{-1}(E(K_n))$, we can pick a point x in H_n such that the arc $[x, e]$ does not intersect $f^{-1}(E(K_n))$. Let $H_n' = H_n - (x, e]$. We have that H_n' is a proper subtree of H_n and $f(H_n') = K_n$, a contradiction.

We now show that $H_i \subseteq H_j - E(H_j)$ if $j > i \geq 1$ and $H_i \cap H_j \neq \emptyset$. Suppose there is a point a in $E(H_n) \cap H_m$ for some $n > m \geq 1$. Then $f(a)$ is in $E(K_n) \cap K_m$; but, by our choice of the sequence $\{K_i\}_{i=1}^{\infty}$, this is a contradiction. Hence, if $j > i \geq 1$ and H_j intersects H_i, then $H_i \subseteq H_j - E(H_j)$.

For some positive integer j, $\operatorname{diam}(K_j) > 0$; and, for each $i \geq j$, $\operatorname{diam}(K_i) \geq \operatorname{diam}(K_j)$. Thus, there does not exist an infinite subcollection of $\{H_i\}_{i=1}^{\infty}$ which is pairwise mutually exclusive. Therefore, we can choose a subsequence $\{H_{u(i)}\}_{i=1}^{\infty}$ of $\{H_i\}_{i=1}^{\infty}$ such that, for each $i \geq 1$, $H_{u(i)}$ intersects $H_{u(1)}$ and consequently, $H_{u(1)} \subseteq H_{u(i)}$. So, in fact, for each $i \geq 1$, $H_{u(i+1)}$ intersects $H_{u(i)}$, implying that $H_{u(i)}$ is a subset of $H_{u(i+1)}$. We have that $\{H_{u(i)}\}_{i=1}^{\infty}$ is monotonic increasing. We may assume, without loss of generality, that the sequence $\{H_{u(i)}\}_{i=1}^{\infty}$ is the sequence $\{H_i\}_{i=1}^{\infty}$.

Let $H = \overline{\bigcup_{i=1}^{\infty} H_i}$. We will show that H satisfies properties 1–4. By the compactness of X and Y and the continuity of f, it follows that $f(H) = K$. This establishes 1.

To prove property 2, we use an argument of Eberhart, Fugate, and Gordh [3, Theorem 2.1]. The proof is included here since we must make a couple of modifications.

Suppose that $B(H)$ is empty. Then H is an arc, as is H_i, for each $i \geq 1$. Since, for each $i \geq 1$, $E(H_{i+1}) \cap E(H_i)$ is empty, $H_{i+1} - H_i$ has exactly two components, C_{i+1} and D_{i+1}. Now, $\lim_{i\to\infty}\{\operatorname{diam}(C_i)\} = 0$, and $\lim_{i\to\infty}\{\operatorname{diam}(D_i)\} = 0$. Let ε be a positive number which is less than the length of each edge of K. Let δ be a positive number such that if x_1 and x_2 are in X and $d(x_1, x_2) < \delta$, then $d(f(x_1), f(x_2)) < \varepsilon$. Since $B(K)$ is not empty, $|E(K)| \geq 3$. Let a, b, and c be points of $E(K)$. For each $i \geq 1$, choose a_i, b_i, and c_i in $E(K_i)$ such that $\lim_{i\to\infty}\{a_i\} = a$, $\lim_{i\to\infty}\{b_i\} = b$, and $\lim_{i\to\infty}\{c_i\} = c$. Since, for each $i \geq 1$, $f(H_i) = K_i$ and $E(K_{i+1}) \cap K_i$ is empty, there must be at least two of $f^{-1}(a_{i+1})$, $f^{-1}(b_{i+1})$, and $f^{-1}(c_{i+1})$ which intersect the same component of $H_{i+1} - H_i$. Let n be a positive integer such that $\operatorname{diam}(C_n) < \delta$, $\operatorname{diam}(D_n) < \delta$, $d(a_n, a) < \varepsilon/2$, $d(b_n, b) < \varepsilon/2$, and $d(c_n, c) < \varepsilon/2$. Suppose, without loss of generality, that $x_1 \in C_n \cap f^{-1}(a_n)$ and $x_2 \in C_n \cap f^{-1}(b_n)$. So, $d(x_1, x_2) < \delta$, which implies that $d(f(x_1), f(x_2)) = d(a_n, b_n) < \varepsilon$. Since a and b are in $E(K)$ and ε was chosen less than the length of each edge of K, we get that $2\varepsilon < d(a, b) \leq d(a, a_n) + d(a_n, b_n) + d(b_n, b)$. Thus, $2\varepsilon < \varepsilon/2 + d(a_n, b_n) + \varepsilon/2$; yielding $\varepsilon < d(a_n, b_n)$, a contradiction. Hence, $B(H)$ is not empty. This completes the verification of property 2.

It is easy to see that $E(H) = H - \bigcup_{i=1}^{\infty} H_i$ and $E(K) = K - \bigcup_{i=1}^{\infty} K_i$. With this observation, we are ready to establish property 3.

Suppose that $b \in H - \bigcup_{i=1}^{\infty} H_i$ and $f(b) \in K_n$ for some $n \geq 1$. Let W be an open set in Y containing $f(b)$ such that $W \cap K_{n+1} = W \cap K_i$ for each $i \geq n+1$. Let U be an open set in X containing b such that $f(U) \subseteq W$. Choose an integer $j > n+1$ such that $E(H_j) \cap U$ is not empty. Finally, by picking a point $x \in E(H_j) \cap U$, we get that $f(x) \in E(K_j) \cap W$ and thus, $f(x) \in K_j \cap W = K_{n+1} \cap W$. So we have that $f(x) \in E(K_j) \cap K_{n+1}$, which contradicts our choice of the K_i's. Hence, it must be the case that $f(H - \bigcup_{i=1}^{\infty} H_i) \subseteq f(H) - \bigcup_{i=1}^{\infty} K_i$. We are now able to establish the following set inclusion,

$$f(E(H)) = f\left(H - \bigcup_{i=1}^{\infty} H_i\right) \subseteq f(H) - \bigcup_{i=1}^{\infty} K_i = K - \bigcup_{i=1}^{\infty} K_i = E(K).$$

For the opposite inclusion, we let $e \in E(K)$ and suppose that there is a point $a \in \bigcup_{i=1}^{\infty} H_i$ such that $f(a) = e$. Let n be a positive integer for which $a \in H_n$. Then e belongs to $K_n \cap E(K)$, a contradiction. Thus, $E(K) \subseteq f(E(H))$. The two inclusions yield property 3.

We need only prove property 4 in order to complete the proof of the theorem.

Suppose there is a point $v \in B(H)$ such that $f(v) \notin B(K)$. Then part c of property 4 is satisfied. Hence, we assume that $f(B(H)) \subseteq B(K)$. Suppose that parts a and b of property 4 are not satisfied; i.e., $|B(H)| < |B(K)|$ and $f|_{B(H)}$ is one-to-one. We have already verified property 3. Therefore, by Lemma 3.2, there is a point $v \in B(H)$ such that $o_H(v) > o_K(f(v))$.

For trees X and Y and a weakly confluent mapping $f\colon X \to Y$, Eberhart, Fugate, and Gordh [4] have shown that the branchpoints of Y must be covered by the branchpoints of X; i.e., $B(Y) \subseteq f(B(X))$. We point out that this result also follows from Theorem 4.1. To see this, let $w \in B(Y)$ and choose a decreasing sequence of subtrees $\{K_i\}_{i=1}^{\infty}$ such that w is a branchpoint of each

K_i and $\bigcap_{i=1}^{\infty} K_i = \{w\}$. For each $i \geq 1$, there is a tree H_i in X such that f maps H_i onto K_i in the manner of Theorem 4.1; in particular, $B(H_i)$ is not empty. Choose a tree H in X that is a sequential limit of some subsequence of $\{H_i\}_{i=1}^{\infty}$. It follows that $f(H) = \{w\}$ and $H \cap B(X)$ is not empty. So, the branchpoints of Y are covered by the branchpoints of X.

Corollary 4.2. *Suppose that in part a of property* 4, $|B(H)| = |B(K)|$. *Then either* $f|_{B(H)}$ *is not one-to-one or there is a point* $v \in B(H)$ *such that* $o_H(v) \geq o_K(f(v))$.

Proof. Suppose that $f(B(H)) \subseteq B(K)$ and $f|_{B(H)}$ is one-to-one. Then $f(B(H)) = B(K)$; therefore, by Lemma 3.3, there is a point $v \in B(H)$ such that $o_H(v) \geq o_K(f(v))$.

Corollary 4.3. *Suppose* $f\colon X \to Y$ *is a weakly confluent mapping from a tree* X *onto a tree* Y *such that* $f|_{B(X)}$ *is one-to-one. Then,* $o_X(v) \geq o_Y(f(v))$ *for each point* v *in* $B(X)$.

Proof. We mentioned earlier that the branchpoints of Y must be covered by the branchpoints of X; i.e., $B(Y) \subseteq f(B(X))$. Let v be a branchpoint of X. If $f(v)$ is not in $B(Y)$, then clearly $o_X(v) > o_Y(f(v))$. Suppose that $f(v) \in B(Y)$. Let K be a tree in Y such that K intersects each component of $Y - \{f(v)\}$ and $K \cap f(B(X)) = \{f(v)\}$. By Theorem 4.1, there is a tree H in X such that $f(H) = K$, $f(E(H)) = E(K)$, and $B(H)$ is not empty. Since $f|_{B(X)}$ is one-to-one and $K \cap f(B(X)) = \{f(v)\}$, it follows that $B(H) = \{v\}$. Hence,

$$o_Y(f(v)) = o_K(f(v)) = |E(K)| \leq |E(H)| = o_H(v) \leq o_X(v).$$

Corollary 4.4. *Suppose that* X *is a tree and* $f\colon X \to X$ *is a weakly confluent map of* X *onto itself. Then,* $o(v) = o(f(v))$ *for each* $v \in B(X)$.

Proof. Again, we have that $B(X) \subseteq f(B(X))$. Therefore, f must be one-to-one on $B(X)$. Let $v \in B(X)$. Since $B(X)$ is finite and $f|_{B(X)}$ is one-to-one, there is a positive integer n such that $f^n(v) = v$. By Corollary 4.3, for each $u \in B(X)$, $o(u) \geq o(f(u))$. Thus, $o(v) \geq o(f(v)) \geq o(f^n(v)) = o(v)$. Hence, $o(v) = o(f(v))$.

Before stating the next theorem, we need a few definitions. If H is a subtree or a point of a tree X, we define $st(H)$ to be the union of all edges of X that intersect H. The arc s is said to be a *leg* of $st(H)$ provided that s is the closure of some component of $st(H) - H$. Notice that each leg of $st(H)$ contains an endpoint of $st(H)$.

Suppose $f\colon X \to Y$ is a mapping between trees, $w \in B(Y)$, $\{t_i\}_{i=1}^{n}$ are the legs of $st(w)$, and $[u, v]$ is an arc in X so that $f(u) = w$, but $f([u, v]) \neq \{w\}$. We will say that $[u, v]$ *has an initial image under* f if there is an integer $j \in \{1, \ldots, n\}$ and a point $x \in [u, v]$ such that $f(x) \in t_j - \{w\}$ and, $f(x') \in t_j$

for every $x' \in [u, x]$. In this case, we will also say that t_j *is the initial image of* $[u, v]$ under f. The reference to f will be omitted if such reference is clear.

The next theorem generalizes, in the case of finite trees, Lemma II.5 of Eberhart, Fugate, and Gordh [4].

Theorem 4.5. *Suppose that* $f: X \to Y$ *is a weakly confluent map of a tree* X *onto a tree* Y, $w \in B(Y)$, *and* $st(w)$ *has legs* $\{t_i\}_{i=1}^n$. *Let* $\{M_i\}_{i=1}^m$ *be the collection of components of* $f^{-1}(w)$ *that contain a branchpoint. Suppose also that, for each* $i \in \{1, \dots, m\}$ *and each leg* s *of* $st(M_i)$, s *has an initial image. Then there is an integer* $i \in \{1, \dots, m\}$ *such that for each* $j \in \{1, \dots, n\}$, *there is a leg* s *of* $st(M_i)$ *whose initial image is* t_j.

Proof. For each integer $i \in \{1, \dots, m\}$, let $\{M_r(i)\}_{r=1}^\infty$ be a decreasing sequence of trees in X such that $\bigcap_{r=1}^\infty M_r(i) = M_i$, and, for each $r \geq 1$,

(i) $M_r(i)$ intersects each component of $X - M_i$,
(ii) $M_r(i) \cap B(X) = M_i \cap B(X)$,
(iii) $(E(M_r(i)) - E(X)) \cap f^{-1}(w)$ is empty, and
(iv) $(E(M_r(i)) - E(X)) \cap M_{r+1}(i)$ is empty.

In addition, for each $i \in \{1, \dots, m\}$, let k_i be a positive integer such that if $r \geq k_i$ and if s is the closure of a component of $M_r(i) - M_i$, then $f(s)$ meets at most one component of $Y - \{w\}$. Such an integer exists since each leg of $st(M_i)$ has an initial image.

Now let K be a tree in Y such that K intersects each component of $Y - \{w\}$, $K \cap B(Y) = \{w\}$, $K \cap f(B(X)) = \{w\}$, and $K \cap f(E(M_{k_i}(i)) - E(X))$ is empty for each $i \in \{1, \dots, m\}$. Also choose K so that for each interior edge $[v_1, v_2]$ of X with the property that $f(v_1) = w$, $f(v_2) = w$, but $f([v_1, v_2]) \neq \{w\}$, $f([v_1, v_2])$ intersects the complement of K. Applying Theorem 4.1, we choose a tree H in X such that properties 1–4 hold. By choice of K, if $v \in B(X) \cap H$, then $f(v) = w$. Suppose that v_1 and v_2 are in $B(X) \cap H$, and that $f([v_1, v_2]) \neq \{w\}$. Now, $f(v_1) = w = f(v_2)$, and thus, $f([v_1, v_2])$ must intersect the complement of K; but, $[v_1, v_2] \subseteq H$, which yields a contradiction. Hence, for each pair of points v_1 and v_2 in $B(X) \cap H$, $f([v_1, v_2]) = \{w\}$. Let M be the unique minimal tree (perhaps degenerate) in H such that $B(X) \cap H \subseteq M$. Then $f(M) = \{w\}$. Let i be the integer in $\{1, \dots, m\}$ for which $M \subseteq M_i$.

We are now ready to show that the conclusion holds for the integer i chosen above. Let t_j be a leg of $st(w)$. Let $[w, b]$ be the terminal edge of K such that $[w, b] \subseteq t_j$. Choose a point $a \in E(H)$ for which $f(a) = b$. Finally, let v be the nearest point of $B(X) \cap H$ to a. Then $v \in M_i$ and $[v, a]$ is a subset of some terminal edge s of $st(M_i)$. Suppose that $f([v, a]) \nsubseteq [w, b]$. Since $[v, a] \subseteq H$, it follows that $f([v, a]) \subseteq K$ and $f([v, a])$ must intersect two components of $Y - \{w\}$. Hence, by our choice of k_i, $[v, a] \nsubseteq M_{k_i}(i)$. Let $e \in [v, a] \cap E(M_{k_i}(i))$ and let $[w, c]$ be the terminal edge of K such that $f([v, e]) \subseteq [w, c]$. Now, $e \notin E(X)$. So, we have that $f(e) \in K$ and $f(e) \in$

$f(E(M_{k_i}(i)) - E(X))$, which contradicts our choice of K. Thus, $f([v, a]) \subseteq [w, b]$. Therefore, t_j is the initial image of the leg of $st(M_i)$ which is contained in s.

We point out that the assumption

(∗) "for each $i = 1, \ldots, m$ and each leg s of $st(M_i)$, s has an initial image"

cannot be eliminated from the hypothesis of Theorem 4.5. Also, we cannot generalize this result even to a branchpoint covering theorem by assuming that Y is a finite tree and X is a fan (see Ex. III.1 of [4]). We do, however, have the following corollaries.

Corollary 4.6. *Property* (∗) *may be replaced by the assumption that* $f^{-1}(w)$ *has finitely many components.*

Corollary 4.7. *Property* (∗) *may be replaced by the assumption that* f *is piecewise linear.*

Next we wish to show that if f is a weakly confluent mapping of a tree onto itself, then f is universal. It is easy to construct examples of weakly confluent (even semiconfluent) mappings between different trees that are not universal.

Lemma 4.8. *If* $f: T \to T$ *is a weakly confluent map from a tree onto itself and* $[w_1, w_2]$ *is an edge of* T *with* $w_1 \in B(T)$, *then there is a unique branchpoint* v_1 *in* T *and an arc* $[v_1, v_2]$ *such that* $f(v_1) = w_1$, $f(v_2) = w_2$, *and* $f([v_1, v_2]) = [w_1, w_2]$. *Moreover, if* $w_2 \in B(T)$, *then* v_2 *may be chosen from* $B(T)$.

Proof. By Theorem 4.1 (or [4]) and the finiteness of $B(T)$, it follows that for each branchpoint w of T, there is exactly one branchpoint w' of T such that $f(w') = w$. Thus, there is a one-to-one correspondence induced by f on the branchpoints of T. Also, by Corollary 4.4, $o(w) = o(w')$ for each $w \in B(T)$.

Let $v_1 = w_1'$. Assume that $w_2 \in E(T)$. For each open connected set U containing w_1, there is a tree H_U as in Theorem 4.1 such that $f(H_U) = \overline{U}$ and $B(H_U)$ is not empty. It follows that there is an arc $[v_1, v_2]$ such that $f(v_2) = w_2$ and $f([v_1, v_2]) = [w_1, w_2]$.

We now suppose that $w_2 \in B(T)$. Let $v_2 = w_2'$. It is clear that $[w_1, w_2] \subseteq f([v_1, v_2])$. Suppose that $f([v_1, v_2]) \nsubseteq [w_1, w_2]$. We assume without loss of generality that $f([v_1, v_2])$ intersects some component of $T - [w_1, w_2]$ which has w_1 in its closure. Let x_1 and x_2 be points in $[v_1, v_2]$ such that $f(x_1) \neq f(x_2)$, $f(x_1) \notin B(T)$, w_1 separates $f(x_1)$ from w_2, and $f(x_1)$ separates $f(x_2)$ from w_1. Let K be the closure of the component of $T - \{f(x_1)\}$ that contains w_1. So, K is a tree which contains the points w_1, w_2, and $f(x_1)$. Choose a subtree H of T as in Theorem 4.1. Now, $f|_{B(H)}$ is one-to-one, and $o_H(v) \leq o_T(v) = o_T(f(v)) = o_K(f(v))$ for each $v \in B(H)$. Thus, by property 4 of Theorem 4.1, we must have that $|B(H)| \geq |B(K)|$. Since $B(K) = B(T) \cap K$ and $f|_{B(H)}$ is one-to-one, it follows that v_1 and v_2 are in H. But $f(x_2) \notin K$. So $x_2 \notin H$, a contradiction.

Theorem 4.9. *If $f: T \to T$ is a weakly confluent map of a tree onto itself, then f is universal.*

Proof. Assume that $g: T \to T$ is a mapping and $f(x) \neq g(x)$ for $x \in T$. For each branchpoint u of T, let u' be the unique branchpoint of T such that $f(u') = u$ and let $\hat{g}(u) = g(u')$. So, for each $u \in B(T)$, $\hat{g}(u) \neq u$. It follows that (see Lemma in [11]) there exists neighboring vertices v and w in T such that $\hat{g}(v) = g(v')$ is in the component of $T - \{v\}$ that contains w and $\hat{g}(w) = g(w')$ is in the component of $T - \{w\}$ that contains v. By Lemma 4.8, $f([v', w']) = [v, w]$. It follows that $g|_{[v', w']}$ and $f|_{[v', w']}$ have a coincidence point, which contradicts our assumption.

The following fixed point theorem follows from Theorems 4.9 and 1.1.

Theorem 4.10. *Suppose that T is a tree and $X = \varprojlim \{X_i, g_i^{i+1}\}$, where, for each $i \geq 1$, $X_i = T$ and g_i^{i+1} is a weakly confluent mapping of X_{i+1} onto X_i. Then X has the fixed point property.*

5. Weakly confluent simple folds

In [17], S. Young introduces the following definitions. Let T_1 and T_2 be trees, $p \in T_1$, and T_a and T_b subtrees of T_1 with $T_a \cup T_b = T_1$ and $T_a \cap T_b = \{p\}$.

If $\beta: T_1 \to T_2$ is not a homeomorphism but each restriction $\beta|_{T_a}$ and $\beta|_{T_b}$ is a homeomorphism, then β is called a *fold.* If one of the trees T_a or T_b is the closure of a component of $T_1 - \{p\}$, then β is called a *simple fold.*

In Theorem 1 of [17], Young gives a procedure for factoring light maps of trees into simple folds; he later proves the following result [17, Corollary 2].

Theorem 5.1. *If X is a tree-like continuum, then X is the limit of an inverse system $T_1 \leftarrow T_2 \leftarrow \cdots \leftarrow X$, where each of the bonding maps is a simple fold.*

With Cook's question in mind, it is natural to ask the following two questions.

(i) Do light weakly confluent maps between trees factor (using Young's technique) into weakly confluent simple folds?

(ii) Are weakly confluent simple folds universal?

An affirmative answer to both of these questions would give an affirmative answer to Cook's question (recall Theorem 1.1). Corollary 5.3 below gives an affirmative answer to question (ii). In regard to question (i), suppose that we have a weakly confluent map $f: T_1 \to T_2$ between trees that factors into a finite sequence of weakly confluent simple folds. It follows from Theorem 5.2 below that some restriction of f must be a homeomorphism with image T_2. Hence, the only weakly confluent maps of trees that we can hope to factor into weakly confluent simple folds are those that have restrictions which are homeomorphisms onto the image tree. It is well known that such mappings are universal. Hence, this approach to answering Cook's question yields only a well-known special case.

Theorem 5.2. *Suppose that $f: T_1 \to T_2$ is a weakly confluent simple fold. Then some restriction of f is a homeomorphism onto T_2.*

Proof. Let p, T_a, and T_b be given as in the definition of a simple fold. Suppose that T_a is the closure of a component of $T_1 - \{p\}$. By Obs. (3) in [17], on some open set U with $p \in U$, f identifies exactly two components of $(T_1 \cap U) - \{p\}$. Let H_1 and H_2 be these components and let $K = f(H_1) = f(H_2)$. Now, one of H_1 or H_2 is a subset of T_a; for otherwise, $H_1 \cup H_2 \subseteq T_b$ implying that $f|_{T_b}$ is not a homeomorphism. Assume that $H_1 \subseteq T_a$. Similarly, it follows that $H_2 \subseteq T_b$. Since $f|_{T_a}$ and $f|_{T_b}$ are homeomorphisms and f is weakly confluent, one of C_1 or C_2 maps onto D, where C_i $(i = 1, 2)$ is the component of $T_1 - \{p\}$ containing H_i and D is the component of $T_2 - \{f(p)\}$ containing K. Say $f(C_k) = D$, where k is either 1 or 2. Let $R = T_b - C_2$. Again, from the weak confluence of f, it follows that $f(R) = T_2 - D$. But now we have that $f(C_k) = D$, $f(R) = T_2 - D$, and each of $f|_{C_k}$ and $f|_R$ is a homeomorphism (notice that C_k is either a subset of T_a or of T_b, and R is a subset of T_b). It follows that $f|_{C_k \cup R}$ is a homeomorphism with image T_2.

Corollary 5.3. *If $f: T_1 \to T_2$ is a weakly confluent simple fold, then f is universal.*

Recall that the proof of Theorem 1.1 shows that if the bonding mappings are universal, then so are the projection mappings. Actually, one only needs that the projection mappings are weakly universal to show that the inverse limit has the fixed point property. A mapping $f: X \to Y$ is *weakly universal* if for each mapping $g: X \to X$, there is a point $x \in X$ such that $f(x) = fg(x)$. In this regard, we ask the following question.

Question 5.4. *If X is an inverse limit of trees with weakly confluent bonding mappings, must the projection mappings be weakly universal?*

Other questions pertaining to universal, weakly universal, and related mappings can be found in the Sacramento State Topology Conference Problem Session Notes (April 1987).

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