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REALIZATION OF THE LEVEL TWO STANDARD $\mathfrak{sl}(2k+1,\mathbf{C})^{\sim}$-MODULES

KAILASH C. MISRA

ABSTRACT. In this paper we study the level two standard modules for the affine special linear Lie algebras. In particular, we give the vertex operator realizations of all level two standard modules for the affine special linear Lie algebras of odd rank.

INTRODUCTION

The existence of explicit constructions of some nontrivial "standard" modules is one of the most interesting features of the representation theory of affine Lie algebras. In recent years it has been established that the existence of such constructions gives rise to a number of important connections of affine Lie algebras with different branches of mathematics and physics.

The study of level two standard $\mathfrak{sl}(n,\mathbf{C})^{\sim}$-module was started in [8, 9]. In [8, 9], among other results, the level two standard modules for the affine Lie algebras $\mathfrak{sl}(3,\mathbf{C})^{\sim}$ and $\mathfrak{sl}(5,\mathbf{C})^{\sim}$ were explicitly constructed. However, the arguments involved some explicit computations which could not be generalized to arbitrary rank case.

The main result of this paper is Theorem 3.15, which gives an explicit basis of all level two standard $\mathfrak{sl}(2k+1,\mathbf{C})^{\sim}$-modules. In order to accomplish this we have strongly used the fact that the "principal" characters of these modules differ from the product sides of the generalized Rogers-Ramanujan identities by a simple factor. Furthermore, throughout the paper, our arguments are very much influenced by two recent papers, [6] and [7]. We are thankful to Robert L. Wilson for some valuable suggestions.

1. PRELIMINARIES

In this section we will recall some notation and facts from [9]. There will be minor changes in the notation which will be self-explanatory. For more details see [9].

Consider the (complex) simple Lie algebra $\mathfrak{g}=\mathfrak{sl}(n+1,\mathbf{C})$, with $n\geq 2$. The associated affine Lie algebra $\hat{\mathfrak{g}}$ has a basis (see [2, 9]),

(1.1) $S=\{c,B(j),X(m,i)\mid i,j\in\mathbf{Z},j\not\equiv 0 \bmod(n+1),m=1,2,\dots,n\}$,

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where c is a central element (suitably normalized), $B(j) = E^j \otimes t^j$, and $X(m,i) = D^m E^i \otimes t^i$. Here t is an indeterminate, $D =$

$$\mathrm{diag}(\omega, \omega^2, \dots, \omega^n, 1), \quad \text{and} \quad E = (\delta_{(i+1)',j})_{i,j=0}^n,$$

where ω is a primitive $(n+1)$th root of unit and prime denotes reduction modulo $(n+1)$. Denote

$$\mathfrak{s}^{\pm} = \prod_{\substack{j>0 \\ j \not\equiv 0 \mod (n+1)}} \mathbf{C}B(\pm j)$$

and $\mathfrak{s} = \mathfrak{s}^- \oplus \mathbf{C}c \oplus \mathfrak{s}^+$. Then $\mathfrak{s}$ is a Heisenberg subalgebra of $\hat{\mathfrak{g}}$, called a *principal Heisenberg subalgebra.* For an indeterminate ζ, define the formal power series:

$$(1.2) \qquad \begin{cases} B(\zeta) = \sum_{j \not\equiv 0 \mod (n+1)} B(j)\zeta^j, \\ X(m,\zeta) = \sum_{i \in \mathbf{Z}} X(m,i)\zeta^i, \qquad m = 1, 2, \dots, n, \\ \delta(\zeta) = \sum_{i \in \mathbf{Z}} \zeta^i, \quad \text{and} \\ D\delta(\zeta) = \sum_{i \in \mathbf{Z}} i\zeta^i. \end{cases}$$

Proposition 1.1 [9]. *Let* ζ_1 *and* ζ_2 *be two commuting indeterminates. Then for* $j \not\equiv 0 \mod (n+1)$ *and* $m, l = 1, 2, \dots, n$, *we have*

$$(1.3) \qquad [B(j), X(m,\zeta_1)] = (\omega^{mj} - 1)\zeta^{-j} X(m,\zeta_1),$$

and

$$(1.4)$$

$$[X(m,\zeta_1), X(l,\zeta_2)] = \begin{cases} \delta(\omega^l \zeta_1/\zeta_2) X((m+l)', \zeta_2) - \delta(\omega^{-m}\zeta_1/\zeta_2) X((m+l)', \omega^m \zeta_2), \\ \qquad\qquad\qquad\qquad\qquad \text{for } m+l \neq n+1, \\ cD\delta(\omega^l \zeta_1/\zeta_2) + \delta(\omega^l \zeta_1/\zeta_2)[B(\zeta_2) - B(\omega^m \zeta_2)], \quad \text{for } m+l = n+1. \end{cases}$$

Let $\mathfrak{a} = C(E)$ denote the centralizer of E in $\mathfrak{g}$. Then

$$\mathfrak{a} = \mathrm{span}_{\mathbf{C}}\{E, E^2, \dots, E^n\}$$

and $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{h}$ denote the Cartan subalgebra of $\mathfrak{g}$ in apposition to $\mathfrak{a}$ (cf. [5]). Note that $\mathfrak{h}$ is the usual Cartan subalgebra spanned by the $(n+1)\times(n+1)$ diagonal matrices of trace zero. Let E_i, F_i, H_i, $i = 1, 2, \dots, n$, denote the usual canonical generators of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$. Let E_0 be a lowest root vector and F_0 a highest root vector suitably normalized such that $[H_0, E_0] = 2E_0$ and $[H_0, F_0] = -2F_0$, where $H_0 = [E_0, F_0]$. In $\hat{\mathfrak{g}}$ for $i = 0, 1, 2, \dots, n$, set

$$(1.5) \qquad e_i = E_i \otimes t, \quad f_i = F_i \otimes t^{-1}, \quad \text{and} \quad h_i = H_i \otimes 1 + (n+1)^{-1}c.$$

Then $\{e_i, f_i, h_i \mid 0 \le i \le n\}$ form a system of canonical generators for the affine Lie algebra $\hat{\mathfrak{g}} = \mathfrak{sl}(n+1,\mathbf{C})^{\wedge}$. Observe that $c = \sum_{i=0}^{n} h_i$. Let d be the derivation of $\hat{\mathfrak{g}}$ given by

$$d(e_i) = e_i, \quad d(f_i) = -f_i, \quad \text{and} \quad d(h_i) = 0, \tag{1.6}$$

for $i = 0, 1, \ldots, n$. Observe that

$$d(B(j)) = jB(j), \quad d(X(m,i)) = iX(m,i), \quad \text{and} \quad d(c) = 0. \tag{1.7}$$

Form the semidirect product Lie algebra

$$\tilde{\mathfrak{g}} = \mathfrak{sl}(n+1,\mathbf{C})^{\sim} = \hat{\mathfrak{g}} \oplus \mathbf{C}d. \tag{1.8}$$

Denote

$$\tilde{\mathfrak{g}}_i = \{x \in \tilde{\mathfrak{g}} \mid [d,x] = ix\}, \qquad i \in \mathbf{Z}.$$

Then

$$\tilde{\mathfrak{g}} = \coprod_{i\in\mathbf{Z}} \tilde{\mathfrak{g}}_i,$$

which gives a $\mathbf{Z}$-gradation (called the principal gradation) for $\tilde{\mathfrak{g}}$, in the sense that $[\tilde{\mathfrak{g}}_i, \tilde{\mathfrak{g}}_j] \subseteq \tilde{\mathfrak{g}}_{i+j}$. This induces naturally a $\mathbf{Z}$-gradation in the universal enveloping algebra $\mathscr{U}(\tilde{\mathfrak{g}})$,

$$\mathscr{U}(\tilde{\mathfrak{g}}) = \coprod_{i\in\mathbf{Z}} \mathscr{U}(\tilde{\mathfrak{g}})_i. \tag{1.9}$$

Set

$$\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbf{C}c \oplus \mathbf{C}d \subseteq \tilde{\mathfrak{g}}.$$

Then $\tilde{\mathfrak{h}}$ is an abelian subalgebra of $\tilde{\mathfrak{g}}$ which is spanned by $\{h_0, h_1, \ldots, h_n, d\}$.

Let $\lambda \in \tilde{\mathfrak{h}}^*$. A $\tilde{\mathfrak{g}}$-module V generated by vector $v_\lambda \ne 0$ such that $e_i \cdot v_\lambda = 0$ for $i = 0, 1, \ldots, n$ and $h \cdot v_\lambda = \lambda(h)v_\lambda$ for all $h \in \tilde{\mathfrak{h}}$ is called a *highest weight module* with *highest weight* λ. Such a vector v_λ is called a *highest weight vector* and is unique up to a scalar multiple. The scalar $\lambda(c)$ is said to be the *level* of V. A highest weight $\tilde{\mathfrak{g}}$-module with highest weight λ and corresponding highest weight vector v_λ is called a *standard* (or *integrable highest weight*) $\tilde{\mathfrak{g}}$-module if there is an integer $r \ge 1$ such that $f_i^r \cdot v_\lambda = 0$, $0 \le i \le n$, which in turn implies that λ is *dominant integral*, that is, $\lambda(h_i) \in \mathbf{N}$, for $0 \le i \le n$ (see [1, 3]). For each dominant integral $\lambda \in \tilde{\mathfrak{h}}^*$, there is a unique (up to isomorphism) standard $\tilde{\mathfrak{g}}$-module $L(\lambda)$ and it is irreducible (see [1, 3]). For convenience we will restrict our attention to $L(\lambda)$ when $\lambda(d) = 0$, so that $\lambda \in \operatorname{span}\{h_i^* \mid 0 \le i \le n\} \subseteq \tilde{\mathfrak{h}}^*$ where $h_i^*(h_j) = \delta_{ij}$ and $h_i^*(d) = 0$, $0 \le i, j \le n$.

Denote by $L_i \subseteq L(\lambda)$ the eigenspace of d with eigenvalue $i \in \mathbf{Z}$. Then

$$L(\lambda) = \coprod_{i\le 0} L_i \tag{1.10}$$

with $L_0 = \mathbf{C}v_\lambda$, $L_i = \mathscr{U}(\tilde{\mathfrak{g}})_i v_\lambda$, and $\dim L_i < \infty$. Note that for $i \le 0$, $v \in L_i$, we have $d \cdot v = iv$. Hence we call (see [5]) the set of elements in L_i the set of *homogeneous elements of degree* i. In particular, we say that $T \in \operatorname{End}(L(\lambda))$

is *homogeneous of degree* $z \in \mathbf{C}$ if $[d, T] = zT$. Observe that $T \in \mathrm{End}(L(\lambda))$ has degree $z \in \mathbf{C}$ if and only if $TL_i \subset L_{i+z}$ for all $i \in \mathbf{Z}$. Define the *principal character* $\chi(L(\lambda))$ of $L(\lambda)$ by

$$\chi(L(\lambda)) = \sum_{i \geq 0} (\dim L_{-i}) q^i \tag{1.11}$$

where q is an indeterminate. Then $\chi(L(\lambda))$ has a known product expansion (for example, see [9, Formula 1.1]).

In this paper we will focus our attention on the level two standard $\tilde{\mathfrak{g}}$-modules $L(\lambda)$ with $\lambda(d) = 0$. To be more precise, we will study the standard $\tilde{\mathfrak{g}}$-modules $L(\lambda)$ where $\lambda = h_0^* + h_l^*$, $0 \leq l \leq t-1$, where $t = [(n+1)/2] + 1$, $[\cdot]$ denoting the greatest integer. Note that up to an isomorphism of the Dynkin diagram of $\tilde{\mathfrak{g}}$ it is enough to consider only the cases $\lambda = h_0^* + h_l^*$ with $0 \leq l \leq t-1$. Other level two standard $\tilde{\mathfrak{g}}$-modules can be studied similarly. By direct computation it can easily be checked that for $\lambda = h_0^* + h_l^*$, $0 \leq l \leq t-1$,

$$\chi(L(\lambda)) = F \prod_{\substack{k>0 \\ k \not\equiv 0, \pm(l+1) \bmod (n+3)}} (1 - q^k)^{-1} \tag{1.12}$$

if $l < (n+1)/2$, and

$$\chi(L(\lambda)) = F \prod_{\substack{k>0 \\ k \not\equiv 0, (l+1) \bmod (n+3)}} (1 - q^k)^{-1} \prod_{\substack{k>0 \\ k \equiv (l+1) \bmod (n+3)}} (1 - q^k), \tag{1.13}$$

if $l = (n+1)/2$ (i.e., $(n+1)$ is even), where

$$F = \chi(\mathscr{U}(\mathfrak{s}^-)) = \prod_{\substack{k>0 \\ k \not\equiv 0 \bmod (n+1)}} (1 - q^k)^{-1}. \tag{1.14}$$

It is important to observe that $\chi(L(\lambda))$ differs from the product side of the generalized Rogers-Ramanujan identities due to Gordon, Andrews, and Bressoud (cf. [4]) by a simple factor.

For a formal indeterminant ζ, denote by $\mathrm{End}(L(\lambda))\{\zeta\}$ the $\mathbf{C}$-vector space of formal Laurent series in ζ with coefficients in $\mathrm{End}(L(\lambda))$. Define

$$E^{\pm}(k, \zeta) = \exp\left(\pm \sum_j (\omega^{\mp kj} - 1) B(\pm j) \zeta^{\pm j} / j \right), \tag{1.15}$$

for $k \in \mathbf{Z}$, where j ranges through the positive integers $\not\equiv 0 \bmod (n+1)$ and exp means the formal exponential series. Also recall the formal Laurent series $B(\zeta)$, $X(m, \zeta)$, $m = 1, 2, \ldots, n$ (see (1.2)). We can and do view $B(\zeta)$, $X(m, \zeta)$, and $E^{\pm}(k, \zeta)$ as elements of $\mathrm{End}(L(\lambda))\{\zeta\}$. Note that if we designate any of the elements $X(m, \zeta)$ or $E^{\pm}(k, \zeta)$ by $Y = \sum_{i \in \mathbf{Z}} Y^i \zeta^i$, then

$$[d, Y] = \sum_{i \in \mathbf{Z}} [d, Y_i] \zeta^i = \sum_{i \in \mathbf{Z}} i Y_i \zeta^i.$$

Therefore, Y_i is a homogeneous operator of degree i for each $i \in \mathbf{Z}$. We will need the following facts in the sequel.

Lemma 1.2 [9]. *In* $\mathrm{End}(L(\lambda))\{\zeta\}$ *for* $m, k \in \mathbf{Z}$,

$$E^{\pm}(m, \omega^{-k}\zeta)E^{\pm}(k, \zeta) = E^{\pm}(m+k, \zeta).$$

In particular,

$$E^{\pm}(-m, \omega^{-m}\zeta)E^{\pm}(m, \zeta) = 1.$$

Proposition 1.3 [9]. *Let* ζ_1, ζ_2 *be two commuting indeterminates. Then on a standard* $\tilde{\mathfrak{g}}$*-modules of level one we have for* $1 \le m \le n$, *and* $k, l \in \mathbf{Z}$,

(a) $X(m, \zeta_1)E^-(k, \zeta_2) = E^-(k, \zeta_2)X(m, \zeta_1)P(m, k, \zeta_1, \zeta_2)^{-1}$,

(b) $E^+(k, \zeta_1)X(m, \zeta_2) = X(m, \zeta_2)E^+(k, \zeta_1)P(k, m, \zeta_1, \zeta_2)^{-1}$, *and*

(c) $E^+(k, \zeta_1)E^-(l, \zeta_2) = E^-(l, \zeta_2)E^+(k, \zeta_1)P(k, l, \zeta_1, \zeta_2)$, *where*

$$P(k, l, \zeta_1, \zeta_2) = \frac{(1-\omega^{l-k}\zeta_1/\zeta_2)(1-\zeta_1/\zeta_2)}{(1-\omega^{-k}\zeta_1/\zeta_2)(1-\omega^{l}\zeta_1/\zeta_2)}.$$

Proposition 1.4 [2]. *On the level one standard* $\tilde{\mathfrak{g}}$*-module* $L(h_l^*)$, $0 \le l \le n$, *we have*

$$X(m, \zeta) = c_l^{(m)}(E^-(m, \zeta))^{-1}(E^+(m, \zeta))^{-1},$$

where

$$c_l^{(m)} = \omega^{(l+1)m}/(\omega^m - 1).$$

From here on, for convenience, we will denote the operators $E^{\pm}(-1, \zeta)$, $X(1, \zeta)$, $X(1, i)$ and the constants $c_l^{(1)}$, $0 \le l \le n$, by $E^{\pm}(\zeta)$, $X(\zeta)$, $X(i)$, and c_l, respectively.

2. Generating function identities

Let $L(\lambda)$ be any standard $\tilde{\mathfrak{g}}$-module with highest weight λ and highest weight vector v_λ. Let $A(\zeta)$ and $C(\xi)$ be the following Laurent series in commuting indeterminates ζ and ξ with coefficients in $\mathrm{End}(L(\lambda))$:

$$\text{(2.1)} \qquad \begin{cases} A(\zeta) = \sum_{i \in \mathbf{Z}} A(i)\zeta^i, \\ C(\xi) = \sum_{i \in \mathbf{Z}} C(i)\xi^i, \end{cases}$$

with $[d, A(i)] = iA(i)$ and $[d, C(i)] = iC(i)$ for $i \in \mathbf{Z}$ (i.e., $A(i)$ and $C(i)$ are homogeneous elements of $\mathrm{End}(L(\lambda))$ of degree i). Then

$$\text{(2.2)} \qquad A(\zeta)C(\xi) = \sum_{i, j \in \mathbf{Z}} A(i)C(j)\zeta^i\xi^j$$

is a well-defined Laurent series in two indeterminates ζ and ξ, with coefficients in $\mathrm{End}(L(\lambda))$. However, if we set $\zeta = \xi$ in (2.2), the product

$$\text{(2.3)} \qquad A(\zeta)C(\zeta) = \sum_{k \in \mathbf{Z}} \Big(\sum_{i+j=k} A(i)C(j) \Big) \zeta^k$$

is not defined in general. We can write (2.3) if one of the following three conditions holds (see [7]).

$$C(j) = 0, \quad \text{for } j < 0, \tag{2.4}$$

$$A(i) = 0, \quad \text{for } i > 0, \tag{2.5}$$

$$[A(i), C(j)] = 0, \quad \text{for } i, j \in \mathbf{Z}. \tag{2.6}$$

Whenever (2.3) is well defined, we will sometimes also write (see [7])

$$A(\zeta)C(\zeta) = \lim_{\xi \to \zeta} A(\zeta)C(\xi). \tag{2.7}$$

Now for products of type (2.2) involving more than two commuting indeterminates extend this definition of limit inductively.

Theorem 2.1. *For commuting indeterminates* $\zeta_1, \zeta_2, \ldots, \zeta_p$, *set*

$$\begin{aligned} F_p(\zeta_1, \zeta_2, \ldots, \zeta_p) \\ = \prod_{1 \le i < j \le p} (1 - \omega\zeta_i/\zeta_j)(1 - \omega^{-1}\zeta_i/\zeta_j) \cdot X(\zeta_1)X(\zeta_2) \cdots X(\zeta_p). \end{aligned} \tag{2.8}$$

Then for every permutation $\sigma \in S_p$, *we have*

$$F_p(\zeta_{\sigma(1)}, \ldots, \zeta_{\sigma(p)}) = F_p(\zeta_1, \ldots, \zeta_p). \tag{2.9}$$

Proof. First observe that by [9, Lemma 2.7] we have

$$(1 - \omega\zeta_1/\zeta_2)\delta(\omega\zeta_1/\zeta_2) = 0$$

and

$$(1 - \omega^{-1}\zeta_1/\zeta_2)\delta(\omega^{-1}\zeta_1/\zeta_2) = 0.$$

Hence for $r = 1, 2, \ldots, p-1$ the commutation relation (1.4) implies

$$\prod_{1 \le i < j \le p} (1 - \omega\zeta_i/\zeta_j)(1 - \omega^{-1}\zeta_i/\zeta_j) \cdot X(\zeta_1) \cdots [X(\zeta_r), X(\zeta_{r+1})] \cdots X(\zeta_p) = 0,$$

and the result follows. □

Corollary 2.2. *For* $p \ge 2$ *the limit*

$$X^{[p]}(\zeta) = \lim_{\zeta_i \to \omega^{2b_i}\zeta} F_p(\zeta_1, \ldots, \zeta_p) \tag{2.10}$$

exists, where $b_1 = 0$, $b_2 = 1, \ldots, b_{t-1} = t-2$, *and for* $k > t-1$, $b_k = b_{k'}$ *where* $k = k' \bmod(t-1)$, $k' < t-1$.

Proof. This corollary follows from Theorem 2.1 and definition of limit by an argument similar to Corollary 5.8 in [7]. □

Proposition 2.3. *On a level one standard* $\tilde{\mathfrak{g}}$*-module* $L(h_l^*)$, $l = 0, 1, \ldots, n$, *we have*

$$X^{[t]}(\zeta) = 0, \tag{2.11}$$

and for $2 \le p < t$,

$$X^{[p]}(\zeta) = c_l \prod_{1 \le k \le p-1} (1 - \omega^{-2k})^2 E^-(\omega^n \zeta) \cdot X^{[p-1]}(\omega^2 \zeta) E^+(\omega^n \zeta). \tag{2.12}$$

Proof. Using Proposition 1.4, Lemma 1.2, and Proposition 1.3, it can easily be seen that on $L(h_l^*)$

$$\begin{aligned} F_p(\zeta_1, \dots, \zeta_p) &= (c_l)^p \prod_{1 \le i < j \le p} (1 - \zeta_i/\zeta_j)^2 \prod_{i=1}^{p} E^-(\omega^n \zeta_i) \prod_{i=1}^{p} E^+(\omega^n \zeta_i) \\ &= c_l \prod_{2 \le k \le p} (1 - \zeta_1/\zeta_k)^2 E^-(\omega^n \zeta_1) F_{p-1}(\zeta_2, \dots, \zeta_p) E^+(\omega^n \zeta_1). \end{aligned}$$

Hence the proposition follows. □

As indicated in §1, we will be interested in the level two standard $\tilde{\mathfrak{g}}$-modules $L(\lambda)$ with highest weights $\lambda = h_0^* + h_l^*$, $0 \le l \le t-1$. Suppose that the fundamental modules $L(h_l^*)$, $0 \le l \le n$, are generated by the highest weight vectors $v_l \ne 0$, $0 \le l \le n$, respectively. Then observe that $L(h_0^*) \otimes L(h_l^*)$ is a $\tilde{\mathfrak{g}}$-module of level two with highest weight $\lambda = h_0^* + h_l^*$ and highest weight vector $(v_0 \otimes v_l)$. Set $v_\lambda = v_0 \otimes v_l$. Then by uniqueness of standard modules the standard $\tilde{\mathfrak{g}}$-module $L(\lambda)$ with highest weight $\lambda = h_0^* + h_l^*$ will be isomorphic to the submodule of $L(h_0^*) \otimes L(h_l^*)$ generated by v_λ. Hence from here on we can and do view the level two standard $\tilde{\mathfrak{g}}$-modules $L(\lambda)$ with highest weight $\lambda = h_0^* + h_l^*$, $0 \le l \le t-1$, as submodules of $L(h_0^*) \otimes L(h_l^*)$ respectively.

Theorem 2.4. *On the standard* $\tilde{\mathfrak{g}}$*-module* $L(\lambda)$, *with highest weight* $\lambda = h_0^* + h_l^*$, $0 \le l \le t-1$, *we have*

$$X^{[t]}(\zeta) = a_l E^-(\omega^n \zeta) X^{[t-2]}(\zeta) E^+(\omega^n \zeta) \tag{2.13}$$

where $a_l = (-2\omega^{l+1}) \prod_{1 \le k \le t-2} (1 - \omega^{-2k})^2 (1 - \omega^{2k+1})(1 - \omega^{2k-1})$.

Proof. We will prove that the identity (2.13) holds on the $\tilde{\mathfrak{g}}$-module $L(h_0^*) \otimes L(h_l^*)$, hence it holds on the standard $\tilde{\mathfrak{g}}$-module $L(\lambda)$ with highest weight $\lambda = h_0^* + h_l^*$, $0 \le l \le t-1$.

First observe that on $L(h_i^*) \otimes L(h_l^*)$ we have

$$E^{\pm}(\zeta) = E^{\pm}(\zeta) \otimes E^{\pm}(\zeta)$$

and

$$X(\zeta) = X(\zeta) \otimes 1 + 1 \otimes X(\zeta).$$

Hence

(2.14)

$$\begin{aligned} F_t(\zeta_1, \dots, \zeta_t) = &\prod_{1 \le i < j \le t} (1 - \omega \zeta_i/\zeta_j)(1 - \omega^{-1} \zeta_i/\zeta_j) \\ &\cdot \sum_{k=0}^{t} \sum_{1 \le i_1 < \cdots < i_k \le t} X(\zeta_1) \cdots X(\widehat{\zeta}_{i_1}) \cdots X(\widehat{\zeta}_{i_k}) \cdots X(\zeta_t) \otimes X(\zeta_{i_1}) \cdots X(\zeta_{i_k}) \end{aligned}$$

(continues)

(continued)

$$= \sum_{k=0}^{t} \sum_{1\le i_1<\cdots<i_k\le t} \prod_{\substack{1\le i<j\le t \\ |\{i,j\}\cap\{i_1,\dots,i_k\}|=1}} (1-\omega\zeta_i/\zeta_j)(1-\omega^{-1}\zeta_i/\zeta_j)$$

$$\cdot F_{t-k}(\zeta_1,\dots,\widehat{\zeta}_{i_1},\dots,\widehat{\zeta}_{i_k},\dots,\zeta_t)\otimes F_k(\zeta_{i_1},\dots,\zeta_{i_k})$$

$$= \sum_{k=0}^{[(t+1)/2]} \sum_{1\le i_1<\cdots<i_k\le t} \prod_{\substack{1\le i<j\le t \\ |\{i,j\}\cap\{i_1,\dots,i_k\}|=1}} (1-\omega\zeta_i/\zeta_j)(1-\omega^{-1}\zeta_i/\zeta_j)$$

$$\cdot \{F_{t-k}(\zeta_1,\dots,\widehat{\zeta}_{i_1},\dots,\widehat{\zeta}_{i_k},\dots,\zeta_t)\otimes F_k(\zeta_{i_1},\dots,\zeta_{i_k})$$

$$+ F_k(\zeta_{i_1},\dots,\zeta_{i_k})\otimes F_{t-k}(\zeta_{i_1},\dots,\widehat{\zeta}_{i_1},\dots,\widehat{\zeta}_{i_k},\dots,\zeta_t)\},$$

where " $\widehat{\ }$ " means "delete the corresponding term" and for any set S, $|S|$ denotes the cardinality of the set S. Now take limit of (2.14) as $\zeta_i \to \omega^{2b_i}\zeta$. Then using Lemma 1.2, Proposition 1.3, 1.4, 2.3, and Theorem 2.1 and reindexing after neglecting the zero terms, we have

$$X^{[t]}(\zeta) = 2\sum_{r=0}^{[(t-1)/2]} \sum_{1\le j_1<\cdots<j_r\le t-2} \prod_{0\le i\le t-2} (1-\omega^{2i+1})(1-\omega^{2i-1})$$

$$\cdot \prod_{\substack{1\le i<j\le t-2 \\ |\{i,j\}\cap\{j_1,\dots,j_r\}|=1}} (1-\omega^{2i-2j+1})(1-\omega^{2i-2j-1})$$

$$\cdot \{F_{t-r-1}(\zeta,\omega^2\zeta,\dots,\omega^{\hat{2}j_1}\zeta,\dots,\omega^{\hat{2}j_r}\zeta,\dots,\omega^{2(t-2)}\zeta)$$

$$\otimes F_{r+1}(\zeta,\omega^{2j_1}\zeta,\dots,\omega^{2j_r}\zeta) + F_{r+1}(\zeta,\omega^{2j_1}\zeta,\dots,\omega^{2j_r}\zeta)$$

$$\otimes F_{t-r-1}(\zeta,\omega^2\zeta,\dots,\omega^{\hat{2}j_1}\zeta,\dots,\omega^{\hat{2}j_r}\zeta,\dots,\omega^{2(t-2)}\zeta)\}$$

$$= (2c_0c_l)(1-\omega)(1-\omega^{-1})\prod_{1\le k\le t-2}(1-\omega^{-2k})^2(1-\omega^{2k+1})(1-\omega^{2k-1})$$

$$\cdot \sum_{r=0}^{[(t-1)/2]} \sum_{1\le j_1<\cdots<j_r\le t-2} \prod_{\substack{1\le i<j\le t-2 \\ |\{i,j\}\cap\{j_1,\dots,j_k\}|=1}} \cdot (1-\omega^{2i-2j+1})(1-\omega^{2i-2j-1})$$

$$\cdot \{E^-(\omega^n\zeta)F_{t-r-2}(\omega^2\zeta,\dots,\omega^{\hat{2}j_1}\zeta,\dots,\omega^{\hat{2}j_r}\zeta,\dots,\omega^{2(t-2)}\zeta)E^+(\omega^n\zeta)$$

$$\otimes E^-(\omega^n\zeta)F_r(\omega^{2j_1}\zeta,\dots,\omega^{2j_r}\zeta)E^+(\omega^n\zeta)$$

$$+ E^-(\omega^n\zeta)F_r(\omega^{2j_1}\zeta,\dots,\omega^{2j_r}\zeta)E^+(w^n\zeta)$$

$$\otimes E^-(\omega^n\zeta)F_{t-r-2}(\omega^2\zeta,\dots,$$

$$\omega^{\hat{2}j_1}\zeta,\dots,\omega^{\hat{2}j_r}\zeta,\dots,\omega^{2(t-2)}\zeta)E^+(\omega^n\zeta)\}$$

$$= (-2\omega^{l+1})\prod_{1\le k\le t-2}(1-\omega^{-2k})^2(1-\omega^{2k+1})(1-\omega^{2k-1})$$

$$\cdot E^-(\omega^n\zeta)X^{[t-2]}(\omega^2\zeta)E^+(\omega^n\zeta),$$

since

$$X^{[t-2]}(\omega^2\zeta) = \lim_{\zeta_i\to\omega^{2b_i+2}\zeta} F_{t-2}(\zeta_1, \dots, \zeta_{t-2}) = \lim_{\zeta_i\to\omega^{2i}\zeta} F_{t-2}(\zeta_1, \dots, \zeta_{t-2})$$

$$= \lim_{\zeta_i\to\omega^{2i}\zeta} \sum_{k=0}^{[(t-1)/2]} \sum_{1\le i_1<\cdots<i_k\le t-2} \prod_{\substack{1\le i<j\le t-2 \\ |\{i,j\}\cap\{i_1,\dots,i_k\}|=1}} (1-\omega\zeta_i/\zeta_j)(1-\omega^{-1}\zeta_i/\zeta_j)$$

$$\cdot \{F_{t-2-k}(\zeta_1, \dots, \widehat{\zeta}_{i_1}, \dots, \widehat{\zeta}_{i_k}, \dots, \widehat{\zeta}_{t-2}) \otimes F_k(\zeta_{i_1}, \dots, \zeta_{i_k})$$

$$+ F_k(\zeta_{i_1}, \dots, \zeta_{i_k}) \otimes F_{t-2-k}(\zeta_1, \dots, \widehat{\zeta}_{i_1}, \dots, \widehat{\zeta}_{i_k}, \dots, \zeta_{t-2})\}$$

$$= \sum_{k=0}^{[(t-1)/2]} \sum_{1\le i_1<\cdots<i_k\le t-2} \prod_{\substack{1\le i<j\le t-2 \\ |\{i,j\}\cap\{i_1,\dots,i_k\}|=1}} (1-\omega^{2i-2j+1})(1-\omega^{2i-2j-1})$$

$$\cdot \{F_{t-k-2}(\omega^2\zeta, \dots, \omega^{\hat{2i_1}}\zeta, \dots, \omega^{\hat{2i_k}}\zeta, \dots, \omega^{2(t-2)}\zeta)$$

$$\otimes F_k(\omega^{2i_1}\zeta, \dots, \omega^{2i_k}\zeta) + F_k(\omega^{2i_1}\zeta, \dots, \omega^{2i_k}\zeta)$$

$$\otimes F_{t-k-2}(\omega^2\zeta, \dots, \omega^{\hat{2i_1}}\zeta, \dots, \omega^{\hat{2i_k}}\zeta, \dots, \omega^{2(t-2)}\zeta)\}. \quad \square$$

3. Bases for level two standard $\mathfrak{sl}(2k+1, \mathbf{C})^{\sim}$-modules

Let $L = L(\lambda) \subseteq L(h_0^*) \otimes L(h_l^*)$, $0 \le l \le k$ (note here $n = 2k$ and $t = k+1$), be the level two standard $\mathfrak{sl}(2k+1, \mathbf{C})^{\sim}$-module with highest weight $\lambda = h_0^* + h_l^*$ and highest weight vector $v_\lambda = v_0 \otimes v_l$, where v_0 and v_l are highest weight vectors in $L(h_0^*)$ and $L(h_l^*)$, respectively. For any sequence of integers $\mu = (m_1, m_2, \dots, m_p)$, $p > 0$, define the elements $X(\mu) = X(m_1, m_2, \dots, m_p)$ in $\operatorname{End} V$ by the equation

$$F_p(\zeta_1, \zeta_2, \dots, \zeta_p) = \sum X(m_1, \dots, m_p)\zeta_1^{m_1} \cdots \zeta_p^{m_p}, \tag{3.1}$$

where the summation ranges over all integers $m_1, m_2, \dots, m_p$. Note that for $p = 0$ we have the unique sequence $\mu = \varnothing$ (empty sequence) and in this case we define $X(\varnothing) = 1$. Now by Theorem 2.1, for $\sigma \in S_p$ we have

$$X(m_1, \dots, m_p) = X(m_{\sigma(1)}, \dots, m_{\sigma(p)}). \tag{3.2}$$

For any sequence $\mu = (m_1, m_2, \dots, m_p) \in \mathbf{Z}^p$, $p > 0$, define $l(\mu) = p$, $|\mu| = m_1 + m_2 + \cdots + m_p$, and write $\mu(i) = m_i$, $1 \le i \le p$. Also define $l(\varnothing) = 0$. For two sequences of integers $\mu = (m_1, m_2, \dots, m_p)$ and $\nu = (n_1, n_2, \dots, n_p)$, $p > 0$, we define $\mu \ge_T \nu$ if and only if

$$m_p \ge n_p;\, m_{p-1} + m_p \ge n_{p-1} + n_p;\, \dots;\, m_1 + \cdots + m_p \ge n_1 + \cdots + n_p.$$

Then clearly for any sequence $\theta = (r_1, r_2, \dots, r_q)$, $q \ge 0$, of integers we have

$$\mu \ge_T \nu \Rightarrow \mu\circ\theta \ge_T \nu\circ\theta \quad \text{and} \quad \theta\circ\mu \ge_T \theta\circ\nu \tag{3.3}$$

where the composition is defined by juxtaposition, i.e., $\mu\circ\theta = (m_1, \dots, m_p, r_1, \dots, r_q)$.

Lemma 3.1. *Let us define the coefficients* $a(\mu)$, $b(\mu)$ *by the formal identities*

$$\prod_{1\le i<j\le p}(1-\omega\zeta_i/\zeta_j)(1-\omega^{-1}\zeta_i/\zeta_j)=\sum_{\mu\in\mathbf{Z}^p}a(\mu)\zeta_1^{\mu(1)}\cdots\zeta_p^{\mu(p)}$$

and

$$\prod_{1\le i<j\le p}(1-\omega\zeta_i/\zeta_j)^{-1}(1-\omega^{-1}\zeta_i/\zeta_j)^{-1}=\sum_{\mu\in\mathbf{Z}^p}b(\mu)\zeta_1^{\mu(1)}\cdots\zeta_p^{\mu(p)}.$$

Then $a(0)=1=b(0)$ *and* $a(\mu)=0=b(\mu)$ *unless* $\mu\le_T \mathbf{0}$, *where* $\mathbf{0}=(0,0,\dots,0)$. *Furthermore,*

$$X(m_1,\dots,m_p)=\sum_{\mu\in\mathbf{Z}^p}a(\mu)X(m_1-\mu(1))\cdots X(m_p-\mu(p)) \tag{1}$$

and

$$X(m_1)\cdots X(m_p)=\sum_{\mu\in\mathbf{Z}^p}b(\mu)X(m_1-\mu(1),\dots,m_p-\mu(p)). \tag{2}$$

Proof. For (1) compare the coefficient of $\zeta_1^{m_1}\zeta_2^{m_2}\cdots\zeta_p^{m_p}$ in Equation (3.1). For (2) multiply Equation (3.1) by $\prod_{1\le i<j\le p}(1-\omega\zeta_i/\zeta_j)^{-1}(1-\omega^{-1}\zeta_i/\zeta_j)^{-1}$ and then compare the coefficient of $\zeta_1^{m_1}\zeta_2^{m_2}\cdots\zeta_p^{m_p}$.

The next corollary follows immediately from Lemma 3.1.

Corollary 3.2. *For* $\mu=(m_1,m_2,\dots,m_p)\in\mathbf{Z}^p$, *we have*

$$X(\mu)=X(m_1)\cdots X(m_p)+\sum_{\nu>_T\mu}a(\mu-\nu)X(\nu(1))\cdots X(\nu(p)), \tag{1}$$

$$X(m_1)\cdots X(m_p)=X(\mu)+\sum_{\nu>_T\mu}b(\mu-\nu)X(\nu). \tag{2}$$

Corollary 3.3. *We have*

$$X(m_1,\dots,m_p)X(n_1,\dots,n_r)=X(m_1,\dots,m_p,n_1,\dots,n_r)+\sum_{\nu>_T(m_1,\dots,m_p,n_1,\dots,n_r)}c(\nu)X(\nu)$$

for some scalars $c(\nu)$.

Proof. It is clear from Corollary 3.2 (1) and (3.3) that

$$X(m_1,\dots,m_p)X(n_1,\dots,n_r)=X(m_1)\cdots X(m_p)X(n_1)\cdots X(n_r)+\sum_{\mu>_T(m_1,\dots,m_p,n_1,\dots,n_r)}d(\nu)X(\nu(1))\cdots X(\nu(p+r))$$

for some scalars $d(\nu)$. Now the result follows from Corollary 3.2 (2). □

For any two sequences $\mu,\nu\in\mathbf{Z}^p$, we say $\mu<\nu$ if and only if $l(\mu)>l(\nu)$ or $\mu<_T\nu$. Then, using equation (3.3), it follows that

$$\mu<\nu\Rightarrow\mu\circ\theta<\nu\circ\theta \quad\text{and}\quad \theta\circ\mu<\theta\circ\nu, \tag{3.4}$$

for any sequence θ. Let $\mathscr{P}$ denote the set of all sequences $\mu = (m_1, m_2, \dots, m_p) \in \mathbf{Z}^p$, $p \geq 0$, such that $m_1 \leq m_2 \leq \cdots \leq m_p < 0$. For the standard $\mathfrak{sl}(2k+1, \mathbf{C})^{\sim}$-module $L = L(\lambda)$, $\lambda = h_0^* + h_l^*$, $0 \leq l \leq k$, define

$$L(\mu) = \sum_{\substack{\nu > \mu \\ \nu \in \mathscr{P}}} \mathscr{U}(\mathfrak{s}^-) X(\nu) v_\lambda, \qquad \mu \neq \varnothing, \tag{3.5}$$

for all $\mu \in \mathscr{P}$ and define $L(\varnothing) = \{0\}$. Clearly, we have

$$L_{(\mu)} \supseteq L_{(\nu)}, \quad \text{for } \mu \leq \nu,\ \mu, \nu \in \mathscr{P}. \tag{3.6}$$

Proposition 3.4. $L = L(\lambda) = \bigcup_{\mu \in \mathscr{P}} L_{(\mu)}$.

Proof. By the Poincaré-Birkhoff-Witt theorem, Proposition 1.1, and (3.2), we have

$$\begin{aligned} L &= \operatorname{span}\{\mathscr{U}(\mathfrak{s}^-) X(\mu) v_\lambda \mid \mu \in \mathbf{Z}^p,\ p \geq 0\} \\ &= \operatorname{span}\{\mathscr{U}(\mathfrak{s}^-) X(\mu) v_\lambda \mid \mu = (m_1, \dots, m_p) \in \mathbf{Z}^p, p \geq 0, m_1 \leq \cdots \leq m_p\}. \end{aligned}$$

Let

$$V = \operatorname{span}\{\mathscr{U}(\mathfrak{s}^-) X(\nu) v_\lambda \mid \nu \in \mathscr{P}\}.$$

Clearly $V \subseteq L$. To show $L \subseteq V$ we will use induction on the ordering of the sequences $\mu = (m_1, m_2, \dots, m_p)$, $p \geq 0$. Fix $\mu = (m_1, m_2, \dots, m_p) \in \mathbf{Z}^p$, $m_1 \leq m_2 \leq \cdots \leq m_p$. Assume that $X(\mu')v_\lambda \in V$ for all $\mu' > \mu$. It is enough to show that $X(\mu)v_\lambda \in V$. Suppose $\mu = (m_1, \dots, m_r, m_{r+1}, \dots, m_p)$, where $m_1 \leq \cdots \leq m_r < 0$ and $0 \leq m_{r+1} \leq \cdots \leq m_p$. By Corollary 3.3 we have

$$X(\mu)v_\lambda = X(m_1, \dots, m_r) X(m_{r+1}, \dots, m_p) v_\lambda + \sum_{\mu' > \mu} c(\mu') X(\mu') v_\lambda.$$

Hence, by assumption, it is enough to show that

$$X(m_1, \dots, m_r) X(m_{r+1}, \dots, m_p) v_\lambda \in V.$$

If $r \gneqq 0$, then this follows from the induction hypothesis by using Proposition 1.1 and Corollary 3.3 since $(m_{r+1}, \dots, m_p) > \mu$. Now suppose $r = 0$. Then we have $0 \leq m_1 \leq \cdots \leq m_p$. By Corollary 3.2,

$$X(\mu)v_\lambda = X(m_1) \cdots X(m_p) v_\lambda + \sum_{\mu' > \mu} d(\mu') X(\mu') v_\lambda,$$

where $d(\mu')$ are some scalars. But since v_λ is a highest weight vector and $0 \leq m_1 \leq \cdots \leq m_p$, so $X(m_1) \cdots X(m_p) v_\lambda$ is a scalar. Hence it follows that $X(\mu)v_\lambda \in V$. □

For $p \geq 1$ and $n \in \mathbf{Z}$ denote by $(p; n)$ the unique sequence of integers (see [7])

$$(p; n) = (m_1, m_2, \dots, m_p)$$

such that

$$n = m_1 + m_2 + \cdots + m_p ,$$
$$m_1 \le m_2 \le \cdots \le m_p ,$$

and

$$0 \le m_p - m_1 \le 1 .$$

The following lemma is clear (see [7, Lemma 8.2]).

Lemma 3.5. *Let* $\mu = (n_1 , \dots , n_p) \in \mathscr{P}$, $p > 0$, *and* $|\mu| = n$. *Then we have*
(1) $(p ; n) \le \mu$,
(2) *if* $\mu \ne (p ; n)$, *then* $n_1 \le -2 + n_p$.

For $p \ge 2$, let

$$X^{[p]}(\zeta) = \sum_{n \in \mathbf{Z}} X^{[p]}(n) \zeta^n . \tag{3.7}$$

The next proposition follows immediately from Corollary 2.2 and Lemma 3.5 since $\omega^{2k+1} = 1$.

Proposition 3.6. *For* $n \in \mathbf{Z}$ *and* $p \ge 2$, *we have*

$$X^{[p]}(n) \in aX(p ; n) + \sum_{\nu > (p ; n)} \mathscr{U}(\mathfrak{s}^-) X(\nu) ,$$

for some scalar $a \ne 0$, *where* $X(p ; n)$ *denotes* $X((p ; n))$.

For a sequence $\mu = (m_1 , \dots , m_r)$, $m_1 \le \cdots \le m_r$, we say that μ satisfies the *difference two condition* if for every $i \in \{1 , \dots , r - t + 1\}$ we have $m_i \le -2 + m_{i+t-1}$. Observe that for $n \in \mathbf{Z}$ the sequence $(t ; n)$ does not satisfy the difference two condition.

Theorem 3.7. *If* $\mu \in \mathscr{P}$ *does not satisfy the difference two condition, then*

$$X(\mu) v_\lambda \in L_{(\mu)} .$$

Proof. Since $\mu \in \mathscr{P}$ does not satisfy the difference two condition, it must be of the form

$$\mu = (m_1 , \dots , m_r , (t ; n) , m_{r+t+1} , \dots , m_s) .$$

Let

$$(t ; n) = (m_{r+1} , \dots , m_{r+t}) .$$

Then Theorem 2.4 implies that for any $v \in L$,

$$X^{[p]}(n) v \in \sum_{l(\nu) < t} \mathscr{U}(\mathfrak{s}^-) X(\nu) v .$$

This together with Proposition 3.6 implies that

$$X(t ; n) v \in \sum_{\nu > (t ; n)} \mathscr{U}(\mathfrak{s}^-) X(\nu) v . \tag{3.8}$$

Now set $v = X(m_{r+t+1}, \ldots, m_s)v_\lambda$, and multiply equation (3.8) from the left by $X(m_1, \ldots, m_r)$. Now thanks to Proposition 1.1, Corollary 3.3, and equations (3.1), (3.4) we have the desired result. □

Observe that (see Proposition 1.4)

$$X(0)v_\lambda = c_0(1 + \omega^l)v_\lambda \tag{3.9}$$

for $\lambda = h_0^* + h_l^*$, $0 \leq l \leq t-1$, where $c_0 = \omega/(\omega - 1)$. For $r \geq 0$, write

$$\mu(r) = (-1, \ldots, -1) \in \mathbf{Z}^r. \tag{3.10}$$

For $p \geq r$, write

$$\mu_p(r) = (-1, \ldots, -1, 0, \ldots, 0) \in \mathbf{Z}^p \tag{3.11}$$

with r entries equal to -1. The next lemma follows from (3.9) and Corollary 3.2.

Lemma 3.8. *For $p \geq r$, we have*

$$X(\mu_p(r))v_\lambda \in (c_0(1+\omega^l))^{p-r} X(\mu(r))v_\lambda + L_{(\mu(r))}.$$

Lemma 3.9. *For $p \geq r$, $\lambda = h_0^* + h_l^*$, $0 \leq l \leq t-1$, there is a polynomial $f_{t,p,r}(x)$ of degree $p - r$ which is independent of l such that the coefficient of $\zeta_1^{-1} \cdots \zeta_r^{-1}$ in $F_p(\zeta_1, \ldots, \zeta_p)v_\lambda$ belongs to*

$$f_{t,p,r}(c_0(1+\omega^l))X(\mu(r))v_\lambda + L_{(\mu(r))}.$$

Proof. From Theorem 2.1 we have

$$F_p(\zeta_1, \ldots, \zeta_p) = \sum_{\sigma \in S_p} \left(\sum_{m_1 \leq \cdots \leq m_p} X(m_1, \ldots, m_p) \right) \zeta_1^{m_{\sigma(1)}} \cdots \zeta_p^{m_{\sigma(p)}}. \tag{3.12}$$

Hence the coefficient of $\zeta_1^{-1} \cdots \zeta_r^{-1}$ in $F_p(\zeta_1, \ldots, \zeta_p)v_\lambda$ is a linear combination of terms of the form $X(\mu)v_\lambda$ where $\mu = (m_1, \ldots, m_s)$, $s \leq p$, $m_1 \leq \cdots \leq m_s$ and $m_1 + \cdots + m_s = -r$. Since the coefficient of $X(\mu(r))v_\lambda$ is nonzero and also if $\mu >_T \mu_s(r)$ then $X(\mu)v_\lambda \in L_{\mu(r)}$; therefore Lemma 3.8 and Corollary 3.2 give the result. □

Proposition 3.10. *For $r \leq t$, $\lambda = h_0^* + h_l^*$, $0 \leq l \leq t-1$, there is a polynomial $g_{t,r}(x)$ of degree $t - r$, independent of l, such that*

$$g_{t,r}(c_0(1+\omega^l))X(\mu(r))v_\lambda \in L_{\mu(r)}.$$

Proof. From Theorem 2.4, we have

$$\lim_{\zeta_t \to \omega^{2b_t}\zeta} [F_t(\zeta_1, \ldots, \zeta_t) - a_t E^-(\omega^{2k}\zeta_t)F_{t-2}(\zeta_1, \ldots, \zeta_{t-2})E^+(\omega^{2k}\zeta_t)]v_\lambda = 0 \tag{3.13}$$

where

$$a_l = a\omega^l, \qquad a = (-2\omega) \prod_{1 \leq s \leq t-2} (1 - \omega^{-2s})]^2 (1 - \omega^{2s+1})(1 - \omega^{2s-1}).$$

In (3.13), first collecting the coefficient of $\zeta_1^{-1}\cdots\zeta_r^{-1}$ by using Lemma 3.9 and then taking the limit, we get

$$[f_{t,t,r}(c_0(1+\omega^l)) - a_l f_{t,t-2,r}(c_0(1+\omega^l))]X(\mu(r))v_\lambda \in L_{\mu(r)}. \tag{3.14}$$

But

$$f_{t,t,r}(c_0(1+\omega^l)) - a_l f_{t,t-2,r}(c_0(1+\omega^l)) = f_{t,t,r}(c_0(1+\omega^l)) - ac_0^{-1}(c_0(1+\omega^l))f_{t,t-2,r}(c_0(1+\omega^l)) + af_{t,t-2,r}(c_0(1+\omega^l)).$$

Hence setting

$$g_{t,r}(x) = f_{t,t,r}(x) - ac_0^{-1}xf_{t,t-2,r}(x) + af_{t,t-2,r}(x)$$

we have the desired result from (3.14). □

The following lemma is an immediate consequence of formulas (1.12) and (1.14) (see [6, Corollary 13.14]) since here $(n+3)-(l+1) = 2k+3-l-1 > l+1$ for $0 \le l \le t-1 = k$.

Lemma 3.11. *If* $r \le l$, $\lambda = h_0^* + h_l^*$, $0 \le l \le t-1$, *then*

$$X(\mu(r))v_\lambda \notin L_{\mu(r)}.$$

Proposition 3.12. *For* $\lambda = h_0^* + h_l^*$, $0 \le l \le t-1$, *we have*

$$X(\mu(l+1))v_\lambda \in L_{(\mu(l+1))}.$$

Proof. If $r \le t$, then by Proposition 3.10 and Lemma 3.11 we have

$$g_{t,r}(c_0(1+\omega^l)) = 0 \quad \text{for } l = r, r+1, \dots, t-1.$$

But since $g_{t,r}$ has degree $t-r$, this implies that

$$g_{t,r}(c_0(1+\omega^{r-1})) \ne 0. \tag{3.15}$$

Now setting $r = l+1$ in (3.15), we have $g_{t,l+1}(c_0(1+\omega^l)) \ne 0$. Hence by Proposition 3.10 we have

$$X(\mu(l+1))v_\lambda \in L_{\mu(l+1)}$$

as desired. □

We say that a sequence $\mu = (m_1, \dots, m_s) \in \mathscr{P}$ satisfies the *initial condition* if at most l $(0 \le l \le t-1)$ elements m_i are equal to -1, that is, if $m_{s-l} \le -2$.

Theorem 3.13. *If* $\mu \in \mathscr{P}$ *does not satisfy the initial condition, then*

$$X(\mu)v_\lambda \in L_{(\mu)}.$$

Proof. Since $\mu \in \mathscr{P}$ does not satisfy the initial condition, it must be of the form

$$\mu = (m_1, \dots, m_{s-l-1}, m_{s-l}, \dots, m_s)$$

where

$$m_{s-l} = \cdots = m_s = -1.$$

Let $\nu = (m_1, \ldots, m_{s-l-1})$. Then $\mu = \nu \circ \mu(l+1)$. By Corollary 3.3 we have

$$X(\mu)v_\lambda \in X(\nu)X(\mu(l+1))v_\lambda + L_\mu .$$

Now the theorem follows by (3.4), Corollary 3.3, and Propositions 1.1 and 3.12. □

Denote by $\mathscr{C}_\lambda$ the set of all $\mu \in \mathscr{P}$ such that μ satisfies the difference two condition and the initial condition.

Denote by $\mathscr{J}$ the set of all $\mu = (m_1, \ldots, m_s)$ in $\mathscr{P}$ such that $m_i \not\equiv 0 \bmod(2k+1)$. For $\mu = (m_1, \ldots, m_s) \in \mathscr{J}$ we denote

$$B(\mu) = B(m_1) \cdots B(m_s) \quad \text{and} \quad B(\emptyset) = 1 .$$

Theorem 3.14. *The set* $\{B(\nu)X(\mu)v_\lambda \mid \nu \in \mathscr{J}, \mu \in \mathscr{C}_\lambda\}$ *spans* $L(\lambda)$.

Proof. By Proposition 3.4 and the Poincaré-Birkhoff-Witt theorem the set of vectors

$$\{B(\nu)X(\mu)v_\lambda \mid \nu \in \mathscr{J}, \mu \in \mathscr{P}\}$$

is a spanning set of L. Now using Theorems 3.7 and 3.13, the desired result follows by induction on the ordering of $\mathscr{P}$. □

Now since for $\nu \in \mathscr{J}$ and $\mu \in \mathscr{C}_\lambda$

$$B(\nu)X(\mu)v_\lambda \in L_{|\nu|+|\mu|},$$

formula (1.12) for the principal character of $L = L(\lambda)$, together with the generalized Rogers-Ramanujan identities (cf. [4]) due to Gordon, Andrews, and Bressoud, implies the following theorem.

Theorem 3.15. *The set* $\{B(\nu)X(\mu)v_\lambda \mid \nu \in \mathscr{J}, \mu \in \mathscr{C}_\lambda\}$ *is a basis of* $L(\lambda)$.

REFERENCES

1. V. G. Kac, *Infinite-dimensional Lie algebras* (2nd ed.), Cambridge Univ. Press, Cambridge, Mass., 1985.
2. V. G. Kac, D. A. Kazhdan, J. Lepowsky, and R. L. Wilson, *Realization of basic representations of the Euclidean Lie algebras*, Adv. in Math. **42** (1981), 83–112.
3. J. Lepowsky, *Lectures on Kac-Moody Lie algebras*, Université Paris VI, Spring, 1978.
4. J. Lepowsky and R. L. Wilson, *A Lie theoretic interpretation and proof of the Rogers-Ramanujan identities*, Adv. in Math. **45** (1982), 21–72.
5. ——, *The structure of standard modules*, I: *Universal algebras and the Rogers-Ramanujan identities*, Invent. Math. 77 (1984), 199–290.
6. ——, *The structure of standard modules* II: *The case* $A_1^{(1)}$, *principal gradation*, Invent. Math. **79** (1985), 417–442.
7. A. Meurman and M. Primc, *Annihilating ideals of standard modules of* $\mathfrak{sl}(2, \mathbf{C})^{\sim}$ *and combinatorial identities*, Adv. in Math. **64** (1987), 177–240.
8. K. C. Misra, *Structure of the standard modules for* $A_n^{(1)}$ *and* $C_1^{(1)}$, Ph.D. Thesis, Rutgers University, 1982.
9. ——, *Structure of certain standard modules for* $A_n^{(1)}$ *and the Rogers-Ramanujan identities*, J. Algebra **88** (1984), 196–227.

DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NORTH CAROLINA 27695–8205