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ON THE DISSIPATIVE EVOLUTION EQUATIONS ASSOCIATED WITH THE ZAKHAROV-SHABAT SYSTEM WITH A QUADRATIC SPECTRAL PARAMETER

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ABSTRACT. In this paper we derive some results for the Zakharov-Shabat system of the form $dm/dx = z^2[J, m] + (zQ + P)m$; J is diagonal and skew-Hermitian [8, 10, 12]. Following the idea of R. Beals and R. R. Coifman, we estimate the wedge products of the columns of m by L^2-norm of the potential (Q, P) [4]. By this result we have the global existence of the dissipative evolution equations associated with this spectral problem if the generic initial data $(Q(x, 0), P(x, 0)) = (Q_0, P_0)$ is of Schwartz class.

1. Direct and inverse scattering for the Z-S system with a quadratic parameter

By a potential here we mean a pair of functions $(Q, P)\colon \mathbf{R} \to M_n(\mathbf{C})$ = set of $n \times n$ complex matrices; Q is off-diagonal and the diagonal part of P equals the diagonal part of $Q(\operatorname{ad} J)^{-1}Q$, and Q, Q_x, P, $P_x \in L^1$. We consider the following spectral problem: Given $z \notin \Sigma = \{z\colon \operatorname{Im}(z^2) = 0\}$, find $m(\cdot, z)\colon R \to M_n(\mathbf{C})$ with

(1.1) $$\partial m(z, z)/\partial x = z^2[J, m(x, z)] + (zQ + P)m(x, z), \text{where}$$
$$J = \operatorname{diag}(id_1, id_2, \dots, id_n),\ d_1 < d_2 < d_3 < \cdots < d_n;$$

(1.2) $$m(\cdot, z) \text{ bounded},\ m(x, z) \to I \text{ as } x \to -\infty.$$

Let
$$\Omega_+ = \{z\colon \operatorname{Im}(z^2) > 0\}, \quad \Omega_- = \{z\colon \operatorname{Im}(z^2) < 0\}.$$

For a certain set of potentials, called generic, m has the following properties [8, 10]:

(1.3) For any $z \in \Sigma$, there is a unique matrix $v(z)$ such that for all x, $m^+(x, z) = m^-(x, z)\exp(xz^2J)v(z)\exp(-xz^2J)$, where $m^{\pm}(x, z)$ = limit of m on Σ from $\Omega_{\pm}$;

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(1.4) $m(x,\cdot)$ has a finite number of poles at $D = \{z_1, z_2, \ldots, z_N\}$ (which do not depend on x); for any z_j, there is a matrix $v(z_j)$ such that

$$\operatorname{Res}(m(x,\cdot), z_j) = \lim_{z\to z_j} m(x,z)\exp(xz_j^2 J)v(z_j)\exp(-xz_j^2 J);$$

(1.5) The map $(Q,P) \to v = \{v(z); z_1, z_2, \ldots, z_N, v(z_1), v(z_2,),$ $\ldots, v(z_N)\}$ is injective. We denote this map by sd; i.e., $\mathrm{sd}(Q,P) = v$.

By a similar argument in [2], we can show that generic Schwartz class potentials form an open and dense set in Schwartz class potentials. To make our exposition clearer, from now on we assume (Q,P) is of Schwartz class. If (Q,P) is generic, we call the associated function $v\colon \Sigma \cup D \to M_n(\mathbf{C})$ the scattering data. We introduce some more notation. For any matrix A, we let $d_k^+(A)$ and $d_k^-(A)$ denote the upper and lower $k\times k$ principal minors. The scattering data satisfies the following constraints:

(1.6) $d_k^+(v) = 1$, $1 \le k \le n$; $d_k^-(v) \ne 0$, $1 \le k \le n$.

(1.7) If z_j is a pole, then $v(z_j) = c_j e_{k,k+1}$ for some k if $z_j \in \Omega_+$; $v(z_j) = c_j e_{k+1,k}$ for some k if $z_j \in \Omega_-$.

(1.8) The winding number of $d_k^-(v) = \beta_k^+ - \beta_k^-$, where β_k^+ is the number of z_j in Ω_+ such that the $n-k+1$ column of $v(z_j) \ne 0$ and β_k^- is the number of z_j in Ω_- such that the $n-k$ column of $v(z_j) \ne 0$; i.e., $\beta_k^+ - \beta_k^- = \int_\Sigma d[\arg d_k^-(v)]$ and Σ is oriented such that Ω_+ is in the left side.

(1.9) $v(z) - I$ is of Schwartz class in Σ.

Let SD $=$ set of v satisfying (1.6)–(1.9). Given $v \in \mathrm{SD}$, then the inverse problem amounts to solving an analytic factorization problem (Riemann-Hilbert problem) with one parameter x. The set $v \in \mathrm{SD}$ such that m is solvable is open and dense in SD, and such v is called generic [2, 8, 12]. Note that $v \in \mathrm{SD}$ is generic if and only if there exists a potential (Q,P) such that $\mathrm{sd}(Q,P) = v$.

More precisely, the inverse problem is solved following the argument of Beals and Coifman in the case of the first-order system [2]. For $x \le 0$ the solution m has the form $m = rm^{\#}\exp(xz^2J)u\exp(-xz^2J)$, where u is a piecewise rational function, $m^{\#}$ satisfies an equation of the form (1.1) with $v^{\#}$ and $v^{\#} - 1$ small, and r is rational in z. The solution for $x \ge 0$ has the same form with different u, $m^{\#}$, r. Given $v \in \mathrm{SD}$, the function u and $m^{\#}$ above can be determined. Determination of r amounts to solving a finite system of linear algebraic equations with parameter x, and $v = \mathrm{sd}(Q,P)$ if and only if these equations are solvable for all $x \in R$.

2. Estimate of m in L^2-norm of (Q, P)

Recall that the spectral problem is of the form

$$\text{(2.0)} \qquad \begin{aligned} &dm/dx = z^2[J, m] + (zQ + p)m, && z \notin \Sigma, \\ &m(x, z) \to I, \quad \text{as } x \to \infty, && P^{\text{diag}} = (Q(\operatorname{ad} J)^{-1}Q)^{\text{diag}}. \end{aligned}$$

In Lee's dissertation a technique of integration by parts enables one to control the term zQ for large z [8]. An alternative method is given in [9] to control the z-dependence. We may solve $m(x, z)$ for $|z| \le N$ by the argument of Beals and Coifman [2]. In (2.0) the constraint on the diagonal part of P enables us to pose the inverse problem easily. (This constraint implies $\lim_{|z|\to\infty} m(x, z) = I$, so in the inverse probem we may solve m normalized at $z = \infty$.) For technical reasons, we need the following transformation:

Let $\tilde{m} = \exp(-\int_{-\infty}^{x}(Q(\operatorname{ad} J)^{-1}Q)^{\text{diag}})m$; then $\tilde{m}$ satisfies

$$d\tilde{m}/dx = z^2[J, \tilde{m}] + z\tilde{Q}m + \tilde{P}\tilde{m}, \quad z \notin \Sigma, \ \tilde{m}(x, z) \to I \text{ as } x \to \infty \text{ where}$$

$$\text{(2.1)} \qquad \begin{aligned} \tilde{Q} &= \exp\left(-\int_{-\infty}^{x}(Q(\operatorname{ad} J)^{-1}Q)^{\text{diag}}\right) Q \exp\left(\int_{-\infty}^{x}(Q(\operatorname{ad} J)^{-1}Q)^{\text{diag}}\right), \\ \tilde{P} &= \exp\left(-\int_{-\infty}^{x}(Q(\operatorname{ad} J)^{-1}Q)^{\text{diag}}\right) P^{\text{off}} \exp\left(\int_{-\infty}^{x}(Q(\operatorname{ad} J)^{-1}Q)^{\text{diag}}\right). \end{aligned}$$

Here $\tilde{Q}$ and $\tilde{P}$ are both off-diagonal. Note that

$$(Q(\operatorname{ad} J)^{-1}Q)^{\text{diag}} = (\tilde{Q}(\operatorname{ad} J)^{-1}\tilde{Q})^{\text{diag}},$$
$$\|\tilde{Q}\|_{L^2} \le \exp(c\|Q\|_{L^2})\|Q\|_{L^2}, \qquad \|\tilde{P}\|_{L^2} \le \exp(c\|Q\|_{L^2})\|P\|_{L^2},$$
$$|m_{ij}(x, z)| \le |\tilde{m}_{ij}(x, z)| \exp(c\|Q\|_{L^2}),$$
$$\left|\lim_{x\to\infty} \tilde{m}_{ij}(x, z)\right| \le \exp(\|Q\|_{L^2}) \left|\lim_{x\to\infty} m_{ij}(x, z)\right|, \quad \text{where } \|Q\|_{L^2} \equiv \sum_{i,j} \|Q_{ij}\|_{L^2}.$$

If $Q^* = -Q$, then $(Q(\operatorname{ad} J)^{-1}Q)^{\text{diag}}$ is purely imaginary and $|\tilde{m}_{ij}(x, z)| = |m_{ij}(x, z)|$. From now on we write m, Q, P instead of $\tilde{m}$, $\tilde{Q}$, $\tilde{P}$. We may convert (2.1) into an integral equation

$$\text{(2.2)} \qquad \frac{dm_{k,l}(x, z)}{dx} = \begin{cases} \delta_{kl} + \int_{-\infty}^{x} e^{(x-y)z^2 i(d_k - d_l)}[(zQ(y) + P(y))m(x, z)]_{kl}\, dy & \text{for } k \ge l, \\ -\int_{x}^{\infty} e^{(x-y)z^2 i(d_k - d_l)}[(zQ(y) + P(y))m(y, z)]_{kl}\, dy & \text{for } k < l. \end{cases}$$

The first column m_1 of m satisfies a Volterra equation. But the remaining columns m_j of m satisfy Fredholm equations. Following the idea of Beals and Coifman, the wedge products $m_1 \wedge m_2$, $m_1 \wedge m_2 \wedge m_3, \dots, m_1 \wedge m_2 \wedge \cdots \wedge m_{n-1}$ satisfy Volterra equations. If we normalize m at $x = \infty$, then m_n, $m_n \wedge m_{n-1}$, $m_n \wedge m_{n-1} \wedge m_{n-2}, \dots$ also satisfy Volterra equations.

(2.3) **Lemma.** *If $a \in M_n(\mathbf{C})$ is an invertible matrix with columns $a_1, a_2, \dots, a_n$, then a is uniquely determined by the wedge products*

$$a_1, a_1 \wedge a_2, a_1 \wedge a_2 \wedge a_3, \dots, a_1 \wedge a_2 \wedge \cdots \wedge a_n,$$
$$a_n, a_{n-1} \wedge a_n, a_{n-2} \wedge a_{n-1} \wedge a_n, \dots, a_1 \wedge a_2 \wedge \cdots \wedge a_n.$$

Proof. See Beals and Coifman [2].

(2.4) **Lemma.** *Given $Q = (Q_{ij})_{n\times n}$, $Q_{ii} = 0$, $Q_{ij} \in L^2(R)$, $\|Q\|_{L^2} \equiv \sum_{imj} \|Q_{ij}\|_{L^2}$, $c_1 = 0$, $c_i \in \mathbf{C}$, $\operatorname{Re} c_i^2 < 0$, and $|c_i^2| \le K|\operatorname{Re} c_i^2|$, $i \ge 2$, then there are unique absolutely continuous bounded functions u_i satisfying the following ordinary differential equations*:

$$(2.5)\quad \begin{cases} du_1/dx = \sum_{j\neq 1} Q_{1j}u_j, \\ du_i/dx = c_i^2 u_i + (c_i^2 Q_{i1}u_1 + \sum_{j\neq 1,i} c_i Q_{ij}u_j), & i = 2, 3, \dots, n, \\ u_1(-\infty) = 1, \quad u_i(-\infty) = 0, & i \ge 2, \end{cases}$$

and $\|u_1\|_{L^\infty} \le C = C(K, \|Q\|_{L^2})$, $\|u_i\|_{L^2} \le C = C(K, \|Q\|_{L^2})$, $\|u_i/c_i\|_{L^\infty} \le C = C(K, \|Q\|_{L^2})$, $i \ge 2$.

Proof. We convert equation (2.5) into an integral equation

$$(*)\quad \begin{cases} u_1(x) = 1 + \int_{-\infty}^x \left(\sum_{j\neq 1} Q_{1j}\right) u_j, \\ u_i(x) = \int_{-\infty}^x e^{c_i^2(x-y)} \left[c_i^2 Q_{i1}u_1 + \sum_{j\neq 1,i} c_i Q_{ij}u_j\right], & i = 2, 3, \dots, n. \end{cases}$$

At first we look for the solution $u_1 \in L^\infty$, $u_i \in L^2$, $i \ge 2$. Let X be the Banach space $\{u = (u_1, u_2, u_3, \dots, u_n) : u_1 \in L^\infty, u_i \in L^2, i \ge 2\}$, with the norm $|||u||| = \operatorname{Max}_{2\le i\le n}\{\|u_1\|_{L^\infty}, \|u_i\|_{L^2}\}$.

Case 1. Note that

$$h_i(x) = \begin{cases} e^{c_i^2 x}, & x \ge 0 \quad \text{is in } L^1 \cap L^2, \\ 0, & x < 0. \end{cases}$$

and $\|h_i\|_{L^1} = 1/|\operatorname{Re} c_i^2|$, $\|h_i\|_{L^2} = 1/\sqrt{2|\operatorname{Re} c_i^2|}$. If $u_1 \in L^\infty$, $u_i \in L^2$, $i \ge 2$, then $Q_{ij}u_j \in L^1$ for $j \neq 1$, $Q_{i1}u_1 \in L^2$, and

$$\left|\int_{-\infty}^x \left(\sum_{j\neq 1} Q_{1j}\right) u_j\right| \le \sum_{j\neq 1} \|Q_{1j}\|_{L^2}\|u_j\|_{L^2} \le \|Q\|_{L^2}|||u|||;$$

$$\begin{aligned} \|h_i * c_i^2 Q_{1l}u_1\|_{L^2} &\le \|h_i\|_{L^1}|c_i^2|\|Q_{i1}\|_{L^2}\|u_1\|_{L^\infty} \\ &\le (|c_i^2|/|\operatorname{Re} c_i^2|)\|Q_{i1}\|_{L^2}\|u_1\|_{L^\infty} \\ &\le K\|Q_{i1}\|_{L^2}|||u|||; \end{aligned}$$

for $j \neq 1, i$, $Q_{ij}u_j \in L_1$,

$$\begin{aligned}\|h_i * c_i Q_{ij} u_j\|_{L^2} &\le \|h_i\|_{L^2}|c_i|\|Q_{ij}\|_{L^2}\|u_j\|_{L^2} \\ &\le (|c_i|/\sqrt{2|\operatorname{Re} c_i^2|})\|Q_{ij}\|_{L^2}\|u_j\|_{L^2} \\ &\le \sqrt{K/2}\|Q_{ij}\|_{L^2}|||u|||.\end{aligned}$$

Hence $\|h_i * c_i^2 Q_{i1} u_1 + \sum_{j\neq 1,i} h_i * c_i Q_{ij} u_j\|_{L^2} \le (K + \sqrt{K/2})\|Q\|_{L^2}|||u|||$. The integral equation $(*)$ is of the form $u = (1, 0, 0, \dots, 0) + Ku$, where $|||Ku||| \le \alpha\|Q\|_{L^2}|||u|||$ and $\alpha = \operatorname{Max}\{1, K + \sqrt{K/2}\}$. If $\|Q\|$ is small enough such that $\alpha\|Q\| < 1$, we may solve $u \in X$ and $\|u\| \le 1/(1 - \alpha\|Q\|)$; i.e., $\|u_1\|_{L^\infty} \le 1/(1 - \alpha\|Q\|)$, $\|u_i\|_{L^2} \le 1/(1 - \alpha\|Q\|)$, $i \ge 2$. Recall that

$$u_i(x) = h_i * c_i^2 Q_{i1} u_1(x) + \int_{-\infty}^{x} e^{c_i^2(x-y)} \left(\sum_{j\neq 1,i} c_i Q_{ij} u_j\right) \quad \text{for } i \ge 2,$$

$$\begin{aligned}|u_i(x)| &\le \|h_i\|_{L^2}|c_i^2|\|Q_{i1}\|_{L^2}\|u_1\|_{L^\infty} + \sum_{j\neq 1,i} |c_i|\|Q_{ij}\|_{L^2}\|u_j\|_{L^2} \\ &\le (|c_i^2|/\sqrt{2|\operatorname{Re} c_i^2|} + |c_i|)\|Q\|_{L^2}|||u|||,\end{aligned}$$

$$|u_i(x)| \le |c_i|(\sqrt{K/2} + 1)\|Q\|_{L^2}(1/(1 - \alpha\|Q\|)).$$

Hence

$$\|u_i/c_i\|_{L^\infty} \le (\sqrt{K/2} + 1)\|Q\|_{L^2}(1/(1 - \alpha\|Q\|)) \quad \text{for } i \ge 2.$$

Case 2. Let N be a positive integer satisfying $\alpha\|Q\|_{L^2}/N < 1$. There exists a fine number of points $x_0 = -\infty$, x_1, $x_2, \dots, x_M = \infty$ such that $\sum_{i,j}(\int_{x_k}^{x_{k+1}} |Q_{ij}|^2)^{1/2} \le \|Q\|_{L^2}/N$.

By Case 1, the solutions u_i exist up to the point x_1. Then we consider the following equations with initial values at x_1:

$$\begin{cases} u_1(x) = u_1(x_1) + \int_{x_1}^{x} \left(\sum_{j\neq 1} Q_{1j}\right) u_j, \\ u_i(x) = e^{c_i^2(x-x_1)} u_i(x_1) + \int_{x_1}^{x} e^{c_i^2(y-x)} \left[c_i^2 Q_{i1} u_1 + \sum_{j\neq 1,i} c_i Q_{ij} u_j\right], \qquad i \ge 2. \end{cases}$$

since $\alpha \sum_{ij}(\int_{x_1}^{x_2} |Q_{ij}|^2)^{1/2} < \alpha\|Q\|_{L^2}/N < 1$. Again by Case 1 we may extend the solutions u_i to the points x_2. If we continue, we obtain u_i defined on the whole line and $\|u_1\|_{L^\infty} \le C = C(K, \|Q\|_{L^2})$, $\|u_i\|_{L^2} \le C = C(K, \|Q\|_{L^2})$, $\|u_i/c_i\|_{L^\infty} \le C = C(K, \|Q\|_{L^2})$ for $i \ge 2$. We are done.

Let m be the solution of

$$\text{(2.6)} \qquad \begin{aligned} &dm/dx = z^2[J, m] + (zQ + P)m, \qquad Q, P \text{ are off-diagonal}, \\ &m(x, z) \to I \quad \text{as } x \to -\infty,\ z \notin \Sigma. \end{aligned}$$

For Q, Q_x, P, $P_x \in L^1$, m is solved in [8, 9]. Note that $m(x, z) \to \exp(-\int_{-\infty}^{\infty}(Q(\operatorname{ad} J)^{-1}Q)^{\text{diag}})\delta(z)$, $\delta(z) = \operatorname{diag}(\delta_1(z), \delta_2(z), \dots, \delta_n(z))$ as $x \to +\infty$.

(2.7) **Theorem.** *Let* $Q = (Q_{ij})$, $P = (P_{ij})$ *be off-diagonal and of Schwartz class. If* $|z| \geq 1$ *and* $\frac{1}{2} \leq |\operatorname{Re} z|/|\operatorname{Im} z| \leq 2$, *we may control the solution* $m(x,z)$ *in* (2.6) *by the* L^2*-norm of* (Q,P); *i.e.,* $\|m_{ij}(\cdot,z)\|_{L^\infty} \leq C = C(\|Q\|_{L^2}, \|P\|_{L^2}, |\delta_1(z)|, |\delta_2(z)|, \dots, |\delta_n(z)|)$.

Proof. It suffices to consider the case $\operatorname{Im} z^2 > 0$. The first column of m satisfies

$$(2.8)\qquad \begin{cases} m_{11} = 1 + \int_{-\infty}^{x} \sum_{j\neq 1}(zQ_{1j} + P_{1j})m_{j1}, \\ m_{l1} = \int_{-\infty}^{x} e^{z^2(y-x)(id_1 - id_l)} \sum_{j\neq l}(zQ_{lj} + P_{lj})m_{j1}, \qquad l \geq 2. \end{cases}$$

Multiplying by z on both sides of the second equation of (2.8), we have

(2.9)

$$\begin{cases} m_{11} = 1 + \int_{-\infty}^{x} \sum_{j\neq 1}(Q_{1j} + P_{1j}/z)(zm_{j1}), \\ zm_{l1} = \int_{-\infty}^{x} e^{z^2(y-x)(id_1 - id_l)}[(z^2Q_{l1} + zP_{l1})m_{11} + \sum_{j\neq l,1}(zQ_{lj} + P_{lj})(zm_{j1})]. \end{cases}$$

Let $m_{11} = u_1$, $zm_{j1} = u_j$, $j \geq 2$. By Lemma (2.4), we have $\|m_{11}(\cdot,z)\|_{L^\infty} \leq C(\|Q\|_{L^2}, \|P\|_{L^2})$ and $\|zm_{j1}(\cdot,z)\|_{L^\infty} \leq |z|C(\|Q\|_{L^2}, \|P\|_{L^2})$ for $j \geq 2$. Hence $\|m_{j1}(\cdot,z)\|_{L^\infty} \leq C(\|Q\|_{L^2}, \|P\|_{L^2})$ for all j. Let $\Lambda^k(\mathbf{C}^n)$ denote the space of alternating k-forms on $\mathbf{C}^n$, $1 \leq k \leq n$. It has a standard basis $\{e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_k} : 1 \leq j_1 < j_2 < \cdots < j_k \leq n\}$. The wedge product $f_k(\cdot,z) = m_1(\cdot,z) \wedge m_2(\cdot,z) \wedge \cdots \wedge m_k(\cdot,z)$ is a $\Lambda^k(\mathbf{C}^n)$-valued function and satisfies the differential equation:

(2.10)

$$\begin{aligned} \frac{\partial f_k}{\partial k} &= z^2 \sum_{j=1}^{k} m_1 \wedge m_2 \wedge \cdots \wedge m_{j-1} \wedge Jm_j \wedge m_{j+1} \wedge \cdots \wedge m_k \\ &\quad - z^2(id_1 + id_2 + \cdots + id_k)f_k \\ &\quad + \sum_{j=1}^{k} m_1 \wedge m_2 \wedge \cdots \wedge m_{j-1} \wedge (zQ + P)m_j \wedge m_{j+1} \wedge \cdots \wedge m_k. \end{aligned}$$

Let

$$N = C\binom{n}{k} = n!/(k!(n-k)!),$$

$$\{C_1 = 0, C_2, C_3, \dots, C_N\} = \{i(d_{j_1} + d_{j_2} + \cdots + d_{j_k} - d_1 - d_2 - \cdots - d_k) : 1 \leq j_1 < j_2 < \cdots < j_k \leq n\},$$

$$m_1 \wedge m_2 \wedge \cdots \wedge m_k = \sum_{j_1 < j_2 < \cdots < j_k} u_{\sigma\{j_1, j_2, \dots, j_k\}} e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_k}.$$

Here σ is a one-one correspondence from $\{(j_1, j_2, \dots, j_k), 1 \leq j_1 < j_2 < \cdots < j_k \leq n\}$ to $\{1, 2, \dots, N\}$ with $\sigma(1,2,3,\dots,k) = 1$. Then u_i satisfies

$$\frac{du_i}{dx} = z^2 C_i u_i + \sum_{j\neq i}(z\widetilde{Q}_{ij} + \widetilde{P}_{ij})u_j.$$

Let

$$i = \sigma(i_1, i_2, \dots, i_k), \quad \alpha \notin \{i_1, i_2, \dots, i_k\},$$
$$l = \sigma(i_1, i_2, \dots, i_{j-1}, \alpha, i_j, \dots, i_k).$$

Then $\widetilde{Q}_{il} = Q_{i_j\alpha}$, $\widetilde{P}_{il} = P_{i_j\alpha}$, $\widetilde{Q}_{ij} = \widetilde{P}_{ij} = 0$ otherwise. Hence $\|\widetilde{Q}\|_{L^2} \leq \|Q\|_{L^2}$, $\|\widetilde{P}\|_{L^2} \leq \|P\|_{L^2}$. $C_{\sigma\{j_1, j_2, \dots, j_k\}} = i(d_{j_1} + d_{j_2} + \dots + d_{j_k} - d_1 - d_2 - \dots - d_k)$, $j_1 < j_2 < \dots < j_k$, $C_1 = 0$, $\operatorname{Im} C_j > 0$ for $j \geq 2$. We have the same type of equation as the first column of m. Hence $\|u_i(\cdot, z)\|_{L^\infty} \leq C(\|Q\|_{L^2}, \|P\|_{L^2})$ for $|z| \geq 1$. Note that

$$m \to \exp\left(-\int_{-\infty}^{\infty} (Q(\operatorname{ad} J)^{-1} Q)^{\text{diag}}\right) \delta(z), \quad \delta(z) = \operatorname{diag}(\delta_1(z), \delta_2(z), \dots, \delta_n(z))$$

as $x \to \infty$, and we have Volterra equations for m_n, $m_{n-1} \wedge m_n$, $m_{n-2} \wedge m_{n-1} \wedge m_n, \dots$ normalized at $x = \infty$. Since $\det m = 1$,

$$\|m_i(x, z)\| \leq \|m_1(x, z) \wedge m_2(x, z) \wedge \dots \wedge m_i(x, z)\|$$
$$\cdot \|m_i(x, z) \wedge m_{i+1}(x, z) \wedge \dots \wedge m_n(x, z)\|.$$

Finally, we have $\|m_{ij}(\cdot, z)\|_{L^\infty} \leq C(\|Q\|_{L^2}, \|P\|_{L^2}, |\delta_1(z)|, |\delta_2(z)|, \dots, |\delta_n(z)|)$ for $|z| \geq 1$.

(2.11) *Remark.* The existence of m in (2.6) was proved in [8, 9]. The point of Theorem (2.7) is to control $m(x, z)$ by the L^2-norm of (Q, P) for z large.

3. Evolution equations

Suppose the scattering data $v(z, t)$ evolves as

$$(3.1) \quad \begin{cases} dv(z, t)/dt = z^{k-1}[J, v(z, t)], \\ dv(z_j, t)/dt = z_j^{k-1}[J, v(z_j, t)], & k \text{ is an odd positive integer}, \\ & z_j \text{ is fixed for each } j. \end{cases}$$

Assume $v(z, 0) \in \mathrm{SD}$ is generic. Then $v(z, t)$ is generic for t small (since the set of the generic potentials is open in SD) [2, 8, 10]. The corresponding potential $(Q(x, t), P(x, t))$ satisfies the following evolution equation:

$$(3.2) \quad \begin{cases} Q_t = [J, F_k], \\ P_t = [Q, F_k] + [J, F_{k+1}], \end{cases}$$

where $F_k = F_k(Q, P)$ are computed by the recurrence formula

$$dF_k/dx - [P, F_k] = [Q, F_{k+1}] + [J, F_{k+2}], \quad F_0 = J \qquad [8, 10, 12].$$

Let $q = (Q, P)$; equation (3.2) is of the form $q_t(\cdot, t) = F(q(t, \cdot))$.

(3.3) **Definition.** We say that the evolution is dissipative if $\operatorname{Re} \int \operatorname{tr}[q^* F(q)\, dx] \leq 0$ for all q.

Let $\|a\| = \operatorname{tr}(a^* a)^{1/2}$ be a norm of $M_n(\mathbf{C})$. $\|q\|_2^2 = \int \operatorname{tr}(q^*(x,t)q(x,t))\,dx$. (Note that this norm is equivalent to $\|q\|_{L^2} = \sum_{ij}(\int |q_{ij}|^2\,dx)^{1/2}$.)

$$\frac{d}{dt}\int \operatorname{tr}(q^* q)\,dx = 2\operatorname{Re}\int \operatorname{tr}(q^* q_t)\,dx.$$

Therefore if $q(x,t)$ is the solution of a dissipative evolution equation, then $\|q(\cdot,t)\|_2 \le c\|q(\cdot,0)\|_2$.

(3.4) **Theorem.** *If* $(Q(x,0),P(x,0)) = (Q_0,P_0)$ *is a generic potential of Schwartz class and the evolution equation* (3.1) *is "dissipative", then* $(Q(x,t), P(x,t))$ *exists for all* $t \in R$.

Proof. Let m be the eigenfunction associated with (Q,P). Note that $m \to \delta(z) = \operatorname{diag}(\delta_1(z),\delta_2(z),\ldots,\delta_n(z))$ as $x \to \infty$, $\delta(z)$ is invariant under the evolution (3.1) [8, 10], and $\delta(z) \to I$ as $|z| \to \infty$. By Theorem (2.7), $\|m_{ij}(\cdot,t,z)\|_{L^\infty} \le C(\|Q_0\|_{L^2}, \|P_0\|_{L^2})$ for z large.

$$m(x,t,\cdot) = r(x,t,\cdot)m^{\#}(x,t,\cdot)\exp(xz^2J)u(x,t,\cdot)\exp(-xz^2J). \tag{3.5}$$

Let $\operatorname{sd}(Q_0,P_0) = v(z,0)$. Since the generic scattering data is open in SD, there exists $T > 0$, $v(z,t)$ is generic for $t < T$. By the same argument as in [3, 11],

$$\|r(x,t_\nu,z)\| \le c \quad \text{for } x \in R, \quad \tfrac{1}{2} \le |\operatorname{Im} z|/|\operatorname{Re} z| \le 2,$$
$$z \text{ large}, \ t_\nu < T, \ t_\nu \to T.$$

Since $\|m_{ij}(\cdot,t,z)\|_{L^\infty} \le C(\|Q_0\|_{L^2}, \|P_0\|_{L^2})$, $\|r(x,t_\nu,z)\| \le C$ for $t_\nu < T$, z large. Passing to a subsequence, we deduce that $r(x,t_\nu,\cdot) \to r(x,T,\cdot)$, where the residues of $r(x,T,\cdot)$ solve the requisite linear equation at $t = T$. Therefore $v(\cdot,T)$ is generic. Since the set of of generic data is open in SD, the solution $(Q(x,t),P(x,t))$ of equation (3.2) exists for $0 \le t < T + \varepsilon$. Obviously this implies the global existence of equation (3.1).

4. Examples

By a slight modification the argument in §3 works for the case

$$\frac{dm}{dx} = z^2[J,m] + zQm, \quad \text{where} \tag{4.1}$$

$$J = \operatorname{diag}(-Ni, i, i, \ldots, i), \qquad Q = \begin{pmatrix} 0 & q_1 & q_2 & \cdots & q_n \\ r_1 & & & & \\ r_2 & & & & \\ \vdots & & & & \\ r_n & & & & \end{pmatrix}, \qquad r_i = \pm q_i^*.$$

The associated evolution is an N-component derivative nonlinear Schrödinger equation:

$$(q_j)_t = i(q_j)_{xx} + \varepsilon\alpha\left(\sum_k (q_k q_k^*) q_j\right)_x, \qquad \varepsilon = \pm 1, \ j = 1, 2, \ldots, n. \tag{4.2}$$

Since $\frac{d}{dt}\int\sum_j(q_j^* q_j) = 0$, the L^2-norm of Q is invariant under the evolution (4.2). We have global existence for the N-component derivative nonlinear Schrödinger equation if $Q(x, 0)$ is generic and of Schwartz class. For $n = 1$, (4.1) becomes the derivative nonlinear Schrödinger equation (DNLS). The global existence of DNLS was obtained in [11]. Let $\psi = m\exp(xz^2 J)$; then (4.1) becomes

$$d\psi/dx = (z^2 J + zQ)\psi. \tag{4.3}$$

For $n = 2$ Kaup and Newell obtained soliton solutions for the spectral problem (4.3) [7]. Gerzhikov et al. also considered this case [6]. Morris and Dodd considered the two-component derivative nonlinear Schrödinger equation using a larger scattering problem (i.e., $n = 3$) [15]. Sasaki derived a Hamiltonian structure for the evolution (4.2) [16].

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