

SOME WEIGHTED INEQUALITIES ON PRODUCT DOMAINS

HENRY LIN

ABSTRACT. We extend the results of R. Fefferman [3] on the bidisc to higher product domains via induction. As an application, we extend the weighted inequality for Calderon-Zygmund operators on the bidisc to higher product domains, and we also extend the result of the Littlewood-Paley operator corresponding to the arbitrary disjoint rectangles to the weighted case.

0. INTRODUCTION

In [3], R. Fefferman discovered a general technique which can be used to prove weighted results in $R^1 \times R^1$ (the product domain) for product weights $A_{p/2}(R^1 \times R^1)$ with $2 < p < \infty$. His technique, however, explicitly used the special geometric properties of R^2 . In this paper we generalize his technique to higher dimension, i.e. $R^1 \times R^1 \times R^1$ and beyond. To be more specific, we reduce Fefferman's technique to a verification of one main estimate for a linear operator T :

$$\int_{\Omega} |Q_t(Tf)(x)|^2 dx \frac{dt}{t} \leq C \int_{\Omega} M(f^2) dx.$$

We will explain the notation in a later section. Having this estimate, we can conclude that T is a bounded map on $L^p(w)$ for $2 < p < \infty$ and a product weight $w \in A_{p/2}$.

Verifying the main estimate for various operators T is not entirely trivial. We work on two operators; we prove the $L^p(w)$ boundedness for $2 < p < \infty$, $w \in A_{p/2}$ for the product domain generalization of Calderon-Zygmund operators, and lastly we prove the following Littlewood-Paley result:

Theorem. Let $\{R_l\}$ be a set of disjoint rectangles with sides parallel to the coordinate axes. Let $\{\chi_{R_l}\}$ be their characteristic functions. Define $S_l(f)^\wedge(\xi) = \chi_{R_l}(\xi) \hat{f}(\xi)$. We have the operator

$$\Delta(f)(x) = \left(\sum_l |S_l(f)(x)|^2 \right)^{1/2}$$

bounded on $L^p(w)$, $2 < p < \infty$, and $w \in A_{p/2}(R^1 \times R^1)$.

Received by the editors February 9, 1988.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 42B20; Secondary 42B25.

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0002-9947/90 \$1.00 + \$.25 per page

In [1] Rubio de Francia first proved the above result in one dimension, then Jean-Lin Journé, in his paper [2], extended the unweighted result to higher dimensions. We are able to fill in the weighted part. For this result, many of our techniques are composites of methods developed by Rubio de Francia [1], Journé [2], and R. Fefferman [3]. We will assume that the reader is familiar with the marvelous paper of Rubio de Francia [1].

We remark that even though our product domain is labeled as $R^1 \times \cdots \times R^1$, it can also be $R^{d_1} \times \cdots \times R^{d_n}$ as this requires only a change in notation.

We wish to thank the reviewer for his helpful suggestions and corrections in making this paper more intelligible.

1. PRELIMINARY DEFINITIONS

The proof is going to be in the language of product domains; we first review some terminology.

We say $w \in A_p(R^1 \times \cdots \times R^1)$ if w is a nonnegative measurable function on R^n , and as a function of each x_i variable (with other variables fixed) it belongs to $A_p(R^1)$, the classical A_p class, with its $A_p(R^1)$ norm uniformly bounded with respect to the other variables.

By a dyadic rectangle we mean a rectangle with sides parallel to the coordinate axes and whose sides are dyadic intervals. We say an interval is dyadic if it is of the form $[x, x + x^k)$, $x \in 2^k \mathbb{Z}$ for some integer k . Let cR be the rectangle with the same center as the rectangle R but each side is lengthened by the factor c .

Finally if $x \in R^n$, $t = (t_1, \dots, t_n)$, $t_i > 0$, we denote by $R_t(x)$ the rectangle centered at x with sides parallel to the axes and having side length t_1, t_2, \dots, t_n for each respective side. In practice, we need to associate each rectangle $R_t(x)$ to a dyadic rectangle. We outline the procedure. For each interval I , $2^k \leq |I| \leq 2^{k+1}$ for an integer k , we associate a dyadic interval I^d such that $|I^d \cap I| > \frac{1}{2}|I|$ and $|I^d| = 2^k$. We can also have $2I \supset I^d$ and $4I^d \supset I$. Applying this to each side of $R_t(x)$, we can select a dyadic rectangle R^d so that $|R^d \cap R_t(x)| > |R_t(x)|/2^n$, and $2R_t(x) \supset R^d$, $4R^d \supset R_t(x)$.

We always let Ω be a bounded open set in R^n , and let \sim be the set enlarging operator defined by

$$\Omega_1 = \tilde{\Omega} = \{x | M_s(\chi_\Omega)(x) > 1/4^n\}, \quad \Omega_{i+1} = (\tilde{\Omega}_i)$$

where M_s is the strong maximal operator.

Next we give some appropriate extension of definitions found in [3] and [6].

Definition 1. We are in R^n . By a maximal dyadic rectangle of Ω in the i th dimension we mean a dyadic rectangle $R \subset \Omega$ whose i th side cannot be stretched to an even larger dyadic interval so that the resulting rectangle is still contained in Ω . We call $M^i(\Omega)$ the collection of maximal dyadic rectangles of Ω in the i th dimension.

Definition 2. For $R \in M^n(\Omega)$, $R = I_1 \times \cdots \times I_n$, we let $\widehat{R} = \widehat{I}_1 \times \cdots \times I_n$ where \widehat{I}_1 is the largest dyadic interval containing I_1 so that $\widehat{R} \in M^1(\widetilde{\Omega})$. We define $\gamma_1(R) = |\widehat{I}_1|/|I_1|$. Similarly we can define $\widehat{\widehat{R}} = \widehat{I}_1 \times \widehat{I}_2 \times I_3 \times \cdots \times I_n$ in the same way, and here $\gamma_2(\widehat{R}) = |\widehat{I}_2|/|I_2|$; similarly for $\gamma_3, \gamma_4, \dots$. We have $\gamma_i = 2^m$ for some positive integer m . We call these γ_i “the stretching factors of R over Ω .”

Definition 3. Let I be a dyadic interval, and let the R' be dyadic rectangles in R^{n-1} . We denote

$$E_I = \bigcup \{R' \subset R^{n-1} \mid I \times R' \subset \Omega\},$$

$$A_{I,m} = \bigcup \{R' \subset R^{n-1} \mid R = I \times R' \in M^n(\Omega), \gamma_1(R) = 2^m\}.$$

We sometimes write $A_{I,m}$ as A_I if no confusion exists.

The next lemma could be found in [6]. For the convenience of the readers we include a proof.

Lemma 4. Let I be a dyadic interval, let I' be the unique dyadic interval containing I and $|I'| = 2^{m+1}|I|$. Then $A_{I,m} \subset \widetilde{E_I \setminus E_{I'}}$.

Proof. Obviously $A_{I,m} \subset E_I$. We show that $E_{I'}$ does not take up too much of $A_{I,m}$. Let R' be one of the rectangles making up $A_{I,m}$. $\gamma_1 = 2^m$ implies $I' \times R' \not\subset \widetilde{\Omega}$, that is

$$\frac{1}{4^n} \geq \frac{|I' \times R' \cap \Omega|}{|I' \times R'|} \geq \frac{|I' \times (R' \cap E_{I'})|}{|I' \times R'|} = \frac{|E_{I'} \cap R'|}{|R'|}.$$

Therefore

$$|R' \cap (E_I \setminus E_{I'})| \geq (1 - 4^{-n})|R'| > (1/4^{n-1})|R'|$$

and the result follows.

Lemma 5. Let Ω and $\widetilde{\Omega}$ be as above. Then for all $g \geq 0$ we have

$$\int_{\widetilde{\Omega}} g \, dx \leq C \int_{\Omega} M_1 \cdots M_n(g) \, dx$$

where M_i is the Hardy-Littlewood maximal operator acting on the i th dimension.

Proof. This is a consequence of a lemma by C. Fefferman and E. Stein [4] which states for $r > 1$ and $f, g \geq 0$

$$\int_{R^1} (M_1 f)^r g \, dx \leq C_r \int f^r M_1 g \, dx.$$

We apply the above inequality in each dimension to get

$$\int_{R^n} (M_1 \cdots M_n f)^r g \, dx \leq C \int_{R^n} f^r M_1 \cdots M_n g \, dx.$$

With $f = \chi_\Omega$ and $\Omega^1 = \{x | M_1 \cdots M_n(\chi_\Omega)(x) > 1/4^n\}$, we have from $\tilde{\Omega} \subset \Omega^1$

$$\int_{\tilde{\Omega}} g \leq \int_{\Omega^1} g \leq (4^n C) \int_{\Omega} M_1 \cdots M_n g. \quad \text{Q.E.D.}$$

We can now prove our version of Journé's covering lemma. The two parameter version is due to R. Fefferman [3].

Lemma (Journé's covering lemma). *If $R \in M^n(\Omega)$, let $\gamma_1 = \gamma_1(R)$, $\gamma_2 = \gamma_2(\hat{R})$, Then for all $g \geq 0$, we have for any $\delta > 0$*

$$\sum_{R \in M^n(\Omega)} |R| \gamma_1^{-\delta} \cdots \gamma_{n-1}^{-\delta} \inf_R g(x) \leq C_\delta \int_{\Omega} M g dx,$$

where M is our generic notation for finitely many compositions of one-dimensional Hardy-Littlewood maximal operators.

Proof. We prove this by induction on the dimension. When $n = 1$, it is obvious since the maximal dyadic intervals are disjoint. We assume the lemma for $n - 1$. For n we write $R = I \times R'$, $x = (x_1, x')$, $x' \in R'^{n-1}$. We are going to follow a technique of R. Fefferman [3] in sorting out R . We first group R according to $\gamma_1(R) = 2^m$, $1 \leq m \leq \infty$. Having done this we further sort out R according to the length $|I|$; we write $|I| = 2^{i+j(m+1)}$ where i, j are integers and $1 \leq i \leq (m+1)$ and group R according to the double indices i, j . We write the left side of the formula as

$$\sum_{m=1}^{\infty} \sum_{i=1}^{m+1} \sum_{j=-\infty}^{\infty} \sum' |I| 2^{-\delta m} \sum_{R' \in M^n(A_{I,m})} |R'| \gamma_2^{-\delta} \cdots \gamma_{n-1}^{-\delta} \inf_{x_1 \in I} \inf_{x' \in R'} g(x_1, x')$$

where \sum' extends over all I with length $2^{i+j(m+1)}$.

Since $\Omega \supset I \times A_I$, the γ_i , $i \geq 2$, are greater than γ'_i , the stretching factors of R' over A_I . Thus the last sum above is dominated by

$$\begin{aligned} \inf_{x_1 \in I} \sum_{R' \in M^n(A_I)} |R'| (\gamma'_2 \cdots \gamma'_{n-1})^{-\delta} \inf_{x' \in R'} g(x_1, x') &\leq C \inf_{x_1 \in I} \int_{A_I} M g(x_1, x') dx' \\ &\leq C \inf_{x_1 \in I} \int_{E_I \setminus E_{I'}} M g(x_1, x') dx', \end{aligned}$$

where I' is the unique dyadic interval containing I with $|I'| = 2^{m+1}|I|$. We have used Lemmas 4 and 5 in the last line above. Thus the entire expression is dominated by

$$\begin{aligned} &\sum_{m=1}^{\infty} 2^{-\delta m} \sum_{i=1}^{m+1} \left(\sum_j \sum' |I| \inf_{x_1 \in I} \int_{E_I \setminus E_{I'}} M g dx' \right) \\ &\leq C \sum_m 2^{-\delta m} \sum_i \int_{\Omega} M g dx \\ &\leq C \left(\sum_{m=1}^{\infty} 2^{-\delta m} (m+1) \right) \int_{\Omega} M g dx. \end{aligned}$$

The first inequality holds because the sets $\{I \times E_I \setminus E_{I'}\}$ as indexed by j (which appeared in $|I|$ and $|I'|$) are disjoint. This gives the covering lemma.

2. THE GENERAL PROCEDURE

We outline here a procedure which can be in general used to yield the weighted norm result for $2 < p < \infty$, and a weight $w \in A_{p/2}$ for product domains. This is taken from R. Fefferman [3] with minor adjustments.

We begin by assuming a main estimate. In practice this is the step that requires checking. The proof of this estimate for certain operators T will be carried out in later sections.

Assumption (the main estimate). Let Ω be open and bounded in R^n . Let $t = (t_1, \dots, t_n)$, $t_i > 0$, and let Q_t be the convolution operator with $\prod_{i=1}^n (1/t_i) \psi(x_i/t_i)$ where $\int \psi = 0$ and ψ is supported on $[-1/2, 1/2]$. Let $R_t(x)$ be a rectangle centered at x with i th side having length t_i . Define $\bar{\Omega} = \{(x, t) | R_t(x) \subset \Omega\}$. Then we have

$$\int_{\bar{\Omega}} |Q_t(Tf)|^2 dx \frac{dt}{t} \leq C \int_{\Omega} M(f^2) dx$$

where $dx = dx_1 \cdots dx_n$, $dt/t = dt_1/t' \cdots dt_n/t_n$, and M is finitely many compositions of one dimensional Hardy-Littlewood maximal operators.

Theorem. *If the main estimate holds, then the linear operator T is bounded on $L^p(w)$, for $2 < p < \infty$ and $w \in A_{p/2}(R^1 \times \cdots \times R^1)$.*

Proof of the theorem. We use the product Lusin area integral

$$S(f)(x) = \left(\int_{\Gamma(x)} |Q_t f|^2 dy \frac{dt}{t^{n+1}} \right)^{1/2}$$

where $\Gamma(x)$ is the tensor product of the cones $\{(y_i, t_i) | |y_i - x_i| < t_i\}$. It is known that $\|Sf\|_{L^p(w)}$ is equivalent to $\|f\|_{L^p(w)}$ for this range of p . Thus for $2 < p < \infty$, $w \in A_{p/2}$,

$$\|T(f)\|_{L^p(w)} \leq C \|S(Tf)\|_{L^p(w)}.$$

We need to show that for $\phi \in C_c^\infty(R^n)$, $\|\phi\|_{L^{(p/2)'}(w)} \leq 1$,

$$\int \{S(Tf)\}^2 \phi w dx \leq C \left(\int f^p w dx \right)^{2/p}.$$

Explicitly writing out $S(Tf)^2$ and interchanging the integrals, we get

$$(1) \quad \int S(Tf)^2 \phi w dx = \int_{(R_+^2)^n} |Q_t(Tf)(y)|^2 \left(\frac{1}{|R_t(y)|} \int_{R_t(y)} \phi w dx \right) dy \frac{dt}{t}.$$

Let us define

$$O_k = \{x | M_s(\phi w)(x) > 2^k\},$$

$$A_k = \{(y, t) | R_t(y) \subset O_k, R_t(y) \not\subset O_{k+1}\}.$$

We remark that since ϕ has compact support, O_k is always open and bounded. The expression in (1) is dominated by

$$(2) \quad \sum_{k=-\infty}^{+\infty} 2^{k+1} \int_{A_k} |Q_t(Tf)(y)|^2 dy \frac{dt}{t}.$$

Now $A_k \subset \overline{O_k}$, so we can replace A_k by $\overline{O_k}$ and apply the main estimate. (2) then is bounded by

$$\sum_{k=-\infty}^{\infty} 2^{k+1} \int_{O_k} M(f^2) dx.$$

And this is equivalent to

$$\int M(f^2) M_s(\phi w) dx,$$

which means (1) becomes

$$\int S(Tf)^2 \phi w dx \leq C \int M(f^2) M_s(\phi w) dx.$$

Finally, we get the weighted inequality by multiplying the left-hand side above by $w^{2/p}/w^{2/p}$ and applying Hölder's inequality with exponents $p/2$, $(p/2)'$. The left side is dominated by

$$\left(\int M(f^2)^{p/2} w dx \right)^{2/p} \cdot \left(\int M_s(\phi w)^{(p/2)'} w^{1-(p/2)'} dx \right)^{1/(p/2)'}$$

From the fact that M and M_s are bounded with respect to weights w and $w^{1-(p/2)'} \in A_{(p/2)'}$, we have

$$\int S(Tf)^2 \phi w dx \leq C \left(\int f^p w dx \right)^{2/p}.$$

Taking the supremum in ϕ yields the theorem. Q.E.D.

Remark. This theorem applies even when T is a Hilbert space valued operator.

3. WEIGHTED NORM BOUNDEDNESS OF $CZ_e^2(R^1 \times \cdots \times R^1)$

3.1. The class $CZ_e^2(R^1 \times \cdots \times R^1)$. We are going to give the definition of $CZ_e^2(R^1 \times \cdots \times R^1)$ inductively, beginning with $CZ_e^2(R^1)$. We say $T \in CZ_e^2(R^1)$ if T is a linear operator bounded on $L^2(R^1)$ and has the following properties:

(1) there exists a kernel $k(x, y)$ defined on $(R^1 \times R^1) \setminus \Delta$ where $\Delta = \{(x, y) | x = y\}$, and such that if $f, g \in C_c^\infty(R^1)$ have disjoint supports, then

$$\int g(x) T(f)(x) dx = \iint g(x) k(x, y) f(y) dx dy.$$

(2) The kernel $k(x, y)$, in addition, satisfies the condition

$$\left(\int_{|x-y| > \gamma |x-x'|} |k(x, y) - k(x', y)|^2 dy \right)^{1/2} \leq C \gamma^{-(1/2+\epsilon)} |x - x'|^{-1/2}, \quad \gamma \geq 2.$$

We define a norm on this class $CZ_\varepsilon^2(R^1)$ by

$$\|T\|_{CZ} = \|T\|_{2,2} + \inf_{C \text{ in (2)}} C.$$

For $T \in CZ_\varepsilon^2(R^1 \times R^1)$, we require T to be a bounded linear operator on $L^2(R^2)$ with

(1') for each $(x_1, y_1), (x_2, y_2) \in R^1 \times R^1$, we have operators $k^1(x_1, y_1), k^2(x_2, y_2)$ belonging in $CZ_\varepsilon^2(R^1)$, such that if $f_i, g_i \in C_c^\infty(R^1)$, $i = 1, 2$, then

$$\begin{aligned} & \iint g_1(x_1) g_2(x_2) T(f_1 f_2)(x_1, x_2) dx_1 dx_2 \\ &= \iint g_1(x_1) \langle g_2, k^1(x_1, y_1)(f_2) \rangle f_1(y_1) dx_1 dy_1 \end{aligned}$$

whenever g_1, f_1 have disjoint supports. And the dual condition

$$\begin{aligned} & \iint g_1(x_1) g_2(x_2) T(f_1 f_2)(x_1, x_2) dx_1 dx_2 \\ &= \iint g_2(x_2) \langle g_1, k^2(x_2, y_2)(f_1) \rangle f_2(y_2) dx_2 dy_2 \end{aligned}$$

holds whenever g_2, f_2 have disjoint supports.

(2') The operators $k^1(x_1, y_1), k^2(x_2, y_2)$ satisfy

$$\begin{aligned} & \left(\int_{|x_i - y_i| > \gamma |x_i - x'_i|} \|k^i(x_i, y_i) - k^i(x'_i, y_i)\|_{CZ}^2 dy_i \right)^{1/2} \\ & \leq C_i \gamma^{-(1/2+\varepsilon)} |x_i - x'_i|^{-1/2}, \quad \gamma \geq 2, \end{aligned}$$

for $i = 1, 2$. We define a norm on T by

$$\|T\|_{CZ} = \|T\|_{2,2} + \sum_{i=1}^2 \inf_{C_i \text{ in (2')}} C_i.$$

The class $CZ_\varepsilon^2(R^1 \times \cdots \times R^1)$ is then defined inductively. (See [2].)

We remark here that $CZ_\varepsilon^2(R^1)$ includes the classical Calderon-Zygmund operator on R^1 , that is, those linear operators bounded on $L^2(R^1)$ with kernels $k(x, y)$ satisfying

$$|k(x, y)| \leq C/|x - y|,$$

$$|k(x, y) - k'(x', y)| \leq C \frac{|x - x'|^\delta}{|x - y|^{1+\delta}}, \quad 2|x - x'| \leq |x - y|, \quad \delta > 0.$$

In fact the product domain generalization of these classical Calderon-Zygmund operators (see Journé [2]) are in $CZ_\varepsilon^2(R^1 \times \cdots \times R^1)$.

3.2. Main estimate for operators in $CZ_e^2(R^1 \times \cdots \times R^1)$.

Theorem. *If $T \in CZ_e^2(R^1 \times \cdots \times R^1)$ then T is bounded on $L^p(\omega)$ for $2 < p < \infty$, $\omega \in A_{p/2}(R^1 \times \cdots \times R^1)$.*

Proof. We only need to show the main estimate

$$\int_{\Omega} |Q_t(Tf)(x)|^2 dx \frac{dt}{t} \leq C \|T\|_{CZ}^2 \int_{\Omega} M(f^2)(x) dx.$$

We are going to use a method composed of techniques developed in [2] and [3]. We first chop f into an inside and outside part with the outside part supported far away from Ω , that is,

$$f(z) = f\chi_{\Omega_{n+3}}(z) + f\chi_{\Omega_{n+3}^c}(z) = f_I + f_O.$$

For f_I , we use the L^2 boundedness of T and Plancherel equality, so that

$$\begin{aligned} \int_{\Omega} |Q_t T(f_I)(x)|^2 dx \frac{dt}{t} &\leq \int_{\bar{R}^n} |Q_t T(f_I)(x)|^2 dx \frac{dt}{t} \\ &= \int_{R^n} \prod_{i=1}^n \int_0^\infty |\hat{\psi}(t_i \xi_i)|^2 |T(f_I)^\wedge(\xi)|^2 \frac{dt_i}{t_i} d\xi \\ &= \left(\int_0^\infty |\hat{\psi}(t_1)|^2 \frac{dt_1}{t_1} \right)^n \int_{R^n} |T(f_I)(x)|^2 dx \\ &\leq C \|T\|_{CZ}^2 \int_{\Omega_{n+3}} f^2(x) dx \leq C \|T\|_{CZ}^2 \int_{\Omega} M(f^2)(x) dx. \end{aligned}$$

The last inequality is due to Lemma 5.

For the term involving f_O , we need to decompose further. We start by decomposing $\bar{\Omega}$, the region of integration. For each $(x, t) \in \bar{\Omega}$, we have a rectangle $R_t(x) \subset \Omega$. To this rectangle we associate a dyadic rectangle R in the manner indicated in §1. We also have $4R \supset R_t(x)$ and $2R_t(x) \supset R$, and thus, as a consequence, $R \subset \Omega_1$. We enlarge the n th side of R so that $R \in M^n(\Omega_1)$. Thus we have

$$\bar{\Omega} \subset \bigcup_{R \in M^n(\Omega_1)} \overline{4R}.$$

For each $R = I_1 \times I_2 \times \cdots \times I_n$ in the above, we have $\hat{I}_1 \times I_2 \times \cdots \times I_n \in M'(\Omega_2)$, $\hat{I}_1 \times \hat{I}_2 \times I_3 \times \cdots \times I_n \in M^2(\Omega_3)$, etc. as in Definition 2 of §1. Now we introduce some notation. We denote S as a subset of $\{1, 2, \dots, n\}$ and S^n as the collection of all nonempty subsets of $\{1, 2, \dots, n\}$. And as in [2], we have, writing $S = \{s_1, s_2, \dots, s_k\}$ with $s_1 < s_2 < \cdots < s_k$ the following: for each $(x, t) \in \bar{\Omega}$, we decompose f_O in the expression $Q_t T(f_O)$ as

$$f_O(z) = \sum_{S \in S^n} f_0(z) (-1)^{k+1} \chi_{(4\hat{I}_{s_1})^c}(z) \cdots \chi_{(4\hat{I}_{s_k})^c}(z).$$

Thus we only need to show for one of these terms above,

$$(1) \quad \int_{\Omega} \left| Q_t T \left(f_0 \prod_{v=1}^k \chi_{(4\hat{I}_{s_v})^c} \right) (x) \right|^2 dx \frac{dt}{t} \leq C \|T\|_{CZ}^2 \int_{\Omega} M(f^2)(x) dx,$$

where the intervals \hat{I}_{s_v} depend on the value of (x, t) .

To show (1), let us first work for the case $n = 1$. We have

$$(2) \quad \int_{\Omega} |Q_t T(f_0 \chi_{(4\hat{I})})(x)|^2 dx \frac{dt}{t} \leq \sum_{I \in M'(\Omega_1)} \int_{4I} |Q_t T(f_0 \chi_{(4\hat{I})^c})(x)|^2 dx \frac{dt}{t},$$

where on the right side we lumped all those (x, t) having the same \hat{I} (these (x, t) must be necessarily in $4\hat{I}$). Next we write $Q_t T(f_0 \chi)$ as

$$(2.1) \quad \int \psi_t(x - y) \sum_{l=1}^{\infty} \int_{|z-x| \sim 2^l |I|} (k(y, z) - k(x, z)) f_0 \chi_{(4\hat{I})^c}(z) dz dy,$$

where $|x - z| \sim 2^l |I|$ means $2^l |I| \leq |x - z| \leq 2^{l+1} |I|$. Applying Hölder's inequality, $Q_t T(f_0 \chi)$ is less than

$$\begin{aligned} & C \|T\|_{CZ} \frac{1}{t} \int_{|x-y| < t} |I|^{-\varepsilon} |x - y|^{\varepsilon} \sum_{l=1}^{\infty} 2^{-l\varepsilon} \left[\frac{1}{2^l |I|} \int_{|x-z| \sim 2^l |I|} |f(z)|^2 dz \right]^{1/2} dy \\ & \leq C \|T\|_{CZ} t^{\varepsilon} \inf_{x \in I} M(f^2)^{1/2}(x) |I|^{-\varepsilon}. \end{aligned}$$

Substituting the above into the corresponding term in (2), we get

$$\begin{aligned} & \int_{\Omega} |Q_t T(f_0 \chi)|^2(x) dx \frac{dt}{t} \\ & \leq C \|T\|_{CZ}^2 \sum_{I \in M'(\Omega_1)} |I|^{-2\varepsilon} \int_{4I} \int_0^{2|I|} t^{2\varepsilon} \frac{dt}{t} dx \inf_{x \in I} M(f^2)(x) \\ & \leq C \|T\|_{CZ}^2 \sum \int_{4I} M(f^2)(x) dx \\ & \leq C \|T\|_{CZ}^2 \sum \int_I M(f^2)(x) dx \quad (\text{Lemma 5}) \\ & \leq C \|T\|_{CZ}^2 \int_{\Omega} M(f^2)(x) dx. \end{aligned}$$

The last inequality is a result of the fact that on R^1 , the I in $M^1(\Omega_1)$ are disjoint (a fact not true for higher dimensions).

To prove the rest, we use induction on n . However, as in [2], we shall need more definitions and a lemma. Let $T \in CZ_{\varepsilon}^2$. We denote the operator $[Q_{t_v} T]_{x_v, z_v}$ acting on the variables $(z_1, \dots, z_{v-1}, z_{v+1}, \dots, z_n)$ by

$$[Q_{t_v} T]_{x_v, z_v} = Q_{t_v} (k^4(\cdot, z_v) - k^v(x_v, z_v)).$$

Lemma A. *If $T \in CZ_\varepsilon^2(\mathbb{R}^n)$, then $[Q_{t_v} T]_{x_v, z_v} \in CZ_\varepsilon^2(\mathbb{R}^{n-1})$, and for an interval \widehat{I}_v , and a positive integer l , we have*

$$\left[\int_{|x_v - z_v| \sim 2^l |\widehat{I}_v|} \|[Q_{t_v} T]_{x_v, z_v}\|_{CZ}^2 dz_v \right]^{1/2} \leq C \|T\|_{CZ} (2^l |\widehat{I}_v|)^{-1/2-\varepsilon} t_v^\varepsilon.$$

Proof. We write

$$[Q_{t_v} T]_{x_v, z_v} = \int \psi_{t_v}(x_v - y_v) (k^v(y_v, z_v) - k^v(x_v, z_v)) dy_v.$$

Then

$$\|[Q_{t_v} T]_{x_v, z_v}\|_{CZ} \leq \int_{|x_v - y_v|} < t_v \frac{1}{t_v} \|k^v(y_v, z_v) - k^v(x_v, z_v)\|_{CZ} dy_v.$$

Applying Minkowski's integral inequality, Fubini's theorem, and the fact that $T \in CZ_\varepsilon^2$, we have the conclusion of Lemma A. Q.E.D.

Now we assume (1) is true up to $n - 1$, and as a consequence, the main estimate is also verified up to $n - 1$. We remind the readers that the indices $\{s_1, \dots, s_k\}$ are fixed throughout this proof, but the intervals $\{I_{s_1}, \dots, I_{s_k}\}$ depend on the point (x, t) . For n we have, from earlier pages, the following:

- (a) if $(x, t) \in \overline{\Omega}$, then $x \in R_t(x) \subset 4R = 4I_1 \times \dots \times 4I_n$ for some $R \in M^n(\Omega_1)$,
- (b) $\overline{\Omega} \subset \bigcup_{R \in M^n(\Omega_1)} \overline{4R}$,
- (c) for each $R = I_1 \times \dots \times I_n$ from above, let us write $\widehat{R} = \widehat{I}_1 \times \dots \times \widehat{I}_{s_1-1} \times I_{s_1} \times \dots \times I_n$.

We want to decompose $\overline{\Omega}$ into sets so that every (x, t) belonging to a particular set has the same \widehat{I}_{s_1} interval. Since for each $(x, t) \in \overline{\Omega}$, $R_t(x)$ is contained in a $4\widehat{R}$, \widehat{R} as in (c) above, we group \widehat{R} according to the ratio $|\widehat{I}_{s_1}|/|I_{s_1}| = 2^m$. For each group corresponding to a fixed m , we further sort out \widehat{R} according to i, j in the expression $|I_{s_1}| = 2^{i+j(m+1)}$, $1 \leq i \leq m+1$, $-\infty < j < +\infty$. Letting

$$\begin{aligned} A_{I,m} &= \bigcup \{ \widehat{I}_1 \times \dots \times \widehat{I}_{s_1-1} \times I_{s_1+1} \times \dots \times I_n \\ &\quad = \widehat{R}' | \widehat{R} = I_{s_1} \times \widehat{R} \in M^{s_1-1}(\Omega_{s_1}), \gamma_{s_1}(\widehat{R}) = 2^m \}, \end{aligned}$$

$$4A_{I,m} = \bigcup \{ 4\widehat{R}' | \widehat{R}' \text{ are rectangles that appear in } A_{I,m} \},$$

$$\overline{4A}_{I,m} = \bigcup \{ \overline{4\widehat{R}'} | \widehat{R}' \text{ as in } 4A_{I,m} \},$$

we split the integral on the left side of (1) so that it is dominated by

$$(3) \quad \sum_{m=1}^{\infty} \sum_{i=1}^{m+1} \sum_{j=-\infty}^{+\infty} \sum' \int_{\overline{4I}_{s_1} \times \overline{4A}_{I,m}} \left| Q_t T \left(f_0 \prod_{v=1}^k \chi_{(4\widehat{I}_{s_v})^c} \right) (x) \right|^2 dx \frac{dt}{t}$$

where \sum' extends over all dyadic intervals I_{s_1} of length $2^{i+j(m+1)}$. This corresponds to (2) in the $n = 1$ case. Since the Q_t operator is a convolution with a smooth bump function ψ , $\int_\psi = 0$, we write $Q_t T(f_0 \pi \chi)$ in (3) as

$$\sum_{l=1}^{\infty} \int_{|x_{s_1} - z_{s_1}| \sim 2^l |\hat{I}_{s_1}|} Q_{t'} [Q_{t_{s_1}} T]_{x_{s_1}, z_{s_1}} (f_0(z_{s_1}, \cdot) \chi_{(4\hat{I}_{s_1})^c} \cdots)(x') dz_{s_1},$$

where $x = (x_{s_1}, x')$, $t = (t_{s_1}, t')$. This corresponds to (2.1). Thus (3) is dominated by

$$\sum \int_{4\bar{I}} \left\{ \sum_{l=1}^{\infty} \int_{|x_{s_1} - z_{s_1}| \sim 2^l |\hat{I}_{s_1}|} \left(\int_{4A_{I,m}} \left| Q_{t'} [Q_{t_{s_1}} T]_{x_{s_1}, z_{s_1}} \left(f_0 \prod_{v=2}^k \chi_{(4\hat{I}_{s_v})^c} \right)(x) \right|^2 \times 8 \frac{dx' dt'}{t'} \right)^{1/2} dz_{s_1} \right\}^2 dx_{s_1} \frac{dt_{s_1}}{t_{s_1}},$$

and the induction hypothesis says the above is dominated by

$$\sum \int_{4\bar{I}_{s_1}} \left\{ \sum_{l=1}^{\infty} \int_{|x_{s_1} - z_{s_1}| \sim 2^l |\hat{I}_{s_1}|} \| [Q_{t_{s_1}} T]_{x_{s_1}, z_{s_1}} \|_{CZ} \times \int_{4A_{I,m}} (M(f^2)(z_{s_1}, x') dx')^{1/2} dz_{s_1} \right\}^2 \frac{dx_{s_1} dt_{s_1}}{t_{s_1}}.$$

Applying Lemma A with Hölder's inequality to the terms inside $\{\cdots\}$ and using

$$4A_{I,m} \subset \tilde{A}_{I,m} \subset \widetilde{E_{I_{s_1}} \setminus E_{I'_{s_1}}} \quad (\text{Lemma 4})$$

where

$$E_{I_{s_1}} = \bigcup \{R' | R' \text{ rectangles in } R^{n-1} \text{ such that } I_{s_1} \times R' \subset \Omega_{s_1}\},$$

we get the above dominated by

$$\begin{aligned} & C \|T\|_{CZ}^2 \sum \int_{4I_{s_1}} \int_0^{8|I_{s_1}|} |\hat{I}_{s_1}|^{-2\epsilon} t_{s_1}^{2\epsilon} dx_{s_1} \frac{dt_1}{t_1} \inf_{x_{s_1} \in I_{s_1}} \int_{E_{I_{s_1}} \setminus E_{I'_{s_1}}} M(f^2)(x_{s_1}, x') dx' \\ & \leq C \|T\|_{CZ}^2 \sum_{m=1}^{\infty} \left(\frac{|I_{s_1}|}{|\hat{I}_{s_1}|} \right)^{m+1} \sum_{i=1}^{m+1} \sum_{j=-\infty}^{+\infty} \sum' \int_{I_{s_1} \times E_{I_{s_1}} \setminus E_{I'_{s_1}}} M(f^2)(x) dx \\ & \leq C \|T\|_{CZ}^2 \int_{\Omega} M(f^2)(x) dx. \end{aligned}$$

The last inequality follows from the same reasoning as in Journé's covering lemma. Q.E.D.

After this paper was written, we were informed that R. Fefferman, in [7], had extended (for the operators on the bidisc) the result to the full range of weights,

i.e., $T: L^p(w) \rightarrow L^p(x)$ for $w \in A_p(R' \times R')$, $1 < p < \infty$. His technique could also be adapted to our case to yield the same result for $w \in A_p(R' \times \cdots \times R')$. The technique is based on the result for the “half” weighted case as we have here. Details will appear in a later paper.

4. THE LITTLEWOOD-PALEY THEOREM

Theorem. *If $W = \{R_l\}$ is a collection of disjoint rectangles with sides parallel to the coordinate axes, and $(S_{R_l}f)^\wedge(\xi) = \chi_{R_l}(\xi)\hat{f}(\xi)$ then the operator*

$$\Delta f(x) = \left(\sum_{R_l \in W} |S_{R_l}(f)(x)|^2 \right)^{1/2}$$

is bounded on $L^p(w)$ for $2 < p < \infty$, $w \in A_{p/2}(R^1 \times \cdots \times R^1)$.

4.1. Reductions. Following Rubio de Francia [1], the theorem can be reduced to the special case when the collection of sets W satisfies the well-distributed condition

$$\sum_{R_l \in W} \chi_{16R_l}(\xi) \leq C.$$

In addition we will treat W to be a finite collection, since a limiting argument will yield the general case.

We dyadicize the rectangles in W according to the process indicated in the preliminary definitions. Thus we associate $R_l \in W$ with a dyadic rectangle R_l^d with $2R_l \supset R_l^d$, $4R_l^d \supset R_l$. We know that for $1 < p < \infty$, $w \in A_p$,

$$(*) \quad \left\| \left(\sum_{R_l \in W} (S_{R_l}f_l)^2 \right)^{1/2} \right\|_{L^p(w)} \leq \left\| \left(\sum_{R_l \in W} f_l^2 \right)^{1/2} \right\|_{L^p(w)}$$

(this is true for $p = 2$, $w \in A_2$; then apply the extrapolation of weights [5]). We let $f_l = S_{4R_l^d}(f)$. $2R_l \supset R_l^d$ and the fact that W is well-distributed imply that each R_l^d can be associated with at most a fixed number of R_l . Thus we may assume that each R_l^d is distinct and

$$\sum_l \chi_{2(4R_l^d)}(\xi) \leq C.$$

This means we can treat W as if composed of these $4R_l^d$. For simplicity, we represent $4R_l^d \in W$ as a point in the lattice $(Z_+^2)^n = (k_1, j_1, \dots, k_n, j_n)$ by the following [1]. For each $1 \leq i \leq n$, let

$$\{I_i\} = \{I_i | I_i = i\text{th side of } R_l^d, 4R_l^d \in W\}.$$

If $|I_i| = 2^k$, we label I_i as I_{k_i} . Since there are at most countably many such I_{k_i} , we label each as $I_{k_i}^{j_i}$, $1 \leq j_i < \infty$. Thus each $4R_l^d \in W$ can be written as

$4R_k^j = 4I_{k_1}^{j_1} \times \cdots \times 4I_{k_n}^{j_n}$ where $j = (j_1, \dots, j_n)$, $k = (k_1, \dots, k_n)$. We define a set of integers $n_k^j = (n_{k_1}^{j_1}, \dots, n_{k_n}^{j_n})$ to be such that $n_{k_i}^{j_i} 2^{k_i} \in I_{k_i}^{j_i}$. Here we remark that the only reason we went through this dyadicizing process is to get the fact that whenever $n_{k_i}^{j_i} \neq n_{k_i}^{j'_i}$ then $j_i \neq j'_i$, a fact we will need.

Next we smooth out our operator in the same manner as [1]. We briefly state it here. We take $\phi \in C^\infty(R^1)$ such that $\widehat{\phi}(\xi) = 1$ for $|\xi| < 2.5$ and $\widehat{\phi}(\xi) = 0$ when $|\xi| > 3$. Then if

$$\phi_k^j(x) = \prod_{i=1}^n 2^{k_i} \phi(2^{k_i} x_i) \exp(2\pi i n_{k_i}^{j_i} 2^{k_i} x_i) = \prod_{i=1}^n \phi_{k_i}^{j_i}(x_i)$$

we have

$$\widehat{\phi}_k^j(\xi) = \begin{cases} 1 & \text{if } \xi \in 4R_k^j, \\ 0 & \text{if } \xi \text{ is outside } 2(4R_k^j). \end{cases}$$

Then the weighted inequality problem turns into (by extrapolation of weights) the weighted problem for the operator

$$G(f) = \left(\sum_{(k,j) \in W} |\Phi_k^j(f)|^2 \right)^{1/2}$$

where Φ_k^j is the operator with kernel

$$k_k^j(x, y) = \prod_{i=1}^n 2^{k_i} \phi(2^{k_i}(x_i - y_i)) \exp(2\pi i n_{k_i}^{j_i} 2^{k_i} y_i).$$

We have trivially the boundedness on L_2 of $G(f)$ by the well-distributed property of W .

4.2. Proof of the main estimate. The theorem follows if we prove the main estimate for G (we are abusing our notation here, actually we treat G as taking f into the l^2 vector-space valued function $\{\Phi_k^j(f)\}$)

$$(*) \quad \int_{\Omega} \sum_{(k,j) \in W} |Q_t \Phi_k^j(f)(x)|^2 dx \frac{dt}{t} \leq C \int_{\Omega} M(f^2) dx.$$

We shall need the following lemma, the proof of which will be delayed to §4.3.

Lemma 6. *If $R = I_1 \times \cdots \times I_n$ is a rectangle in R^n and I'_1, \dots, I'_n are intervals containing I_1, \dots, I_n respectively, then for any $\varepsilon > 0$*

$$\int_R \sum_{k,j} \left| Q_t \Phi_k^j \left(f \prod_{i=1}^n \chi_{(4I'_i)} \right) (x) \right|^2 dx \frac{dt}{t} \leq C_\varepsilon \prod_{i=1}^n \left(\frac{|I_i|}{|I'_i|} \right)^\varepsilon |R| \inf_{z \in R} M(f^2)(z).$$

Notice that the sum is over all (k, j) and not just those in W .

We start again with

$$f = f \chi_{\Omega_{n+3}} + f \chi_{\Omega_{n+3}^c} = f_I + f_0.$$

The term corresponding to f_I is treated via Plancherel's equality and the L^2 boundedness of G , that is,

$$\begin{aligned} \int_{\overline{\Omega}} \sum_{(k,j) \in W} |Q_t \Phi_k^j(f_I)(x)|^2 dx \frac{dt}{t} &\leq \int_{\overline{R}^n} \sum |Q_t \Phi_k^j(f_I)(x)|^2 dx \frac{dt}{t} \\ &\leq C \int_{\Omega_{n+3}} f^2(x) dx \leq C \int_{\Omega} M(f^2)(x) dx. \end{aligned}$$

We decompose the f_0 term as in §3.2. In short, for each (x, t) in $\overline{\Omega}$, we can decompose f_0 in $Q_t \Phi_k^j(f_0)(x)$ into

$$(**) \quad f_0(z) = \sum_{S \in S^n} f_0(z) (-1)^{k+1} \chi_{(4\hat{I}_{s_1})^c} \cdots \chi_{(4\hat{I}_{s_k})^c},$$

where S^n is a collection of all nonempty subsets of $\{1, \dots, n\}$, and $S = \{s_1, \dots, s_k\}$ with $1 \leq s_1 < \dots < s_k \leq n$. To prove the estimate, we only need to show the following for one of those terms above:

$$(1) \quad \int_{\overline{\Omega}} \sum_{(k,j) \in W} \left| Q_t \Phi_k^j \left(f_0 \prod_{v=1}^k \chi_{(4\hat{I}_{s_v})^c} \right) (x) \right|^2 dx \frac{dt}{t} \leq C \int_{\Omega} M(f^2) dx.$$

We decompose $\overline{\Omega}$ in (1) in the same manner as in (3) of §3.2, so that the left-hand side of (1) is less than

$$\sum_{m_1=1}^{\infty} \sum_{i_1=1}^{m_1+1} \sum_{j_1=-\infty}^{+\infty} \sum' \int_{4\hat{I}_{s_1} \times 4\overline{A}_{I_{s_1}, m_1}} \sum_{(k,j) \in W} \left| Q_t \Phi_k^j \left(f_0 \prod_{v=1}^k \chi_{(4\hat{I}_{s_v})^c} \right) (x) \right|^2 dx \frac{dt}{t}.$$

If we let $\sum_1 = \sum_{m_1=1}^{\infty} \cdots \sum'$ in the above, then the above term is dominated by

$$\begin{aligned} &\sum_1 \int_{4\hat{I}_{s_1}} \sum_{k_1, j_1} \int_{4\overline{A}_{I_{s_1}, m_1}} \\ &\times \sum_{(k', j')} \left| Q_{t'} \Phi_{k'}^{j'} \left[Q_{t_{s_1}} \Phi_{k_{s_1}}^{j_{s_1}} \left(f_0 \chi_{(4\hat{I}_{s_1})^c} \right) \prod_{v=2}^k \chi_{(4\hat{I}_{s_v})^c} \right] (x_{s_1}, x') \right|^2 dx' \frac{dt'}{t'} dx_{s_1} \frac{dt_{s_1}}{t_{s_1}}, \end{aligned}$$

where $x = (x_{s_1}, x')$, $t = (t_{s_1}, t')$, $k = (k_{s_1}, k')$, $j = (j_{s_1}, j')$. Let

$$F_{x_1}(z') = Q_{t_{s_1}} \Phi_{k_{s_1}}^{j_{s_1}} \left(f_0 \chi_{(4\hat{I}_{s_1})^c} \right) (z').$$

We see that the term

$$\int_{4\overline{A}_{I_{s_1}, m_1}} \sum_{k', j'} \left| Q_{t'} \Phi_{k'}^{j'} \left(F_{x_{s_1}} \prod_{v=2}^k \chi_{(4\hat{I}_{s_v})^c} \right) (x') \right|^2 dx' \frac{dt'}{t'}$$

is the $n-1$ version. Now if the term $\prod_{v=2}^k \chi_{(4\hat{I}_{s_v})^c} = 1$ (that is, $k = 1$), then we can apply the same kind of decomposition as in the beginning of this section,

that is,

$$F_{x_{s_1}}(x') = F_{x_{s_1}} \chi_{(4A_{I_{s_1}, m_1})_{n+2}} + F_{x_{s_1}} \chi_{(4A_{I_{s_1}, m_1})_{n+2}^c} = F_{x_{s_1}; I} + F_{x_{s_2}; 0}.$$

Then use L^2 boundedness of the operator G ($n-1$ version) for $F_{x_s; I}$, and decompose $F_{x_{s_1}; 0}$ as in $(**)$ for each $(x', t') \in \overline{4A_{I_{s_1}, m_1}}$. If $\prod_{v=2}^k \chi_{(4I_{s_v})^c}$ is not equal to 1 ($k > 1$), then we continue to reduce to a lower dimensional case. Thus we will need to check only the following two cases:

$$(2.1) \quad \sum_1 \cdots \sum_n \int_{\overline{4I_1}} \cdots \int_{\overline{4I_n}} \sum_{(k, j) \in W} \left| Q_t \Phi_k^j \left(f_0 \prod_{v=1}^n \chi_{(4I_v)^c} \right) (x) \right|^2 dx \frac{dt}{t},$$

$$(2.2) \quad \sum_1 \cdots \sum_p \int_{\overline{4I_{s_1}}} \cdots \int_{\overline{4I_{s_p}}} \int_{4A_{I_{s_1}, \dots, I_{s_p}}, m_1, \dots, m_p} \\ \times \sum_{\substack{k_{s_1} \cdots k_{s_p} \\ j_{s_1} \cdots j_{s_p}}} |Q_{t''} \Phi_{k''}^{j''}(f_0)(x'', x''')|^2 ds''' \frac{dt'' dt'''}{t''}$$

where $\{s_1, \dots, s_p\}$ is some subset of $\{1, \dots, n\}$, $x'' = (x_{s_1}, \dots, x_{s_p})$, $x = (x'', x''')$, etc. The case (2.1) is handled by Lemma 6, that is, (2.1) becomes

$$\begin{aligned} & \sum_1 \cdots \sum_n \int_{\overline{4R}} \sum_{(k, j)} \left| Q_t \Phi_k^j \left(f_0 \prod_{v=1}^n \chi_{(4I_v)^c} \right) (x) \right|^2 dx \frac{dt}{t} \\ & \leq C \sum_1 \cdots \sum_n \prod_{v=1}^n \left(\frac{|I_v|}{|\widehat{I}_v|} \right)^\varepsilon |R| \inf_{z \in R} M(f^2)(z), \end{aligned}$$

which, by Journé's covering lemma, is dominated by

$$C \int_{\Omega} M(f^2)(x) dx.$$

(2.2) is equal to

$$\sum_1 \cdots \sum_p \int_{4A_{I, m}} \int_{\overline{4R}} \sum_{k_{s_1} \cdots k_{s_p}} |Q_{t''} \Phi_{k''}^{j''}(f_0)(x'', x''')|^2 dx'' \frac{dt''}{t''} dx''',$$

where $R = I_{s_1} \times \cdots \times I_{s_p}$. Then the inner integral is dominated by (as a consequence of Lemma 6)

$$C \sum_1 \cdots \sum_p \prod_{v=1}^p \left(\frac{|I_{s_v}|}{|\widehat{I}_{s_v}|} \right)^\varepsilon |I_{s_v}| \int_{4A_{I, m}} \inf_{x'' \in I_{s_1} \times \cdots \times I_{s_p}} M(f^2)(x'', x''') dx''.$$

Then applying Journé's covering lemma p times, we get the above dominated by

$$C \int_{\Omega} M(f^2)(x) dx.$$

4.3. Proof of Lemma 6. So we are left with

Lemma 6. *If $R = I_1 \times \cdots \times I_n$ in a rectangle in R^n and I'_1, \dots, I'_n are intervals containing I_1, \dots, I_n respectively, then for any $\varepsilon > 0$*

$$\int_{\bar{R}} \sum_{k,j} \left| Q_t^j \Phi_k \left(f \prod_{i=1}^n \chi_{(4I'_i)^c} \right) (x) \right|^2 dx \frac{dt}{t} \leq C \prod_{i=1}^n \left(\frac{|I_i|}{|I'_i|} \right)^\varepsilon |R| \inf_{z \in R} M(f^2)(z).$$

The sum is over all (k, j) and not just those in W .

Proof. We shall prove it for $n = 2$. The general case is obvious from this proof. We first claim that for $n = 1$ we have (we denote $|x - z| \sim z^l |I'|$ as $2^l |I'| \leq |x - z| \leq 2 \cdot 2^l |I'|$)

$$\sum_{k,j} |Q_t^j \Phi_k^j(f \chi_{(4I')^c})(x)|^2 \leq C t^\varepsilon |I|^{-\varepsilon} \sum_{l=0}^{\infty} 2^{-l\varepsilon/2} \frac{1}{|I'| 2^l} \int_{|x-z| \sim 2^l |I'|} |f(z)|^2 dz.$$

We linearize the expression. Let $H = \{\lambda_{k,j}\}$ with norm $(\sum_{k,j} |\lambda_{k,j}|^2)^{1/2}$. For each fixed (x, t) we pick $\{\lambda_{k,j}(x, t)\}$ with norm 1 such that

$$\begin{aligned} \left(\sum_{k,j} \left| Q_t^j \Phi_k^j(f \chi_{(4I')^c})(x) \right|^2 \right)^{1/2} &= \sum_{k,j} \lambda_{k,j}(x, t) |Q_t^j \Phi_k^j(f \chi)(x)| \\ &= Q_t \left(\sum_{k,j} \lambda_{k,j}(x, t) \Phi_k^j(f \chi) \right) (x). \end{aligned}$$

The latter expression becomes, using $\int \psi = 0$,

$$\int \psi_t(x - y) \int \left\{ \sum_{k,j} (k_{k,j}(y, z) - k_{k,j}(x, z)) \lambda_{k,j}(x, t) \right\} f \chi dz dy$$

which is dominated by (let $K_\lambda(y, z) - K_\lambda(x, z)$ be the expression inside $\{\cdots\}$)

$$\int_{|x-y| < t} \sum_{l=0}^{\infty} \int_{|x-z| \sim 2^l |I'|} |K_\lambda(y, z) - K_\lambda(x, z)| |f(z)| dz \frac{dy}{t}$$

which is dominated by

$$\begin{aligned} \int_{|x-y| < t} \sum_{l=0}^{\infty} \left(\int_{|x-z| \sim 2^l (|I'|/|x-y|)} |K_\lambda(y, z) - K_\lambda(x, z)|^2 dz \right)^{1/2} \\ \times \left(\int_{|x-z| \sim 2^l |I'|} |f(z)|^2 dz \right)^{1/2} \frac{dy}{t}. \end{aligned}$$

Now we use a lemma from Rubio de Francia [1, pp. 8–12], which says whenever we have $j \neq j'$ then $n_k^j \neq n_k^{j'}$. Then

$$\int_{|x-z| \sim \gamma |x-y|} |K_\lambda(y, z) - K_\lambda(x, z)|^2 dx \leq C \frac{\gamma^{-(1+\varepsilon)}}{|x-y|}.$$

Applying this fact we get

$$\begin{aligned} & \left(\sum_{k,j} |\mathcal{Q}_t \Phi_k^j(f\chi_{(4I')^c})(x)|^2 \right)^{1/2} \\ & \leq C t^{\varepsilon/2} |I'|^{-\varepsilon/2} \sum_l 2^{-l\varepsilon/2} \left(\frac{1}{2^l |I'|} \int_{|x-z| \sim 2^l |I'|} |f(z)|^2 dz \right)^{1/2} \\ & \leq C t^{\varepsilon/2} |I'|^{-\varepsilon/2} \left(\sum_l 2^{-l\varepsilon/2} \frac{1}{2^l |I'|} \int_{|x-z| \sim 2^l |I'|} |f(z)|^2 dz \right)^{1/2}. \end{aligned}$$

Now then for the proof of Lemma 6,

$$\begin{aligned} & \int_{\bar{I} \times \bar{J}} \sum_{k_2, j_1} \sum_{k_1, j_1} |\mathcal{Q}_{t_1} \Phi_{k_1}^{j_1}(\mathcal{Q}_{t_2} \Phi_{k_2}^{j_2} f \chi_{(4J')^c} \chi_{(4I')^c})(x_1)|^2 dx \frac{dt}{t} \\ & \leq C \int_{\bar{I} \times \bar{J}} \left(t_1^\varepsilon |I'|^{-\varepsilon} t_2^\varepsilon |J'|^{-\varepsilon} \sum_l 2^{-l\varepsilon/2} \frac{1}{|I'| 2^l} \int_{|x_1-z_1| \sim 2^l |I'|} \right. \\ & \quad \times \left. \sum_{l'} 2^{-l'\varepsilon/2} \frac{1}{|J'| 2^{l'}} \int_{|x_2-z_2| \sim 2^{l'} |J'|} |f(z_1, z_2)|^2 dz_2 dz_1 \right) dx \frac{dt}{t} \\ & \leq C \int_I dx_1 \int_0^{|I|} t_1^{\varepsilon-1} dt_1 \int_J dx_2 \int_0^{|J|} t_2^{\varepsilon-1} dt_2 (|I'| |J'|)^{-\varepsilon} \inf_{z \in R} M(f^2)(z) \\ & \leq C \left(\frac{|I|}{|I'|} \right)^\varepsilon \left(\frac{|J|}{|J'|} \right)^\varepsilon |R| \inf_{z \in R} M(f^2)(z). \end{aligned}$$

And this yields the lemma.

REFERENCES

1. José L. Rubio de Francia, *A Littlewood-Paley inequality for arbitrary intervals*, Mittag-Leffler Institute, Report No. 18, 1983.
2. Jean-Lin Journé, *Calderon-Zygmund operators on product spaces*, Rev. Mat. Iberoamericana **1** (1985), 55–91.
3. Robert Fefferman, *Harmonic analysis in product spaces*, Ann. of Math. (2) **126** (1987), 109–130.
4. C. Fefferman and E. M. Stein, *Some maximal inequalities*, Amer. J. Math. **93** (1971), 107–115.
5. José L. Rubio de Francia, *Factorization and extrapolation of weights*, Bull. Amer. Math. Soc. **7** (1982), 393–396.
6. J. Pipher, *Journé's covering lemma and its extension to higher dimensions*, Duke J. Math. **53** (1986), 683–690.
7. R. Fefferman, *A_p weights and singular integrals*, Amer. J. Math. **110** (1988), 975–987.

DIVISION OF MATHEMATICS, F.A.T., AUBURN UNIVERSITY, 218 PARKER HALL, AUBURN, ALABAMA 36849