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A NOTE ON LOCAL CHANGE OF DIFFEOMORPHISM

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ABSTRACT. Let $D(M)$ be the group of pseudo-isotopy classes of orientation preserving diffeomorphisms of a compact manifold M which restrict to the identity on ∂M. If a compact manifold N of the same dimension as M is embedded in M, then extending maps in $D(N)$ as the identity on the exterior of N defines a homomorphism $E: D(N) \to D(M)$. We ask if the kernel of E is finite and show that this is the case for special cases.

INTRODUCTION

Two diffeomorphisms f_0 and f_1 of a smooth manifold M, which are the identity on the boundary ∂M of M if ∂M is nonempty, are called pseudo-isotopic (or concordant) if there is a diffeomorphism F of $M \times I$ $(I = [0, 1])$ such that $F(x, 0) = (f_0(x), 0)$, $F(x, 1) = (f_1(x), 1)$ for all $x \in M$ and that F is the identity on $\partial M \times I$. The set $D(M)$ of pseudo-isotopy classes of orientation preserving diffeomorphisms of M forms a group under composition of maps.

It is interesting by itself to compute $D(M)$ and moreover it is sometimes related to geometrical problems, e.g. when $M = D^n$ or S^n [KM], $S^p \times S^q$ [B, L], $\mathbf{C}P^k \times D^q$ [BP]. The group $D(M)$ is well understood for some M, but in general it seems not so well understood. For instance, the author does not know a higher dimensional example of M such that $D(M)$ is trivial.

Let N be a compact manifold embedded in M of the same dimension. Then there is defined a homomorphism $E: D(N) \to D(M)$ by extending a map in $D(N)$ as the identity on the exterior of N. Then it is natural to ask

Question. Is $\ker E$ trivial or finite?

An interesting case is when N is an n-disk D^n. As is well known $D(D^n)$ is isomorphic to the Kervaire-Milnor group θ_{n+1} of oriented homotopy $(n+1)$-spheres $(n \geq 4)$ and the group is nontrivial in general if $n \geq 6$. Thus if $\ker E$ is trivial for $N = D^n$, then one can conclude that $D(M)$ is nontrivial in general.

In this paper we consider the case where $N = \mathbf{C}P^k \times D^q$. The group $D(\mathbf{C}P^k \times D^q)$ is fairly well understood by Browder-Petrie [BP] in connection

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with the study of semifree circle actions on homotopy spheres. The purpose of this paper is to prove the following, which gives an evidence supporting the above question.

Theorem A. *Let X be a closed orientable manifold of dimension $q \geq 2$ such that $H^1(X;\mathbf{Z}) = 0$. Then the kernel of*

$$E = E_X : D(\mathbf{C}P^k \times D^q) \to D(\mathbf{C}P^k \times X)$$

is finite, where D^q is any q-disk embedded in X.

Remark. The map E_X depends on the choice of an embedding of D^q into X in general. However the disk theorem tells us that it only depends on a connected component of X into which D^q is embedded and on whether the embedding preserves orientation or not (we fix an orientation on D^q).

The group $D(\mathbf{C}P^k \times D^q)$ is finitely generated abelian. The rank $r_{k,q}$ of the free part is explicitly computed [BP]. In fact, if q is even $r_{k,q} = 0$, i.e. $D(\mathbf{C}P^k \times D^q)$ is finite; so Theorem A is trivial in this case. If q is odd, $r_{k,q}$ is nonzero in most cases. In fact, it is given by

$$r_{k,q} = [k/2] + a_{k,q} + b_{k,q}$$

where

$$a_{k,q} = \begin{cases} 1 & \text{if } k \text{ is odd and } q+1 \equiv 0\ (4), \\ 0 & \text{otherwise}, \end{cases}$$
$$b_{k,q} = \begin{cases} 1 & \text{if } 3 \leq q \leq 2k+1\ (q : \text{odd}), \\ 0 & \text{otherwise}. \end{cases}$$

Let $\operatorname{Diff}_+ M$ be the group (with C^∞ topology) of orientation preserving diffeomorphisms of M which restrict to the identity on ∂M. The connected components $\pi_0(\operatorname{Diff}_+ M)$ are nothing but the isotopy classes of those diffeomorphisms; so there is a natural epimorphism $\Pi: \pi_0(\operatorname{Diff}_+ M) \to D(M)$.

Similarly to E_X, a homomorphism

$$E'_X : \pi_0(\operatorname{Diff}_+ \mathbf{C}P^k \times D^q) \to \pi_0(\operatorname{Diff}_+ \mathbf{C}P^k \times X)$$

is defined. Clearly Π commutes with E_X and E'_X. According to Cerf [C] $\Pi: \pi_0(\operatorname{Diff}_+ \mathbf{C}P^k \times D^q) \to D(\mathbf{C}P^k \times D^q)$ is an isomorphism (when $\dim \mathbf{C}P^k \times D^q = 2k+q \geq 5$) because $\mathbf{C}P^k \times D^q$ is simply connected. Hence we have

Corollary B. *Let X be the same as in Theorem* A. *Then the kernel of E'_X is finite when $2k+q \geq 5$.*

The outline of the proof of Theorem A is as follows. First, by using the Atiyah-Singer invariant [AS, §7], we define an invariant σ on a subgroup of $D(\mathbf{C}P^k \times X)$ containing the image of E_X (§1). Next we see that the composition $\sigma \cdot E_X$ is independent of X (Lemma 2.1). Thirdly we see that $\sigma \cdot E_X$ is a homomorphism (Lemma 3.1). Finally we check the finiteness of the kernel of

$\sigma \cdot E_X$ for $X = S^q$ (Theorem 4.1). The assumption $H^1(X;\mathbf{Z}) = 0$ is used for the well-definedness of the invariant σ.

1. An invariant σ

As remarked in the introduction Theorem A is trivial when $q = \dim X$ is even. Therefore q will be odd throughout this paper unless otherwise stated.

We orient X. The disjoint union $X \amalg -X$ ($-$ denotes the reversed orientation) is null-cobordant in the oriented cobordism. Extending maps in $D(\mathbf{C}P^k \times X)$ as the identity on $\mathbf{C}P^k \times (-X)$ induces a monomorphism

$$D(\mathbf{C}P^k \times X) \to D(\mathbf{C}P^k \times (X \amalg -X)).$$

Thus it suffices to prove Theorem A for X being null-cobordant.

Let α be a generator of $H^2(\mathbf{C}P^k;\mathbf{Z})$ and we also regard α as an element of $H^2(\mathbf{C}P^k \times X)$ via the projection map $\mathbf{C}P^k \times X \to \mathbf{C}P^k$.

Definition. $D_0(\mathbf{C}P^k \times X) = \{[f] \in D(\mathbf{C}P^k \times X) | f^*\alpha = \alpha\}$, where $[f]$ denotes the class of a diffeomorphism f of $\mathbf{C}P^k \times X$.

Clearly $D_0(\mathbf{C}P^k \times X)$ forms a subgroup of $D(\mathbf{C}P^k \times X)$. The image of E_X is contained in $D_0(\mathbf{C}P^k \times X)$. We shall define an invariant

$$\sigma\colon D_0(\mathbf{C}P^k \times X) \to F(S^1)/\mathbf{Z}$$

where $F(S^1)/\mathbf{Z}$ is the quotient group of $F(S^1)$, the fraction field of the complex character (or representation) ring $R(S^1)$ of the circle group S^1, divided by the normal subgroup $\mathbf{Z}$ consisting of integer valued constant functions on S^1. Since $R(S^1)$ is the Laurent polynomial ring $\mathbf{Z}[t, t^{-1}]$ as is well known, $F(S^1)$ is the field of rational functions of t.

Let Y and Y' be a pair of compact connected oriented manifolds which are bounded by X and have the same signature. Let $[f]$ be an element of $D_0(\mathbf{C}P^k \times X)$. We paste together $\mathbf{C}P^k \times Y$ and $-\mathbf{C}P^k \times Y'$ along the boundary by f to get a closed oriented manifold M. The oriented diffeomorphism type of M does not depend on the choice of a representative f of $[f]$.

The S^1 bundle over $\mathbf{C}P^k \times X$ corresponding to α is $S^{2k+1} \times X \to \mathbf{C}P^k \times X$ where S^1 acts on S^{2k+1} linearly and freely. Since $f^*\alpha = \alpha$, f lifts to an S^1 equivariant diffeomorphism $\tilde{f}$ of $S^{2k+1} \times X$. The difference of two liftings of f is measured by a continuous map from X to S^1. The homotopy classes of such maps are exactly $H^1(X;\mathbf{Z})$ and we assume the group vanishes. Hence the lifting is unique up to S^1 equivariant isotopy.

We paste together $S^{2k+1} \times Y$ and $-S^{2k+1} \times Y'$ along the boundary by $\tilde{f}$ to get a closed oriented manifold $\widetilde{M} = \widetilde{M}([f], Y, Y')$ with a free S^1 action. Since q is odd, $\widetilde{M}$ is of odd dimension. Therefore the Atiyah-Singer invariant $\sigma(g, \widetilde{M}) \in \mathbf{C}$ is defined for $g \neq 1 \in S^1$. The function $\sigma(\ , \widetilde{M})$ belongs to the fraction field $F(S^1)$ (see [AS, §7]). Note that the function $\sigma(\ , \widetilde{M})$ is independent of the choice of a representative f because so is the oriented S^1-diffeomorphism type of $\widetilde{M}$.

Lemma 1.1. *The function $\sigma(\ ,\widetilde{M})$, regarded as an element of $F(S^1)/\mathbf{Z}$, depends only on $[f]$, and not on the choice of Y and Y'.*

Proof. Let $A(Y)$ be the closed S^1 manifold defined as

$$A(Y) = D^{2k+2} \times X \cup S^{2k+1} \times Y$$

where $D^{2k+2} \times X$ and $S^{2k+1} \times Y$ are pasted together along the boundary by $\tilde{f}$. We consider the S^1 manifold defined as

$$W = D^{2k+2} \times Y \cup A(Y) \times I \cup D^{2k+2} \times Y'$$

where $D^{2k+2} \times Y$ and $D^{2k+2} \times Y'$ are attached to $A(Y) \times I$ along $D^{2k+2} \times X$ via the identity map at 0- and 1-levels respectively. We orient W suitably so that W is an oriented S^1 cobordism between $\widetilde{M}([f], Y, Y)$ and $\widetilde{M}([f], Y, Y')$.

By definition we have

$$\sigma(g, \partial W) = L(g, W) - \operatorname{Sign}(g, W) \quad \text{for } g \neq 1 \in S^1$$

where $L(g, W)$ is the number occurring on the right-hand side of the G-signature formula and $\operatorname{Sign}(g, W)$ is the equivariant signature of W evaluated on g. Since S^1 is a connected group $\operatorname{Sign}(g, W) = \operatorname{Sign} W$. As for $L(g, W)$ the G-signature formula involves the characteristic classes of TW^g and the normal bundle of W^g, so we have to investigate them. First we note that the S^1 action on W is semifree, so $W^g = W^{S^1}$. The fixed point set W^{S^1} is $Y \cup X \times I \cup (-Y')$. The normal bundle to W^{S^1} admits a complex structure induced from the S^1 action. As easily observed the bundle is trivial. Furthermore we have

$$\operatorname{Sign} W^{S^1} = \operatorname{Sign} Y - \operatorname{Sign} Y' = 0$$

by the additivity property of signature (see [AS, §7]) and the assumption that $\operatorname{Sign} Y = \operatorname{Sign} Y'$. Putting these into the G-signature formula, one can see that $L(g, W) = 0$. Thus we have

$$\sigma(g, \widetilde{M}([f], Y, Y')) - \sigma(g, \widetilde{M}([f], Y, Y)) = \sigma(g, \partial W) = -\operatorname{Sign} W$$

and hence

$$\sigma(\ , \widetilde{M}([f], Y, Y')) = \sigma(\ , \widetilde{M}([f], Y, Y)) \quad \text{in } F(S^1)/\mathbf{Z}.$$

Applying the same argument to $\widetilde{M}([f^{-1}], Y', Y)$, we get

$$\sigma(\ , \widetilde{M}([f^{-1}], Y', Y)) = \sigma(\ , \widetilde{M}, [f^{-1}], Y', Y') \quad \text{in } F(S^1)/\mathbf{Z}.$$

Here we note that $\widetilde{M}([f^{-1}], Y', Y) = -\widetilde{M}([f], Y, Y')$ and $\widetilde{M}([f^{-1}], Y', Y') = -\widetilde{M}([f], Y', Y')$. Since the Atiyah-Singer invariant changes the sign if the orientation of the manifold is reversed, the above identities prove the lemma. Q.E.D.

Definition. We set $\sigma([f])(g) = \sigma(g, \widetilde{M}([f], Y, Y))$ and regard $\sigma([f])$ as an element of $F(S^1)/\mathbf{Z}$.

By Lemma 1.1 $\sigma([f]) \in F(S^1)/\mathbf{Z}$ is an invariant of $[f] \in D_0(\mathbf{C}P^k \times X)$.

2. $\sigma \cdot E_X$ IS INDEPENDENT OF X

As remarked before the image of $E_X \colon D(\mathbf{C}P^k \times D^q) \to D(\mathbf{C}P^k \times X)$ is contained in $D_0(\mathbf{C}P^k \times X)$. Hence the composition $\sigma \cdot E_X \colon D(\mathbf{C}P^k \times D^q) \to F(S^1)/\mathbf{Z}$ is defined. The purpose of this section is to verify

Lemma 2.1. *$\sigma \cdot E_X$ is independent of X.*

Proof. It suffices to prove $\sigma \cdot E_X = \sigma \cdot E_{S^q}$. Let Y be a connected compact oriented manifold bounded by X. We may assume that $H^1(Y; \mathbf{Z}) = 0$, if necessary, by doing surgery to kill the fundamental group of Y. Let $\mathring{Y}$ be the cobordism between X and S^q obtained by removing a small open disk from Y. We choose a smooth simple path in $\mathring{Y}$ connecting X and S^q which is transverse to the boundary. The tubular neighborhood is of the form $D^q \times I$, where I is the path direction and D^q is the normal direction.

Let $[f]$ be an element of $D(\mathbf{C}P^k \times D^q)$ and let id be the identity map of I. The map $f \times \mathrm{id}$ restricts to the identity map on $\mathbf{C}P^k \times S^{q-1} \times I$; so it extends to a diffeomorphism of $\mathbf{C}P^k \times \mathring{Y}$, say F, as the identity on the exterior of $\mathbf{C}P^k \times D^q \times I$. Since $H^1(\mathring{Y}; \mathbf{Z}) = 0$, F lifts uniquely to an S^1 equivariant diffeomorphism $\widetilde{F}$ of $S^{2k+1} \times \mathring{Y}$ up to S^1 equivariant isotopy.

We view $\partial(Y \times I)$ as a triad consisting of three pieces $Y \times \{0\} \cup X \times I$ ($\cong Y$), $\mathring{Y}$, and D^{q+1} ($\subset Y \times \{1\}$) so that $D^q \times I$ is embedded in the piece $\mathring{Y}$. We paste together two copies of $S^{2k+1} \times Y \times I$ along $S^{2k+1} \times \mathring{Y}$ by $\widetilde{F}$. The resulting S^1 manifold V is an S^1 cobordism between $\widetilde{M}(E_X([f]), Y, Y)$ and $\widetilde{M}(E_{S^q}([f]), D^{q+1}, D^{q+1})$. Since the S^1 action on V is free, we have

$$\sigma(g, \widetilde{M}(E_X([f]), Y, Y)) - \sigma(g, \widetilde{M}(E_{S^q}([f]), D^{q+1}, D^{q+1})) = \operatorname{Sign} V$$

up to sign. This implies the lemma. Q.E.D.

3. ADDITIVITY

We shall denote $\sigma \cdot E_X$ again by σ. The purpose of this section is to verify

Lemma 3.1. *$\sigma \colon D(\mathbf{C}P^k \times D^q) \to F(S^1)/\mathbf{Z}$ is a homomorphism.*

Proof. By Lemma 2.1 we may assume $X = S^q$. Let $[f_i]$ $(i = 1, 2)$ be elements of $D(\mathbf{C}P^k \times D^q)$. Since f_i is the identity on the boundary, one can deform f_i via isotopy so that f_i is the identity on the exterior of $\mathbf{C}P^k \times D_i$ where D_i is a small disk in D^q. Furthermore we may assume that D_1 has no intersection with D_2.

Set $E_{S^q}([f_i]) = [F_i] \in D(\mathbf{C}P^k \times S^q)$ and let $\widetilde{F}_i\colon S^{2k+1} \times S^q \to S^{2k+1} \times S^q$ be a lifting of F_i, which is unique up to S^1 equivariant isotopy as $H^1(S^q;\mathbf{Z}) = 0$. We form a closed S^1 manifold

$$\Sigma([f_i]) = D^{2k+2} \times S^q \cup S^{2k+1} \times D^{q+1} \tag{3.2}$$

where $D^{2k+2} \times S^q$ and $S^{2k+1} \times D^{q+1}$ are pasted together by $\widetilde{F}_i$ along the boundary and S^1 acts on D^{2k+2} linearly extending the free S^1 action on S^{2k+1}. The action on $\Sigma([f_i])$ is semifree and the fixed point set is $\{0\} \times S^q$. It turns out that $\Sigma([f_i])$ is a homotopy $(2k+q+2)$-sphere.

We regard $\Sigma([f_i])$ as a homotopy $(2k+q+2)$-sphere with a semifree S^1 action and a decomposition as in (3.2). The connected sum $\Sigma([f_1])\#\Sigma([f_2])$ can be done equivariantly around fixed points using the decompositions and it is also a homotopy $(2k+q+2)$-sphere with a semifree S^1 action and a decomposition. Taking account of decompositions we have

$$\Sigma([f_1])\#\Sigma([f_2]) = \Sigma([f_1 \cdot f_2]) \quad (= \Sigma([f_1][f_2])).$$

We abbreviate $\widetilde{M}(E_{S^q}([f]), D^{q+1}, D^{q+1})$ by $\widetilde{M}([f])$. We note that $\widetilde{M}([f_i])$ agrees with the S^1 manifold obtained from $\Sigma([f_i])$ by doing surgery on the identity map: $D^{2k+2} \times S^q \to D^{2k+2} \times S^q \subset \Sigma([f_i])$. Therefore $\widetilde{M}([f_1][f_2])$ is obtained from $\Sigma([f_1])\#\Sigma([f_2])$ in this way.

Consider the equivariant boundary connected sum $\Sigma([f_1]) \times I \natural \Sigma([f_2]) \times I$ at the 1-level. This yields an S^1 cobordism between $\Sigma([f_1]) \amalg \Sigma([f_2])$ and $\Sigma([f_1])\#\Sigma([f_2]) = \Sigma([f_1][f_2])$. We attach three copies of the handle $D^{2k+2} \times D^{q+1}$ to $\Sigma([f_1]) \times I \natural \Sigma([f_2]) \times I$ via the identity maps from $D^{2k+2} \times S^q$ to $D^{2k+2} \times S^q$ embedded in $\Sigma([f_1]) \times \{1\}\#\Sigma([f_2]) \times \{1\}$, $\Sigma([f_1]) \times \{0\}$, and $\Sigma([f_2]) \times \{0\}$ respectively. This yields a semifree S^1 cobordism V between $\widetilde{M}([f_1]) \amalg \widetilde{M}([f_2])$ and $\widetilde{M}([f_1][f_2])$. As easily observed the complex normal bundle to V^{S^1} is trivial and $\operatorname{Sign} V^{S^1} = 0$. Hence, similarly to the proof of Lemma 1.1, we have

$$\sigma(\ ,\widetilde{M}([f_1])) + \sigma(\ ,\widetilde{M}([f_2])) = \sigma(\ ,\widetilde{M}([f_1][f_2])) \quad \text{in } F(S^1)/\mathbf{Z}.$$

This proves the lemma as $\sigma([f]) = \sigma(\ ,\widetilde{M}([f]))$. Q.E.D.

4. Review of the work of Browder-Petrie

In this section and the next section we prove the following theorem from which Theorem A follows immediately.

Theorem 4.1. *The kernel of* $\sigma\colon D(\mathbf{C}P^k \times D^q) \to F(S^1)/\mathbf{Z}$ *is finite.*

We need some knowledge about the group $D(\mathbf{C}P^k \times D^q)$ for the proof of Theorem 4.1. The group is analysed by Browder-Petrie [BP]. In this section we shall review their work.

There is a long exact sequence of groups:

$$(4.2)\qquad \pi_{q+1}(G\mathbf{C}P^k) \xrightarrow{\psi} hS(\mathbf{C}P^k \times (D^{q+1}, S^q)) \xrightarrow{\partial} D(\mathbf{C}P^k \times D^q) \xrightarrow{\lambda} \pi_q(G\mathbf{C}P^k).$$

Here $G\mathbf{C}P^k$ is the identity component of the space of self-maps of $\mathbf{C}P^k$ and $hS(\mathbf{C}P^k \times (D^{q+1}, S^q))$ is the set of equivalence classes of pairs $[Q, h]$ where Q is an oriented smooth manifold and $h: [Q, \partial Q] \to \mathbf{C}P^k \times (D^{q+1}, S^q)$ is a homotopy equivalence preserving orientation and $h|\partial Q: \partial Q \to \mathbf{C}P^k \times S^q$ is a diffeomorphism. Two pairs $[Q_1, h_1]$ and $[Q_2, h_2]$ are equivalent if there is a diffeomorphism $d: Q_1 \to Q_2$ such that the composition $h_2 \cdot d$ is homotopic to h_1 relative boundary.

We shall explain the homomorphism ∂ and λ. First we note that Q is diffeomorphic to $\mathbf{C}P^k \times D^{q+1}$. This can be seen as follows. Since $h|\partial Q$ is a diffeomorphism, $\mathbf{C}P^k$ can be embedded in the interior of Q so that the embedding induces a homotopy equivalence and that the normal bundle is trivial. The complement $\mathring{Q}$ of a small open tubular neighborhood of the embedded $\mathbf{C}P^k$ in Q turns out to be an h-cobordism between $\mathbf{C}P^k \times S^q$ and $\partial Q = \mathbf{C}P^k \times S^q$. Hence $\mathring{Q}$ is diffeomorphic to $\mathbf{C}P^k \times S^q \times I$ by the h-cobordism theorem and hence Q is diffeomorphic to $\mathbf{C}P^k \times D^{q+1}$. Thus any class in $hS(\mathbf{C}P^k \times (D^{q+1}, S^q))$ can be represented by a pair $[\mathbf{C}P^k \times D^{q+1}, h]$. Moreover, a similar argument shows that one can choose h so that $h|\mathbf{C}P^k \times D^q_-$ is the identity where D^q_- is the lower hemisphere of S^q.

With this understood $\partial: hS(\mathbf{C}P^k \times (D^{q+1}, S)) \to D(\mathbf{C}P^k \times D^q)$ is defined by

$$\partial([\mathbf{C}P^k \times D^{q+1}, h]) = [h|\mathbf{C}P^k \times D^q_+].$$

Note that the image of $\partial([\mathbf{C}P^k \times D^{q+1}, h])$ through the homomorphism

$$E_{S^q}: D(\mathbf{C}P^k \times D^q) \to D(\mathbf{C}P^k \times S^q)$$

is nothing but $[h|\mathbf{C}P^k \times S^q]$.

Let $\rho: \mathbf{C}P^k \times D^q \to \mathbf{C}P^k$ be the projection. Then $\lambda: D(\mathbf{C}P^k \times D^q) \to \pi_q(G\mathbf{C}P^k)$ is defined by

$$(\lambda([f])(x))(u) = \rho(f(u, x))$$

where $[f] \in D(\mathbf{C}P^k \times D^q)$, $x \in D^q$, and $u \in \mathbf{C}P^k$.

The linear action of the unitary group $U(k+1)$ on $\mathbf{C}P^k$ induces a semihomomorphism $i: U(k+1)/\Delta \to G\mathbf{C}P^k$, where Δ is the subgroup of $U(k+1)$ consisting of scalar multiples of the identity matrix. It is known that

$$i_*: \pi_q(U(k+1)/\Delta) \otimes \mathbf{Q} \to \pi_q(G\mathbf{C}P^k) \otimes \mathbf{Q}$$

is an isomorphism (see [S] for example). In particular

$$\pi_q(G\mathbf{C}P^k) \otimes \mathbf{Q} = \begin{cases} \mathbf{Q} & \text{if } 3 \le q \le 2k+1 \ (q : \text{odd}), \\ 0 & \text{otherwise.} \end{cases}$$

There is another homomorphism

$$\mu\colon \pi_q(U(k+1)/\Delta) \to D(\mathbf{C}P^k \times D^q)$$

defined by

$$\mu([h])(u,x) = ((h(x))(u), x)$$

where $h\colon (D^q, S^{q-1}) \to (U(k+1)/\Delta, \mathrm{Id})$. It is easy to see that $\lambda \cdot \mu = i_*$. The exact sequence (4.2) together with the above observation shows

Lemma 4.3. *The subgroup of $D(\mathbf{C}P^k \times D^q)$ generated by the subgroups*

$$\partial(hS(\mathbf{C}P^k \times (D^{q+1}, S^q))) \quad \textit{and} \quad \mu(\pi_q(U(k+1)/\Delta))$$

is of finite index in $D(\mathbf{C}P^k \times D^q)$.

5. Proof of Theorem 4.1

We abbreviate $\widetilde{M}(E_{S^q}([f]), D^{q+1}, D^{q+1})$ by $\widetilde{M}$ and let M be the S^1 orbit space of $\widetilde{M}$. Since the S^1 action on $\widetilde{M}$ is free, the projection $\widetilde{M} \to M$ is an S^1 bundle. Let α be the first Chern class of it. The associated disk bundle D_α supports a semifree S^1 action rotating fibers, the zero section M being the fixed point set. Since $\partial D_\alpha = \widetilde{M}$, the Atiyah-Singer invariant $\sigma(g, \widetilde{M})$ can be described using D_α. In fact, since the normal bundle to M in D_α is the complex line bundle with α being the first Chern class, we have

$$\sigma(g, \widetilde{M}) = 2^m \frac{ge^\alpha + 1}{ge^\alpha - 1} L(M)[M] - \operatorname{Sign} D_\alpha$$

where $2m = 2k + q + 1$ and L denotes the Atiyah-Singer L-class. Hence, since $\sigma([f])(g) = \sigma(g, \widetilde{M})$ and $\sigma([f])$ is considered in $F(S^1)/\mathbf{Z}$, we have

$$\sigma([f]) = 2^m \frac{te^\alpha + 1}{te^\alpha - 1} L(M)[M] \tag{5.1}$$

where $t \in F(S^1)/\mathbf{Z}$ is the image of the standard complex 1-dimensional S^1 representation to $F(S^1)/\mathbf{Z}$.

Expanding $(te^\alpha + 1)/(te^\alpha - 1)$ with respect to $e^\alpha - 1$, we have

$$\frac{te^\alpha + 1}{te^\alpha - 1} = 1 - 2\sum_{r \geq 0} \frac{t^r (e^\alpha - 1)^r}{(1-t)^{r+1}}. \tag{5.2}$$

It says that $t = 1$ is the only pole of $\sigma([f])$.

Lemma 5.3. *If $[f] \in \partial(hS(\mathbf{C}P^k \times (D^{q+1}, S^q)))$, then the highest degree of the pole of $\sigma([f])$ is at most $k+1$.*

Proof. Let $[f] = \partial([\mathbf{C}P^k \times D^{q+1}, h])$. Then $E_{S^q}([f]) = [h|\mathbf{C}P^k \times S^q]$ as remarked before. Since M is obtained by pasting together two copies of $\mathbf{C}P^k \times D^{q+1}$ along the boundary by $h|\mathbf{C}P^k \times S^q$, M is homotopy equivalent to $\mathbf{C}P^k \times S^{q+1}$. Therefore $\alpha^r = 0$ and hence $(e^\alpha - 1)^r = 0$ for $r > k$. This together with (5.1) and (5.2) implies the lemma. Q.E.D.

Since Δ is a circle group, the projection map $U(k+1) \to U(k+1)/\Delta$ induces an isomorphism $\pi_q(U(k+1)) \to \pi_q(U(k+1)/\Delta)$ for $q \geq 2$. Suppose $3 \leq q \leq 2k+1$ (q : odd). Then $\pi_q(U(k+1))$ sits in the stable range; so it is infinite cyclic and is detected by the $(q+1)/2$th Chern classes of the complex vector bundles over S^{q+1} corresponding to $\pi_q(U(k+1))$. We shall denote by $c([f]) \in H^{q+1}(S^{q+1}; \mathbf{Z})$ the $(q+1)/2$th Chern class of

$$[f] \in \pi_q(U(k+1)) = \pi_q(U(k+1)/\Delta).$$

Hereafter we identify $\pi_q(U(k+1))$ with a subgroup of $D(\mathbf{C}P^k \times D^q)$ via μ when $3 \leq q \leq 2k+1$.

Lemma 5.4. *Let* $3 \leq q \leq 2k+1$ *and* $[f] \in \pi_q(U(k+1))$. *Then the highest degree of the pole of* $\sigma([f])$ *is at most* $k+1+(q+1)/2$ *and the coefficient at the pole of degree* $k+1+(q+1)/2$ *is* $2^{m+1}c([f])[S^{q+1}]$ *up to sign.*

Proof. In this case M is the total space of the complex projective bundle $\pi\colon M \to S^{q+1}$ associated with the complex vector bundle over S^{q+1} corresponding to $[f]$. According to Borel-Hirzebruch [BH, p. 516] we have

$$\alpha^{k+1} + (-1)^{(q+1)/2}\pi^*(c([f]))\alpha^{k+1-(q+1)/2} = 0$$

and hence

$$\alpha^{k+(q+1)/2} = -(-1)^{(q+1)/2}\pi^*(c([f]))\alpha^k. \tag{5.5}$$

On the other hand (5.1) and (5.2) show that the highest degree of the pole of $\sigma([f])$ is at most $k+1+(q+1)/2$ and the coefficient at the pole of degree $k+1+(q+1)/2$ is $2^{m+1}\alpha^{k+(q+1)/2}[M]$ up to sign. Here we have

$$\begin{aligned}\alpha^{k+(q+1)/2}[M] &= \pm\pi^*(c([f]))\alpha^k[M] \quad \text{(by (5.5))}\\ &= \pm c([f])[S^{q+1}]\end{aligned}$$

where the latter identity is the so-called integration along the fiber. This proves the lemma. Q.E.D.

To prove Theorem 4.1 it suffices to show that the kernel of σ is finite when σ is restricted to the subgroup of $D(\mathbf{C}P^k \times D^q)$ which consists of all elements $[f]$ of the form $[f] = [f_1] + [f_2]$ where $[f_1] \in \partial(hS(\mathbf{C}P^k \times (D^{q+1}, S^q)))$ and $[f_2] \in \mu(\pi_q(U(k+1)/\Delta))$. In fact, it suffices to show that each element of this group is of finite order because $D(\mathbf{C}P^k \times D^q)$ is finitely generated abelian.

Lemma 5.6. *If* $\sigma([f_1] + [f_2]) = 0$, *then* $[f_2]$ *is of finite order.*

Proof. We may assume $3 \leq q \leq 2k+1$ (q : odd) because otherwise $\pi_q(U(k+1)/\Delta)$ is a finite group. Hence we may view $[f_2]$ as an element of $\pi_q(U(k+1)) = \pi_q(U(k+1)/\Delta)$. By Lemma 3.1, we have $\sigma([f_1]) + \sigma([f_2]) = 0$.

Lemmas 5.3 and 5.4 tell us that the coefficient at the pole of degree $k+1+(q+1)/2$ is $2^{m+1}c([f_2])[S^{q+1}]$ up to sign. Since it vanishes, $c([f_2])=0$ and hence $[f_2]=0$. Q.E.D.

To show Theorem 4.1 we need to show only that σ has a finite kernel when restricted to $\partial(hS(\mathbf{C}P^k\times(D^{q+1},S^q)))$, or any element in this kernel is of finite order. Remember that M is then homotopy equivalent to $\mathbf{C}P^k\times S^{q+1}$.

Lemma 5.7. *Let* $[f]\in\partial(hS(\mathbf{C}P^k\times(D^{q+1},S^q)))$. *If* $\sigma([f])=0$, *then the total Pontrjagin class* $p(M)$ *of* M *is of the same form as* $\mathbf{C}P^k\times S^{q+1}$, *i.e.* $p(M)=(1+\alpha^2)^{k+1}$.

Proof. Since M is homotopy equivalent to $\mathbf{C}P^k\times S^{q+1}$, one can express

$$L(M)=u(\alpha)+v(\alpha)\beta$$

where $u(\alpha)$ and $v(\alpha)$ are polynomials of α of degree at most k and β is a generator of $H^{q+1}(M)$ corresponding to the factor S^{q+1}.

Remember that $M=\mathbf{C}P^k\times D^{q+1}\cup\mathbf{C}P^k\times D^{q+1}$; so $\mathbf{C}P^k$ is naturally embedded in M with the trivial normal bundle. This means that the restriction of $L(M)$ to the embedded $\mathbf{C}P^k$ is of the same form as $\mathbf{C}P^k$. Hence $u(\alpha)$ is determined. The identities (5.1) and (5.2) tell us that $\sigma([f])$ determines the values $\alpha^r L(M)[M]$ for $0\le r\le k$, which determine $v(\alpha)$, in fact $v(\alpha)=0$.

It is known that the total L-classes determine the total Pontrjagin classes and vice versa. Consequently $\sigma([f])$ determines $p(M)$. Since $p(M)=(1+\alpha^2)^{k+1}$ is a solution of the equation $\sigma([f])=0$, the lemma follows. Q.E.D.

Let $[f]=\partial([Q_f,h_f])$. Remember that we may assume $Q_f=\mathbf{C}P^k\times D^{q+1}$. Since $\pi_{q+1}(G\mathbf{C}P^k)$ is finite as $q+1$ is even, the map ∂ has finite kernel by (4.2). Therefore to complete our proof of Theorem 4.1, it suffices to show that $[Q_f,h_f]$ is of finite order. This we shall do now. Petrie [P] defined a map

$$\gamma\colon hS(\mathbf{C}P^k\times(D^{q+1},S^q))\to hS(\mathbf{C}P^k\times S^{q+1})$$

as follows: $\gamma([Q,h])=[\gamma(Q),\gamma(h)]$ where $\gamma(Q)$ is the manifold obtained by pasting together Q and $\mathbf{C}P^k\times D^{q+1}$ along the boundary by $h|\partial Q$, and $\gamma(h)|Q=h$, $\gamma(h)|\mathbf{C}P^k\times D^{q+1}$ is the identity. We note that $M=\gamma(Q_f)$. By Lemma 5.7 $\gamma(Q_f)$ has the Pontrjagin classes of the same form as $\mathbf{C}P^k\times S^{q+1}$. This fact seems to imply that $[\gamma(Q_f),\gamma(h_f)]$ is of finite order in $hS(\mathbf{C}P^k\times S^{q+1})$. However this argument does not work because $hS(\mathbf{C}P^k\times S^{q+1})$ does not admit a natural group structure.

To avoid this trouble we shall consider the following commutative diagram of surgery exact sequences:

(5.8)

$$\begin{array}{ccccccc} 0 & \longrightarrow & hS(\mathbf{C}P^k \times (D^{q+1}, S^q)) & \xrightarrow{\eta} & [\mathbf{C}P^k \times D^{q+1}/\mathbf{C}P^k \times S^q, G/O] & \longrightarrow \\ & & \downarrow \gamma & & \downarrow \kappa & \\ 0 & \longrightarrow & hS(\mathbf{C}P^k \times S^{q+1}) & \xrightarrow{\eta'} & [\mathbf{C}P^k \times S^{q+1}, G/O] & \end{array}$$

Here G/O is the homotopy fiber of the classifying map $BO \to BG$ (BG is the classifying space of stable spherical fibrations), and $[A, G/O]$ is the set of homotopy classes of continuous maps from A to G/O. In fact, induced from the H-space structure of G/O, the set $[A, G/O]$ forms an abelian group. The vertical map κ^* is a homomorphism induced from the quotient map $\kappa\colon \mathbf{C}P^k \times S^{q+1} \to \mathbf{C}P^k \times D^{q+1}/\mathbf{C}P^k \times S^q$. It is known that η is a homomorphism.

Let A be a finite CW-complex. The inclusion map $j\colon G/O \to BO$ induces a homomorphism $j_*\colon [A, G/O] \to [A, BO] = \widetilde{K}O(A)$ and there is a functor ph (called Pontrjagin character) from $\widetilde{K}O(A)$ to $\widetilde{H}^{4^*}(A; \mathbf{Q})$. It is well known that

(5.9) $\qquad j_*$ and ph are both isomorphisms when tensored by $\mathbf{Q}$.

As easily checked $\kappa^*\colon \widetilde{H}^{4^*}(\mathbf{C}P^k \times D^{q+1}/\mathbf{C}P^k \times S^q; \mathbf{Q}) \to \widetilde{H}^{4^*}(\mathbf{C}P^k \times S^{q+1}; \mathbf{Q})$ is injective. Hence it follows from (5.9) that κ^* in the diagram (5.8) has finite kernel. In the sequel it suffices to show that $\eta'([\gamma(Q_f), \gamma(h_f)])$ is of finite order since η is a monomorphism.

It is also known that

$$\text{(5.10)} \qquad \begin{aligned} &\mathrm{ph} \cdot j_* \cdot \eta'([\gamma(Q_f), \gamma(h_f)]) \\ &= (\gamma(h_f)^*)^{-1}\, \mathrm{ph}(\gamma(Q_f)) - \mathrm{ph}(\mathbf{C}P^k \times S^{q+1}) \end{aligned}$$

where $\mathrm{ph}(B)$ denotes the Pontrjagin character of the tangent bundle of a manifold B. Since $p(\gamma(Q_f)) = (1 + \alpha^2)^{k+1}$, $(\gamma(h_f)^*)^{-1} p(\gamma(Q_f)) = p(\mathbf{C}P^k \times S^{q+1})$ and hence the right-hand side of (5.10) is zero. Thus $\eta'([\gamma(Q_f), \gamma(h_f)])$ is of finite order by (5.9). Q.E.D.

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