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HARMONIC MEASURE AND RADIAL PROJECTION

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ABSTRACT. Among all curves in the closed unit disk that meet every radius, there is one, γ_0, whose harmonic measure at the origin is minimal. We give an explicit description of γ_0 and compute its harmonic measure. We also give a quadratically convergent algorithm to compute the harmonic measure of one side of a rectangle at its center.

1. INTRODUCTION

Let W be a domain in the plane and let E be a Borel subset of the boundary of W, ∂W. The harmonic measure of E at $z \in W$ (relative to W) is the solution to the Dirichlet problem in W with boundary values 1 on E and 0 on $\partial W \backslash E$. More precisely, let $\chi_E(\varphi) = 1$ for $\varphi \in E$, $\chi_E(\varphi) = 0$ for $\varphi \in \partial W \backslash E$. Then the harmonic measure at z is

$$w(z, E, W) = \sup\left\{u(z) \colon u \text{ is subharmonic in } W \text{ and } \limsup_{z \to \varphi} u(z) \le \chi_E(\varphi) \text{ for } \varphi \in \partial W\right\}.$$

If F is a Borel subset of the closure of W, the harmonic measure of F at z will mean the harmonic measure of $F \cap \partial(W \backslash F)$ with respect to the component of $W \backslash F$ containing z. See Ahlfors [1] for an introduction to this subject.

Harmonic measure is extremely useful for estimating the growth of analytic and harmonic functions, see Garnett [6]. An early example is the Carleman-Milloux problem [4, 12, 13]: Suppose that f is analytic and $|f(z)| \le M$ in the unit disk D. Suppose further that $|f(z)| \le m$ on a curve γ that connects the origin to ∂D. How large can $|f(z_0)|$ be at a given point z_0? By the two constant theorem [1], $|f(z_0)| \le m^w M^{1-w}$ where $w = w(z_0, \gamma, D \backslash \gamma)$. (Milloux [12] attributes a version of this fact to Carleman.) What is needed, therefore, is a lower bound for w which depends only on the fact that γ connects 0 to ∂D. A more general version of this problem was solved independently by Beurling

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[3] and R. Nevanlinna [14]. Kakutani [9] proved later that $w(z, E, W)$ equals the probability that a Brownian traveler starting from the point z first hits ∂W in the set E. Thus the "further away" the set E is from z, the smaller its harmonic measure. Beurling's result fits this intuitive notion. It says that if $z_0 > 0$, then $w(z_0, \gamma, D\backslash\gamma)$ is minimal when γ is the radius $[-1, 0]$.

We wish to consider a variant of this problem, namely: Suppose that f is analytic and $|f(z)| \leq M$ in D. Suppose further that $|f(z)| \leq m$ on a curve γ that meets every radius. In other words, for each θ, $0 \leq \theta \leq 2\pi$, there is a point $re^{i\theta} \in \gamma$, $0 < r < 1$. How large can $|f(0)|$ be? Again, by the two constant theorem, $|f(0)| \leq m^w M^{1-w}$ where $w = w(0, \gamma, D\backslash\gamma)$. If γ is a closed curve, then by the maximum principle, $|f(z)| \leq m$, i.e., $w = 1$. What is needed, again, is a lower bound for $w(0, \gamma, D\backslash\gamma)$.

Theorem 1. *Suppose γ is a continuum in the closed unit disk $\overline{D}$ that meets every radius. Then the harmonic measure at the origin for γ in $D\backslash\gamma$ is at least $c_0 = .9771266984986656 69\ldots$. This lower bound is achieved only for rotations and reflections of the curve γ_0 given by $\gamma_0 = \gamma^1 \cup \gamma^2$ where γ^1 is the lower half of the unit circle, $\{z : |z| = 1, \text{Im}(z) \leq 0\}$, and γ^2 is the image of the half-hyperbola $x^2/3 - y^2 = 1/4$, $y \leq 0$, $x > 0$, under the linear fractional transformation $(1-z)/(1+z)$, $z = x + iy$ (see Figure 3). The constant c_0 is equal to the harmonic measure at the center of a $1:3$ rectangle for the two long sides.*

A version of Hall's lemma [10] states that if $E \subset D$ and if $E^* = \{e^{i\theta} : re^{i\theta} \in E, \text{ some } r > 0\}$ is the radial projection of E on ∂D, then $w(0, E, D\backslash E) \geq cw(0, E^*, D) = c|E^*|/2\pi$, where $|E^*|$ is the Lebesgue measure of E^*. Unlike Beurling's theorem, $c \neq 1$. Fuchs [2, p. 493] has asked what the optimal constant c is. Our result shows how close c is to 1 in a special case.

In §2, we use extremal length to determine the optimal curve γ_0. In §3, we give a more explicit description of γ_0 and give a quadratically convergent algorithm to compute c_0. In a future paper, with different techniques, we will treat the more general problem where the curve γ meets radii $\{re^{i\theta} : 0 < r < 1\}$ with $0 \leq \theta \leq \alpha < 2\pi$. In [5] it is proved that if $\alpha \leq \pi$ then the minimal harmonic measure is $\alpha/2\pi$. These authors pose the problem of determining the maximal α for which this remains true. Our analysis will yield a computation of this extremal value. In §4, we show that the constant c_0 in Theorem 1 is not the optimal constant c in the problem mentioned in the previous paragraph. In the process of showing $c < c_0$, we give an alternate proof of a result of Hayman [7].

We would like to mention that numerical computations were indispensable at various stages of this project, though the proof of Theorem 1 does not depend upon them. The extremal curve [Figure 3] was drawn using the conformal mapping technique given in Marshall and Morrow [11]. We would like to thank J. Morrow for his assistance. It was only after viewing this picture that we discovered the simple formula for it given in §3. It was because that mapping technique is so well suited to map regions $D\backslash\gamma$ to D, that L. Carleson passed this problem on to us, for which we would like to thank him.

2. Proof of Theorem 1

In the course of the proof of Theorem 1, we will need the notion of extremal length. If F is a family of locally rectifiable arcs in a region U and if ρ is a nonnegative Borel measurable function on U (such a function will henceforth be called a metric), we define the ρ-length of $\phi \in F$ to be

$$L(\phi, \rho) = \int_{\phi} \rho |dz|$$

and the ρ-area of U to be

$$A(U, \rho) = \int_{U} \rho^2 \, dA$$

where dA is the Lebesgue measure on U. The extremal length of F in U is defined to be

$$\lambda_U(F) = \sup_{\rho} \inf_{\phi \in F} \left\{ \frac{L(\phi, \rho)^2}{A(U, \rho)} \right\}.$$

See Ahlfors [1] for an introduction to extremal length. We will use only three elementary facts about extremal length. The first is that it is conformally invariant, i.e., if f is a conformal map of U onto an open set U' and if F' is the image of F then $\lambda_U(F) = \lambda_{U'}(F')$. Indeed, if $w = f(z)$, and if ρ is a metric on U' then $\rho(w)|dw|$ is transformed to the metric $\rho(f(z))|f'(z)||dz|$ on U. The second needed fact is a beautiful criterion due to Beurling; see [1] for the extremality of a metric.

Theorem 2 (Beurling). *A metric ρ_0 is extremal for F if F contains a subfamily F_0 with the following properties*:

$$\text{(1)} \qquad \int_{\varphi} \rho_0 |dz| = \inf_{\phi \in F} L(\phi, \rho_0) \quad \textit{for all } \varphi \in F_0.$$

(2) *For real-valued h in U: if* $\int_{\varphi} h|dz| \geq 0$ *for all* $\varphi \in F_0$ *then* $\int_U h\rho_0 \, dA \geq 0$.

Moreover, in this case, the metric ρ_0 is (a.e. dA) the unique extremal metric, up to multiplication by a positive constant.

The major difficulty in extremal length problems is to discover the extremal metric. Once such a metric is found, Beurling's criterion is usually used to prove it is extremal. We suggest the reader use Beurling's criterion to prove that $\rho \equiv 1$ is the extremal metric for curves that connect opposite sides of a rectangle.

Since the uniqueness portion of this theorem is not explicitly stated in [1], we shall include the proof for completeness.

Let ρ be a metric normalized by

$$\inf_{\varphi\in F} L(\varphi, \rho) = \inf_{\varphi\in F} L(\varphi, \rho_0).$$

Then

$$\int_\varphi \rho|dz| \geq \int_\varphi \rho_0|dz| \quad \text{for all } \varphi \in F_0.$$

Let $h = \rho - \rho_0$. By (2)

$$\int_U \rho_0^2\, dA \leq \int_U \rho_0\rho\, dA.$$

By the Cauchy-Schwarz inequality

$$\int_U \rho_0^2\, dA \leq \int_U \rho^2\, dA.$$

This proves that ρ_0 is extremal. If $\int_U \rho^2\, dA = \int_U \rho_0^2\, dA$, then

$$\int_U (\rho_0 - \rho)^2\, dA = 2\int_U (\rho_0^2 - \rho_0\rho)\, dA \leq 0.$$

Hence $\rho_0 = \rho$ a.e. dA.

The third fact needed in the proof is given in the following elementary lemma.

Lemma 3. *Suppose T is a one-to-one, conformal map (either analytic or anti-analytic) from U onto U such that $T \circ T(z) = z$ on U. Suppose further that the curve family F satisfies $T(F) = F$. To compute the extremal length of the family F, it suffices to consider metrics ρ with $\rho \circ T|T'| = \rho$.*

Proof. Let $\tilde\rho = \rho \circ T|T'|$ and let $\rho_1 = (\rho + \tilde\rho)/2$. Note that

$$\int_U \rho^2\, dA = \int_U \tilde\rho^2\, dA \quad \text{and} \quad \int_{T(\varphi)} \rho|dz| = \int_\varphi \tilde\rho|dz|.$$

This implies that

$$\int_U \rho_1^2\, dA = \int_U \rho^2/2\, dA + \int \rho\tilde\rho/2\, dA \leq \int_U \rho^2\, dA$$

and

$$\int_\varphi \rho_1|dz| \geq \min\left[\int_\varphi \rho|dz|, \int_{T(\varphi)} \rho|dz|\right].$$

Thus $\inf L(\varphi, \rho_1)^2/A(U, \rho_1) \geq \inf L(\varphi, \rho)^2/A(U, \rho)$. Since $T \circ T(z) = z$, we conclude that $\rho_1 \circ T|T'| = \rho_1$.

We will now prove Theorem 1. To obtain a lower bound for $w(0, \gamma, D\backslash\gamma)$, we may suppose, by an approximation, that γ is a piecewise smooth Jordan arc. We will later show there is an extremal curve γ_0 which is piecewise smooth. Indeed, if f is the conformal map of the component of $D\backslash\gamma$ containing 0 (which is necessarily simply connected) onto the disk D, then there is an arc $J \subset \partial D$ such that $f^{-1}(z)$ tends to γ and z approaches the interior of J and $f^{-1}(z)$ approaches ∂D as z approaches the interior of $\partial D\backslash J$. Let $\Gamma = \{z \in$

$D: w(z, J, D) = 1-\varepsilon\}$, and let $\gamma_1 = f^{-1}(\Gamma)$. Clearly γ_1 is smooth, meets each radius and $w(0, \gamma_1, D\backslash\gamma_1) = w(0, \gamma, D\backslash\gamma)/(1-\varepsilon)$. By altering Γ slightly near the ends of J, and rotating D, we may suppose that γ is a smooth map from the interval $[0, 1]$ into $\overline{D}$ with $\gamma(0) > 0$. By taking a subarc of γ, if necessary, we may suppose that γ intersects the interval $(0, 1]$ in exactly two points: $\gamma(0)$ and $\gamma(1)$. By reparameterizing γ, we may suppose $\gamma(0) > \gamma(1)$.

There is another reduction, which comes from [5]. The proof can be completed without it, however it makes the proof a little easier. Let $r = \min\{a \in [-1, 0): a = \gamma(t)$ for some $t\}$, let $E = [-1, r] \cup [\gamma(0), 1]$, and let $W = D\backslash E$. Finally, let k be the conformal map of $D\backslash E$ onto D with $k(0) = 0$ and k real-valued on $(-1, 1)$. Clearly,

$$w(0, k(\gamma), D\backslash k(\gamma)) = w(0, \gamma, D/(\gamma \cup E)) \leq w(0, \gamma, D\backslash\gamma).$$

Note that $k(\gamma)$ meets every radius, and meets ∂D in at least two points. If $\gamma(t_0) \in \partial D$ and $\gamma(t_1) \in \partial D$, $t_0 < t_1$, then we may replace γ on the interval $[t_0, t_1]$ by an arc on ∂D from $\gamma(t_0)$ to $\gamma(t_1)$, resulting in smaller harmonic measure, by the maximum principle. The resulting curve will still meet every radius since γ begins and ends on the interval $(0, 1]$. Replacing γ with $\overline{\gamma} = \{\overline{z}: z \in \gamma\}$, if necessary, we may suppose that $\operatorname{Im}(\gamma(t)) < 0$ for $t < \varepsilon$ and $\operatorname{Im}(\gamma(t)) > 0$ for $t > 1-\varepsilon$. In other words, we may suppose that the curve γ begins at 1, follows the unit circle clockwise at least as far as the point -1, ends at $\gamma(1) > 0$, meets each radius, and crosses the interval $(-1, 1)$ between -1 and 0 (if at all). (See Figure 1.)

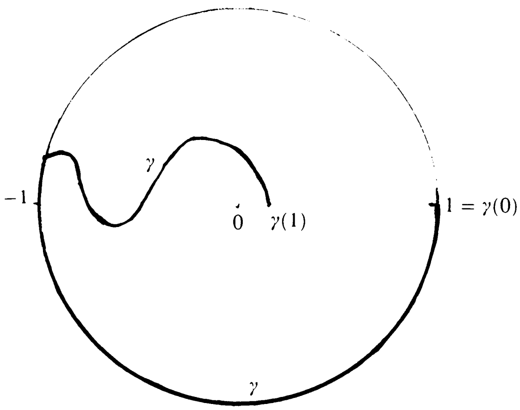

FIGURE 1

Let f be the conformal map of $D\backslash\gamma$ onto D, with $f(0) = 0$, where $f(\partial D\backslash\gamma)$ consists of an arc α on ∂D centered at 1. By the conformal invariance of harmonic measure, and the Poisson integral formula,

$$w(0, \gamma, D\backslash\gamma) = 1 - |\alpha|/2\pi$$

where $|\alpha|$ denotes the length of the arc α. So to minimize $w(0, \gamma, D\backslash\gamma)$, we must maximize $|\alpha|$. Let F denote the family of rectifiable arcs ϕ, defined for

$0 \le t \le 1$, with $\phi(0) \in \alpha$, $\phi(1) \in \alpha$, $\phi(t) \in D\backslash\{0\}$ for $0 < t < 1$ and such that 0 and -1 are not in the same component of $\overline{D}\backslash(\phi \cup \alpha)$. In other words, each ϕ encloses 0 and begins and ends on α. Clearly if we increase α, we will increase the family F and hence decrease the corresponding extremal length. Let G_γ denote the family $\{f^{-1}(\phi): \phi \in F\}$. These are curves that separate 0 from γ and begin and end on $\partial D\backslash\gamma$. By the conformal invariance of λ, our problem is to find the curve that minimizes $\lambda_{D\backslash\gamma}(G_\gamma)$.

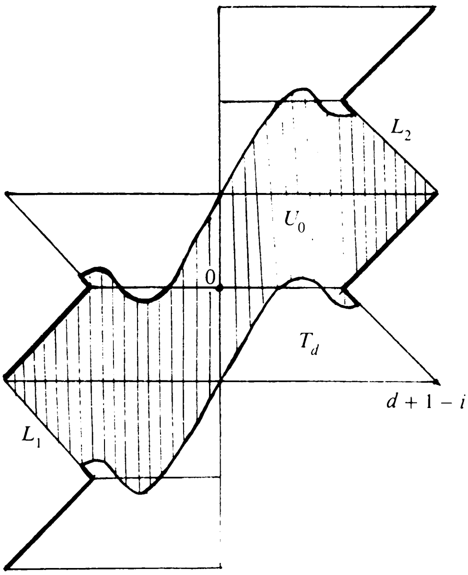

FIGURE 2

Let T_d be the trapezoid with vertices $v_1 = 0, v_2 = d > 0$, $v_3 = d+1-i$, and $v_4 = -i$. There is a unique number $d > 0$ and corresponding conformal map h_d with maps T_d onto the upper half disk $D^+ = \{z: |z| < 1 \text{ and } \mathrm{Im}(z) > 0\}$ with $h(v_1) = 0$, $h(v_2) = -1$, $h(v_3) = +1$ and $h(v_4) = \gamma(1)$. Let S_1 be the union of T_d together with its reflection about the line $\mathrm{Im}(z) = 0$. Let S_2 be the reflection of S_1 about the line $\mathrm{Im}(z) = 1$, let S_3 be the reflection of S_1 about the line $\mathrm{Re}(z) = 0$ and let S_4 be the reflection of S_3 about the line $\mathrm{Im}(z) = -1$. Finally, let $S = S_1 \cup S_2 \cup S_3 \cup S_4$ (see Figure 2). The region S is the union of 8 trapezoids congruent to T_d. The map h_d extends analytically to S by the Schwarz reflection principle. Let U_0 be the component of $\mathrm{int}(S)\backslash\{z \in S: h(z) \in \gamma\}$ containing the origin, where $\mathrm{int}(S)$ is the interior of S. Since γ crosses $(-1, 1)$ only between -1 and 0, the boundary of U_0 consists of the two curves $\{z \in S: h(z) \in \gamma\}$ together with two line segments L_1 and L_2 with $L_1 \in \partial S_4$ and $L_2 \in \partial S_2$, each with slope -1. On the region $\overline{U}_0$, the map h is a two-to-one cover of $D\backslash\gamma$. To see this, note that h is two-to-one on $S_1 \cup S_3$, and that if $z \in S$ and $z \pm 2i \in S$, then $h(z) = h(z \pm 2i)$. Moreover, if $z \in U_0$, exactly one of the points $z, z+2i, z-2i$ is in $S_1 \cup S_3$, call it $u(z)$. The function u extends to a one-to-one map of $\overline{U}_0$ onto $S_1 \cup S_3$.

Thus h is two-to-one; in fact, $h(-z) = h(z)$. Thus for each $\phi \in F$, there is a rectifiable arc ψ in U_0 with endpoints on $L_1 \cup L_2$ that either "encloses" the origin or else connects L_1 and L_2. Let H_d be the family in U_0 of all such curves. To compute the length of the family H_d, by Lemma 3, we need only consider metrics ρ on U_0 with $\rho(-z) = \rho(z)$. Any such metric can be written in the form $\rho(z) = q(h_d(z))|h_d'(z)|$ where q is a metric on $D\backslash\gamma$. Clearly $A(U_0, \rho) = 2A(U_0, q)$ and $L(\psi, \rho) = L(h_d(\psi), q)$, so that $\lambda_{U_0}(H_d) = \lambda_{D\backslash\gamma}(G_\gamma)/2$. Thus we wish to minimize $\lambda_{U_0}(H_d)$.

Let $\rho \equiv 1$ on U_0. The ρ-area of T_d is $d + 1/2$, so by use of the above map u, $\int_{U_0} \rho\, dA = 4(d + 1/2)$. Note that the distance from L_1 to the line through the origin with slope -1 equals $\sqrt{2} + d/\sqrt{2}$. Thus every curve $\psi \in H_d$ has length

$$\int_\psi |dz| \geq 2\left(\sqrt{2} + \frac{d}{\sqrt{2}}\right).$$

We conclude

$$\lambda_{U_0}(H_d) \geq \frac{[2(\sqrt{2} + d/\sqrt{2})]^2}{4(d + 1/2)} \geq 3.$$

The latter inequality is strict, unless $d = 1$. Now let $d = 1$ and let σ be the straight line from v_2 to v_4. Let τ be the bottom half of the unit circle $= \{z : |z| = 1 \text{ and } \operatorname{Im}(z) \leq 0\}$. Let $\gamma_0 = \tau \cup h(\sigma)$ (see Figure 3). The corresponding U_0 is then clearly a 3 by 1 rectangle whose sides have slope $+1$ and -1 (see Figure 4). Let H_0 consist of the line segments of slope 1 connecting opposite sides of U_0. By Beurling's criterion, $\rho_0 \equiv 1$ on U_0 is the unique extremal metric for this family and hence $\lambda_{U_0}(H_0) = 3$. This proves that $\min \lambda_{D\backslash\gamma}(G_\gamma) = 6$ and that γ_0 is extremal. Moreover, any other extremal curve would necessarily have $d = 1$. The proof also shows that the Euclidean metric on U_0 must be extremal. By Beurling's criterion, the extremal metric always comes from the conformal map of U_0 onto a rectangle. Hence U_0 must be a rectangle in the extremal case.

So far, we have supposed that the candidates for the extremal curve are piecewise smooth. We would now like to prove that γ_0 is the only continuum in $\overline{D}$, meeting every radius, for which the harmonic measure is minimal. Let γ be such a continuum. Then γ may be approximated, as indicated above, by curves γ_n, $n = 1, 2, \ldots$, with corresponding regions $U_0(\gamma_n)$ and conformal maps h_n of $U_0(\gamma_n)$ onto $D\backslash\gamma_n$. Let k_n be the conformal map of $U_0(\gamma_n)$ onto a rectangle R_n so that $\rho_n = |k_n'|$ is the extremal metric for $\lambda_{U_0(\gamma_n)}(H_{d_n})$. Since $\lambda_{U_0(\gamma_n)}(H_{d_n}) = 3$, we may choose a normalization for k_n so that R_n converges to the 3 by 1 rectangle $U_0(\gamma_0)$. Thus there is a subsequence $\{k_{n_j}^{-1}\}$ of $\{k_n\}$ converging uniformly on compact subsets of $U_0(\gamma_0)$ to the map $k(z) = z$. We conclude that the corresponding maps h_{n_j} must converge uniformly on compact subsets of U_0 to the map h_0 of U_0 onto $D\backslash\gamma_0$. Thus γ_0 is the unique extremal continuum. We note that the curve in Figure 3 beginning at

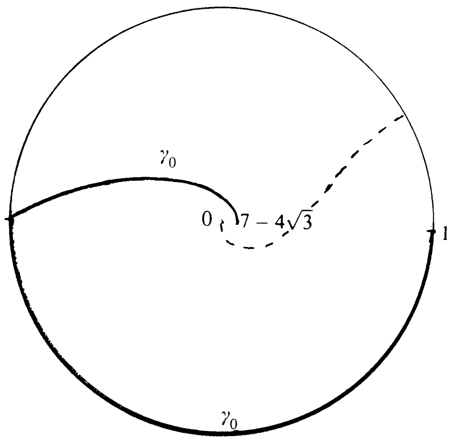

FIGURE 3

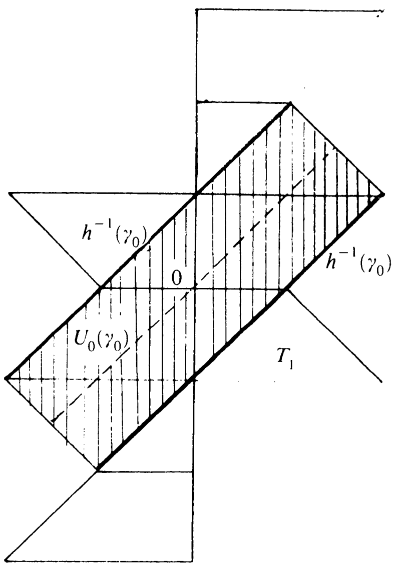

FIGURE 4

the origin and extending to ∂D is, in some sense, the most efficient path from the origin to $\partial D\backslash\gamma$. It is the image under the map h of the straight line in U_0 through 0 and with slope 1.

We remark here that our discovery of the extremal metric was motivated by the deep work of Jenkins [8]. It is not hard to show that Jenkins' theorem implies that the extremal curve (suitably normalized) must consist of an arc from 1 to a point $e^{i\theta}$ on $|z| = 1$ then an arc in D from $e^{i\theta}$ to a point $x \in D$,

$x > 0$. Moreover, the extremal metric is given by

$$\rho^2 = c\frac{(z - e^{i\theta})}{z(z-1)(z-x)(z-1/x)}$$

where c is a constant. Jenkins show that for each x there is a unique θ so that this metric corresponds to the curve, of minimal harmonic measure, that ends at x and meets every radius. He does not have a formula for θ in terms of x. The difficulty is in deciding which x corresponds to the extreme case in our problem. It may be instructive to note that the trapezoids T_d that we constructed do *not* correspond to these metrics, unless $d = 1$. Indeed, one can show that $e^{i\theta} \neq -1$ if x is not the endpoint of our external γ_0, yet the metrics associated with the rectangles formed from T_d will have a singularity at -1. We do not know of any "nice" conformal maps corresponding to the metrics of Jenkins. We will reexamine Jenkins' metrics in another paper, as mentioned in the introduction.

3. The extremal curve

Let γ_0 be the extremal curve in §2, let h be the associated map of the rectangle U_0 onto $D\backslash\gamma_0$, and let f be the conformal map of $D\backslash\gamma_0$ onto D with $f(0) = 0$, and with $f(\partial D)$ equal to an arc α on ∂D centered at 1. If φ_1 is a conformal map of D onto U_0 with $f \circ h \circ \varphi_1(0) = 0$ then $\phi = f \circ h \circ \varphi_1$ is a two-to-one map of D onto D with $\phi(0) = \phi'(0) = 0$. An elementary argument shows that $\phi(z) = cz^2$ where c is a constant. Hence $\sqrt{f \circ h}$ is a conformal map of the rectangle U_0 onto D, which maps the four vertices to points $\beta, \overline{\beta}, -\beta, \overline{\beta}$. Note that the endpoints of the arc α are β^2 and $\overline{\beta}^2$, and hence $w(0, \gamma_0, D\backslash\gamma_0) = 1 - 2|\theta|/\pi$ where $\beta = e^{i\theta}$. Note that U_0 has side ratio $3:1$. The quantity $|\theta|/\pi$ is the harmonic measure, at the center, of one of the short sides of a $3:1$ rectangle. By the Schwarz-Christoffel formula, the number θ is the solution to the integral equation $(0 < \theta < \pi/4)$:

$$3\int_0^\theta \left[\frac{1}{\cos 2s - \cos 2\theta}\right]^{1/2} ds = \int_\theta^{\pi/2} \left[\frac{1}{\cos 2\theta - \cos 2s}\right]^{1/2} ds.$$

The solution of this equation by standard methods of numerical integration, while possible, is not trivial and converges slowly. Trefethen [15] gives a rapidly converging algorithm to compute θ based on a different method. An algorithm to compute θ seems to be of some numerical interest. For example, Trefethen [16] has shown the use of θ in the design of a resistor. This is equivalent to finding the parameter k, $0 \leq k \leq 1$, for which the Jacobi elliptic function $sn(z, k)$ maps a 1 by 3 rectangle onto the upper half-plane. We give here a method that is very simple and converges even faster (quadratic convergence) than Trefethen's method. It is based on the following observation. Suppose R_L is a rectangle centered at the origin with sides parallel to the axes, with height 1 and length $L \geq 1$. Let S_L be the semi-infinite strip $\{z\colon \mathrm{Re}(z) > -L/2, |\mathrm{Im}(z)| < 1/2\}$. Let $w_{R_L}(z)$ be the harmonic measure of the left-hand edge of R_L, $\{z \in \overline{R}_L \colon \operatorname{Re}(z) = -L/2\}$, at the point z and let $w_{S_L}(z)$ be the harmonic measure of the left edge of S_L, $\{z \in \overline{S}_L \colon \operatorname{Re}(z) = -L/2\}$, at the point z. Clearly, $w_{R_L}(z) \le w_{S_L}(z)$ by the maximum principle. On $\operatorname{Re}(z) = L/2$, $w_{S_L}(z) \le w_{S_L}(L/2)$. Thus for $z \in \partial R_L$

$$w_{S_L}(z) + w_{S_L}(-z) \le \{1 + w_{S_L}(L/2)\}\{w_{R_L}(z) + w_{R_L}(-z)\}.$$

By the maximum principle, this inequality holds at $z = 0$. We conclude that

$$\frac{w_{S_L}(0)}{1 + w_{S_L}(L/2)} \le w_{R_L}(0) \le w_{S_L}(0). \tag{1}$$

Note that $w_{S_L}(L/2) = w_{S_{2L}}(0)$. By explicitly mapping S_L onto the upper-half plane and using elementary trigonometric identities, it is easy to show that

$$w_{S_L}(0) = \tfrac{4}{\pi} \tan^{-1}(e^{-\pi L/2}).$$

Hence

$$\frac{\frac{4}{\pi}\tan^{-1}(e^{-\pi L/2})}{1 + \frac{4}{\pi}\tan^{-1}(e^{-\pi L})} \le w_{R_L}(0) \le \tfrac{4}{\pi}\tan^{-1}(e^{-\pi L/2}) \tag{2}$$

Note that (2) says $w_{R_L}(0) \sim \frac{4}{\pi} e^{-\pi L/2}$ as L tends to ∞. We can improve this estimate in the following way. The quantities $w_{R_L}(0)$ and $w_{R_{2L}}(0)$ are related as follows: Let φ_L be the conformal map of R_L onto D so that the images of the four vertices are $e^{i\theta_L}, e^{-i\theta_L}, -e^{i\theta_L}, -e^{-i\theta_L}$, for some number θ_L, $0 < \theta_L < \pi/2$, and $\varphi_L'(0) > 0$. Note that φ_L is real on $\mathbf{R} \cap R_L$ and $\varphi_L(0) = 0$. Moreover, $w_{R_L}(0) = \theta_L/\pi$. Note that φ_L maps the top half of R_L onto the upper half disk D^+. Let χ be the conformal map of D^+ onto D so that $\chi(e^{i\theta_L}) = \overline{\chi(1)} = -\chi(-1) = -\overline{\chi(-e^{-i\theta_L})}$. The top half of R_L is clearly the image under the map $z \mapsto (z + i/2)/2$ of R_{2L}. Hence $\varphi_{2L}(z) = \chi(\varphi_L((z + i/2)/2))$ and hence $e^{i\theta_{2L}} = \chi(e^{i\theta_L})$. An explicit computation of the map χ^{-1} shows that

$$\sin \pi w_{R_L}(0) = \frac{2\sqrt{\sin \pi w_{R_{2L}}(0)}}{1 + \sin \pi w_{R_{2L}}(0)}.$$

Hence, if we let

$$x_n = (\sin \pi w_{R_{2^n L}}(0))^{-1/2}$$

we have

$$x_{n-1}^2 = \tfrac{1}{2}(x_n + 1/x_n).$$

By (1)

$$0 \le \left[\frac{1}{w_{R_{2^n L}}(0)} - \frac{1}{w_{S_{2^n L}}(0)}\right] \le \frac{w_{S_{2^{n+1} L}}(0)}{w_{S_{2^n L}}(0)}.$$

By elementary calculus, if

$$y = (\sin \pi w_{S_{2^n L}}(0))^{-1/2}$$

then $0 \le x_n - y \le c(e^{-\pi L/2})^{2^n}$, for some constant c. Note that $x_n > x_{n-1} > 1$ for all n, and that if $y > 1$,

$$|x_{n-1} - \sqrt{\tfrac{1}{2}(y + \tfrac{1}{y})}| \le c|x_n - y|$$

for some constant c ($c = \frac{1}{4}$ will suffice). Thus if we let $y_0 = y$ and $y_k = \sqrt{\frac{1}{2}(y_{k-1} + \frac{1}{y_{k-1}})}$, $1 \le k \le n$, then by induction

$$|(\sin \pi w_{R_L}(0))^{-1/2} - y_n| \le C(e^{-\pi L/2})^{2^n} \tag{3}$$

where C is a constant which can be determined by elementary calculus. In other words, the number of correct digits in the decimal expansion at least doubles when n is increased by 1. To simplify the algorithm further, note that

$$y_0 = \left(\frac{1 + e^{-\pi 2^n L}}{2}\right) e^{\pi 2^{n-2} L}$$

and hence

$$\left| y_0 - \frac{1}{2} e^{\pi 2^{n-2} L} \right| < \frac{e^{-\pi 2^n L}}{2}.$$

Thus we may replace y_0 with $\frac{1}{2}e^{\pi 2^{n-2} L}$ and obtain the estimate (3) above with slightly larger constant C. Finally

$$|w_{R_L}(0) - \tfrac{1}{\pi} \sin^{-1}(y_n^{-2})| \le C(e^{-\pi L/2})^{2^n}$$

for a slightly larger constant C. Notice that

$$\sin^{-1}(y_n^{-2}) = \tan^{-1}\left(\frac{2}{y_{n-1} - 1/y_{n-1}}\right).$$

Thus we obtain the following quadratically convergent algorithm:

Given n and L, let

$$y_0 = \tfrac{1}{2} e^{\pi L 2^{n-2}},$$
$$y_k = \sqrt{\tfrac{1}{2}(y_{k-1} + 1/y_{k-1})}, \qquad 1 \le k \le n-1,$$

and

$$w = \tfrac{1}{\pi} \tan^{-1}\left(\frac{2}{y_{n-1} - 1/y_{n-1}}\right).$$

Then

$$|w_{R_L}(0) - w| \le C(e^{-\pi L/2})^{2^n}.$$

For example, with $L = 3$ and $n = 2$ the computed value for $c_0 = 1 - 2w = .977126698498665669\ldots$ is correct to 18 decimal places. This is virtually a formula for c_0. We carried out the above estimate more precisely to obtain an error less than 10^{-17} for any $L \geq 1$, when $n = 4$. Of course, for large L one could (and should) reduce the number of steps. The astute reader can use the above technique to give a quadratically convergent algorithm to compute the conformal map of D to any rectangle if he so desires.

The optimal curve γ_0 (see Figure 3) was drawn using the conformal mapping technique given in Marhsall and Morrow [11]. Although the map from the trapezoid T_1 to the upper half disk cannot be written in terms of elementary conformal maps, the picture of γ_0 led us to believe that there might be a simple formula for γ_0. Recall that T_1 is the trapezoid with vertices $0, 1, 2 - i$ and $-i$. Let S_1 be the square with vertices $1, 2 - i, 1 - 2i$ and $-i$. Let f_1 be the conformal map of S_1 onto D such that $f_1(2 - i) = 1, f_1(1) = i, f_1(-i) = -1, f_1(1 - 2i) = -i$. Then f_1 maps $S_1 \cap T_1$ onto the upper half disk D^+ and the right triangle T_2 with vertices $-i, 1$ and $1 - i$ onto the left quarter circle $Q = \{z\colon |z| < 1, \mathrm{Re}(z) < 0, \mathrm{Im}(z) > 0\}$. Reflecting T_2 about the line $\mathrm{Re}(z) = \mathrm{Im}(z)$ and Q about the quarter circle on its boundary, we obtain a conformal map of T_1 onto the union of D^+ and $\{z \in \mathbf{C}\colon \mathrm{Re}(z) < 0, \mathrm{Im}(z) > 0\}$. Applying the conformal map $f_2 = -1/z^2$ to this region, we obtain a conformal map to $\mathbf{C}\backslash(\overline{D^+} \cup (-\infty, -1]$. Apply the conformal map $f_3(z) = (1+z)/(1-z)$ we obtain the region $C\backslash(\{z\colon \mathrm{Re}(z) \geq 0$ and $\mathrm{Im}(z) \geq 0\} \cup [-1, 0]$. Applying the map $f_4(z) = (-iz)^{2/3}$ we obtain the region $\{z\colon \mathrm{Im}(z) > 0\}\backslash L_3$ where L_3 is a straight line segment from the origin to the point $e^{i\pi/3}$. The function $g_1(z) = 3(z - 1/3)^{1/3}(z + 2/3)^{2/3}/2^{2/3}$ maps the upper half plane conformally onto this latter region (see Marshall and Morrow [11] for details). Note that $g_1(-1) = -1$. Let

$$g_2(z) = (1 - 4z^2)/3 \quad \text{and} \quad g_3(z) = (1 - z)/(1 + z).$$

The upper half-plane is the image of the quarter plane $Q = \{z\colon \mathrm{Re}(z) > 0, \mathrm{Im}(z) < 0\}$ under the map g_2 and $Q = g_3(D^+)$. Thus $h = g_3^{-1} \circ g_2^{-1} \circ g_1^{-1} \circ f_4 \circ f_3 \circ f_2 \circ f_1$ is the conformal map of T_1 onto D^+ such that $h(0) = 0$, $h(1) = -1$ and $h(2-i) = 1$. A direct computation shows that $h(-i) = 7 - 4\sqrt{3}$. The curve $\gamma_0 \cap D$ is the image of the straight line segment L_4 from $-i$ to 1 in T_1, under the map h. To find this curve more explicitly, first note that the image of L_4 under the map $f_4 \circ f_3 \circ f_2 \circ f_1$ is the ray $\{re^{i2\pi/3}\colon 0 < r < \infty\}$. To find the image of this curve under the map $g_2^{-1} \circ g_1^{-1}$, we must find all $z \in Q$ such that $\{g_1 \circ g_2(z)\}^3$ is positive. In other words, $4z^2(1 - \frac{4}{3}z^2)^2 < 0$. Taking a square root, we want $\mathrm{Re}(4z(z^2 - 3/4)) = 0$. Let $z = \cos(u) = (e^{iu} + e^{-iu})/2$. Then $4z(z^2 - 3/4) = \cos(3u) = \cos 3x \cosh 3y + i \sin 3x \sinh 3y$. Thus we seek those u for which $\mathrm{Re}(e^{3iu}) = 0$. Hence $e^{iu} = re^{i\theta}$ where $\theta = \pm\pi/6, \pm\pi/2, \pm 5\pi/6$, $r > 0$. Since $z = \cos\theta(r + 1/r)/2 + i \sin\theta(r - 1/r)/2 \in Q$, we may write $z = \sqrt{3}(r + 1/r)/4 + i(r - 1/r)/4$ where $0 < r < 1$. If we write $z = x + iy$ then $4x^2/3 - 4y^2 = 1$ and $y < 0$. Finally we compute the image of this half-hyperbola under the map $g_3^{-1} = (1-z)/(1+z)$. In terms of r it is

$$\frac{4r - \sqrt{3}(r^2+1) - i(r^2-1)}{4r + \sqrt{3}(r^2+1) + i(r^2-1)}.$$

Thus the optimal curve γ_0 begins at 1, follows the unit circle clockwise to the point -1, then enters the unit disk D, making an angle of $\pi/6$ with the positive x-axis, continues in the upper half disk D^+, until it meets $\mathbf{R}$ at right angles at the point $7 - 4\sqrt{3} \cong .0718$.

4. On Fuchs' problem

In this section, we show that if we remove the hypothesis that γ is connected, the constant c_0 is no longer the lower bound for the harmonic measure at the origin. The proof below is motivated by the example in [7].

Suppose E is a Borel subset of the closure of the right half-plane, $\overline{R}$, and suppose E contains a free boundary arc on the imaginary axis. In other words, there exists a open disk $D(ia, r)$ with center at ia, a real, and radius $r > 0$ so that $D(ia, r) \cap E = \{it: a - r < t < a + r\}$. If $0 < \delta < r$, let l_1 denote the line segment from $i(a+\delta)$ to $\delta + ia$, let l_2 denote a line segment from $i(a+\delta/2)$ to $i(a+\delta)$, and let $E_\delta = E \cup l_1 \backslash l_2$. In the case when E is the imaginary axis, I, and $a = 0$, we denote this set by I_δ. The idea of the argument below is that $w(z, I, R) \equiv 1$ and for z fixed $w(\delta z, I_\delta, R) = c < 1$, where c is independent of δ. So for an arbitrary set E, at a point in R near ia, with δ extremely small, E "looks like" $I + ia$ and E_δ "looks like" $I_\delta + ia$, and thus $w(z, E_\delta, R\backslash E_\delta)$ should be less than $w(z, E, R\backslash E)$.

Proposition. *For each* $z \in R\backslash E$, *there is a* $\delta_0 > 0$ *so that* $w(z, E_\delta, R\backslash E_\delta) < w(z, E, R\backslash E)$ *when* $\delta < \delta_0$.

Proof. Without loss of generality, $a = 0$. By the maximum principle, it suffices to show the inequality on $R \cap \{z: |z| = 3\delta\}$. Let $\sigma_\varepsilon = \{e^{i\theta}: |e^{2i\theta} + 1| < \varepsilon, -\pi/2 \le \theta \le \pi/2\}$ and let $\sigma_0 = \{e^{i\theta}: |e^{i\theta} - 1| < 1/2\}$. Recall that I_1 is the set formed from the imaginary axis with $a = 0$ and $\delta = 1$, by the process described above. Let

$$\varepsilon_1 = \min_{e^{i\theta} \in \sigma_0} (1 - w(2e^{i\theta}, I_1, R\backslash I_1)).$$

By using an explicit conformal map of $R\backslash D(0,2)$ onto R, it is not hard to show that

$$w(z, 2\sigma_\varepsilon, R\backslash D(0,2)) < \tfrac{1}{3}\varepsilon_1 w(z, 2\sigma_0, R\backslash D(0,2)) \tag{4}$$

when $|z| = 3$, and ε is sufficiently small.

Now for $z = 2\delta e^{i\theta}$ with θ fixed, $w(z, E, R\backslash E) \to 1$ as $\delta \to 0$ and $w(z, E_\delta, R\backslash E_\delta) \to w(2e^{i\theta}, I_1, R\backslash I_1) < 1$. Thus we may choose $\delta > 0$ so that when $z = 2\delta e^{i\theta}$ with $e^{i\theta} \in \sigma_0$,

$$w(z, E, R\backslash E) - w(z, E_\delta, R\backslash E_\delta) > \tfrac{1}{2}\varepsilon_1$$

and so that when $z = 2\delta e^{i\theta}$ with $e^{i\theta} \in \partial D(0,1)\backslash\sigma_\varepsilon$

$$w(z, E, R\backslash E) - w(z, E_\delta, R\backslash E_\delta) > 0.$$

Thus for $|z| = 2\delta$

$$\begin{aligned} &w(z, E, R\backslash E) - w(z, E_\delta, R\backslash E_\delta) \\ &\quad \geq -w(z, 2\delta\sigma_\varepsilon, R\backslash D(0, 2\delta)) + \tfrac{1}{2}\varepsilon_1 w(z, 2\delta\sigma_0, R\backslash D(0, 2\delta)). \end{aligned}$$

We conclude that for $|z| = 3\delta$, by (4) and the above,

$$w(z, E, R\backslash E) - w(z, E_\delta, R\backslash E_\delta) \geq (\tfrac{1}{2}\varepsilon_1 - \tfrac{1}{3}\varepsilon_1) w(z, 2\delta\sigma_0, R\backslash D(0, 2\delta)) > 0.$$

This proves the proposition.

An easy consequence of this proposition is the following result of Hayman [7]. If E is a Borel subset of the right half-plane R, let $E^* = \{iy : y > 0$ and $e^{i\theta} y \in E$ for some $\theta\}$.

Corollary 1. *There is a set* F *with* $F^* = \{iy : y \geq 0\}$ *and* $w(1, F^*, R) = \frac{1}{2} > w(1, F, R)$.

To see this, apply the proposition with $F = \{I_\delta + i\}\backslash\{iy : y < 0\}$ for δ sufficiently small. We remark that this example doesn't really depend on the exact nature of the set I_δ near 0.

If we let E denote the image in R of the extremal curve given in the previous section, under the linear fractional transformation $(1+z)/(1-z)$, and then apply the proposition, we obtain:

Corollary 2. *If we do not assume that the set* γ *in the statement of Theorem* 1 *is connected, then the harmonic measure of* γ *at the origin can be smaller than* c_0.

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