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ZERO INTEGRALS ON CIRCLES AND CHARACTERIZATIONS OF HARMONIC AND ANALYTIC FUNCTIONS

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ABSTRACT. We determine the kernels of two circular Radon transforms of continuous functions on an annulus and use this to obtain a characterization of harmonic functions in the open unit disc which involves Poisson averages over circles computed at only one point of the disc and to obtain a version of Morera's theorem which involves only the circles which surround the origin.

1. INTRODUCTION

Suppose that f is a continuous function on the open unit disc Δ in the complex plane. For each simple closed curve $\Gamma \subset \Delta$ bounding a domain D, $0 \in D$, let F_Γ be the function which is continuous on $D \cup \Gamma$, harmonic in D and which coincides with f on Γ. It is known that if $F_\Gamma(0) = f(0)$ for each smooth curve Γ bounding a strictly convex domain then f is harmonic in Δ. This is a special case of the main result of [5].

The starting point of the present investigation was the fact that the function f is not necessarily harmonic if one assumes only that $F_\Gamma(0) = f(0)$ for each circle $\Gamma \subset \Delta$ surrounding the origin. In fact, for each $k \in N$ there is a function f of class $\mathscr{C}^k$ on Δ such that $F_\Gamma(0) = f(0)$ for each circle $\Gamma \subset \Delta$ surrounding the origin and which is not harmonic in Δ ([5], see also §9).

One of the results of the present paper is that if $F_\Gamma(0) = f(0)$ for each circle $\Gamma \subset \Delta$ which surrounds the origin then f is harmonic in Δ under the additional assumption that it is infinitely differentiable at the origin, that is, if for each $n \in N$ there is a polynomial P_n of degree n such that

$$f(z) = P_n(z, \bar{z}) + Q_n(z) \qquad (z \in \Delta)$$

where $\lim_{z \to 0} |z|^{-n} Q_n(z) = 0$. In fact, to get harmonicity it is enough to assume only that $F_\Gamma(0) = f(0)$ for each circle Γ belonging to a family which is only slightly larger than the family of circles in Δ centered at the origin. Note that f is infinitely differentiable at the origin if it is of class $\mathscr{C}^\infty$ in a neighbourhood of the origin, or more generally, if it belongs to $\mathscr{C}^\infty(\{0\})$, that is, if for each $k \in N$ there is a neighbourhood of the origin in which f is of class $\mathscr{C}^k$.

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This characterization of harmonic functions is a consequence of our first main result which describes the Fourier coefficients of the continuous functions on an annulus $\Omega = \{\zeta \in C: R_1 < |\zeta| < R_2\}$ which have zero average on each circle $\Gamma \subset \Omega$ that surrounds the origin. The second main result describes the Fourier coefficients of the continuous functions f on Ω such that $\int_\Gamma f(z)\,dz = 0$ for each circle that surrounds the origin. Its consequence is a version of Morera's theorem: If f is continuous on Δ and infinitely differentiable at the origin and if $\int_\Gamma f(z)\,dz = 0$ for each circle $\Gamma \subset \Delta$ which surrounds the origin then f is analytic in Δ. Again, one cannot drop the smoothness requirement at the origin since for each $k \in N$ there is a function f of class $\mathscr{C}^k$ on Δ such that $\int_\Gamma f(z)\,dz = 0$ for each circle $\Gamma \subset \Delta$ surrounding the origin and which is not analytic in Δ ([3], see also §9).

2. The main results

Let $0 \le R_1 < R_2$ and let f be a continuous function on the annulus $\Omega = \{\zeta \in C: R_1 < |\zeta| < R_2\}$. For each r, $R_1 < r < R_2$, and for each $n \in Z$, let

$$f_n(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(re^{i\theta})\,d\theta.$$

Thus, for each r, $R_1 < r < R_2$, $\sum_{-\infty}^{\infty} f_n(r)e^{in\theta}$ is the Fourier series of $f(re^{i\theta})$.

Let $\Gamma = \{\zeta \in C: |\zeta - z| = r\}$ and let $\varepsilon > 0$. The ε-neighbourhood of the circle Γ is the family of all circles of the form $\{\zeta \in C: |\zeta - w| = \rho$ with $|w - z| < \varepsilon$ and $|r - \rho| < \varepsilon$, $\rho > 0$. Let $\mathscr{G}$ be a family of circles in Ω which surround the origin such that $\mathscr{G}$ contains a neighbourhood of each circle $\Gamma \subset \Omega$ centered at the origin. Note that $\mathscr{G}$ may be only slightly larger that the family of all circles in Ω centered at the origin.

If $\Gamma = \{a + e^{i\theta}b: 0 \le \theta < 2\pi\}$ and if $\int_0^{2\pi} f(a + e^{i\theta}b)\,d\theta = 0$ then we will say that f has zero average on Γ.

Theorem 1. *Let f be a continuous function on Ω. The following are equivalent:*

(i) *f has zero average on each circle $\Gamma \in \mathscr{G}$,*
(ii) *f has zero average on each circle $\Gamma \subset \Omega$ which surrounds the origin,*
(iii) *$f_0(r) = 0$ $(R_1 < r < R_2)$ and for each $n \in Z$, $n \ne 0$, there are numbers $a_{n0}, a_{n1}, \dots, a_{n,|n|-1}$ such that*

$$f_n(r) = r^{-|n|}(a_{n0} + a_{n1}r^2 + \dots + a_{n,|n|-1}r^{2(|n|-1)}) \qquad (R_1 < r < R_2).$$

Theorem 2. *Let f be a continuous function on Ω. The following are equivalent:*

(i) *$\int_\Gamma f(z)\,dz = 0$ for each circle $\Gamma \in \mathscr{G}$,*
(ii) *$\int_\Gamma f(z)\,dz = 0$ for each circle $\Gamma \subset \Omega$ which surrounds the origin,*
(iii) *$f_{-1}(r) = 0$ $(R_1 < r < R_2)$ and for each $n \in Z$, $n \ne -1$, there are numbers $b_{n0}, b_{n1}, \dots, b_{n,|n+1|-1}$ such that*

$$f_n(r) = r^{-|n|}(b_{n0} + b_{n1}r^2 + \dots + b_{n,|n+1|-1}r^{2(|n+1|-1)}) \qquad (R_1 < r < R_2).$$

Let $\mathscr{H}$ be a family of circles in Δ which surround the origin such that $\mathscr{H}$ contains a neighbourhood of each circle $\Gamma \subset \Delta$ centered at the origin.

Corollary 1. *Let f be a continuous function in Δ which is infinitely differentiable at the origin. If $F_\Gamma(0) = f(0)$ for each circle $\Gamma \in \mathscr{H}$ then f is harmonic in Δ. In particular, if $F_\Gamma(0) = f(0)$ for each circle $\Gamma \subset \Delta$ surrounding the origin then f is harmonic in Δ.*

Corollary 2. *Let f be a continuous function in Δ which is infinitely differentiable at the origin. If $\int_\Gamma f(z)\,dz = 0$ for each circle $\Gamma \in \mathscr{H}$ then f is analytic in Δ. In particular, if $\int_\Gamma f(z)\,dz = 0$ for each circle $\Gamma \subset \Delta$ surrounding the origin then f is analytic in Δ.*

An easy consequence of Theorems 1 and 2 is the following support theorem for the circular Radon transforms $f \mapsto \int_\Gamma f(z)\,ds$ and $f \mapsto \int_\Gamma f(z)\,dz$ which probably has a simple direct proof:

Corollary 3. *Let f be a continuous function on Ω such that for each $k \in N$ the function $z \mapsto (R_2 - |z|)^{-k} f(z)$ is bounded as $|z| \to R_2$. If f has zero average on each circle $\Gamma \in \mathscr{G}$ then f vanishes identically on Ω. If $\int_\Gamma f(z)\,dz = 0$ for each circle $\Gamma \in \mathscr{G}$ then f vanishes identically on Ω.*

The paper is organized as follows. We first prove that Corollaries 1–3 follow from Theorems 1 and 2 (§3). Then we show that $\int_\Gamma f(z)\,ds = 0$ or $\int_\Gamma f(z)\,dz = 0$ for each circle $\Gamma \in \Omega$ surrounding the origin if and only if the same holds for each function $re^{i\theta} \mapsto f_n(r)e^{in\theta}$ (§4). This happens if and only if in each case f_n satisfies a Volterra integral equation of the first kind whose kernel has a weak singularity on the diagonal (§5). We look at the properties of these equations and iterate the kernels to get equations for the functions $r \mapsto f_n(r)(r_0 - r)^{-1/2}$ with analytic kernels having zeros of order $|n|$ and $|n+1|$ on the diagonal (§6). Since only the structure of bounded solutions of such equations has been studied in detail [9, 12] with only a remark being made in [9] about the general case we revisit [9] to show that the approximation procedure used there for bounded analytic solutions can be used also for unbounded smooth solutions (§§7, 8). We then present examples of functions satisfying $\int_\Gamma f(z)\,ds = 0$, $\int_\Gamma f(z)\,dz = 0$ and show that using these examples one gets all solutions of the original integral equations and thus complete the proofs of Theorems 1 and 2 (§9).

3. Proofs of the corollaries

Proposition 1. *Let $D \subset C$ be an open disc, $0 \in D$, and let Γ be its boundary. Assume that F is a continuous function on $D \cup \Gamma$ which is harmonic in D. Then $F(0) = 0$ if and only if the function $z \mapsto F(z)/|z|^2$ has zero average on Γ.*

Proof. Let $\Gamma = \{w + Re^{i\theta}: 0 \le \theta < 2\pi\}$, $w = re^{i\alpha}$. If $0 \le \rho < 1$ and $0 \le \varphi < 2\pi$ then the Poisson formula gives

$$F(w + \rho Re^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{F(w + Re^{it})(1-\rho^2)}{1 - 2\rho\cos(t-\varphi) + \rho^2}\,dt.$$

Putting $\rho = r/R$ and $\varphi = \alpha + \pi$ we get

$$\begin{aligned} F(0) &= F(re^{i\alpha} + (r/R)Re^{i(\alpha+\pi)}) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{F(re^{i\alpha} + Re^{it})(1-\rho^2)}{1 - 2(r/R)\cos(t-\alpha-\pi) + r^2/R^2}\,dt \\ &= (R^2 - r^2)\frac{1}{2\pi}\int_0^{2\pi} \frac{F(re^{i\alpha} + Re^{it})}{|re^{i\alpha} + Re^{it}|^2}\,dt. \end{aligned}$$

This completes the proof.

Proof of Corollary 1, *assuming Theorem* 1. Suppose that f is a continuous function on Δ which is infinitely differentiable at the origin and which satisfies $F_\Gamma(0) = f(0)$ for each circle $\Gamma \in \mathscr{H}$. It is enough to prove that f is harmonic in $R\Delta$ for each $R < 1$ so we may assume with no loss of generality that f extends continuously to the closure $\overline{\Delta}$ of the unit disc.

Let g be the function which is continuous on $\overline{\Delta}$, harmonic in Δ and which coincides with f on the unit circle $b\Delta$ and put $h = f - g$. The function h is continuous on $\overline{\Delta}$ and vanishes identically on $b\Delta$. Since g is harmonic in Δ it follows that h is infinitely differentiable at the origin and that $H_\Gamma(0) = 0$ for each circle $\Gamma \in \mathscr{H}$. Proposition 1 now implies that the function $z \mapsto w(z) = h(z)/|z|^2$ has zero average on each circle $\Gamma \in \mathscr{H}$. By Theorem 1 it follows that $w_0(r) = 0$ $(0 < r < 1)$ and that for each $n \in Z$, $n \ne 0$, there are numbers a_{ni}, $0 \le i \le |n| - 1$, such that

$$w_n(r) = r^{-|n|}(a_{n0} + a_{n1}r^2 + \cdots + a_{n,|n|-1}r^{2(|n|-1)}) \qquad (0 < r < 1).$$

Since $h_n(r) = r^2 w_n(r)$ $(0 < r < 1)$ it follows that

$$r^{-|n|}h_n(r) = r^{-2|n|}(a_{n0}r^2 + a_{n1}r^4 + \cdots + a_{n,|n|-1}r^{2|n|}) \qquad (0 < r < 1).$$

Fix $n \in Z$, $n \ne 0$. Since h is infinitely differentiable at the origin there is a polynomial $P_{|n|}$ of degree $|n|$ such that $h(z) = P_{|n|}(z, \bar{z}) + Q_{|n|}(z)$ where $\lim_{z\to 0}|z|^{-|n|}Q_{|n|}(z) = 0$. This implies that there is a number α_n such that

$$h_n(r) = \alpha_n r^{|n|} + \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}Q_{|n|}(re^{i\theta})\,d\theta$$

and

$$\lim_{r\to 0} r^{-|n|}h_n(r) = \alpha_n.$$

Thus $a_{nj} = 0$ $(0 \le j \le |n| - 2)$ and $h_n(r) = a_{n,|n|-1}r^{|n|}$ $(0 < r \le 1)$. Since h vanishes identically on $b\Delta$ it follows that $h_n(1) = 0$ which implies that

$h_n(r) = 0$ $(0 < r \leq 1)$. As this holds for each $n \in Z$ it follows that h vanishes identically on Δ, that is, f coincides with g on Δ which implies that f is harmonic in Δ. This completes the proof.

Proof of Corollary 2, *assuming Theorem* 2. Suppose that f is a continuous function in Δ which is infinitely differentiable at the origin and which satisfies $\int_\Gamma f(z)\,dz = 0$ for each circle $\Gamma \in \mathscr{H}$. By Theorem 2, $f_{-1}(r) = 0$ $(0 < r < 1)$ and for each $n \in Z$, $n \neq -1$, there are numbers b_{ni}, $0 \leq i \leq |n+1| - 1$, such that

$$f_n(r) = r^{-|n|}(b_{n0} + b_{n1}r^2 + \cdots + b_{n,|n+1|-1}r^{2(|n+1|-1)}) \qquad (0 < r < 1).$$

As in the proof of Corollary 1, since f is infinitely differentiable at the origin it follows that for each $n \in Z$, $n \neq 0$, $\lim_{r\to 0} r^{-|n|} f_n(r)$, exists, that is,

$$\lim_{r\to 0}(b_{n0}r^{-2|n|} + b_{n1}r^{-2|n|+2} + \cdots + b_{n,|n+1|-1}r^{-2|n|+2|n+1|-1})$$

exists. It follows that if $n < -1$ then $f_n(r) = 0$ $(0 < r < 1)$ and if $n \geq 0$ then $f_n(r) = b_{n,|n+1|-1}r^n$ $(0 < r < 1)$. Now, for each r, $0 < r < 1$, $\sum_{-\infty}^{\infty} f_n(r)e^{in\theta}$ is the Fourier series of $f(re^{i\theta})$ so on $[0, 2\pi]$ the function $f(re^{i\theta})$ is the uniform limit of its Cesàro means

$$\begin{aligned}\sigma_m(re^{i\theta}) &= m^{-1}\left(f_0(r) + \sum_{k=-1}^{1} f_k(r)e^{ik\theta} + \cdots + \sum_{k=-(m-1)}^{m-1} f_k(r)e^{ik\theta}\right)\\ &= m^{-1}\left(b_{00} + \sum_{k=0}^{1} b_{k,k}(re^{i\theta})^k + \cdots + \sum_{k=0}^{m-1} b_{k,k}(re^{i\theta})^k\right).\end{aligned}$$

The usual proof of the Fejér theorem [8] shows that the convergence is also uniform in r, $\rho_1 \leq r \leq \rho_2$, for each ρ_1, ρ_2, $0 < \rho_1 < \rho_2 < 1$, since f is uniformly continuous in $\{\zeta\colon \rho_1 \leq |\zeta| \leq \rho_2\}$ [3]. Since each σ_m is analytic in Δ it follows that f is analytic in $\Delta\backslash\{0\}$ and being continuous at 0, f is analytic in Δ. This completes the proof.

Remark. The referee has kindly pointed out that if $f \in \mathscr{C}^\infty(\{0\})$ then the second part of Corollary 2 can be easily derived from the following consequence of an old result of A. M. Cormack: If $g \in \mathscr{C}^\infty(\Delta)$ has zero average on each circle $\Gamma \subset \Delta$ which passes through the origin then g vanishes identically (for the proof see [1]). The reason is that once $f \in \mathscr{C}^\infty(\{0\})$ then it can be uniformly approximated by $\mathscr{C}^\infty$ functions with the same vanishing properties as f. Let $f \in \mathscr{C}^\infty(\Delta)$ and let $\int_\Gamma f(z)\,dz = 0$ for each circle Γ surrounding the origin. By continuity we have that $\int_{bD} f(z)\,dz = 0$ for every disc $D \subset \Delta$, $0 \in bD$. By Green's formula, $\iint_D \partial f/\partial\bar{z}\,dz \wedge d\bar{z} = 0$. It is easy to conclude that $\int_{bD} \partial f/\partial\bar{z}\,ds = 0$ for all such discs D. It follows that $\partial f/\partial\bar{z}$ vanishes identically so f is analytic on Δ.

Proof of Corollary 3, *assuming Theorems* 1 *and* 2. The assumption implies that for each $n \in Z$, $k \in N$, the function $r \mapsto (R_2 - r)^{-k} f_n(r)$ is bounded as

$r \to R_2$. If f has zero average on each circle $\Gamma \in \mathcal{G}$ then by Theorem 1 each f_n extends to an analytic function in a neighbourhood of $(0, R_2]$ which is possible only if f_n vanishes identically so f vanishes identically in Ω. In the same way, using Theorem 2, we see that if $\int_\Gamma f(z)\,dz = 0$ for each $\Gamma \in \mathcal{G}$ then f vanishes identically in Ω. This completes the proof.

4. Zero integrals and Fourier coefficients

Let $\Gamma \subset C$ be a circle which surrounds the origin and whose center is not the origin. Let A be the closed annulus obtained by rotating Γ around the origin, that is,

$$A = \bigcup_{0 \le \alpha < 2\pi} e^{i\alpha}\Gamma.$$

Let R_1, R_2 be such that $A = \{\zeta \in C : R_1 \le |\zeta| \le R_2\}$. Let f be a continuous function on A. For each $n \in Z$, let

$$\Phi_n(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(e^{i\theta} z)\,d\theta \qquad (R_1 \le |z| \le R_2).$$

Note that if $z = re^{i\alpha}$ then $\Phi_n(z) = f_n(r)e^{in\alpha}$.

Lemma 1. *The following are equivalent*:

(i) *f has zero average on each circle $e^{i\alpha}\Gamma$, $0 \le \alpha < 2\pi$,*
(ii) *for each $n \in Z$ the function Φ_n has zero average on Γ.*

Proof. Suppose that (i) holds. Let $\Gamma = \{a + e^{i\theta} b : 0 \le \theta < 2\pi\}$. Let $n \in Z$. By the assumption, $\int_0^{2\pi} f(e^{i\alpha}(a + e^{i\theta} b))\,d\theta = 0$ $(0 \le \alpha < 2\pi)$ so by the Fubini theorem

$$\int_0^{2\pi}\left[\frac{1}{2\pi}\int_0^{2\pi} e^{-in\alpha} f(e^{i\alpha}(a + e^{i\theta} b))\,d\alpha\right] d\theta = 0$$

which proves (ii). Conversely, assume that (ii) holds. Note first that (ii) implies that for each $n \in Z$ the function Φ_n has zero average on each circle $e^{i\alpha}\Gamma$, $0 \le \alpha < 2\pi$. For each r, $R_1 \le r \le R_2$, $\sum_{-\infty}^{\infty} f_n(r)e^{in\theta} = \sum_{-\infty}^{\infty} \Phi_n(re^{i\theta})$ is the Fourier series of $f(re^{i\theta})$. As in the proof of Corollary 2 we see that its Cesàro means

$$\sigma_m(re^{i\theta}) = m^{-1}\left(\Phi_0(re^{i\theta}) + \sum_{k=-1}^{1} \Phi_k(re^{i\theta}) + \cdots + \sum_{k=-(m-1)}^{m-1} \Phi_k(re^{i\theta})\right)$$

converge to $f(re^{i\theta})$ uniformly on A which implies (i). This completes the proof.

In almost the same way we prove the following lemma.

Lemma 1′. *The following are equivalent*:

(i) $\int_{e^{i\alpha}\Gamma} f(z)\,dz = 0$ *for each* α, $0 \le \alpha < 2\pi$,
(ii) $\int_\Gamma \Phi_n(z)\,dz = 0$ *for each* $n \in Z$.

5. The integral equations for the Fourier coefficients

For each ρ, R, $0 < \rho < R$, let $\Gamma_{\rho R}$ be the circle whose diameter is the segment $[-R, \rho]$ on the real axis. Further, for each $n \in N \cup \{0\}$ let T_n be the Čebyšev polynomial of the first kind of degree n [10]. We have $T_n(\cos\alpha) = \cos n\alpha$ for each α.

Lemma 2. *Let* $n \in Z$ *and let* f_n *be a continuous function on* $[\rho, R]$. *Let* $\Phi_n(re^{i\theta}) = f_n(r)e^{in\theta}$ $(\rho \le r \le R,\ 0 \le \theta < 2\pi)$ *and let* $s = \rho/R$. *Write*

$$P_n(s,t) = T_{|n|}\left(\frac{s-t^2}{t(1-s)}\right)(t^2-s^2)^{-1/2}(1-t^2)^{-1/2}t \qquad (s<t<1),$$

$$Q_n(s,t) = \left[2tT_{|n+1|}\left(\frac{s-t^2}{t(1-s)}\right) + (1-s)T_{|n|}\left(\frac{s-t^2}{t(1-s)}\right)\right] \times (t^2-s^2)^{-1/2}(1-t^2)^{-1/2}t \qquad (s<t<1).$$

Then

(i) Φ_n *has zero average on* $\Gamma_{\rho R}$ *if and only if*

$$\int_s^1 f_n(Rt)P_n(s,t)\,dt = 0,$$

(ii) $\int_{\Gamma_{\rho R}} \Phi_n(z)\,dz = 0$ *if and only if*

$$\int_s^1 f_n(Rt)Q_n(s,t)\,dt = 0.$$

Proof. In polar coordinates r, the circle $\Gamma_{\rho R}$ is given by the equation

$$\cos\varphi = \frac{R\rho - r^2}{r(R-\rho)} \tag{5.1}$$

Differentiating this we get

$$\frac{d\varphi}{dr}\sin\varphi = \frac{R\rho + r^2}{(R-\rho)r^2} \qquad (\rho<r<R).$$

If $0 < \varphi < \pi$ then (5.1) gives

$$\sin\varphi = (1-\cos^2\varphi)^{1/2} = (r^2-\rho^2)^{1/2}(R^2-r^2)^{1/2}r^{-1}(R-\rho)^{-1}$$

so

$$\frac{d\varphi}{dr} = \frac{R\rho + r^2}{r(r^2-\rho^2)^{1/2}(R^2-r^2)^{1/2}} \qquad (\rho<r<R).$$

Similarly, if $-\pi < \varphi < 0$,

$$\frac{d\varphi}{dr} = -\frac{R\rho + r^2}{r(r^2-\rho^2)^{1/2}(R^2-r^2)^{1/2}} \qquad (\rho<r<R).$$

In both cases we have

$$\left(1+\left(r\frac{d\varphi}{dr}\right)^2\right)^{1/2} = r(R+\rho)(r^2-\rho^2)^{-1/2}(R^2-r^2)^{-1/2} \qquad (\rho < r < R).$$

Now, writing

$$\varphi(r) = \arccos\frac{R\rho - r^2}{r(R-\rho)} \qquad (\rho < r < R)$$

we have $\Gamma_{\rho R} = \{re^{i\varphi(r)}: \rho \le r \le R\} \cup \{re^{-i\varphi(r)}: \rho \le r \le R\}$ so if Ψ is a continuous function on $\Gamma_{\rho R}$ its average on $\Gamma_{\rho R}$ is

$$\pi^{-1}\int_\rho^R [\Psi(re^{i\varphi(r)}) + \Psi(re^{-i\varphi(r)})](r^2-\rho^2)^{-1/2}(R^2-r^2)^{-1/2} r\, dr.$$

To prove (i), let $\Psi = \Phi_n$ and put $r = tR$. To prove (ii), observe that $\int_{\Gamma_{\rho R}} \Phi_n(z)\, dz = 0$ if and only if the average of $\Psi(z) = \Phi_n(z)(z-(\rho-R)/2)$ on $\Gamma_{\rho R}$ is zero and put $r = tR$. This completes the proof.

Lemma 3. *Let $n \in Z$, let $0 < R_1 < R_2$ and assume that f_n is a continuous function on $[R_1, R_2]$. Let $\Psi_n(re^{i\theta}) = f_n(r)e^{in\theta}$ $(R_1 \le r \le R_2,\ 0 \le \theta < 2\pi)$. Let $R_1/R_2 < q < 1$.*

(i) *If for each ρ, R, $R_1 \le \rho < R \le R_2$, the function Ψ_n has zero average on $\Gamma_{\rho R}$ then on $[q, 1]$ the function $t \mapsto f_n(tR_2)$ can be uniformly approximated by functions Φ of class $\mathscr{C}^\infty$ which satisfy*

$$\int_s^1 \Phi(t)P_n(s,t)\, dt = 0 \qquad (q \le s < 1). \tag{5.2}$$

(ii) *If for each ρ, R, $R_1 \le \rho < R \le R_2$, $\int_{\Gamma_{\rho R}} \Psi_n(z)\, dz = 0$, then on $[q, 1]$ the function $t \mapsto f_n(tR_2)$ can be uniformly approximated by functions Φ of class $\mathscr{C}^\infty$ which satisfy*

$$\int_s^1 \Phi(t)Q_n(s,t)\, dt = 0 \qquad (q \le s < 1). \tag{5.3}$$

Proof. Suppose that for each ρ, R, $R_1 \le \rho < R \le R_2$, Ψ_n has zero average on $\Gamma_{\rho R}$. By Lemma 2

$$\int_s^1 f_n(tR)P_n(s,t)\, dt = 0 \qquad (R_1/R \le s < 1)$$

holds for each R, $R_1 < R \le R_2$. In particular, if $R_1 < R_0 < R_2$ then for each R, $R_0 < R < R_2$, the function $t \mapsto \Phi(t) = f_n(tR)$ satisfies

$$\int_s^1 \Phi(t)P_n(s,t)\, dt = 0 \qquad (R_1/R_0 < s < 1). \tag{5.4}$$

Choose $R_0 < R_2$ so close to R_2 that $R_1/R_0 < q$ and choose $\delta > 0$ so small that $(1-\delta)R_2 > R_0$.

Let χ be a nonnegative $\mathscr{C}^\infty$ function on R with support in $(1-\delta, 1)$ and such that $\int \chi(\omega)\,d\omega = 1$. Since each function $t \mapsto f_n(\omega t R_n)$, $1-\delta < \omega < 1$, satisfies (5.4) it follows by the Fubini theorem that also the function $t \mapsto \Phi(t) = \int \chi(\omega) f_n(\omega t R_2)\,d\omega$ satisfies (5.4). The function Φ is of class $\mathscr{C}^\infty$ on $R\backslash\{0\}$ and if δ is chosen small enough then $|\Phi(t) - f_n(tR_2)|$ will be uniformly small on $[R_1/R_0, 1]$. This completes the proof of (i). We prove (ii) in the same way with P_n replaced by Q_n.

6. Properties of the integral equations

To prove Theorems 1 and 2 we will need all smooth solutions of (5.2) and (5.3). We first mention two trivial special cases:

Proposition 2. *Let $0 < R_1 < R_2$ and assume that Φ is a continuous function on (R_1, R_2). If $\Psi(re^{i\theta}) = \Phi(r)$ $(R_1 < r < R_2,\ 0 \le \theta < 2\pi)$ and if Ψ has zero average on each circle $|\zeta| = r$, $R_1 < r < R_2$, then $\Phi(r) = 0$ $(R_1 < r < R_2)$. If $\Psi(re^{i\theta}) = \Phi(r)e^{-i\theta}$ $(R_1 < r < R_2,\ 0 \le \theta < 2\pi)$ and if $\int_{|\zeta|=r} \Psi(z)\,dz = 0$ for each r, $R_1 < r < R_2$, then $\Phi(r) = 0$ $(R_1 < r < R_2)$.*

This shows that we will only have to consider (5.2) if $n \neq 0$ and (5.3) if $n \neq -1$.

Let P_n and Q_n be as in Lemma 3.

Lemma 4. *Let $0 < \tau < 1$ and let Φ be a continuous function on $[1-\tau, 1]$.*

(i) *If $n \in Z$, $n \neq 0$, and if Φ satisfies*

$$\int_s^1 \Phi(t) P_n(s, t)\,dt = 0 \qquad (1-\tau < s < 1), \tag{6.1}$$

then

$$\int_0^p K_n(p, t) g(t)\,dt = 0 \qquad (0 < p < \tau)$$

where $g(t) = h(1-t)$, $h(t) = t^{-|n|+1}(1-t^2)^{-1/2}\Phi(t)$ and where K_n is analytic in a neighbourhood of zero and is of the form

$$K_n(p, t) = \sum_{i=0}^{|n|} b_i t^{|n|-i} p^i + \textit{higher order terms in } t, p$$

where $\sum_{i=0}^{|n|} b_i \neq 0$.

(ii) *If $n \in Z$, $n \neq -1$, and if Φ satisfies*

$$\int_s^1 \Phi(t) Q_n(s, t)\,dt = 0 \qquad (1-\tau < s < 1), \tag{6.2}$$

then

$$\int_0^p L_n(p, t) g(t)\,dt = 0 \qquad (0 < p < \tau)$$

where $g(t) = h(1-t)$, $h(t) = t^{-|n+1|+1}(1-t^2)^{-1/2}\Phi(t)$ *and where* L_n *is analytic in a neighbourhood of zero and is of the form*

$$L_n(p,t) = \sum_{i=0}^{|n+1|} c_i t^{|n+1|-i} p^i + \text{ higher order terms in } t, p$$

where $\sum_{i=0}^{|n+1|} c_i \neq 0$.

Proof. Let $n \in Z$, $n \neq 0$, and let Φ satisfy (6.1). Let g be as in (i). Multiplying (6.1) by $(1-s)^n$ we get

$$\int_s^1 L(s,t)h(t)(t-s)^{-1/2}\,dt = 0 \qquad (1-\tau < s < 1) \tag{6.3}$$

where

$$L(s,t) = (t+s)^{-1/2}\sum_{i=0}^{|n|} p_i(s-t^2)^{|n|-i}(t(1-s))^i$$

and where $T_{|n|}(x) = \sum_{i=0}^{|n|} p_i x^{|n|-i}$. Put $Q(s,t) = L(1-s, 1-t)$ and replace t by $1-t$ and s by $1-s$ in (6.3) to get

$$\int_0^s Q(s,t)g(t)(t-s)^{-1/2}\,dt = 0 \qquad (0 < s < \tau) \tag{6.4}$$

where

$$Q(s,t) = 2^{-1/2}\sum_{i=0}^{|n|} p_i(2t-s)^{|n|-i}s^i + \text{ higher order terms in } s, t.$$

Since $T_{|n|}(\cos x) = \cos nx$ we have $\sum_{i=0}^{|n|} p_i = 1$ so $Q(s,s) = 2^{-1/2}s^{|n|} +$ higher order terms in s. Thus we have shown that

$$Q(s,t) = \sum_{i=0}^{|n|} a_i t^{|n|-i}s^i + \text{ higher order terms in } s, t$$

where $\sum_{i=0}^{|n|} a_i \neq 0$. One completes the proof of (i) by iterating the kernel as in [9, p. 155].

To prove (ii) let first $n \geq 0$. Let

$$T_n(x) = \sum_{i=0}^{n} p_i x^{n-i}, \qquad T_{n+1}(x) = \sum_{i=0}^{n+1} q_i x^{n+1-i}.$$

Multiplying (6.2) by $(1-s)^{n+1}$ we get (6.3) where

$$L(s,t) = (t+s)^{-1/2}\left[2t\sum_{i=0}^{n+1} q_i(s-t^2)^i(t(1-s))^{n+1-i} + t(1-s)^2\sum_{i=0}^{n} p_i(s-t^2)^i(t(1-s))^{n-i}\right].$$

As before, this implies (6.4) where

$$Q(s,t) = 2^{-1/2}\left[2\sum_{i=0}^{n+1} q_i(2t-s)^i s^{n+1-i} + \text{ higher order terms in } s,t\right]$$
$$+ 2^{-1/2}s^2\left[\sum_{i=0}^{n} p_i(2t-s)^i s^{n-i} + \text{ higher order terms in } s,t\right]$$
$$= 2^{1/2}\sum_{i=0}^{n+1} q_i(2t-s)^i s^{n+1-i} + \text{ higher order terms in } s,t.$$

Again, $Q(s,s) = 2^{1/2}s^{n+1} +$ higher order terms in s, so

$$Q(s,t) = \sum_{i=0}^{n+1} a_i t^{n+1-i} s^i + \text{ higher order terms in } s,t$$

where $\sum_{i=0}^{n+1} a_i \neq 0$. Together with [9, p. 155] this completes the proof of (ii) if $n \geq 0$. Now, let $n \leq -2$. Write $n = -m$. Since $T_m(x) = 2xT_{m-1}(x) - T_{m-2}(x)$ [10] we have

$$2tT_{m-1}\left(\frac{s-t^2}{t(1-s)}\right) + (1-s)T_m\left(\frac{s-t^2}{t(1-s)}\right)$$
$$= 2\frac{s}{t}T_{m-1}\left(\frac{s-t^2}{t(1-s)}\right) - (1-s)T_{m-2}\left(\frac{s-t^2}{t(1-s)}\right)$$

so multiplying (6.2) by $(1-s)^{m-1}$ we get (6.3) where

$$L(s,t) = (t+s)^{-1/2}\left[2\frac{s}{t}\sum_{i=0}^{m-1} p_i(s-t^2)^{m-1-i}(t(1-s))^i\right.$$
$$\left. - (1-s)^2 t\sum_{i=0}^{m-2} q_i(s-t^2)^{m-2-i}(t(1-s))^i\right]$$

and where $T_{m-1}(x) = \sum_{i=0}^{m-1} p_i x^{m-1-i}$ and $T_{m-2}(x) = \sum_{i=0}^{m-2} q_i x^{m-2-i}$. As before, (6.1) follows where

$$Q(s,t) = 2^{1/2}\sum_{i=0}^{m-1} p_i(2t-s)^{m-1-i}s^i + \text{ higher order terms in } s,t.$$

Again, $Q(s,s) = 2^{1/2}s^{m-1} +$ higher order terms in s, so

$$Q(s,t) = \sum_{i=0}^{m-1} a_i t^{m-1-i} s^i + \text{ higher order terms in } s,t$$

where $\sum_{i=0}^{m-1} a_i \neq 0$. One completes the proof of (ii) as in [9, p. 155].

7. The integro-differential equation

Fix $n \in N$ and let K be analytic in a neighbourhood of zero and of the form

$$K(x,s) = \sum_{i=0}^{n} \beta_i s^{n-i} x^i + \text{ higher order terms in } s, x$$

where $\sum_{i=0}^{n} \beta_i \neq 0$. For the proofs of Theorems 1 and 2 we have to prove that if $\tau > 0$ is small then the dimension of the space of all smooth solutions g of the equation

$$\int_0^x K(x,s)g(s)\,ds = 0 \qquad (0 < x < \tau) \tag{7.1}$$

such that $s^{1/2}g(s)$ is bounded on $(0,\tau)$, does not exceed n. For continuous functions g on $[0,\tau)$ this follows from [12]. For bounded analytic functions g this also follows from [9] where also a remark about the unbounded case was made. We follow [9] to show that the approximation procedure used there works also in our case.

Write $K(s,x) = \sum_{i=0}^{\infty} a_i(s)(x-s)^i/i!$ where a_i are analytic in a neighbourhood of zero and where the series converges in a neighbourhood of zero. By the properties of K,

(7.2) a_0 has zero of order n at the origin and each a_i, $1 \le i \le n-1$, has zero of order at least $n-i$ at the origin.

Suppose that g is a smooth solution of (7.1) such that

$$|g(s)| \le Cs^{-1/2} \qquad (0 < s < \tau) \tag{7.3}$$

for some constant C. Since g is smooth on $(0,\tau)$ and satisfies (7.1) and (7.3) one can differentiate (7.1) $n+1$ times to see that g satisfies

$$(Dg)(x) = \int_0^x K_n(x,s)g(s)\,ds \qquad (0 < x < \tau) \tag{7.4}$$

where K_n is analytic in a neighbourhood of zero and where

$$Dg = (a_0 g)^{(n)} + (a_1 g)^{(n-1)} + \cdots + (a_n g).$$

By (7.2) zero is a singular point of D which is of Fuchsian type, that is, it is a regular singular point of D.

Lemma 5. *There are k, $0 \le k \le n$, complex numbers r_i, $1 \le i \le n$, $\mathrm{Re}\, r_i > 0$ $(1 \le i \le k)$, $\mathrm{Re}\, r_i \le 0$ $(k+1 \le i \le n)$, and functions Ω_i, H_i, of the form*

$$\Omega_i(x) = \sum_{j=0}^{k_i} \Omega_{ij}(x)(\log x)^j, \quad H_i(x) = \sum_{j=0}^{k_i} H_{ij}(x)(\log x)^j \tag{7.5}$$

where each Ω_{ij} and each H_{ij} is analytic in a neighbourhood of zero such that for small $\tau > 0$ the following holds:

If Ψ *is a continuous function on* $[0,\tau]$ *such that* $|\Psi(x)| \le Cx^{\eta}$ $(0 < x < \tau)$ *for some* $\eta > 0$ *and for some constant* C *then*

$$y(x) = \sum_{i=1}^{k} \Omega_i(x) x^{r_i} \int_{\tau/2}^{x} t^{-r_i-1} H_i(t)\Psi(t)\,dt + \sum_{i=k+1}^{n} \Omega_i(x) x^{r_i} \int_{0}^{x} t^{-r_i-1} H_i(t)\Psi(t)\,dt$$

satisfies $(Dy)(x) = \Psi(x)$ $(0 < x < \tau)$.

Proof. Each nonzero solution of $Dy = 0$ has the form

$$y(x) = x^r \sum_{j=0}^{m} \Phi_j(x)(\log x)^j$$

where each Φ_j is analytic in a neighbourhood of zero and at least one of the numbers $\Phi_j(0)$, $0 \le j \le m$, is different from zero [2]. Choose a fundamental system $y_i(x) = x^{r_i}\Omega_i(x)$, $1 \le i \le n$, where Ω_i are as in (7.5). The Wronskian has the form $W(x) = x^{r_1+\cdots+r_n-n(n-1)/2}R(x)$ where R is analytic in a neighbourhood of zero and satisfies $R(0) \ne 0$ [2, p. 77]. We complete the proof by using the variation of constants.

8. Successive approximations and the dimension of the space of solutions

We keep the notation from §7. Define the operator

$$(L_n\varphi)(x) = \sum_{i=1}^{k} \Omega_i(x) x^{r_i} \int_{\tau/2}^{x} t^{-r_i-1} H_i(t) \left[\int_0^t K_n(t,s)\varphi(s)\,ds\right] dt + \sum_{i=k+1}^{n} \Omega_i(x) x^{r_i} \int_{0}^{x} t^{-r_i-1} H_i(t) \left[\int_0^t K_n(t,s)\varphi(s)\,ds\right] dt.$$

If $\tau > 0$ is small and if φ is smooth on $(0,\tau)$ and such that $\varphi(x)x^{1/2}$ is bounded on $(0,\tau)$ then $L_n\varphi$ is well defined and smooth on $(0,\tau)$. Further, if τ is small then all the functions involved in the definition of L_n are analytic in $\Sigma_\tau = \{re^{i\alpha} : 0 < r < \tau, |\alpha| < \pi/4\}$ so if φ is analytic in Σ_τ and such that $\varphi(x)x^{1/2}$ is bounded on Σ_τ then $L_n\varphi$ is well defined and analytic in Σ_τ.

Using elementary estimates of the integrals we get

Lemma 6. *There is a* $\tau_0 > 0$ *such that for each* τ, $0 < \tau < \tau_0$, *and for each positive constant* C *the following hold*:

(i) *if* φ *is smooth on* $(0,\tau)$ *and if* $|\varphi(x)| \le Cx^{-1/2}$ $(0 < x < \tau)$ *then* $L_n\varphi$ *is bounded on* $(0,\tau)$

(ii) *if* φ *is smooth on* $(0,\tau)$ *and if* $|\varphi(x)| \le C$ $(0 < x < \tau)$ *then* $|(L_n\varphi)(x)| \le C/2$ $(0 < x < \tau)$

(iii) *if* φ *is analytic in* Σ_τ *and if* $|\varphi(x)| \le C|x|^{-1/2}$ $(x \in \Sigma_\tau)$ *then* $L_n\varphi$ *is bounded in* Σ_τ

(iv) *if* φ *is analytic in* Σ_τ *and if* $|\varphi(x)| \le C$ $(x \in \Sigma_\tau)$ *then* $|(L_n\varphi)(x)| \le C/2$ $(x \in \Sigma_\tau)$.

Now we use successive approximations:

Lemma 7. *If* $\tau > 0$ *is small enough then the following are equivalent*
(8.1)
(i) g *is a smooth solution of* (7.4) *such that* $g(s)s^{1/2}$ *is bounded on* $(0, \tau)$,
(ii) $g = g_0 + L_n g_0 + L_n^2 g_0 + \cdots$, *where* g_0 *satisfies* $Dg_0 = 0$ *on* $(0, \tau)$ *and is such that* $g_0(x)x^{1/2}$ *is bounded on* Σ_τ *and where the series converges uniformly on* Σ_τ.

Proof. Let $\tau > 0$ be so small that Lemma 6 holds and that in $2\tau\Delta$ every solution of $Dy = 0$ has the form $x^r \sum_{j=0}^m P_j(x)(\log x)^j$ where P_j, $0 \le j \le m$, are analytic in $2\tau\Delta$. Suppose that (i) holds. By Lemma 6, $g_0 = g - L_n g$ is smooth on $(0, \tau)$ and such that $g_0(s)s^{1/2}$ is bounded on $(0, \tau)$. By the definition of L_n we have

$$\begin{aligned}(Dg_0)(x) &= (Dg)(x) - (D(L_n g))(x) \\ &= (Dg)(x) - \int_0^x K_n(x, s)g(s)\,ds = 0 \qquad (0 < x < \tau)\end{aligned}$$

since g satisfies (7.4). Since $g_0(x)x^{1/2}$ is bounded on $(0, \tau)$ it follows that it is bounded in Σ_τ. Since g_0 is analytic there Lemma 6 implies that the series (8.1) converges uniformly in Σ_τ and since $g_0 = g - L_n g$ it follows by Lemma 6(ii) that the sum of the series (8.1) is g.

Conversely, suppose that (ii) holds. Write $L_n^i g_0 = g_i$ so that $g = \sum_{i=0}^\infty g_i$. By the uniform convergence of the series in Σ_τ we have $Dg = \sum_{i=0}^\infty Dg_i$ on $(0, \tau)$ and

$$\int_0^x K_n(x, s)g(s)\,ds = \sum_{i=0}^\infty \int_0^x K_n(x, s)g_i(s)\,ds \qquad (0 < x < \tau).$$

Since for each i,

$$(Dg_{i+1})(x) = \int_0^x K_n(x, s)g_i(s)\,ds \qquad (0 < x < \tau)$$

it follows that g is a solution of (7.4). That $g(s)s^{1/2}$ is bounded on $(0, \tau)$ follows from Lemma 6. This completes the proof.

Lemma 8. *If* τ *is small enough then the dimension of the space of all smooth solutions* g *of* (7.4) *such that* $g(s)s^{1/2}$ *is bounded on* $(0, \tau)$ *does not exceed* n.

Proof. Let $\tau > 0$ be so small that Lemmas 6 and 7 hold. If g_0 is a solution of $Dg_0 = 0$ such that $g_0(x)x^{1/2}$ is bounded in Σ_τ then by Lemma 6 the series (8.1) converges uniformly in Σ_τ. Let $y_1, y_2, \dots, y_m$ be the basis of the space of all solutions of $Dy = 0$ for which $y(x)x^{1/2}$ is bounded on Σ_τ. For each i, $1 \le i \le m$, let $\Phi_i = y_i + L_n y_i + L_n^2 y_i + \cdots$. By Lemma 7 a smooth function g on $(0, \tau)$ such that $g(s)s^{1/2}$ is bounded on $(0, \tau)$ satisfies (7.4) if and only if it is a linear combination of the functions Φ_i, $1 \le i \le m$. This completes the proof.

Now, using Lemma 4 and §7 we get the following consequence:

Lemma 9. *Let $n \in N$, $n \neq 0$. There is some $q_0 < 1$ such that for each q, $q_0 < q < 1$, the dimension of the space of smooth functions Φ on $[q, 1]$ which satisfy* (5.2) *does not exceed $|n|$.*

Let $n \in N$, $n \neq -1$. There is some $q_0 < 1$ such that for each q, $q_0 < q < 1$, the dimension of the space of smooth functions Φ on $[q, 1]$ which satisfy (5.3) *does not exceed $|n+1|$.*

9. Examples and the proofs of Theorems 1 and 2

Proposition 3. *Let $n \in N$. Each of the functions $z \mapsto z^{n-k}\bar{z}^{-k}$, $1 \le k \le n$, has zero average on each circle that surrounds the origin.*

Proof [5]. Fix k, $1 \le k \le n$, and let $\varphi(z) = z^{n-k}\bar{z}^{-k}$. Let $0 \le |a| < |b|$. Then

$$\begin{aligned}\int_0^{2\pi} \varphi(a + e^{i\theta}b)\, d\theta &= \int_0^{2\pi} (a + e^{i\theta}b)^{n-k}(\bar{a} + e^{-i\theta}\bar{b})^{-k}\, d\theta \\ &= -i\int_0^{2\pi} (\bar{a}e^{i\theta} + \bar{b})^{-k}(e^{i\theta})^{k-1}(a + e^{i\theta}b)^{n-k} ie^{i\theta}\, d\theta \\ &= -i\int_{b\Delta} (\bar{a}w + \bar{b})^{-k} w^{k-1}(a + bw)^{n-k}\, dw.\end{aligned}$$

Since $|b| > |a|$, since $k \ge 1$ and since $n - k \ge 0$ the integrand in the last integral is analytic in a neighbourhood of $\overline{\Delta}$ so the last integral is zero. This completes the proof.

Example [5]. If $n \in N$ then the function

$$\varphi(z) = \begin{cases} \bar{z}^{-1} z^{n+2} & (z \neq 0), \\ 0 & (z = 0) \end{cases}$$

is of class $\mathscr{C}^n$ on C. By Propositions 1 and 3 we have $\Phi_\Gamma(0) = \varphi(0)$ for every circle Γ that surrounds the origin, yet φ is not harmonic in Δ.

Proposition 4. *Let $0 < a < b \le 1$ and let $n \in N$. The uniform limit on (a, b) of a sequence of polynomials of the form $a_0 + a_1 x^2 + \cdots + a_n x^{2n}$ is a polynomial of the same form.*

The proof is easy and we omit it.

Proof of Theorem 1. It is enough to prove the equivalence of (ii) and (iii). Let f be a continuous function on Ω. Assume that f satisfies (iii). By Proposition

3 for each $n \in Z$ the function $re^{i\theta} \mapsto f_n(r)e^{in\theta}$ has zero average on each circle $\Gamma \subset \Omega$ surrounding the origin so by Lemma 1 f satisfies (ii).

Suppose that f satisfies (ii). By Proposition 2, $f_0(r) = 0$ $(R_1 < r < R_2)$. Let $n \in Z$, $n \neq 0$. By Proposition 3 and Lemma 2 each of the functions $\Phi_k(t) = t^{|n|-2k}$, $1 \leq k \leq n$, satisfies (5.2). By Lemma 9 it follows that if $q_0 < q < 1$ then each smooth solution of (5.2) is a linear combination of the functions Φ_k, $1 \leq k \leq n$. By Proposition 4 the uniform limit on $[q, 1]$ of a sequence of linear combinations of Φ_k, $1 \leq k \leq n$, is again a linear combination of Φ_k, $1 \leq k \leq n$. If $R_1 < R < R_2$ and if $q > R_1/R$ then it follows by Lemma 3 that on $[q, 1]$ the function $t \mapsto f_n(tR)$ is a linear combination of Φ_k, $1 \leq k \leq n$. As this holds for every R, $R_1 < R < R_2$, the proof is complete.

Proposition 5. *If $n \in Z$, $n \geq 0$, then each of the functions $F_k(z) = z^{n-k}\bar{z}^{-k}$, $0 \leq k \leq n$, satisfies $\int_\Gamma F_k(z)\,dz = 0$ for each circle Γ surrounding the origin. If $n \in Z$, $n \leq -2$ then each of the functions $G_k(z) = \bar{z}^k z^{n+k}$, $0 \leq k \leq -n-2$, satisfies $\int_\Gamma G_k(z)\,dz = 0$ for each circle Γ surrounding the origin.*

Proof. Let $n \geq 0$ and let $0 \leq k \leq n$. Let $0 < |a| < |b|$ and let $\Gamma = \{a + e^{i\theta}b : 0 \leq \theta < 2\pi\}$. If $F_k(z) = z^{n-k}\bar{z}^{-k}$ then

$$\begin{aligned}\int_\Gamma F_k(z)\,dz &= \int_0^{2\pi} (a + e^{i\theta}b)^{n-k}(\bar{a} + e^{-i\theta}\bar{b})^{-k} ibe^{i\theta}\,d\theta \\ &= b\int_0^{2\pi} e^{ik\theta}(a + e^{i\theta}b)^{n-k}(\bar{a}e^{i\theta} + \bar{b})^{-k} ie^{i\theta}\,d\theta \\ &= b\int_{b\Delta} w^k(a + wb)^{n-k}(\bar{a}w + \bar{b})^{-k}\,dw = 0\end{aligned}$$

since the integrand in the last integral is analytic in a neighbourhood of $\overline{\Delta}$. Let $n \leq -2$. Write $n = -m$ and let $G_k(z) = \bar{z}^k z^{-m+k}$, $0 \leq k \leq m-2$. We have

$$\begin{aligned}\overline{\int_\Gamma G_k(z)\,dz} &= \overline{\int_0^{2\pi} (\bar{a} + e^{-i\theta}\bar{b})^k (a + e^{i\theta}b)^{-m+k} ibe^{i\theta}\,d\theta} \\ &= -\int_0^{2\pi} (a + e^{i\theta}b)^k(\bar{a} + e^{-i\theta}\bar{b})^{-m+k} i\bar{b}e^{-i\theta}\,d\theta \\ &= -\int_0^{2\pi} (a + e^{i\theta}b)^k e^{i(m-k)\theta}(\bar{a}e^{i\theta} + \bar{b})^{-m+k} i\bar{b}e^{-i\theta}\,d\theta \\ &= -\bar{b}\int_{b\Delta} (a + wb)^k w^{m-k-2}(\bar{a}w + \bar{b})^{-m+k}\,dw = 0\end{aligned}$$

since the integrand in the last integral is analytic in a neighbourhood of $\overline{\Delta}$.

Example [3]. If $n \in N$ then the function

$$\varphi(z) = \begin{cases} \bar{z}^{-1}z^{n+2} & (z \neq 0), \\ 0 & (z = 0) \end{cases}$$

is of class $\mathscr{C}^n$ on C. By Proposition 5 we have $\int_\Gamma \varphi(z)\,dz = 0$ for every circle Γ that surrounds the origin, yet φ is not analytic in Δ.

Proof of Theorem 2. It is enough to prove the equivalence of (ii) and (iii). Let f be a continuous function on Ω. If f satisfies (iii) then by Proposition 5 for each $n \in Z$ the function $re^{i\theta} \mapsto \Psi_n(re^{i\theta}) = f_n(r)e^{in\theta}$ satisfies $\int_\Gamma \Psi_n(z)\,dz = 0$ for each circle $\Gamma \subset \Omega$ surrounding the origin so by Lemma 1′ f satisfies (ii).

Suppose that f satisfies (ii). By Proposition 2, $f_{-1}(r) = 0$ $(R_1 < r < R_2)$. Let $n \in Z$, $n \neq -1$. If $n \geq 0$ then by Proposition 5 and Lemma 2 each of the functions $\Phi_k(t) = t^{-|n|+2k}$, $0 \leq k \leq n$, satisfies (5.3). As in the proof of Theorem 1 it follows that if $R_1 < R < R_2$ and if $q > R_1/R$ then on $[q, 1]$ the function $t \mapsto f_n(tR)$ is a linear combination of Φ_k, $0 \leq k \leq n$. As this holds for every R, $R_1 < R < R_2$, (iii) follows for $n \geq 0$. If $n \leq -2$ then by Proposition 5 and Lemma 2 each of the functions $\Phi_k(t) = t^{-|n|+2k}$, $0 \leq k \leq |n| - 2$, satisfies (5.3) and again (iii) follows for $n \leq -2$. This completes the proof.

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REFERENCES

1. A. M. Cormack and E. T. Quinto, *A Radon transform on spheres through the origin in* R^n *and applications to the classical Darboux equation*, Trans. Amer. Math. Soc. **260** (1980), 575–581.
2. A. R. Forsyth, *Theory of differential equations*, Vol. 4, Dover, New York, 1959.
3. J. Globevnik, *Analyticity on rotation invariant families of curves*, Trans. Amer. Math. Soc. **280** (1983), 247–254.
4. ——, *Testing analyticity on rotation invariant families of curves*, Trans. Amer. Math. Soc. **306** (1988), 401–410.
5. J. Globevnik and W. Rudin, *A characterization of harmonic functions*, Indag. Math. **91** (1988), 419–426.
6. E. Goursat, *Cours d'analyse mathématique*, Vol. II, Gauthier-Villars, Paris, 1933.
7. S. Helgason, *The Radon transform*, Progress in Math., No. 5, Birkhäuser, Boston, Mass., Basel and Stuttgart, 1980.
8. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N. J., 1962.
9. T. Lalesco, *Sur l'equation de Volterra*, J. Math. (6) **4** (1908), 125–202.
10. W. Magnus, F. Oberhettinger and R. P. Soni, *Formulas and theorems for the special functions of mathematical physics*, Grundlagen. Math. Wiss., Bd. 52, Springer, Berlin, 1966.

11. V. G. Romanov, *Integral geometry and inverse problems for hyperbolic equations*, Tracts in Natural Philos., Vol. 26, Springer, Berlin, 1974.

12. V. Volterra and J. Pérès, *Théorie générale des fonctionelles*, Gauthier-Villars, Paris, 1936.

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