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ON A PROBLEM OF S. MAZUR

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ABSTRACT. In this work a generalization of Mazur's problem concerning the continuity of linear functionals is given.

S. Mazur asked (about 1935) the following question [7, Problem 24]: In a Banach space E an additive functional f is given with the property that, for any continuous function $\varphi\colon [0,1] \to E$ the function $f \circ \varphi$ is Lebesgue-measurable. Is f continuous? This question was answered affirmatively by I. Labuda and R. D. Mauldin in [3] by the following theorem:

Theorem 1. *Let E be a Banach space, F a Hausdorff topological vector space, $f\colon E \to F$ an additive operator. If $f \circ \varphi$ is Lebesgue-measurable for any continuous function $\varphi\colon [0,1] \to E$, then f is continuous.*

The following more general theorem is due to Z. Lipecki [4].

Theorem 2. *Let G, H be Hausdorff topological abelian groups, G is metrizable, complete, connected and locally arcwise connected, and let $f\colon G \to H$ be a homomorphism. If $f \circ \varphi$ is Lebesgue-measurable for any continuous function $\varphi\colon [0,1] \to G$, then f is continuous.*

Recently, R. Ger presented similar results concerning convex functionals [2].

The aim of this paper is to give another generalization of Mazur's problem. Namely, we show, that if in the original problem f is an exponential polynomial, then the statement remains valid.

First we collect some necessary facts about polynomials and exponential polynomials on groups. Most of these results can be found in [5, 6]. Let G be an abelian group, H a complex linear space. The function $p\colon G \to H$ is called a polynomial if for some nonnegative integer N we have

$$\Delta^{N+1}_{y_1,\dots,y_{N+1}} p(x) = 0 \tag{1}$$

for all $x, y_1, \dots, y_{N+1}$ in G. The smallest integer N with this property is called the degree of p and is denoted by $\deg p$. It is well known [1] that any

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function p satisfying (1) can uniquely be represented in the form

$$p(x) = A_N(x, \dots, x) + A_{N-1}(x, \dots, x) + \dots + A_1(x) + A_0$$

for all x in G, where $A_k\colon G^k \to H$ is a k-additive and symmetric function $(k = 1, 2, \dots, N)$ and A_0 is in H. For the sake of simplicity we shall use the notation

$$A^{(k)}(x) = A_k(x, \dots, x)$$

for all x in G, that is $A^{(k)}$ is the diagonalization of the k-additive and symmetric function A_k, $(k = 1, 2, \dots, N)$.

Let C denote the set of complex numbers. The function $m\colon G \to C$ is called an exponential if for all x, y in G we have

$$m(x + y) = m(x)m(y)$$

and m is not identically zero. That is, exponentials are just the homomorphisms of G into the multiplicative group of nonzero complex numbers.

The function $f\colon G \to H$ is called an exponential polynomial if it has a representation

$$f(x) = \sum_{k=1}^{n} p_k(x)m_k(x) \tag{2}$$

for all x in G, where $p_k\colon G \to H$ is a polynomial and $m_k\colon G \to C$ is an exponential $(k = 1, \dots, n)$. It is well known [6] that if in (2) we have $m_i \neq m_j$ for $i \neq j$ then the representation (2) for f is unique.

In order to prove our main theorem for exponential polynomials we first consider the case of polynomials.

Theorem 3. *Let G be a metrizable topological abelian group which is complete, connected and locally arcwise connected, further let H be a metrizable locally convex complex topological vector space and $p\colon G \to H$ a polynomial. If $p \circ \varphi$ is Lebesgue-measurable for any continuous function $\varphi\colon [0, 1] \to G$, then p is continuous.*

Proof. Let $p = A^{(N)} + q$, where $N \geq 1$ is an integer, $A_N\colon G^N \to H$ is N-additive and symmetric and $q\colon G \to H$ is a polynomial of degree at most $N - 1$. It is enough to show that A_N is continuous, then by induction we have the statement. It is well known [1] that

$$\begin{aligned} A_N(x_1, x_2, \dots, x_N) &= \frac{1}{N!}\Delta^N_{x_1, x_2, \dots, x_N} p(0) \\ &= \frac{1}{N!} \sum_{i_1 < \dots < i_k} (-1)^{N-k} p(x_{i_1} + \dots + x_{i_k}) \end{aligned}$$

which implies that the function $t \to A_N(\varphi(t), x_2, \dots, x_N)$ is Lebesgue-measurable for any continuous function $\varphi\colon [0, 1] \to G$, and for any fixed $x_2, \dots, x_N$ in G. Using the symmetry of A_N and Theorem 2 we have that A_N is continuous in each variable. From the theorem of Baire it follows that A_N is continuous at least at one point. Then, using the connectedness of G, it follows from Theorem 4.2 in [5] that A_N is continuous.

Theorem 4. *Let G be a metrizable topological abelian group which is complete, connected and locally arcwise connected, further let H be a metrizable locally convex complex topological vector space and $f: G \to H$ an exponential polynomial. If $f \circ \varphi$ is Lebesgue-measurable for any continuous function $\varphi: [0,1] \to G$, then f is continuous.*

Proof. Let $f = \sum_{k=1}^{n} p_k m_k$, where $n \geq 1$ is an integer, $p_k: G \to H$ is a polynomial and $m_k: G \to C$ is an exponential $(k = 1, 2, \ldots, n)$, $m_i \neq m_j$ for $i \neq j$ and $p_k = A_k^{(N_k)} + q_k$, where $A_{k,N_k}: G^{N_k} \to H$ is N_k-additive and symmetric, $q_k: G \to H$ is a polynomial of degree at most $N_k - 1$, $A_{k,N_k} \neq 0$ $(k = 1, 2, \ldots, n)$. We show that $m_k, A_k^{(N_k)}$ is continuous $(k = 1, 2, \ldots, n)$. By induction on n, first let $n = 1$, $f = p_1 m_1$. Here we use induction on the degree of p_1. If $\deg p_1 = 0$, then p_1 is constant and $p_1 \neq 0$. It is very easy to see, that in this case the property of f implies that $m_1 \circ \varphi$ is Lebesgue-measurable for any continuous function $\varphi: [0,1] \to G$, hence by Theorem 2, m_1 is continuous. Then $p_1 \circ \varphi$ is Lebesgue-measure for any continuous function $\varphi: [0,1] \to G$, and by Theorem 3, p_1 is continuous and hence f is continuous. If $\deg p_1 \geq 1$ then p_1 is nonconstant, hence there exists y for which $\Delta_y p_1$ is not identically zero and $\deg \Delta_y p_1 < \deg p_1$. On the other hand

$$\begin{aligned} \Delta_y p_1(x) m_1(x) &= m_1(-y) p_1(x+y) m_1(x+y) - p_1(x) m_1(x) \\ &= m_1(-y) f(x+y) - f(x) \end{aligned}$$

that is the function $(\Delta_y p_1 m_1) \circ \varphi$ is Lebesgue-measurable for any continuous function $\varphi: [0,1] \to G$ which implies the statement of the theorem for $n = 1$. Let now $n \geq 2$. We show, that, for example, m_2 and $A_2^{(N_2)}$ are continuous. Let y be an element for which $m_1(y) \neq m_2(y)$. Then we have

$$\Delta_y^{N_1+1}(f m_1^{-1})(x) = m_1(x)^{-1} \sum_{j=0}^{N_1+1} \binom{N_1+1}{j} (-1)^{N_1+1-j} m_1(y)^{-j} f(x+jy)$$

by the definition of difference operators. From this equation we infer that the function $[m_1 \Delta_y^{N_1+1}(f m_1^{-1})] \circ \varphi$ is Lebesgue-measurable for any continuous function $\varphi: [0,1] \to G$, by the same property of f. On the other hand

$$(f m_1^{-1})(x) = p_1(x) + \sum_{k=2}^{n} p_k(x)(m_k m_1^{-1})(x)$$

holds for all x in G. By taking differences we have

$$\begin{aligned}\Delta_y^{N_1+1}(fm_1^{-1})(x) &= \sum_{k=2}^{n} \Delta_y^{N_1+1}[(m_k m_1^{-1})(A_k^{(N_k)} + q_k)](x) \\ &= \sum_{k=2}^{n}\sum_{j=0}^{N_1+1} \binom{N_1+1}{j}(-1)^{N_1+1-j} m_k(x)m_1(x)^{-1}m_k(y)^j m_1(y)^{-j} \\ &\quad \times (A_k^{(N_k)}(x+jy) + q_k(x+jy)) \\ &= m_1(x)^{-1}\sum_{k=2}^{n} m_k(x)\left[\sum_{j=0}^{N_1+1}\binom{N_1+1}{j}(-1)^{N_1+1-j}m_k(y)^j\right. \\ &\qquad\qquad \left. \times m_1(y)^{-j}(A_k^{(N_k)}(x) + q_{k,j,y}^*(x))\right] \\ &= m_1(x)^{-1}\sum_{k=2}^{n} m_k(x)[(m_k(y)m_1(y)^{-1} - 1)^{N_1+1}A_k^{(N_k)}(x) + q_{k,y}^*(x)],\end{aligned}$$

where $q_{k,j,y}^*: G \to H$ is a polynomial of degree at most $N_k - 1$ and $q_{k,y}^*: G \to H$ is a polynomial of degree at most $N_k - 1$ $(k = 2, \dots, n;\ j = 0, 1, \dots, N_1 + 1)$. We have a representation for the exponential polynomial $m_1\Delta_y^{N_1+1}(fm_1^{-1})$ from which we infer—by the above consideration—that m_k and its polynomial coefficient is continuous. It follows that $(m_k(y)m_1(y)^{-1} - 1)^{N_1+1}A_k^{(N_k)}$ must be continuous, and especially—as $m_2(y) \neq m_1(y)$—the function $A_2^{(N_2)}$ is continuous. Hence the theorem is proved.

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