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QUANTIZATION OF CURVATURE OF HARMONIC TWO-SPHERES IN GRASSMANN MANIFOLDS

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ABSTRACT. Various pinching theorems for curvature of minimal two-spheres in Grassmann manifolds have been proved. In particular, we show that when the curvature is large, then the minimal map from S^2 into $G(m, N)$ must be either holomorphic or antiholomorphic. Also, minimal two-spheres of curvature $\kappa \geq 2$ in $G(2, 4)$ have been classified.

0. INTRODUCTION

It is known that starting from a harmonic map f from a Riemann surface into a complex Grassmann manifold, one can define sequences of harmonic maps by using the ∂-transform and the $\overline{\partial}$-transform. These sequences are called harmonic sequences [CW]. When f is holomorphic, the sequence generated by f via the ∂-transform is called a pseudoholomorphic sequence, and the corresponding maps are called pseudoholomorphic maps.

In 1980 A. M. Din and W. J. Zakrzewski [DZ], and then in 1983 J. Eells and J. C. Wood [EC], showed that all harmonic maps from S^2 to the complex projective space CP^n are, in fact, pseudoholomorphic. Recently, F. Burstall, J. Rawnsley and S. Salamon announced in [BRS] that all stable harmonic maps from S^2 into irreducible Hermitian symmetric spaces are either holomorphic or antiholomorphic. It is also known that when the target space is a general complex Grassmann manifold, holomorphic curves do not generate all harmonic maps [CW], [Ra].

On the other hand, M. Rigoli showed in [R] that for a linearly full nonsingular holomorphic curve from S^2 into CP^n, if its curvature $\kappa \geq 4/n$, then $\kappa = 4/n$ is constant. Then J. Bolton et al. [BJRW] generalized this result to pseudoholomorphic maps, and showed that for a linearly full isometric pseudoholomorphic map of position r from S^2 into CP^n, if its curvature $\kappa \geq 4/(2r(n-r)+n)$, then $\kappa = 4/(2r(n-r)+n)$ is constant.

In this paper, we prove various pinching theorems of curvature for harmonic maps and show that for any nonsingular harmonic map from S^2 into a complex Grassmann manifold, if the curvature of its induced metric is large, then

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it is actually a holomorphic or antiholomorphic curve of constant curvature. Precisely, we show that for any harmonic isometric immersion from S^2 to $G(m, N)$, if it is not antiholomorphic (or not holomorphic), and if its curvature $\kappa \geq 4/\operatorname{rank}(\partial f)$ (or $\kappa \geq 4/\operatorname{rank}(\overline{\partial} f)$), then f must be a holomorphic map (or antiholomorphic map) of constant curvature $4/\operatorname{rank}(\partial f)$ (or $4/\operatorname{rank}(\overline{\partial} f)$). When f generates an orthogonal harmonic sequence and if its curvature $\kappa \geq \kappa(f)$, a number associated to the sequence generated by f, then $\kappa = \kappa(f)$ is constant, and f is a holomorphic map which essentially generates a Frenet pseudoholomorphic sequence. We also show that for a harmonic map f in a Frenet harmonic sequence of position j generated by a harmonic map g, if its curvature $\kappa \geq \kappa(g, j)$, a number associated to the sequence and j, then again $\kappa = \kappa(g, j)$ is constant and the directrix g is holomorphic. In particular, f is a pseudoholomorphic map of constant curvature $\kappa(g, j)$. Notice that when the directrix is holomorphic and the target space is CP^n, we recover the pinching theorem of J. Bolton, et al.

Some relations between the pinching conditions of curvature and the ranks of the ∂-transform and the $\overline{\partial}$-transform are discussed in this paper. As examples of using the technique developed in this paper, we prove some pinching theorems for harmonic two-spheres in $G(2, 4)$ and $G(2, 6)$. As a by-product, we show that for a minimal two-sphere in $G(2, 4)$, if its curvature $\kappa \geq 2$, then κ is either 2 or 4, and all these maps are explicitly classified up to $u(4)$-congruences.

The main tool used is the method of moving frames. Extensively using this method, we are able to develop a system of partial differential equations associated to a family of invariants. The study of these invariants plays a key role in proving the main results of this paper.

Definitions and basic formulas are given in §1. General pinching theorems are proved in §2. In §3, we give a detail discussion of harmonic two-spheres in $G(2, 4)$ and $G(2, 6)$.

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1. Harmonic maps, harmonic sequences and pseudoholomorphic maps

The complex Grassmann manifold $G(m, N)$ is the set of all m-dimensional complex linear subspaces of C^N through the origin, which can be realized as the homogeneous space $U(N)/(U(m) \times U(N-m))$, where

$$\pi: U(N) \rightarrow G(m, N)$$
$$g \rightarrow [g \cdot 0]$$

is a principal $U(m) \times U(N-m)$-bundle, 0 is the $N \times m$ matrix

$$\begin{bmatrix} I_m \\ 0 \end{bmatrix},$$

and I_m is the $m \times m$ identity matrix.

Let $W = (W_{AB}) = g^{-1} dg$ be the Maurer-Cartan form of $U(N)$, where $1 \le A, B \le N$, and let e be a local section of $\pi: U(N) \to G(m, N)$. Set

$$\Psi_{AB} = e^*(W_{AB}) \tag{1.1}$$

and

$$ds^2_{G(m,N)} = \sum_{\substack{1 \le i \le m \\ m < \alpha \le N}} \Psi_{i\alpha} \overline{\Psi}_{i\alpha}. \tag{1.2}$$

Then $ds^2_{G(m,N)}$ is a $U(N)$-invariant Hermitian metric on $G(m,N)$. When $m = 1$ and $N = n+1$, this is just the Fubini-Study metric on CP^n of constant holomorphic curvature 4.

Suppose M is a Riemann surface with a Riemannian metric ds^2_M. Then locally

$$ds^2_M = \theta\overline{\theta} \tag{1.3}$$

and

$$d\theta = iw \wedge \theta, \tag{1.4}$$

where θ is a local unitary coframe of bidegree $(1,0)$ which is determined up to a $U(1)$-valued function, and w is the real-valued Levi-Civita connection form associated to the coframe θ. Also,

$$dw = \frac{i}{2}\kappa\theta \wedge \overline{\theta}, \tag{1.5}$$

where κ is the Gaussian curvature of ds^2_M.

Let f be a smooth map from M to $G(m,N)$. Choose a local unitary frame $e = (e_1, \dots, e_N)$ along f such that $e_1, \dots, e_m$ span f. Then for $1 \le A \le N$

$$de_A = \sum_{B=1}^{n} e_B w_{BA}, \tag{1.6}$$

where $w_{AB} = f^*(\Psi_{AB}) = e^*(W_{AB})$ are the entries of the pull back of the Maurer-Cartan form of $U(N)$ via e.

The Maurer-Cartan structure equations give

$$dw_{AB} = -\sum_{c=1}^{N} w_{AC} \wedge w_{CB} \tag{1.7}$$

and

$$w_{AB} + \overline{w_{BA}} = 0. \tag{1.8}$$

For $1 \le j \le m$ and $m+1 \le \alpha \le N$, set

$$w_{\alpha j} = a_{\alpha j}\theta + b_{\alpha j}\overline{\theta}. \tag{1.9}$$

Then f is an isometric immersion if and only if

$$\sum_{\substack{1\le j\le m\\ m<\alpha\le N}} a_{\alpha j}b_{\alpha j}=0 \tag{1.10}$$

and

$$\sum_{\substack{1\le j\le m\\ m<\alpha\le n}} (a_{\alpha j}\overline{a_{\alpha j}}+b_{\alpha j}\overline{b_{\alpha j}})=1\,. \tag{1.11}$$

The harmonicity condition for f is

$$da_{\alpha j}+ia_{\alpha j}w+\sum_{\beta=m+1}^{n} a_{\beta j}w_{\alpha\beta}-\sum_{t=1}^{m} a_{\alpha t}w_{tj}\equiv 0 \quad \operatorname{mod}\theta\,, \tag{1.12}$$

which is equivalent to

$$db_{\alpha j}-ib_{\alpha j}w+\sum_{\beta=m+1}^{n} b_{\beta j}w_{\alpha\beta}-\sum_{t=1}^{m} b_{\alpha t}w_{tj}\equiv 0 \quad \operatorname{mod}\overline{\theta}\,. \tag{1.13}$$

In terms of matrix notation, we can rewrite these equations as

$$\operatorname{tr}(A^t\overline{B})=0\,, \tag{1.10$'$}$$

$$|A|^2+|B|^2=1\,, \tag{1.11$'$}$$

$$dA=-iwA+A\Phi_{11}-\Phi_{22}A \quad \operatorname{mod}\theta \tag{1.12$'$}$$

and

$$dB=iwB+B\Phi_{11}-\Phi_{22}B \quad \operatorname{mod}\overline{\theta}\,, \tag{1.13$'$}$$

where

$$A=\begin{pmatrix} a_{m+1,1} & \cdots & a_{m+1,m}\\ \vdots & & \vdots\\ a_{N,1} & \cdots & a_{N,m}\end{pmatrix},\qquad B=\begin{pmatrix} b_{m+1,1} & \cdots & b_{m+1,m}\\ \vdots & & \vdots\\ b_{N,1} & \cdots & b_{N,m}\end{pmatrix},$$

$$\Phi_{11}=\begin{pmatrix} w_{1,1} & \cdots & w_{1,m}\\ \vdots & & \vdots\\ w_{m,1} & \cdots & w_{m,m}\end{pmatrix} \quad\text{and}\quad \Phi_{22}=\begin{pmatrix} w_{m+1,m+1} & \cdots & w_{m+1,N}\\ \vdots & & \vdots\\ w_{N,m+1} & \cdots & w_{N,N}\end{pmatrix}.$$

Here we define the norm $|x|$ of a matrix x by $|x|^2=\operatorname{tr}(x^t\overline{x})$ in a standard way.

For a harmonic map f from M into $G(m,N)$, using the ∂-transform and the $\overline{\partial}$-transform S. S. Chern and J. G. Wolfson defined new harmonic maps, $\partial f\colon M\to G(m_1,N)$, if f is not antiholomorphic, and $\overline{\partial} f\colon M\to G(m_{-1},N)$, if f is not holomorphic. These maps satisfy

$$\partial f(x)=\operatorname{span}\left\{\sum_{\alpha=m+1}^{N} a_{\alpha j}(x)e_\alpha(x)\colon 1\le j\le m\right\} \tag{1.14}$$

and

$$\overline{\partial} f(x) = \text{span}\left\{ \sum_{\alpha=m+1}^{N} b_{\alpha j}(x) e_\alpha(x) \colon 1 \le j \le m \right\} \tag{1.15}$$

for $x \in M$, except for at most isolated points. Here m_1 and m_{-1} are positive integers, called the rank of ∂f and the rank of $\overline{\partial} f$, respectively. Sucessively using these transforms they obtained two sequences of harmonic maps

$$f \equiv f_0 \xrightarrow{\partial} f_1 \xrightarrow{\partial} f_2 \to \cdots$$

and

$$f \equiv f_0 \xrightarrow{\overline{\partial}} f_{-1} \xrightarrow{\overline{\partial}} f_{-2} \to \cdots,$$

which are called the harmonic sequences, where for $j \ge 1$, $f_j \equiv \partial f_{j-1} \colon M \to G(m_j, N)$, and for $j \le -1$, $f_j \equiv \overline{\partial} f_{j+1} \colon M \to G(m_j, N)$. We say that f_j and f_i are orthogonal, if for $x \in M$, $f_j(x) \perp f_i(x)$ as linear subspaces of C^N with respect to the standard Hermitian inner product. If $f_j \perp f_i$ for $i \ne j$ in a sequence, we say this sequence is orthogonal. In addition, if m_j's are equal, we have

$$f_0 \underset{\overline{\partial}}{\overset{\partial}{\rightleftarrows}} f_1 \underset{\overline{\partial}}{\overset{\partial}{\rightleftarrows}} f_2 \underset{\overline{\partial}}{\overset{\partial}{\rightleftarrows}} f_3 \cdots$$

which is called a Frenet harmonic sequence. It is a finite sequence, and f_0 is called the directrix of this sequence.

When f is holomorphic, it will generate a harmonic sequence,

$$f \equiv f_0 \xrightarrow{\partial} f_1 \xrightarrow{\partial} f_2 \to \cdots$$

which is orthogonal and therefore is finite. This sequence is called a pseudo-holomorphic sequence, and each map f_j is called a pseudoholomorphic map generated by f with position j.

A locally defined smooth function $V \colon M \supset U \to C^L$ is of analytic type, if either v is identically zero on U or for any local complex coordinate z centered about x, the local representation of v has the form $v(z) = z^r g(z)$ where $g(z)$ is a C^L-valued smooth function such that $g(0) \ne 0$.

The following proposition will be repeatedly used in this paper, whose proof can be found in [Z].

Proposition 1.1. *Let U be an open subset of M, C be a $k \times k$-matrix-valued smooth function defined on U, and θ be a unitary coframe field defined on U. Suppose that C satisfies*

$$dC = C\eta + \phi\eta - C\Psi \mod \theta, \tag{1.16}$$

where η is an imaginary valued 1-form and ϕ and Ψ are $k \times k$-matrix-valued 1-forms with $(\operatorname{tr}\phi - \operatorname{tr}\Psi)$ imaginary valued. Then $\det C$ is a function of analytic type, and

$$(\Delta \log|\det C|)\theta \wedge \overline{\theta} = 2kd\eta + 2d(\operatorname{tr}\phi) - 2d(\operatorname{tr}\Psi) \tag{1.17}$$

on $U \setminus Z$, where Z is the set of zeros of $\det C$ on U.

Remark. When $k=1$ and $\phi=\Psi=0$, this is the formula given in [EGT].

2. Quantization of curvature of harmonic two-spheres in $G(m,N)$

Let f be a harmonic map from a Riemann surface M into $G(m,N)$. Take a local unitary frame $e=(e_1,\dots,e_N)$ along f such that $e_1,\dots,e_m$ span f. Then

$$e^*(W)=\phi=\begin{bmatrix}\phi_{11} & -{}^t\overline{A}\,\overline{\theta} & -{}^t\overline{B}\theta\\ A\theta+B\overline{\theta} & & \phi_{22}\end{bmatrix},$$

where $A, B\in C^\infty(M((N-m)\times m))$, $\phi_{11}\in C^\infty(u(m)\otimes T^*M)$ and $\phi_{22}\in C^\infty(u(N-m)\otimes T^*M)$. Here we denote $C^\infty(M(m_1\times m_2))$ as the set of all locally defined $m_1\times m_2$-matrix-valued smooth functions and $C^\infty(u(k)\otimes T^*M)$ as the set of all locally defined $u(k)$-valued smooth 1-forms, where $u(k)$ is the Lie algebra of $U(k)$.

The frame e is determined up to $U(m)\times U(n-m)$-transformations. Changing such a local frame, $\tilde{e}=e\left[\begin{smallmatrix}K_1&0\\0&K_2\end{smallmatrix}\right]$, where $K_1\in C^\infty(U(m))$ and $K_2\in C^\infty(U(N-m))$, we shall have

$$\tilde{e}^*(W)=\tilde{\phi}=\begin{bmatrix}\tilde{\phi}_{11} & -{}^t\overline{\tilde{A}\theta}-{}^t\overline{\tilde{B}}\theta\\ \tilde{A}\theta+\tilde{B}\overline{\theta} & \tilde{\phi}_{22}\end{bmatrix}$$

where $\tilde{A}=K_2^{-1}AK_1$ and $\tilde{B}=K_2^{-1}BK_1$. So,

$$ {}^t\tilde{A}\overline{\tilde{B}}=\overline{K}_1^{-1}\,{}^tA\overline{B}\,\overline{K}_1. \tag{2.1}$$

Setting $C={}^tA\overline{B}$ and $\tilde{C}={}^t\tilde{A}\overline{\tilde{B}}$, we see that $\det C=\det\tilde{C}$ and $\operatorname{tr}C=\operatorname{tr}\tilde{C}$. Notice that changing a local unitary coframe θ into $\tilde{\theta}$, $\det C$ becomes $u_1\det C$ and $\operatorname{tr}C$ becomes $u_2\operatorname{tr}C$, where u_1 and u_2 are smooth functions of absolute value 1. Thus $|\det C|$ and $|\operatorname{tr}C|$ are globally defined nonnegative invariants on M, which are continuous.

Proposition 2.1. $\det C$ *and* $\operatorname{tr}C$ *are of analytic type. Furthermore, away from zeros of* $|\det C|$ *and* $|\operatorname{tr}C|$, *they satisfy*

$$\Delta\log|\det C|=2m\kappa \tag{2.2}$$

and

$$\Delta\log|\operatorname{tr}C|=2\kappa, \tag{2.3}$$

where κ *is the Gaussian curvature of the metric* ds^2_M.

Proof. Taking the exterior derivative of C and using (1.12) and (1.13) we obtain

$$dC=-2iwC+{}^t\phi_{11}C-C\,{}^t\phi_{11}\mod\theta. \tag{2.4}$$

Applying Proposition 1.1 to (2.4),

$$\Delta\log|\det C|\theta\wedge\overline{\theta}=-4mi\,dw. \tag{2.5}$$

(2.2) then follows from using (1.5) to (2.5).

For (2.3), we take the trace on both sides of (2.4). Then

$$d(\operatorname{tr} C) = -2iw(\operatorname{tr} C) \mod \theta, \tag{2.6}$$

and the conclusion again is obtained by using (1.5) and Proposition 1.1 for the case of $k = 1$. Q.E.D.

Following this proposition, we have

Corollary 2.1 (Vanishing Theorem). *If M is a closed Riemann surface, then either the genus g of M is positive or both $|\det c|$ and $|\operatorname{tr} c|$ vanish identically. In particular, if the Gaussian curvature $\kappa \geq 0$, then either $\kappa = 0$ or both $|\det c|$ and $|\operatorname{tr} c|$ are identically zero.*

Proof. Use the Gauss-Bonnet formula and Stokes' Theorem on (2.2) and (2.3). Q.E.D.

Remarks. 1. Corollary 2.1 is known [CW].

2. When $M = S^2$, then $|\operatorname{tr} C| = 0$, which means that harmonic maps from S^2 into $G(m, N)$ are weakly conformal (i.e., (1.10) holds).

We say that the ∂-transform (resp. the $\overline{\partial}$-transform) is degenerate, if $\operatorname{rank}(\partial f) < \operatorname{rank}(f)$ (resp. $\operatorname{rank}(\overline{\partial} f) < \operatorname{rank}(f)$).

Corollary 2.2. *For a harmonic map from S^2 into $G(m, 2m)$, one of the ∂-transform and the $\overline{\partial}$-transform must be degenerate.*

Proof. Observe that in this case $\det C = \det A \cdot \det \overline{B}$, and generically $\operatorname{rank}(A) = \operatorname{rank}(\partial f)$ and $\operatorname{rank}(B) = \operatorname{rank}(\overline{\partial} f)$. Q.E.D.

Now suppose that $M = S^2$ is the Riemann sphere and f is a harmonic isometric immersion so that κ is the Gaussian curvature of the induced metric $f^* ds^2_{G(m,N)}$. The following theorem says that if the curvature is large enough, then f is forced to be holomorphic or antiholomorphic.

Theorem 2.1. *If f is not antiholomorphic, and if $\kappa \geq 4/\operatorname{rank}(\partial f)$, then f is a holomorphic map of constant curvature $4/\operatorname{rank}(\partial f)$. Similarly, if f is not holomorphic, and if $\kappa \geq 4/\operatorname{rank}(\overline{\partial} f)$, then f is an antiholomorphic map of constant curvature $4/\operatorname{rank}(\overline{\partial} f)$.*

Proof. Suppose f is not antiholomorphic. Choose a local unitary frame $e = (e_1, \dots, e_N)$ along f so that $e_1, \dots, e_m$ span f and $e_{m+1}, \dots, e_{m+k^0}$ span ∂f, where $k_0 = \operatorname{rank}(\partial f)$. Furthermore, we can require that $e_{k_0+1}, \dots, e_m$ span the kernel of the ∂-transform. The pull back of the Maurer-Cartan form of $U(N)$ by e is then

$$\phi = \begin{bmatrix} \Omega_{11} & \Omega_{12} & -{}^\tau\overline{A}_{11}\overline{\theta} - {}^\tau\overline{B}_{11}\theta & -{}^\tau\overline{B}_{21}\theta \\ \Omega_{21} & \Omega_{22} & -{}^\tau\overline{B}_{12}\theta & -{}^\tau\overline{B}_{22}\theta \\ A_{11}\theta + B_{11}\overline{\theta} & B_{12}\overline{\theta} & \Omega_{33} & \Omega_{34} \\ B_{21}\overline{\theta} & B_{22}\overline{\theta} & \Omega_{43} & \Omega_{44} \end{bmatrix},$$

where A_{11}, $B_{11} \in C^\infty(u(k_0 \times k_0))$, $B_{21} \in C^\infty(M((N-m-k_0) \times k_0))$, $B_{12} \in C^\infty(M(k_0 \times (m-k_0)))$, $B_{22} \in C^\infty(M(m-k_0) \times (m-k_0))$, Ω_{11}, $\Omega_{33} \in$

$C^{\infty}(u(k_0)\otimes T^*S^2)$, $\Omega_{21}\in C^{\infty}(M((m-k_0)\times k_0)\otimes T^*S^2)$, $\Omega_{12}=-{}^{\tau}\overline{\Omega}_{21}$, $\Omega_{22}\in C^{\infty}(u(m-k_0)\otimes T^*S^2)$, $\Omega_{43}\in C^{\infty}(M((N-m-k_0)\times k_0)\otimes T^*S^2)$, $\Omega_{34}=-{}^{\tau}\overline{\Omega}_{43}$ and $\Omega_{44}\in C^{\infty}(u(N-m-k_0)\otimes T^*S^2)$. From (1.11′), we have

$$|A_{11}|^2+|B_{11}|^2+|B_{12}|^2+|B_{21}|^2+|B_{22}|^2=1. \tag{2.7}$$

It can be easily checked that $|\det A_{11}|$ is a well-defined invariant on S^2.
Harmonicity condition (1.12′) of f gives

$$d\begin{bmatrix}A_{11} & 0\\ 0 & 0\end{bmatrix}=-iw\begin{bmatrix}A_{11} & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}A_{11} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\Omega_{11} & \Omega_{12}\\ \Omega_{21} & \Omega_{22}\end{bmatrix} -\begin{bmatrix}\Omega_{33} & \Omega_{34}\\ \Omega_{43} & \Omega_{44}\end{bmatrix}\begin{bmatrix}A_{11} & 0\\ 0 & 0\end{bmatrix}\mod\theta, \tag{2.8}$$

from which it follows that

$$dA_{11}=-iwA_{11}+A_{11}\Omega_{11}-\Omega_{33}A_{11}\quad\mod\theta, \tag{2.9}$$

$$\Omega_{43}A_{11}=0\quad\mod\theta \tag{2.10}$$

and

$$A_{11}\Omega_{12}=0\quad\mod\theta.$$

(2.10) shows that Ω_{12} and Ω_{43} are matrices of bidegree (1.0) forms. So

$$\Omega_{12}=X_{12}\theta \tag{2.11}$$

and

$$\Omega_{43}=X_{43}\theta, \tag{2.12}$$

where X_{12} and X_{43} are matrix-valued smooth functions.

Applying Proposition 1.1 to (2.9) and using (1.5), we get

$$\Delta\log|\det A_{11}|\theta\wedge\overline{\theta}=k_0\kappa\theta\wedge\overline{\theta}+2d(\operatorname{tr}\Omega_{11}-\operatorname{tr}\Omega_{33}). \tag{2.13}$$

By (1.7), (2.7), (2.10) and (2.11),

$$\begin{aligned}\Delta\log|\det A_{11}|&\\ &=k_0\kappa+2(|X_{12}|^2+|X_{43}|^2-2|A_{11}|^2+2|B_{11}|^2+|B_{12}|^2+|B_{21}|^2)\\ &\geq k_0\kappa-4.\end{aligned} \tag{2.14}$$

Since $\kappa\geq 4/k_0$,

$$\Delta\log|\det A_{11}|\geq 0. \tag{2.15}$$

Notice that (2.15) holds on S^2 except for at most finitely many points, and that $|\det A_{11}|$ assumes its positive maximum value at some point where (2.15) is valid. By the maximum principle of subharmonic functions, it therefore must be constant. Then by (2.7) and (2.14) again, $\kappa=4/k_0$ and $B_{11}=B_{12}=B_{21}=B_{22}=0$, which shows that f is a holomorphic map of constant curvature $4/k_0$.

A similar argument will work for proving the second claim of the statement. Q.E.D.

Recall that a holomorphic map or an antiholomorphic map from the Riemann sphere into a complex Grassmann manifold is always stable. Thus we have

Corollary 2.3. *If* $\mathrm{rank}(\partial f) \neq 0$ (*resp.* $\mathrm{rank}(\bar{\partial} f) \neq 0$, *and if* $\kappa \geq 4/\mathrm{rank}(\partial f)$ (*resp.* $K \geq 4/\mathrm{rank}(\bar{\partial} f)$), *then* f *is stable.*

Remark. One can construct a holomorphic map from S^2 into $G(m, N)$ with $\mathrm{rank}(\partial f) = k_0$ and constant curvature $\kappa = 4/k_0$ in terms of homogeneous coordinates in the following way,

$$f\begin{bmatrix} Z_0 \\ Z_1 \end{bmatrix} = \left[\begin{array}{cccccccc} Z_0 & & & & & & & \\ Z_1 & & & & & & & \\ & Z_0 & & & & & & 0 \\ & Z_1 & & & & & & \\ & & \ddots & & & & & \\ & & & Z_0 & & & & \\ & & & Z_1 & & & & \\ & 0 & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \\ & & & 0 & & & & \end{array}\right] \begin{array}{l} \left.\begin{matrix} \\ \\ \\ \\ \\ \\ \\ \end{matrix}\right\} 2k_0 \\ \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} m-k_0 \\ \left.\begin{matrix} \\ \end{matrix}\right\} N-m-k_0 \end{array}$$

When $k_0 > 0$, a small perturbation of this map will produce a holomorphic curve of nonconstant curvature κ with $\mathrm{rank}(\partial f) = k_0$ and arbitrarily small positive value of $|\kappa - 4/k_0|$.

For example, for any $\varepsilon > 0$ one can define $f_\varepsilon \colon S^2 \to G(m, N)$ by

$$f_\varepsilon\begin{bmatrix} Z_0 \\ Z_1 \end{bmatrix} = \left[\begin{array}{cccccccc} (1+\varepsilon)Z_0 & & & & & & & \\ Z_1 & & & & & & & \\ & Z_0 & & & & & & 0 \\ & Z_1 & & & & & & \\ & & \ddots & & & & & \\ & & & Z_0 & & & & \\ & & & Z_1 & & & & \\ 0 & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \\ & & & 0 & & & & \end{array}\right] \begin{array}{l} \left.\begin{matrix} \\ \\ \\ \\ \\ \\ \\ \end{matrix}\right\} 2k_0 \\ \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} m-k_0 \\ \left.\begin{matrix} \\ \end{matrix}\right\} N-m-k_0 \end{array}$$

It is holomorphic. In terms of the complex coordinate Z on S^2, the induced metric is $ds^2 = h(Z)|dz|^2$, where

$$h(Z) = \frac{k_0 - 1}{(1+|Z|^2)^2} + \frac{(1+\varepsilon)^2}{(|Z|^2 + (1+\varepsilon)^2)^2}.$$

Its curvature is

$$\begin{aligned}\kappa = \frac{4}{h^3(Z)} \Bigg[& \frac{(k_0-1)^2}{1+|Z|^2)^6} + \frac{(1+\varepsilon)^6}{(|Z|^2+(1+\varepsilon)^2)^6} \\ & + \frac{(k_0-1)(1+\varepsilon)^4}{(1+|Z|^2)^2(|Z|^2+(1+\varepsilon)^2)^4} \\ & + \frac{(k_0-1)(1+\varepsilon)^2}{(1+|Z|^2)^4(|Z|^2+(1+\varepsilon)^2)^2} \\ & - \frac{2(k_0-1)(1+\varepsilon)^2|Z|^2(2\varepsilon+\varepsilon)^2}{(1+|Z|^2)^4(|Z|^2+(1+\varepsilon)^2)^4} \Bigg],\end{aligned}$$

which is not constant. Since $\lim_{\varepsilon \to 0} f_\varepsilon = f$, one certainly can make $|\kappa - 4/k_0|$ as small as desired.

Now, if the curvature κ is not that big, the following theorem describes some relations between the pinching conditions of the curvature κ and the ranks of the ∂-transform and the $\overline{\partial}$-transform.

Theorem 2.2. *Let f be a harmonic isometric immersion from S^2 into $G(m, N)$, where $N \geq 2m$. Let $k_1 = \operatorname{rank}(\partial f)$, $k_2 = \operatorname{rank}(\overline{\partial} f)$ and κ be the Gaussian curvature. We have the following.*

(i) *If $0 < k_1 + k_2 \leq m$ and $\kappa \geq 4/(k_1 + k_2)$, then $\kappa = 4/(k_1 + k_2)$ is constant, and $f = [f_1 \oplus f_2 \oplus V]\colon S^2 \to G(m, n)$, where $f_1 \perp f_2 \perp V$ such that $f_1\colon S^2 \to G(k_1, N)$ is holomorphic, $f_2\colon S^2 \to G(k_2, N)$ is antiholomorphic and V is a constant complex vector subspace of C^N of dimension $m - k_1 - k_2$.*

(ii) *If $0 < k_1 < m$, $0 < k_2 < m$ and $k_1 + k_2 > m$, then $\min \kappa < 4/(k_1 + k_2)$.*

(iii) *If $k_1 = m$ and $0 < k_2 < m$ (or $0 < k_1 < m$ and $k_2 = m$) when $N > 2m$, then $\min \kappa < 8/(2k_1 + 3k_2)$; (or $\min \kappa < 8/(2k_2 + 3k_1)$); when $N = 2m$, then $\min \kappa < 4/(k_1 + 2k_2)$ (or $\min \kappa < 4/(k_2 + 2k_1)$).*

(iv) *If $k_1 = k_2 = m$ and $\kappa \geq 1/m$, then $\kappa = 1/m$ is constant and $\partial f \perp \overline{\partial} f$. Furthermore, f is a pseudoholomorphic map in the following sequence,*

$$f_{-1} \xrightarrow{\partial} f \equiv f_0 \xrightarrow{\partial} f_1,$$

where $f_{-1}\colon S^2 \to G(m, N)$ is holomorphic and $f_1\colon S^2 \to G(m, N)$ is antiholomorphic.

Remarks. 1. The hypothesis that $N \geq 2m$ is not crucial, since for any harmonic map f, its orthogonal complement, denoted by $f^\perp$, is also a harmonic map having the same induced metric as that of f. Moreover, $\operatorname{rank}(\partial f)$

$= \operatorname{rank}(\overline{\partial}(f^{\perp}))$ and $\operatorname{rank}(\overline{\partial} f) = \operatorname{rank}(\partial(f^{\perp}))$. So we can replace f by $f^{\perp}$ if it is necessary.

2. (iii) says that when $N > 2m$, if $\kappa \geq 8/(2m + 3\operatorname{rank}(\overline{\partial} f))$ (or $\kappa \geq 8/(2m + 3\operatorname{rank}(\partial f)))$, then the ∂-transform (or $\overline{\partial}$-transform) must be degenerate; when $N = 2m$, if $\kappa \geq 4/(m+2\operatorname{rank}(\overline{\partial} f))$ (or $\kappa \geq 4/(m+2\operatorname{rank}(\partial f)))$, then the ∂-transform (or $\overline{\partial}$-transform) must be degenerate.

Proof. Choosing a unitary frame e as in the proof of Theorem 2.1 and adopting the notations there we have

$$\Delta \log|\det A_{11}| = k_1\kappa + 2(|X_{12}|^2 + |X_{43}|^2 - 2|A_{11}|^2 + 2|B_{11}|^2 + |B_{12}|^2 + |B_{21}|^2). \tag{2.16}$$

Change such a unitary frame e into $\tilde{e}$ such that $\tilde{e} = (\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_N)$, where $\tilde{e}_1, \dots, \tilde{e}_m$ span f, $\tilde{e}_{m+1}, \dots, \tilde{e}_{m+k_2}$ span $\overline{\partial} f$ and $\tilde{e}_{k_2+1}, \dots, \tilde{e}_m$ span the kernel of the $\overline{\partial}$-transform. If we denote the corresponding new quantities by adding tildes, then a similar argument leads to

$$\Delta \log|\det \tilde{B}_{11}| = k_2\kappa + 2(|\tilde{X}_{12}|^2 + 2|\tilde{A}_{11}|^2 + |\tilde{A}_{21}|^2 - 2|\tilde{B}_{11}|^2 + |\tilde{A}_{12}|^2 + |\tilde{X}_{43}|^2) \tag{2.17}$$

Since $\tilde{e} = \left[\begin{smallmatrix} K_1 & 0 \\ 0 & K_2 \end{smallmatrix}\right] e$, where $K_1 \in C^{\infty}(U(m))$ and $K_2 \in C^{\infty}(U(N-m))$, we have $\tilde{A} = K_2^{-1} A K_1$ and $\tilde{B} = K_2^{-1} B K_1$, and therefore $|\tilde{A}|^2 = |A|^2$ and $|\tilde{B}|^2 = |B|^2$.

If $0 < k_1 + k_2 \leq m$ and $\kappa \geq 4/(k_1 + k_2)$, then combining (2.16) and (2.17), together with (1.11′), we shall have

$$\begin{aligned} &\Delta \log(|\det A_{11}||\det \tilde{B}_{11}|) \\ &\quad = (k_1 + k_2)\kappa + 2(|X_{12}|^2 + |\tilde{X}_{12}|^2 + |X_{43}|^2 + |\tilde{X}_{43}|^2 - 2|A_{11}|^2 - 2|\tilde{B}_{11}|^2 \\ &\qquad + 2|B_{11}|^2 + |B_{12}|^2 + |B_{21}|^2 + 2|\tilde{A}_{11}|^2 + |\tilde{A}_{12}|^2 + |\tilde{A}_{21}|^2) \\ &\quad \geq (k_1 + k_2)\kappa - 4(|A_{11}|^2 + |\tilde{B}_{11}|^2) \\ &\quad = (k_1 + k_2)\kappa - 4 \geq 0. \end{aligned} \tag{2.18}$$

Using the same argument as in the proof of Theorem 2.1, we see that the globally defined invariant $|\det A_{11}||\det \tilde{B}_{11}|$ is constant and $\kappa = 4/(k_1+k_2)$ is constant. Moreover, (2.18) implies

$$|X_{12}| = |X_{43}| = |B_{11}| = |B_{12}| = |B_{21}| = 0, \tag{2.19}$$

and

$$|\tilde{X}_{12}| = |\tilde{X}_{43}| = |\tilde{A}_{11}| = |\tilde{A}_{12}| = |\tilde{A}_{21}| = 0$$

which mean that $\partial f \perp \overline{\partial} f$.

Specifying the local frame e further by requiring that $e_{k_1+k_2+1}, \dots, e_m$ span the kernel of the $\overline{\partial}$-transform and using (2.19), the pull back of the Maurer-Cartan form by e is then of the form

$$\phi = \begin{bmatrix} \phi_1 & 0 & 0 & -{}^\tau\overline{A}_{11}\overline{\theta} & 0 & 0 \\ 0 & \phi_2 & 0 & 0 & -{}^\tau\overline{B}_{22}\theta & 0 \\ 0 & 0 & \phi_3 & 0 & 0 & 0 \\ A_{11}\theta & 0 & 0 & \phi_4 & 0 & 0 \\ 0 & B_{22}\overline{\theta} & 0 & 0 & \phi_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \phi_6 \end{bmatrix}$$

where ϕ_i, $i = 1, \dots, 6$, are matrix-valued 1-forms, $A_{11} \in C^\infty(M(k_1 \times k_1))$ and $B_{22} \in C^\infty(M(k_2 \times k_2))$.

Now, we see that $f = [f_1 \oplus f_2 \oplus V]$, where $f_1 \perp f_2 \perp V$, and $f_1 = \operatorname{span}\{e_1, e_2, \dots, e_{k_1}\}: S^2 \to G(k_1, N)$ is holomorphic,

$$f_2 = \operatorname{span}\{e_{k_1+1}, \dots, e_{k_1+k_2}\}: S^2 \to G(k_2, N)$$

is antiholomorphic and $V = \operatorname{span}\{e_{k_1+k_2+1}, \dots, e_m\}$ is a constant vector subspace of complex dimension $m - k_1 - k_2$.

Suppose now $0 < k_1 < m$, $0 < k_2 < m$ and $m < k_1 + k_2$. If $\min \kappa \geq 4/(k_1 + k_2)$, then (2.19) would imply $f = [g_1 \oplus g_2]$, where $g_1 \perp g_2$, $g_1: S^2 \to G(k_1, N)$ is holomorphic and $g_2: S^2 \to G(m - k_1, N)$ is antiholomorphic, with $\operatorname{rank}(\overline{\partial} g_2) = k_2$. But this means that $m - k_1 \geq k_2$ which contradicts the hypothesis that $k_1 + k_2 > m$.

When $k_1 = m$ and $0 < k_2 < m$ (or $0 < k_1 < m$ and $k_2 = m$), there are two cases.

(a) $N > 2m$. In this case, since $k_1 = m$, $X_{12} = B_{12} = B_{22} = 0$, and $|B|^2 = |B_{11}|^2 + |B_{21}|^2$. Thus (2.16) becomes

$$\text{(2.20)} \qquad \begin{aligned} \Delta \log|\det A_{11}| &= k_1\kappa + 2(|X_{43}|^2 - 2|A_{11}|^2 + 2|B_{11}|^2 + |B_{21}|^2) \\ &\geq k_1\kappa + 2(-2|A|^2 + |B|^2). \end{aligned}$$

Combining it with (2.17), we get

$$\text{(2.21)} \qquad \begin{aligned} \Delta \log|\det A_{11}|^2 |\det \tilde{B}_{11}|^3 &\geq (2k_1 + 3k_2)\kappa - 8(|A|^2 + |B|^2) \\ &= (2k_1 + 3k_2)\kappa - 8. \end{aligned}$$

If $\min k \geq 8/(2k_1 + 3k_2)$, then it would follow by (2.17) and (2.21) that $|\tilde{A}_{11}| = |\tilde{A}_{12}| = |\tilde{A}_{21}| = |\tilde{X}_{12}| = 0$, which means that $\operatorname{rank}(\tilde{A}) = \operatorname{rank}(\partial f) < m$, contradicting the assumption on $\operatorname{rank} \partial f$.

(b) $N = 2m$. In this case, $X_{12} = X_{34} = B_{12} = B_{21} = B_{22} = 0$ and $|B|^2 = |B_{11}|^2$. So (2.16) is now

$$\text{(2.22)} \qquad \Delta \log|\det A_{11}| = k_1\kappa + 4(|B|^2 - |A_{11}|^2),$$

from which, together with (2.17), it follows that

$$\Delta \log(|\det A_{11}||\det \tilde{B}_{11}|^2) \geq (k_1 + 2k_2)\kappa - 4. \tag{2.23}$$

If $\min \kappa \geq 4/(k_1 + k_2)$, then again $\operatorname{rank}(\partial f) < m$, which is a contradiction.

Finally, suppose $k_1 = k_2 = m$ and $\kappa = 1/m$. (2.16) and (2.17) become

$$\begin{aligned} \Delta \log |\det A_{11}| &= k_1\kappa + 2(|X_{43}|^2 - 2|A_{11}|^2 + |B|^2 + |B_{11}|^2) \\ &\geq m\kappa - 4|A|^2 + 2|B|^2 \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} \Delta \log |\det \tilde{B}_{11}| &= k_2\kappa + 2(|\tilde{X}_{43}|^2 - 2|\tilde{B}_{11}|^2 + |\tilde{A}|^2 + |\tilde{A}_{11}|^2). \\ &\geq m\kappa - 4|B|^2 + 2|A|^2. \end{aligned} \tag{2.25}$$

So

$$\Delta \log(|\det A_{11}||\det \tilde{B}_{11}|) \geq 2m\kappa - 2 \geq 0.$$

Thus $\kappa = 1/m$ is constant, and $X_{43} = B_{11} = \tilde{X}_{43} = \tilde{A}_{11} = 0$. Now it is easy to see that $\partial f \perp \overline{\partial} f$ and

$$f_{-1} \equiv \overline{\partial} f \xrightarrow{\partial} f \equiv f_0 \xrightarrow{\partial} f_1 \equiv \partial f,$$

where f_{-1} is holomorphic and f_1 is antiholomorphic. Q.E.D.

An interesting situation is when the harmonic map f generates an orthogonal harmonic sequence,

$$f \equiv f_0 \xrightarrow{\partial} f_1 \xrightarrow{\partial} f_2 \rightarrow \cdots \xrightarrow{\partial} f_n,$$

such that: when $0 \leq j \leq k_0$, then the rank $(f_j) = m_0$; and for $1 \leq t \leq s$, when

$$\sum_{i=0}^{t-1} k_i + 1 \leq j \leq \sum_{i=0}^{t} k_i,$$

then the $\operatorname{rank}(f_j) = m_t$; and $\sum_{i=0}^{s} k_i = n$.

For such a map, we can prove the following theorem.

Theorem 2.3. *Let f be a harmonic isometric immersion from S^2 into $G(m, N)$. Suppose that f generates an orthogonal harmonic sequence as above, and the Gaussian curvature κ is bounded below by the constant*

$$\kappa(f) = \frac{4\prod_{j=0}^{s}(k_j + 1)}{\sum_{r=0}^{s}\prod_{j=r}^{s} k_r(k_j + 1)m_r}.$$

Then $k = \kappa(f)$ and f is a holomorphic map which essentially generates a Frenet pseudoholomorphic sequence.

Proof. Suitably choose a smooth unitary frame e so that the pull back of the Maurer-Cartan form via e is

$$\phi = \begin{bmatrix} \phi_0 & -{}^{\tau}\overline{A}_0\overline{\theta} & & 0 & & -{}^{\tau}\overline{B}_1\theta & -{}^{\tau}\overline{B}_2\theta \\ A_0\theta & \phi_1 & -{}^{\tau}\overline{A}_1\overline{\theta} & & & & \\ & 0 & A_{j-1}\theta \quad \phi_j & -{}^{\tau}\overline{A}_j\overline{\theta} & & 0 & \\ B_1\overline{\theta} & & & & A_{n-1}\theta & \phi_n & -{}^{\tau}\overline{A}_n\overline{\theta} \\ B_2\overline{\theta} & & 0 & & & A_n\theta & \phi_{n+1} \end{bmatrix}$$

where $B_1 \in C^\infty(M(m_s \times m_0))$, $B_2 \in C^\infty(M((N - \sum_{i=0}^s k_i m_i - m_0) \times m_0))$; for each $r = 0, \dots, s$, when $k_{-1} + \dots + k_{r-1} < j < k_{-1} + \dots + k_r$, $A_j \in C^\infty(M(m_r \times m_r))$: for $r = 0, \dots, s-1$, $A_{k_0+\dots+k_r} \in C^\infty(M(m_r \times m_{r+1}))$; $A_n \in C^\infty(M((n - \sum_{i=0}^s k_i m_i - m_0) \times m_s))$; for $r = 0, \dots, s$, when $k_{-1} + \dots + k_{r-1} < j < k_{-1} + \dots + k_r$, $\phi_j \in C^\infty(u(m_r) \otimes T^*S^2)$ and $\phi_{s+1} \in C^\infty(u(N - \sum_{i=0}^s k_i m_i - m_0) \otimes T^*S^2)$. Here we set $k_{-1} = 0$ for notational convenience. Also, we can require

$$A_{k_0+\dots+k_r} = [0, \tilde{A}_{k_0+\dots+k_r}] \tag{2.26}$$

where $\tilde{A}_{k_0+\dots+k_r} \in C^\infty(M(m_{r+1} \times m_{r+1}))$. Now (1.11′) becomes

$$|A_0|^2 + |B_1|^2 + |B_2|^2 = 1, \tag{2.27}$$

and the harmonicity condition (1.12′) and the Maurer-Cartan structure equations imply

$$dA_j = -iwA_j + A_j\phi_j - \phi_{j+1}A_j \mod \theta, \tag{2.28}$$

for $j = 0, \dots, n-1$. Thus, for $k_{-1} + \dots + k_{r-1} < j < k_{-1} + \dots + k_r$, where $r = 0, \dots, s$,

$$\Delta \log|\det A_j|\theta \wedge \overline{\theta} = 2d(-im_r w + \operatorname{tr}\phi_j - \operatorname{tr}\phi_{j+1}), \tag{2.29}$$

and so

$$\Delta \log|\det A_j| = m_r\kappa + 2(|A_{j-1}|^2 + |A_{j+1}|^2 - 2|A_j|^2), \tag{2.30}$$

if $j \neq n-1, 0$,

$$\Delta \log|\det A_{n-1}| = m_s\kappa + 2(|A_{n-2}|^2 + |A_n|^2 - 2|A_{n-1}|^2 + |B_1|^2)$$

and

$$\begin{aligned}\Delta \log|\det A_0| &= m_0\kappa + 2(|A_1|^2 - 2|A_0|^2 + |B_1|^2 + |B_2|^2) \\ &\geq m_0\kappa + 2(|A_1|^2 - 2|A_0|^2),\end{aligned}$$

if $k_0 \neq 0$.

For $j = k_0 + \dots + k_r$, set

$$\phi_j = \begin{bmatrix} \Omega_{11}^j & \Omega_{12}^j \\ \Omega_{21}^j & \Omega_{22}^j \end{bmatrix},$$

where $\Omega_{11}^j \in C^\infty(\mu(m_r - m_{r+1}) \otimes T^*S^2)$, $\Omega_{22}^j \in C^\infty(\mu(m_{r+1}) \otimes T^*S^2)$ and $\Omega_{21}^j = -{}^\tau\overline{\Omega_{12}^j} \in C^\infty(M(m_{r+1} \times (m_r - m_{r+1})) \otimes T^*S^2)$. Set also

$$A_{j-1} = \begin{bmatrix} D_1^{j-1} \\ D_2^{j-1} \end{bmatrix},$$

where $D_1^{j-1} \in C^\infty(M((m_r - m_{r+1}) \times m_r))$ and $D_2^{j-1} \in C^\infty(M(m_{r+1} \times m_r))$. Then

$$d\tilde{A}_j = -iw\tilde{A}_j + \tilde{A}_j\Omega_{22}^j - \phi_{j+1}\tilde{A}_j \mod \theta \tag{2.31}$$

and

$$\tilde{A}_j \Omega^j_{21} = 0 \mod \theta, \tag{2.32}$$

from which it follows that

$$\Omega^j_{21} = X^j_{21}\theta, \tag{2.33}$$

$$\Delta \log|\det \tilde{A}_{k_0+\cdots+k_r}| = m_{r+1}\kappa + 2(|D_2^{k_0+\cdots+k_r-1}|^2 + |X_2^{k_0+\cdots+k_r}|^2 - 2|\tilde{A}_{k_0+\cdots+k_r}|^2 + |A_{k_0+\cdots+k_r+1}|^2) \tag{2.34}$$

for $r = 0, \dots, s-1$ and if $k_0 \neq 0$. Whereas if $k_0 = 0$,

$$\Delta \log|\det \tilde{A}_0| \geq m_1\kappa + 2(|X^0_{21}|^2 - 2|\tilde{A}_0|^2 + |A_1|^2). \tag{2.35}$$

Observe that $|A_{k_0+\cdots+k_r}|^2 = |\tilde{A}_{k_0+\cdots+k_r}|^2$. For $l = 0, 1, \dots, s$, set

$$F_l = \sum_{j=k_{-1}+k_0+\cdots+k_{l-1}}^{k_{-1}+\cdots+k_l-1} (k_{-1}+k_0+\cdots+k_l-j)\Delta\log|\det A_j|, \tag{2.36}$$

where for notational convenience we set $\det A_{k_0+\cdots+k_r} = \det \tilde{A}_{k_0+\cdots+k_r}$, even though the first one does not make sense. We set also $D_2^{-1} = 0$. Then F_l, $l = 0, 1, \dots, s$, are well-defined continuous functions on S^2.

A tedious computation shows that

$$\begin{aligned} F_s + \sum_{r=0}^{s-1}\left[\prod_{j=r+1}^{s}(k_j+1)\right] F_r &\geq \frac{1}{2}\sum_{r=0}^{s}\prod_{j=r}^{s} k_r(k_r+1)m_r\kappa + 2|A_{k_0+\cdots+k_s}|^2 \\ &\quad - 2\left[\prod_{j=1}^{s}(k_j+1)\right](k_0+1)|A_0|^2 \\ &\quad + 2\left[k_s|D_2^{k_0+\cdots+k_{s-1}-1}|^2 + \sum_{r=0}^{s-1}\prod_{j=r+1}^{s}(k_j+1)k_r|D_2^{k_0+\cdots+k_{r-1}-1}|^2\right] \\ &\geq \frac{1}{2}\sum_{r=0}^{s}\left[\prod_{j=r}^{s}k_r(k_j+1)\right] m_r\kappa - 2\prod_{j=0}^{s}(k_j+1). \end{aligned} \tag{2.37}$$

Now, if $\kappa \geq k(f)$, then

$$F_s + \sum_{r=0}^{s-1}\left[\prod_{j=r+1}^{s}(k_j+1)F_r\right] \geq 0,$$

which means that the function h defined by

$$h = \prod_{j=0}^{s} h_j \tag{2.38}$$

is a subharmonic function on S^2 except for finitely many points, where for $0 \le l < s$

$$h_l = \log \left[\prod_{j=k_{-1}+\cdots+k_{l-1}}^{k_{-1}+\cdots+k_l-1} |\det A_j|^{(k_{-1}+\cdots+k_l-j)} \right]^{\prod_{i=l+1}^{s}(k_i+1)}$$

and

$$h_s = \log \left[\prod_{j=k_{-1}+\cdots+k_{s-1}}^{k_{-1}+\cdots+k_s-1} |\det A_j|^{(k_{-1}+\cdots+k_s-j)} \right].$$

Therefore, h must be constant. It follows from (2.27) and (2.37) that $\kappa = k(f)$.

$$|B_1| = |B_2| = 0, \quad |A_n| = |A_{k_0+\cdots+k_s}| = 0 \tag{2.40}$$

and

$$k_s|D_2^{k_0+\cdots+k_{s-1}-1}|^2 + \sum_{r=0}^{s-1}\left[\prod_{j=r+1}^{s}(k_j+1)k_r|D_2^{k_0+\cdots+k_{r-1}-1}|^2\right] = 0, \tag{2.41}$$

which imply that f is holomorphic, f_n is antiholomorphic and

$$k_s|D_2^{k_0+\cdots+k_{r-1}-1}|^2 = 0, \tag{2.42}$$

for $r = 1, \dots, s$. Thus, either $k_r = 0$ or $D_2^{k_0+\cdots+k_{r-1}-1} = 0$. But when $r \ge 2$, $D_2^{k_0+\cdots+k_{r-1}-1} = 0$ would imply that $\operatorname{rank}(\partial f_{k_0+\cdots+k_{r-1}-1}) = m_r < m_{r-1}$, which is a contradiction. Therefore $k_r = 0$ for $r \ge 2$. Also, $k_1 = 0$, unless $k_0 = 0$. For the latter case by (2.34) and (2.37),

$$X_{12} = 0. \tag{2.43}$$

The pull back of the Maurer-Cartan form is now the following,

$$\phi = \begin{bmatrix} \Omega_{11}^0 & 0 & & & & & \\ 0 & \Omega_{22}^0 & -{}^\tau\overline{\tilde{A}_0}\overline{\theta} & & & 0 & \\ & \tilde{A}_0\theta & \phi_1 & -{}^\tau\overline{A_1}\overline{\theta} & & & \\ & & A_1\theta & \phi_2 & & & \\ & 0 & & & A_{n-1}\theta & \phi_n & 0 \\ & & & & & 0 & \phi_{n+1} \end{bmatrix}.$$

We see that after a unitary transformation on C^N, f can be decomposed as $f = [\tilde{f} \oplus V_0]\colon S^2 \to G(m, N)$, where V_0 is an $(m - m_1)$-dimensional constant vector subspace of C^N and $\tilde{f}$ is a holomorphic map from S^2 to $G(m_1, N)$, and $\operatorname{rank}(f_j) = m_1$, for $j = 1, \dots, n$. In this sense, f essentially generates a Frenet pseudoholomorphic sequences. Q.E.D.

Remark. Given a harmonic map $f: S^2 \to G(m, N)$, one can define an associated vector bundle of f over S^2 in the following way:

$$\delta_f = \{(x, \nu) \in S^2 \times C^N | \nu \in f(x)\}.$$

If f generates an orthogonal harmonic sequence then the trivial complex vector bundle

$$\pi: S^2 \times C^N \to S^2$$

splits into orthogonal complex vector subbundles,

$$S^2 \times C^N = \delta_f \oplus \delta_{f_1} \oplus \cdots \oplus \delta_{f_n} \oplus \delta,$$

which are, in fact, holomorphic subbundles. Here δ is the orthogonal complement of $\delta_f \oplus \cdots \oplus \delta_{f_n}$. Theorem 2.3 says that if the curvature is large, then the ranks of δ_{f_j}, $j = 1, \ldots, n$, are equal, f is holomorphic and δ is a trivial subbundle.

We say that a map g is in a Frenet harmonic sequence (i.e. constant ranks and orthogonal)

$$f \equiv f_0 \xrightarrow{\partial} f_1 \xrightarrow{\partial} f_2 \to \cdots \to f_n$$

of positive j, if $g = f_j$ for some $j = 0, \ldots, n$, where f is harmonic and $\operatorname{rank}(f_j) = m$. For such a map, we show the following pinching theorem.

Theorem 2.4. *Let $g = f_j: S^2 \to G(m, N)$ be a harmonic isometric immersion, which is in the above Frenet harmonic sequence, and let κ_j be the Gaussian curvature. Suppose that $\kappa_j \geq 4/m[2j(n-j)+n]$, then $\kappa_j = 4/m[2j(n-j)+n]$ is constant, and the directrix f is holomorphic.*

Proof. We use the invariants developed in the proof of Theorem 2.3. These invariants now satisfy

$$\text{(2.44)} \qquad \begin{aligned} \Delta \log|\det A_0| &= m\kappa_j + 2(|A_1|^2 - 2|A_0|^2 + |B_1|^2 + |B_2|^2), \\ \Delta \log|\det A_\tau| &= m\kappa_j + 2(|A_{\tau-1}|^2 + |A_{\tau+1}|^2 - 2|A_\tau|^2) \end{aligned}$$

for $0 < \tau < n-1$, and

$$\Delta \log|\det A_{n-1}| = m\kappa_j + 2(|A_{n-2}|^2 + |A_n|^2 - 2|A_{n-1}|^2 + |B_1|^2).$$

$(1.11')$ becomes

$$\text{(2.45)} \qquad \begin{aligned} |A_{j-1}|^2 + |A_j|^2 &= 1, \quad \text{if } j \neq 0, n-1, \\ |A_0|^2 + |B_1|^2 + |B_2|^2 &= 1, \quad \text{if } j = 0, \end{aligned}$$

and

$$|A_{n-1}|^2 + |A_n|^2 + |B_1|^2 = 1, \quad \text{if } j = n-1.$$

The case when $j = 0$ has been treated in Theorem 2.3 by setting $m_r = m$ for $0 \leq r \leq s$, $k_0 = n$ and $k_1 = k_2 = \cdots = 0$. When $j > 0$, set

$$h_j = \left[\prod_{i=0}^{j-1} |\det A_i|^{(2j+1)(n-i-1)+2i}\right] \cdot \left[\prod_{i=j}^{n-1} |\det A_i|^{(2j+1)(n-i)}\right]. \tag{2.46}$$

Then

$$\begin{aligned}\Delta \log h_j = {} & \frac{n+1}{2}[(2j(n-j)+n)m\kappa_j - 4(|A_j|^2 + |A_{j-1}|^2)] \\ & + 2(2j+1)(n-1)(|B_1|^2 + |B_2|^2) + 2(2j+1)(|A_n|^2 + |B_1|^2) \\ \geq {} & \frac{n+1}{2}[(2j(n-j)+n)m\kappa_j - 4].\end{aligned} \tag{2.47}$$

The assumption on κ_j implies

$$\Delta \log h_j \geq 0,$$

which means that h_j is constant, $\kappa_j = \frac{4}{m[2j(n-j)+n]}$, and $A_n = B_1 = B_2 = 0$. Therefore the directrix f is holomorphic. Q.E.D.

We say that a map f from M into CP^n is linearly full, if $f(M)$ is not contained in some linear subspace $CP^{n'} \subset CP^n$, where $n' < n$. The following corollary was first proved in [BJRW].

Corollary 2.4. *For a linearly full pseudoholomorphic map f_j of positions j from S^2 into CP^n, if it is an isometric immersion, and if the curvature $\kappa_j \geq 4/(2j(n-j)+n)$, then $\kappa_j = 4/(2j(n-j)+n)$ is constant.*

3. Harmonic two-spheres in $G(2,4)$ and $G(2,6)$

In this section, we look at harmonic maps from S^2 into $G(2,4)$ and $G(2,6)$.

Let f be a harmonic isometric immersion from S^2 into $G(2,4)$. By Corollary 2.2, we know that at least one of the ∂-transform and the $\bar{\partial}$-transform is degenerate. Let us say rank $(\partial f) \leq 1$. Choosing a suitable local unitary frame $e = (e_1, e_2, e_3, e_4)$ along f as before, the pull back of the Maurer-Cartan form is

$$\phi = \begin{bmatrix} \phi_{11} & \phi_{12} & -\bar{a}_{31}\bar{\theta} - \bar{b}_{31}\theta & -\bar{b}_{41}\theta \\ \phi_{21} & \phi_{22} & -\bar{b}_{32}\theta & -\bar{b}_{42}\theta \\ a_{31}\theta + b_{31}\bar{\theta} & b_{32}\bar{\theta} & \phi_{33} & \phi_{34} \\ b_{41}\bar{\theta} & b_{42}\bar{\theta} & \phi_{43} & \phi_{44} \end{bmatrix},$$

where

$$|a_{31}|^2 + |b_{31}|^2 + |b_{32}|^2 + |b_{41}|^2 + |b_{42}|^2 = 1. \tag{3.1}$$

From Corollary 2.1, the vanishing of $\operatorname{tr} C = \operatorname{tr}^t A\bar{B}$ implies

$$a_{31}\bar{b}_{31} = 0. \tag{3.2}$$

Harmonicity conditions (1.12′) and (1.13′) imply

$$da_{31} = a_{31}(-iw + \phi_{11} - \phi_{33}) \mod \theta, \tag{3.3}$$

$$\phi_{12} = a_{12}\theta \tag{3.4}$$

and

$$\phi_{43} = a_{43}\theta. \tag{3.5}$$

By Proposition 1.1, (3.3) means that a_{31} is a function of analytic type. Thus either it is identically zero which means that f is an antiholomorphic map or away from its zeros it satisfies

$$\Delta \log |a_{31}| \theta \wedge \overline{\theta} = 2d(-iw + \phi_{11} - \phi_{33}). \tag{3.6}$$

Suppose that f is not antiholomorphic. Then $b_{31} = 0$. Harmonicity condition (1.13) also implies

$$db_{32} = b_{32}(iw + \phi_{22} - \phi_{33}) \mod \overline{\theta}, \tag{3.7}$$

$$db_{41} = b_{41}(iw + \phi_{11} - \phi_{44}) \mod \overline{\theta} \tag{3.8}$$

and

$$db_{42} = b_{42}(iw + \phi_{22} - \phi_{44}) + b_{41}\phi_{12} - b_{32}\phi_{43} \mod \overline{\theta}, \tag{3.9}$$

from which it follows again that b_{32} and b_{41} are functions of analytic type.

If b_{32} is not identically zero, then away from its zeros we get

$$\Delta \log |b_{32}| \theta \wedge \overline{\theta} = 2d(-iw + \phi_{33} - \phi_{22}). \tag{3.10}$$

Combining (3.6) and (3.10) and using the Maurer-Cartan structure equations, we have

$$\begin{aligned} \Delta \log |a_{31} b_{32}| &= 2\kappa + 4|a_{12}|^2 - 2|a_{31}|^2 + 2|b_{41}|^2 \\ &\quad - 2|b_{32}|^2 - 2|b_{42}|^2 \\ &\geq 2\kappa - 2. \end{aligned} \tag{3.11}$$

Notice that the function $|a_{31} b_{32}|$ is a globally defined continous function. If $\kappa \geq 1$, then we see that $\kappa = 1$ and $a_{12} = b_{41} = 0$. Notice also that the local unitary frame e is uniquely determined up to $U(1) \times U(1) \times U(1) \times U(1)$-transformations, and thus for $i = 1, 2, 3$ and 4, each e_i defines a map $[e_i]: S^2 \to CP^3$. In this case since $a_{12} = b_{41} = b_{31} = 0$, $[e_1]$ is holomorphic, $[e_2]$ is antiholomorphic and $f = [e_1 \oplus e_2]$.

The other case is $b_{32} = 0$. Suppose b_{41} is not identically zero. Then by (3.8) and (3.6), we get

$$\begin{aligned} \Delta \log |b_{41} a_{31}| &= 2\kappa + 4|a_{43}|^2 - 2|b_{42}|^2 - 2|b_{41}|^2 - 2|a_{31}|^2 \\ &\geq 2\kappa - 2. \end{aligned} \tag{3.12}$$

Again, if $\kappa \geq 1$, then $\kappa = 1$ and $a_{43} = 0$. In this case, we see that $[e_3]$ is an antiholomorphic map, $[e_4]$ is a holomorphic map and $f = [e_3 \oplus e_4]^{\perp}$.

Finally, if $b_{32} = b_{41} = 0$, but b_{42} is not identically zero, then by (3.9),

$$\Delta \log |b_{42}| \theta \wedge \overline{\theta} = 2d(-iw + \phi_{44} - \phi_{22}). \tag{3.13}$$

Taking the exterior derivatives on both sides of (3.4) and (3.5), and using the structure equations, we get

$$da_{12} = a_{12}(-iw - \phi_{11} + \phi_{22}) \mod \theta \tag{3.14}$$

and

$$da_{43} = a_{43}(-iw + \phi_{33} - \phi_{44}) \mod \theta . \tag{3.15}$$

Repeating the previous argument, we see that both a_{12} and a_{43} are of analytic type and that $|a_{12}|$ and $|a_{43}|$ are globally defined on S^2.

We claim that if $k \geq 1$, then a_{12} and a_{43} are identically zero. There are three cases to be checked.

Case (i). Both a_{12} and a_{43} are not identically zero. Then from (3.14) and (3.15),

$$\Delta \log |a_{12}| \theta \wedge \overline{\theta} = 2d(-iw - \phi_{11} + \phi_{22}) \tag{3.16}$$

and

$$\Delta \log |a_{43}| \theta \wedge \overline{\theta} = 2d(-iw + \phi_{33} - \phi_{44}). \tag{3.17}$$

These, together with (3.6) and (3.13), would give

$$\Delta \log |a_{12}a_{31}a_{43}b_{42}| = 4k \geq 4 > 0 . \tag{3.18}$$

But then $|a_{12}a_{31}a_{43}b_{42}|$ must be constant and $k = 0$, which is a contradiction.

Case (ii). $a_{12} = 0$, but a_{43} is not zero. Then (3.17) holds and the combination of (3.6), (3.13) and (3.17) gives

$$\Delta \log |a_{31}a_{43}b_{42}| = 3k - 2 > 1 , \tag{3.19}$$

which again would imply $k = \frac{2}{3}$, contradicting the hypothesis on k.

Case (iii). $a_{43} = 0$, but a_{12} is not zero. This case can be treated in exactly the same way as Case (ii), and it cannot happen either.

Now since $a_{12} = a_{43} = 0$, by reading the pull back of the Maurer-Cartan form we see that e_1 defines a holomorphic map $[e_1]: S^2 \to CP^1 \subset CP^3$, and that e_2 defines an antiholomorphic map $[e_2]: S^2 \to CP^1 \subset CP^3$. Up to a unitary transformation the map f is then given by $f = h \circ g$ in the following way.

$$S^2 \xrightarrow[g]{} CP^1 \times CP^1 \xrightarrow[h]{} CP^3, \qquad f = h \circ g \tag{3.20}$$

where $g = [e_1] \times [e_2]\colon S^2 \to CP^1 \times CP^1$ and h is the holomorphic embedding given by

$$h([Z_0, Z_1], [Z_2, Z_3]) = \left[\frac{Z_0}{\sqrt{|Z_0|^2 + |Z_1|^2}}, \frac{Z_1}{\sqrt{|Z_0|^2 + |Z_1|^2}}, \frac{Z_2}{\sqrt{|Z_2|^2 + |Z_3|^2}}, \frac{Z_3}{\sqrt{|Z_2|^2 + |Z_3|^2}}\right].$$

In summary, we have shown

Theorem 3.1. *Let f be a harmonic isometric immersion from S^2 into $G(2,4)$ with $\operatorname{rank}(\partial f) = 1$ and $\operatorname{rank}(\bar{\partial} f) \geq 1$. If the curvature $\kappa \geq 1$, then either $f = [e_1 \oplus e_2]$ or $f^\perp = [e_1 \oplus e_2]$ with $[e_1]\colon S^2 \to CP^3$ holomorphic, $[e_2]\colon S^2 \to CP^3$ antiholomorphic and $e_1 \perp e_2$. Moreover, if the curvature κ is not constant 1, then up a unitary transformation f is given by (3.20).*

Remark. Suppose now that $\kappa \geq 2$. Then it happens only when $a_{12} = a_{43} = b_{31} = b_{32} = b_{41} = 0$. From (3.6) and (3.13), we set that

$$\Delta \log |a_{31} b_{42}| = 2k - 4 \geq 0. \tag{3.21}$$

Thus $k = 2$ and $|a_{31} b_{42}|$ is constant. But by (3.1),

$$|a_{31}|^2 + |b_{42}|^2 = 1, \tag{3.22}$$

and thus both $|a_{31}|$ and $|b_{31}|$ are constant. We can specify the unitary frame e so that $a_{31} \geq 0$ and $b_{31} \geq 0$ are constant. Using (3.6), it is easy to see that $a_{31} = b_{42} = \frac{1}{2}$. Thus all invariants are constant. In this case up to a $U(4)$-congruence, f is the map $V_{1,\bar{1}}$ defined by

$$V_{1,\bar{1}} \begin{bmatrix} Z_0 \\ Z_1 \end{bmatrix} = \begin{bmatrix} Z_0 & 0 \\ Z_1 & 0 \\ 0 & \overline{Z}_0 \\ 0 & \overline{Z}_1 \end{bmatrix}. \tag{3.16}$$

Notice that the map $V_{1,0}\colon S^2 \to G(2,4)$, defined by

$$V_{1,0} \begin{bmatrix} Z_0 \\ Z_1 \end{bmatrix} = \begin{bmatrix} Z_0 & 0 \\ Z_1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

is the only holomorphic map from S^2 to $G(2,4)$ with constant curvature 4 up to $U(4)$-congruences. In [CZ], we also classified all holomorphic maps from S^2 to $G(2,4)$ of curvature 2. In fact, the argument in [CZ] actually shows that for a holomorphic isometric immersion f from S^2 to $G(2,4)$, if the curvature $\kappa \geq 2$, then either $\kappa = 2$ or $\kappa = 4$. Theorem 3.1, together with the argument in [CZ], shows the following.

Theorem 3.2. *Up to $U(4)$-congruence, all isometric harmonic maps from S^2 into $G(2,4)$ of curvature $\kappa \geq 2$ are given by: $V_{1,\bar{1}}$, $V_{1,0}$, and the holomorphic curves (and their complex conjugates) of curvature 2 given in* [CZ].

Now suppose that f is a harmonic isometric immersion from S^2 into $G(2,6)$. Then the only nonholomorphic or nonantiholomorphic cases are those when $\operatorname{rank}(\partial f) \geq 1$ and $\operatorname{rank}(\bar{\partial} f) \geq 1$. These cases can be treated by using Theorem 2.2. In fact, we have the following theorem.

Theorem 3.3. *For a harmonic isometric immersion f from S^2 into $G(2,6)$, IF* $\operatorname{rank}(\partial f) = \operatorname{rank}(\bar{\partial} f) = 2$, *and if the curvature $\kappa \geq \frac{1}{2}$, then $\kappa = \frac{1}{2}$ is constant and f is a pseudoholomorphic curve of position one; if* $\operatorname{rank}(\bar{\partial} f) = 2$ *and* $\operatorname{rank}(\bar{\partial} f) = 1$, *or if* $\operatorname{rank}(\partial f) = 1$ *and* $\operatorname{rank}(\bar{\partial} f) = 2$, *then* $\min \kappa < \frac{8}{7}$; *finally, if* $\operatorname{rank}(\partial f) = \operatorname{rank}(\bar{\partial} f) = 1$, *and if $\kappa \geq 2$, then $\kappa = 2$ is constant.*

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