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WEIGHTED NORM ESTIMATES FOR THE FOURIER TRANSFORM WITH A PAIR OF WEIGHTS

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ABSTRACT. We prove weighted norm inequalities of the form

$$\|\hat{f}\|_{L^q_u} \leq C\|f\|_{H^p_v}, \qquad 0 < p \leq q < \infty,$$

for the Fourier transform on $\mathbf{R}^n$. For some weight functions v, the Hardy space H^p_v on the right can be replaced by L^p_v. The proof depends on making an atomic decomposition of f and using cancellation properties of the atoms.

1. INTRODUCTION.

As indicated by Pitt's theorem (see Stein [St]), the following weighted inequality holds for the Fourier transform of a function f on R:

$$\int_R |\hat{f}(y)|^p |y|^{p-2-\gamma} dy \leq C \int_R |f(x)|^p |x|^\gamma dx$$

when $1 < p < \infty$ and $\max(0, p-2) \leq \gamma < p-1$ (and also a similar estimate with an exponent $q > p$ on the left-hand side). Sadosky and Wheeden show in [Sad-W] that the inequality above holds for all $\gamma \geq \max(0, p-2)$ with $\gamma \neq kp-1$, $k = 1, 2, \ldots$, if we assume in addition that the function f has vanishing moments up to a certain order depending on γ and p; in fact they show that if $1 < p \leq q < \infty$, k is a positive integer and the function f has vanishing moments of order less than or equal to $k-1$ then

$$\left(\int_R \left(\frac{|\hat{f}(y)|}{|y|^{k-1}}\right)^q \left(\frac{1}{|y|} w\left(\frac{1}{y}\right)\right)^{q/p} \frac{dy}{|y|}\right)^{1/q} \leq C\left(\int_R |f(x)|^p |x|^{kp} w(x) dx\right)^{1/p}$$

for any weight w satisfying $w^{q/p} \in A_{1+(q/p')}$, $1/p + 1/p' = 1$. (See §2 for the definition of A_p.) In fact a slightly weaker condition on w is sufficient. Other results of this kind are given in Benedetto, Heinig and Johnson [B-H-J] and in Benedetto and Heinig [B-H].

We see that the origin plays a special role in the description of the weights in the inequalities above. This may be natural on the Fourier transform side since

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we assume that $\hat{f}(\xi)$ vanishes at the origin to a certain order. However, on the function side the origin should not play any more role than any other point.

In this paper we consider conditions for a pair (u, v) of weights such that the L^q_u norm of the Fourier transform is dominated by the weighted Hardy space H^p_v norm. The application we have in mind is the case when $v(x) = |Q(x)|^p w(x)$, where Q is a polynomial and $w \in A_p$. If a function f has vanishing moments up to a certain order depending on Q then the H^p_v norm of f is dominated by its L^p_v norm. This will give us an estimate

$$\|\hat{f}\|_{L^q_u} \leq C\|f\|_{L^p_v}$$

which is a generalization of the inequality obtained by Sadosky and Wheeden.

Note that the estimates of the Fourier transform which have been obtained by Jurkat and Sampson [J-Sam], Muckenhoupt [M], and Heinig [H1] are quite different, since the condition there is stated in terms of rearrangements of the weights, and moment conditions do not enter their estimates. See also [B-H-J and H2] for some further results.

The weighted Hardy space norm inequality in this paper is done in R^n and the proof uses Hausdorff-Young's inequality and depends heavily on the atomic decomposition of Hardy spaces. In §2 we give some preliminaries which are used later. In §3 we state the main result with the condition on the weights. In §§4 and 5 we make the technical estimate of the Fourier transform for a finite sum of atoms, and in §6 we use some rather simple limit arguments to obtain our main result from the estimates which are proved in the previous section.

2. Preliminaries

We now introduce some notation. The halfspace R^{n+1}_+ is the set $R^{n+1}_+ = \{(x, t) : x \in R^n \text{ and } t > 0\}$ and the ball $B(x, t)$ in R^n is the set $B(x, t) = \{y \in R^n : |y - x| < t\}$. The Lebesgue measure of a set E in R^n is denoted by $|E|$. A weight function (or a weight) w on R^n is a nonnegative function on R^n. By $w(E)$ we denote the corresponding measure of the set E in R^n, i.e., $w(E) = \int_E w(x)dx$. Let $1 \leq d < \infty$. The weight function w satisfies the doubling condition D_d (i.e., $w \in D_d$) if

$$w(B(x, rt)) \leq Cr^{nd} w(B(x, t)) \quad \text{for all } t > 0, \ r \geq 1, \text{ and } x \in R^n,$$

with the constant C independent of t, r, and x. We say that $w \in D_\infty$ if $w \in D_d$ for some $d \geq 1$. Let ψ be a function in the Schwartz class $\mathscr{S}$, and let $\psi_t(x) = \psi(x/t)t^{-n}$. The nontangential maximal function $N_\psi f$ of a tempered distribution f is defined by $N_\psi f(x) = \sup_{(y,t):|y-x|<t} |f * \psi_t(y)|$. For $\lambda > 0$, we define the tangential maximal function $N^\lambda_\psi f$ by

$$N^\lambda_\psi f(x) = \sup_{(y,t) \in R^{n+1}_+} |f * \psi_t(y)|(1 + |x - y|/t)^{-\lambda},$$

and the grand maximal function $f^* = f^{*M}$ by $f^{*M}(x) = \sup_\psi N_\psi f(x)$, where the supremum is taken over all Schwartz functions with Schwartz norm

$$\|\psi\|_M = \sup_{x\in R^n, |\alpha|\le M} |D^\alpha \psi(x)|(1+|x|)^M \le 1.$$

The weighted Hardy space H^p_w is defined as the space of tempered distributions f with norm $\|f\|_{H^p_w} = \|N_\psi f\|_{L^p_w} < \infty$, where ψ is a function in $\mathscr{S}$ with $\int \psi \neq 0$ and $\|\cdot\|_{L^p_w}$ is defined by $\|g\|_{L^p_w} = (\int_{R^n} |g(x)|^p w(x)dx)^{1/p}$, $0 < p < \infty$. Observe by [Str-T] that if $w \in D_\infty$ this definition is independent of ψ with norm within a constant factor. It would also be equivalent to use the tangential maximal function $N^\lambda_\psi f$ or the grand maximal function f^{*M} instead of $N_\psi f$ when defining the weighted Hardy space norm provided that λ or M, respectively, is large enough depending on the doubling exponent d and on p. In fact, $N^\lambda_\psi f(x) \le C N^\lambda_{\psi_1} f(x)$ for any two functions ψ and ψ_1 in $\mathscr{S}$ with nonzero integrals, with the constant C depending only on ψ, ψ_1, and λ, and similarly $N^\lambda_\psi f(x) \le C f^{*M}(x)$ when $\lambda \ge M$ and $f^{*M}(x) \le C N^\lambda_\psi f(x)$ when $M > M(\lambda)$.

If $p > 1$ and the weight w satisfies Muckenhoupt's A_p condition:

$$\frac{1}{|B|}\int_B w\,dx \left(\frac{1}{|B|}\int_B w^{-\frac{1}{p-1}}dx\right)^{p-1} \le C \quad \text{for all balls } B,$$

then the H^p_w norm and the L^p_w norm are equivalent, and the spaces H^p_w and L^p_w can be identified in the natural way:

$$\langle f, \psi\rangle = \int f(x)\psi(x)dx \quad \text{for } \psi \text{ in } \mathscr{S}.$$

In this paper, however we mainly consider H^p_v spaces where the weight v does not satisfy the A_p condition. One type of weight we have in mind is the following.

Let v be a weight on R which can be written $v(x) = |Q(x)|^p w(x)$ with $w \in A_p$, $p > 1$, and Q a polynomial of nonzero degree. Then v does not satisfy the A_p condition and the spaces H^p_v and L^p_v are not equal under the identification described above. However, H^p_v can still be identified with a subspace of L^p_v defined by some moment conditions (in fact, the subspace is the whole space L^p_v if Q has only real zeros). We refer to Strömberg-Wheeden [Str-W]. (A similar identification between weighted Hardy and Lebesgue spaces holds in higher dimensions for Q generalized in a suitable way. (See Adams [A].)) This identification of a function with a tempered distribution coincides with the natural one in the case when the function is locally integrable and has zero moments of order less than the degree of Q, e.g., for functions in the class $\mathscr{S}_{0,0}$, which is defined as the space of functions in $\mathscr{S}$ whose Fourier transforms are compactly supported away from the origin.

For a function f in $\mathcal{S}$ we define the Fourier transform by

$$\hat{f}(y) = \int_{R^n} f(x) e^{-2\pi i x\cdot y} dx ,$$

and for a tempered distribution f we define its Fourier transform $\hat{f}$ to be the tempered distribution defined by $\langle \hat{f}, \varphi\rangle = \langle f, \hat{\varphi}\rangle$ for all $\varphi \in \mathcal{S}$.

3. Main result

First we introduce a weight condition for a pair (u, v) of weights. Let $1 \le p_1 \le 2$, $1/p_1 + 1/q_1 = 1$, $0 < p \le q < \infty$ and $q \le q_1$. Let φ be a nonnegative compactly supported function on R^n which is not identically zero, and let $\varphi_t(x) = t^{-n}\varphi(x/t)$. A pair of weights u and v, with v satisfying the doubling condition, is said to satisfy the $PW(q, p; q_1, p_1)$ condition if there is a constant C such that for all $t > 0$

$$\left(\int_{|y|\ge t} u^{\frac{q_1}{q_1-q}} dy\right)^{\frac{q_1-q}{q_1 q}} \left(\int_{R^n} (\varphi_{1/t} * v)^{-\frac{p_1}{p-p_1}} dx\right)^{\frac{p-p_1}{p_1 p}} \le C \tag{1}$$

in the case $p_1 < p$, or

$$\left(\int_{|y|\ge t} u^{\frac{q_1}{q_1-q}} dy\right)^{\frac{q_1-q}{q_1 q}} \left(t^{-\frac{n(p_1-p)}{p_1}} \inf_{x\in R^n} (\varphi_{1/t} * v(x))\right)^{-\frac{1}{p}} \le C \tag{1'}$$

in the case $p_1 \ge p$. (When $q_1 = q$ the left expression means $(\sup_{|y|\ge t} u(y))^{1/q}$ and when $q_1 = \infty$ it means $(\int_{|y|>t} u dy)^{1/q}$.)

The notation "PW" used above is an abbreviation for "pair of weights". We note that the condition allows the consideration of weights v which have zeros of large order; e.g., if $v(x) = |x|^\gamma$, $\gamma > -n$, it is easy to see that

$$\varphi_{1/t} * |x|^\gamma \approx (|x| + 1/t)^\gamma$$

and for $r > 0$,

$$\left(\int_{R^n} (\varphi_{1/t} * |x|^\gamma)^{-r} dx\right)^{1/r} \approx \begin{cases} t^{\gamma - n/r} & \text{if } \gamma > n/r, \\ \infty & \text{otherwise.} \end{cases}$$

Furthermore, if $v(x) = |x|^{kp} w(x)$ for $w \in A_p$, $k = 1, 2, \dots$, and $1 < p < \infty$, then

$$\left(\int_{R^n} (\varphi_{1/t} * v(x))^{-\frac{1}{p-1}} dx\right)^{p-1} \approx \left(\int_{|x|>1/t} w(x)^{-\frac{1}{p-1}} \frac{dx}{|x|^{kp'}}\right)^{p-1},$$

$1/p + 1/p' = 1$. If $k \ge n$, this expression is finite and equivalent to

$$t^{(k-n)p} \left(\int_{|x|<1/t} w(x) dx\right)^{-1}.$$

Weights v of this kind were considered in [Sad-W] and are naturally related to the $PW(p,q;\infty,1)$ condition, i.e., to the condition

$$\left(\int_{|x|>t} u\,dx\right)^{\frac{1}{q}} \left(\int_{R^n} (\varphi_{1/t} * v(x))^{-\frac{1}{p-1}} dx\right)^{\frac{p-1}{p}} \leq C,$$

$1 < p < \infty$. In fact, it is not difficult to see that the weights in [Sad-W] satisfy this condition.

The condition $PW(q,p;q_1,p_1)$ is also of interest for values of p_1 other than 1: e.g., the classical result of Hardy that $\|\hat{f}(x)/|x|\|_{L^1} \leq C\|f\|_{H^1}$ when $n=1$ corresponds to the case $p_1 = q_1 = 2$, $p = q = 1$, $u = 1/|x|$, and $v = 1$; note that $(1')$ is the appropriate condition to check in this case since $p_1 > p$.

Remark. The integral expression

$$\left(\int_{R^n} (\varphi_{1/t} * v)^{-\frac{p_1}{p-p_1}} dx\right)^{\frac{p-p_1}{p_1 p}} \qquad (\text{when } p_1 < p)$$

may also be interpreted as a sum

$$\left(\sum_{Q_k} |Q_k| \frac{v(Q_k)}{|Q_k|}^{-\frac{p_1}{p-p_1}}\right)^{\frac{p-p_1}{p_1 p}},$$

where the sum is taken over all cubes Q_k with side $1/t$ in a dyadic decomposition of R^n. Similarly, $\inf_{x\in R^n}(\varphi_{1/t} * v(x))$ in the case $p_1 \geq p$ can be interpreted as $\inf_{y\in R^n} v(B_{1/t}(y))/\,|B_{1/t}(y)|$. In the case $p_1 < p$ the $PW(q,p;q_1,p_1)$ condition can be written

$$\left(\int_{|y|\geq t} u^{\frac{q_1}{q_1-q}} dy\right)^{\frac{q_1-q}{q_1 q}} \leq C\left(\sum_{s(Q_k)=1/t} |Q_k| \left(\frac{v(Q_k)}{|Q_k|}\right)^{-\frac{p_1}{p-p_1}}\right)^{-\frac{p-p_1}{p_1 p}} \quad \text{for all } t>0,$$

where the sum is taken over all cubes Q_k in the dyadic decomposition of R^n into cubes Q_k with side $s(Q_k) = 1/t$. In the case $p_1 \geq p$ the $PW(q,p;q_1,p_1)$ condition can be written

$$\left(\int_{|y|\geq t} u^{\frac{q_1}{q_1-q}} dy\right)^{\frac{q_1-q}{q_1 q}} \leq C\left(t^{-\frac{n(p_1-p)}{p_1}} \inf_{y\in R^n} (v(B_{1/t}(y))/|B_{1/t}(y)|)\right)^{\frac{1}{p}} \quad \text{for all } t>0.$$

Our main result is the following weighted estimate for the Fourier transform.

Theorem 1. *Let $0 < p \leq q < \infty$ and let (u,v) be a pair of weights with v satisfying the doubling condition and the pair (u,v) satisfying the $PW(q,p;p_1,q_1)$ condition for some $1 \leq p_1 \leq q_1 \leq \infty$, $1/p_1 + 1/q_1 = 1$, with $q \leq q_1$. Then*

$$\|\hat{f}\|_{L^q_u} \leq C\|f\|_{H^p_v}$$

for all tempered distributions f in H^p_v.

Remark 1. The doubling property of the weight v is used to get an intrinsic definition of the H_v^p space norm and then mainly used to obtain the left-hand side inequality in Lemma 1 below, namely,

$$\frac{1}{C}\left(\frac{s}{t}\right)^{n(d-1)r}\int_{R^n}(\varphi_s * v(x))^{-r}dx \le \int_{R^n}(\varphi_t * v(x))^{-r}dx \quad \text{for } s \le t.$$

If we only assume that this holds for some d instead of the somewhat stronger condition that v is doubling, then we obtain, for any $\lambda > 0$, the inequality

$$\left(\int_{R^n}|\hat{f}(y)|^q u(y)dy\right)^{1/q} \le C\left(\int_{R^n}|N_\psi^\lambda f(x)|^p v(x)dx\right)^{1/p} \quad \text{for } f \in \mathscr{S}_{0,0}.$$

Remark 2. In the condition on the pair of weights (u, v) given in the theorem it is a problem to choose the exponents p_1 and q_1 in an optimal way. By splitting u into parts $u = u_1 + u_2$ and using different pairs of exponents (q_1, p_1) for u_1 and u_2, we get a weaker condition on the weights u and v. We will not go further into this problem.

Remark 3. If we want to consider the limiting case $q = \infty$ we should first replace the weight u by u^q, both in the PW condition and in the weighted norms before we let q go to infinity. Then we get

$$\sup_{y\in R^n}|\hat{f}(y)|u(y) \le C\|f\|_{H_v^p}$$

when $0 < p < \infty$ with the PW condition

$$\sup_{|y|>t} u(y) \le C\begin{cases}\left(\int_{R^n}(\varphi_{1/t} * v(x))^{-\frac{1}{p-1}}dx\right)^{-\frac{p-1}{p}} & \text{for } p > 1,\\ \left(t^{-n(1-p)}\inf_{x\in R^n}(\varphi_{1/t} * v(x))\right)^{1/p} & \text{for } 0 < p \le 1.\end{cases}$$

If we also replace the weight v by v^p and consider the limiting case when $p = \infty$ we get

$$\sup_{y\in R^n}|\hat{f}(y)|u(y) \le C\sup_{x\in R^n} N_\psi^\lambda f(x)v(x)$$

provided

$$\sup_{|y|>t} u(y) \le C\left(\int_{R^n}\left(\sup_{|z-x|\le c/t} v(z)\right)^{-1}dx\right)^{-1}.$$

We leave the details to the reader.

Let $p > 1$ and assume that the weight v on R can be written $v(x) = |Q(x)|^p w(x)$ where Q is a polynomial and $w \in A_p$. Then $\|f\|_{H_v^p} \le C\|f\|_{L_v^p}$ for all $f \in \mathscr{S}_{0,0}$ (see [Str-W]) and we get the following corollary.

Corollary. *Let $1 < p \le q < \infty$ and let $v = |Q|^p w$ where Q is a polynomial on R and w is a weight on R satisfying the A_p condition. Assume also that* *the pair of weights* (u, v) *satisfies the* $PW(q, p, ; q_1, p_1)$ *condition for some* p_1 *and* q_1, $1/p_1 + 1/q_1 = 1$ *with* $1 \le p_1 \le 2$ *and* $q_1 \ge q$. *Then*

$$\|\hat{f}\|_{L^q_u} \le C\|f\|_{L^p_v} \quad \text{for all } f \in \mathscr{S}_{0,0}.$$

4. Technical proof

We are going to use the atomic decomposition of $f \in H^p_v$. The space $\mathscr{S}_{0,0}$ is dense in H^p_v and a function in $\mathscr{S}_{0,0}$ can be approximated by finite sums of atoms. In this section we state and prove an estimate for the Fourier transform of a finite sum of atoms. In a later section this estimate will be extended to $\mathscr{S}_{0,0}$ and H^p_v by rather simple limiting arguments.

Let f be a finite sum $\sum \lambda_k a_k$ where λ_k are nonnegative numbers and $\{a_k\}$ are atoms satisfying

$|a_k(x)| \le 1$,
$\int_{R^n} a_k(x) x^\alpha dx = 0$, for all $|\alpha| \le m$,
a_k is supported in a cube Q_k.

Set $f^{**} = \sum \lambda_k \chi_{Q_k}$ and, more generally, for $0 < r < \infty$, set $f^{*r*} = (\sum \lambda_k^r \chi_{Q_k})^{\frac{1}{r}}$. We are going to show

Proposition 1. *Let* $0 < p \le q < \infty$. *Let* u *and* v *be weights on* R^n *with* $v \in D_d$ *and suppose that* u *and* v *satisfy the* $PW(q, p; q_1, p_1)$ *condition for some* p_1, q_1, $1 \le p_1 \le 2$, $q_1 \ge q$, *and* $1/p_1 + 1/q_1 = 1$.

If $q \ge 1$ *then we have*

$$\left(\int_{R^n} |\hat{f}(y)|^q u(y) dy\right)^{1/q} \le C \left(\int_{R^n} |f^{**}(x)|^p v(x) dx\right)^{1/p},$$

and more generally, if $0 < r \le 1$ *and* $q \ge r$ *then we have*

$$\left(\int_{R^n} |\hat{f}(y)|^q u(y) dy\right)^{1/q} \le C \left(\int_{R^n} |f^{*r*}(x)|^p v(x) dx\right)^{1/p}$$

for finite sums f *of atoms with vanishing moments of order less than or equal to* m, *provided* $m > -1 + n[d - \min(1, p/p_1)]/p$.

The proof of Theorem 1 is based on Hausdorff-Young's inequality:

$$\|\hat{g}\|_{L^{q_1}} \le C_{p_1} \|g\|_{L^{p_1}}, \tag{2}$$

for $1 \le p_1 \le 2$ and $1/p_1 + 1/q_1 = 1$.

First we are going to break the sum f of atoms into pieces f_j according to the sizes of the corresponding cubes. We need

Lemma 1. *Let* $v(x)$ *be a weight in* D_d *and let* $\varphi_t(x) = \varphi(x/t) t^{-n}$ *where* φ *is a nonnegative compactly supported* C^∞ *function with* $\int \varphi = 1$. *Then for* $r > 0$ *there is a constant* C *depending only on* φ *and* r *such that*

$$\begin{aligned}\frac{1}{C}\left(\frac{s}{t}\right)^{n(d-1)r} \int_{R^n} (\varphi_s * v(x))^{-r} dx &\le \int_{R^n} (\varphi_t * v(x))^{-r} dx \\ &\le C \int_{R^n} (\varphi_s * v(x))^{-r} dx\end{aligned}$$

for all $t \geq s > 0$; *and in the limit case* $r = \infty$ *we have*

$$C\left(\frac{t}{s}\right)^{n(d-1)} \inf_{x \in R^n} \varphi_s * v(x) \geq \inf_{x \in R^n} \varphi_t * v(x) \geq \frac{1}{C} \inf_{x \in R^n} \varphi_s * v(x)$$

for all $t \geq s > 0$.

Proof of Lemma 1. Using the doubling property of the weight v we get $\varphi_t * v(x) \sim v(B(x,t))/|B(x,t)|$ and from this and the D_d condition we see that

$$\frac{1}{C}\varphi_s * \varphi_t * v(x) \leq \varphi_t * v(x) \leq C\left(\frac{t}{s}\right)^{n(d-1)} \varphi_s * v(x) \quad \text{for } s \leq t.$$

From this we get the limit case $r = \infty$ and the left inequality of the lemma in the case $r < \infty$. For the right inequality in the case $r < \infty$ we use Jensen's inequality to obtain $\varphi_t * ((\varphi_s * v)^{-r})(x) \geq (\varphi_t * \varphi_s * v(x))^{-r}$ and by integration

$$\int_{R^n} (\varphi_t * v(x))^{-r} dx \leq C \int_{R^n} (\varphi_t * \varphi_s * v(x))^{-r} dx$$
$$\leq C \int_{R^n} \varphi_t * ((\varphi_s * v)^{-r})(x) dx = C \int_{R^n} (\varphi_s * v(x))^{-r} dx. \quad \square$$

By Lemma 1, we see that if $p > p_1$ and $\int_{R^n} (\phi_{1/s} * v(x))^{-p_1/(p-p_1)} dx$ is infinite for one value of s then it is infinite for all values of s. In this case, if the pair (u, v) satisfies the $PW(q, p; q_1, p_1)$ condition for any weight u then u must be identically zero on R^n, and consequently, Theorem 1 is trivially true. If $p > p_1$ and the integral is finite, we can, by Lemma 1, partition $(0, \infty)$ into intervals with endpoints s_j such that $s_j < s_{j+1}$ and

$$\left(\int_{R^n} (\phi_{1/s} * v(x))^{-\frac{p_1}{p-p_1}} dx\right)^{\frac{p-p_1}{p_1}} \sim \beta^j \quad \text{for } s_j < s < s_{j+1},$$

where $\beta > 1$ is a constant depending on the constant C in Lemma 1. The sequence $\{s_j\}$ might be finite, infinite to the left or to the right, or infinite at both ends. The equivalence above holds for the endpoints $s = s_j$ and $s = s_{j+1}$ provided $0 < s < \infty$. Furthermore it follows from Lemma 1 that $s_j \geq C\beta^{(j-k)/n(d-1)} s_k$ for $j > k$.

To be more precise, if we set

$$F(s) = \left(\int_{R^n} (\varphi_{1/s} * v(x))^{-\frac{p_1}{p-p_1}} dx\right)^{\frac{p-p_1}{p_1}}, \qquad 0 < s < \infty,$$

then by Lemma 1

$$C^{-1} F(s) \leq F(t) \leq C(t/s)^{n(d-1)} F(s), \qquad 0 < s \leq t < \infty.$$

Let $\beta = 2C^2$ and define $\{\sigma_j\}_{-\infty}^{\infty}$ by

$$\sigma_j = \sup\{s : s > 0, F(s) < \beta^j\},$$

with the convention that $\sigma_j = 0$ if $\{s : s > 0, F(s) < \beta^j\}$ is empty. Then it is not difficult to see that

(a) $\sigma_j \le \sigma_{j+1}$,
(b) $\sigma_j < \sigma_{j+1}$ unless $\sigma_j = \infty$ or $\sigma_{j+1} = 0$,
(c) $C^{-2}\beta^j \le F(s) \le C^2\beta^{j+1}$ if $\sigma_j < s < \sigma_{j+1}$,
(d) $C^{-1}\beta^j \le F(\sigma_j) \le C\beta^j$ if $\sigma_j \ne 0, \infty$.

Thus we take $\{s_j\}$ to be the distinct σ_j, $0 \le s_j \le \infty$. Also note that

$$\lim_{s\to\infty} F(s) = \lim_{s\to\infty}\left(\int_{R^n} (\phi_{1/s} * v)^{-\frac{p_1}{p-p_1}} dx\right)^{\frac{p-p_1}{p_1}} = \left(\int_{R^n} v^{-\frac{p_1}{p-p_1}} dx\right)^{\frac{p-p_1}{p_1}}$$

when $p > p_1$, since by Fatou's Lemma and Jensen's inequality

$$\begin{aligned}\int_{R^n} v^{-r} dx &\le \liminf_{s\to\infty} \int_{R^n} (\phi_{1/s} * v)^{-r} dx \\ &\le \limsup_{s\to\infty} \int_{R^n} (\phi_{1/s} * v)^{-r} dx \le \int_{R^n} v^{-r} dx\end{aligned}$$

for $r > 0$. Thus if $s_{j+1} = \infty$ then

$$\left(\int_{R^n} v^{-\frac{p_1}{p-p_1}} dx\right)^{\frac{p-p_1}{p_1}} \sim \beta^{j+1}.$$

Similarly, when $p \le p_1$, if $\inf_{x\in R^n}(\varphi_{1/s} * v(x))^{-1} = \infty$ for one value of s, then the same is true for every s, and Theorem 1 is trivial. Thus, as above, we may assume when $p \le p_1$ that $(0, \infty)$ is partioned into intervals with endpoints s_j such that $s_j < s_{j+1}$ and

$$F(s) = \left(s^{-\frac{n(p_1-p)}{p_1}} \inf_{x\in R^n}(\varphi_{1/s} * v(x))\right)^{-1} \sim \beta^j \quad \text{for } s_j < s < s_{j+1}.$$

This equivalence also holds for the endpoints $s = s_j$ and $s = s_{j+1}$ provided $0 < s < \infty$. In particular, by Lemma 1, we get

$$\frac{1}{C}\beta^{(j-k)/(n(d-\frac{p}{p_1}))} \le \frac{t}{s} \le C\beta^{(j-k)/(n(1-\frac{p}{p_1}))}$$

if $j > k$, $s_k \le s \le s_{k+1}$, and $s_j \le t \le s_{j+1}$ provided $0 < s \le t < \infty$. We may replace t/s in the inequality above by s_j/s_k whenever $0 < s_k < s_j < \infty$. However, when $p < p_1$ we get $0 < s_j < \infty$ for all j as a consequence of the inequality above. When $p = p_1$ we may have $s_{j+1} = \infty$. If this is the case we conclude in a similar way as when $p > p_1$ that

$$\lim_{s\to\infty} F(s) = \left(\inf_{x\in R^n} v(x)\right)^{-1} \sim \beta^{j+1}.$$

We define the index sets I_j by

$$I_j = \{k : 1/s_{j+1} < s(Q_k) \le 1/s_j\},$$

where $s(Q_k)$ denotes the side of the supporting cube Q_k of the atom a_k. With

$$f_j(x) = \sum_{k\in I_j} \lambda_k a_k,$$

we can write f as a sum $f = \sum_j f_j$, and we have for the sum of characteristic functions $f_j^{**} = \sum_{k\in I_j} \lambda_k \chi_{Q_k}$ the identity $f^{**} = \sum_j f_j^{**}$.

We also split the space R^n into parts $\Omega_i = \{y : s_{i-1} \le |y| < s_i\}$. From the definition of $\{s_j\}$ and the condition on the weights u and v we get

$$\|u\|_{L^{\frac{q_1}{q_1-q}}(\Omega_i)} \le C\beta^{-qi/p}.$$

We want to show that

$$\|\hat{f}_j\|_{L^q_u(\Omega_i)} \le C\beta^{-\delta|i-j|}\|f_j^{**}\|_{L^p_v}, \tag{3}$$

with $\delta = \min\{1/p, (m+1)/(n[d-\min(1, p/p_1)]) - 1/p\}$.

By Hölder's inequality we have

$$\|\hat{f}_j\|_{L^q_u(\Omega_i)} \le \|\hat{f}_j\|_{L^{q_1}(\Omega_i)}(\|u\|_{L^{\frac{q_1}{q_1-q}}(\Omega_i)})^{1/q} \le C\beta^{-i/p}\|\hat{f}_j\|_{L^{q_1}(\Omega_i)}. \tag{4}$$

For $j \le i$ we use (2) and the pointwise inequality $|f_j(x)| \le f_j^{**}(x)$ to obtain

$$\|\hat{f}_j\|_{L_{q_1}} \le C\|f_j^{**}\|_{L_{p_1}}. \tag{5}$$

For $j > i$ we will make use of the fact that f_j is a sum of atoms satisfying moment conditions. When $n = 1$ this can be done by finding the antiderivative of order $m+1$ of f_j. In general, when $n \ge 1$ we proceed as follows. For each atom $a = a_k$ we use the following lemma.

Lemma 2. *Let a be an atom with vanishing moments of order less than or equal to m and with support in a cube Q with side s. Then we have*

$$a(x) = \sum_{|\alpha|=m+1} c_\alpha \frac{\partial^\alpha}{\partial x^\alpha} A_\alpha(x),$$

where $s^{-m-1}A_\alpha$, $|\alpha| = m+1$, are atoms supported in Q, without vanishing moments in general, and $c_\alpha = 2^{m+1}\binom{n}{\alpha}$.

We apply Lemma 2 to the atoms a_k, $k \in I_j$, and get the atoms $s_j^{m+1}A_{k,\alpha}$, $|\alpha| = m+1$. These atoms do not have vanishing moments in general. Observe that when $j > i$ we may assume that $0 < s_i < s_j < \infty$ since otherwise I_j is empty and consequently $f_j \equiv 0$, or Ω_i is the empty set. If we write $F_{j,\alpha} = \sum_{k\in I_j}(s_j^{-m-1}\lambda_k)s_j^{m+1}A_{k,\alpha}$ then

$$f_j(x) = \sum_{|\alpha|=m+1} c_\alpha \frac{\partial^\alpha}{\partial x^\alpha} F_{j,\alpha}(x),$$

and with $f_j^{**} = s_j^{m+1} F_{j,\alpha}^{**} = \sum_{k \in I_j} \lambda_k \chi_{Q_k}$ we get

$$\sum_{j,0<s_j<\infty} s_j^{m+1} F_{j,\alpha}^{**} = \sum_{j,0<s_j<\infty} f_j^{**} \leq \sum_{j,0\leq s_j<\infty} f_j^{**} = f^{**}.$$

Lemma 2 is obtained by repeated use of the following lemma.

Lemma 3. *Let a be an atom with vanishing moments of order less than or equal to m and with support in a cube Q with side s. Then there are atoms $s^{-1}A_l$, $l = 1, \dots, n$, supported in Q with vanishing moments of order less than or equal to $m-1$ such that*

$$a(x) = \sum_{l=1}^{n} 2\frac{\partial}{\partial x_l} A_l(x).$$

Proof of Lemma 3. By translation and dilation we may assume that $s = 1$ and $Q = \{x : 0 \leq x_i \leq 1, i = 1, \dots, n\}$. Define $b_0(x) = a(x)$ and for $l = 1, \dots, n$, define

$$b_l(x_1, \dots, x_n) = \begin{cases} \int_{R^l} a(y_1, \dots, y_l, x_{l+1}, \dots, x_n)dy_1 \cdots dy_l & \text{for } x \in Q, \\ 0 & \text{for } x \notin Q. \end{cases}$$

Observe that b_n is identically zero by the moment condition of a. Set $a_l = \frac{1}{2}(b_{l-1} - b_l)$. Then, of course, $a = 2\sum_{l=1}^{n} a_l$ and we claim that each function a_l is also an atom supported in Q whose moments of order less than or equal to m all vanish, and furthermore that

$$\int a_l(x_1, \dots, x_{l-1}, y_l, x_{l+1}, \dots, x_n)dy_l = 0 \tag{6}$$

for each $(x_1, \dots, x_{l-1}, x_{l+1}, \dots, x_n)$. It is obvious that a_l is supported in Q and that $\|a_l\|_\infty \leq 1$. We also get (6) directly by performing the multiple integration. The moment conditions can be shown by induction as follows. Since $b_0 = a$ we have b_0 satisfying all moment conditions of order less than or equal to m by assumption. Let us assume that we have shown all moment conditions of order less than or equal to m for $b_0, \dots, b_{l-1}$. We will then show that b_l satisfies all these moment conditions. Fix a monomial x^α with $|\alpha| \leq m$, and split it into parts $x^\alpha = h_1(x) + h_2(x)$ where h_1 is a monomial independent of x_l: $h_1(x) = \int_0^1 x^\alpha dx_l = \frac{1}{\alpha_l+1} x_1^{\alpha_1} \cdots x_{l-1}^{\alpha_{l-1}} x_{l+1}^{\alpha_{l+1}} \cdots x_n^{\alpha_n}$, and $h_2(x)$ is a polynomial of degree less than or equal to m satisfying $\int_0^1 h_2(x)dx_l = 0$. Since $b_l(x)$ as a function of x_l is a multiple of the characteristic function $\chi_{[0,1]}$ for each fixed $(x_1, \dots, x_{l-1}, x_{l+1}, \dots, x_n)$ we get $\int_0^1 b_l(x)h_2(x)dx_l = 0$. By (6) we have $\int_0^1 (b_l(x) - b_{l-1}(x))h_1(x)dx_l = 0$, and since by our assumption $\int b_{l-1}(x)h_1(x)dx = 0$, we get $\int b_l(x)h_i(x)dx = 0$ for $i = 1, 2$. Thus b_l and hence also a_l satisfies the moment conditions.

Finally we set

$$A_l(x) = \int_{-\infty}^{x_l} a_l(x_1, \dots, x_{l-1}, y_l, x_{l+1}, \dots, x_n)dy_l.$$

It is readily verified that A_l is supported in Q with $\|A_l\|_\infty \le 1$ and, by integrating by parts, that A_l satisfies all moment conditions of order less than or equal to $m-1$, and that $a(x) = \sum_1^n 2a_l(x) = \sum_1^n 2\partial A_l(x)/\partial x_l$. □

In order to estimate the $L_{q_1}(\Omega_i)$-norm of $\hat{f}_j$ for $j > i$, we use Hausdorff-Young's inequality (2) on the functions $F_{j,\alpha}$ together with the pointwise estimate $|F_{j,\alpha}(x)| \le F_{j,\alpha}^{**}(x) = s_j^{-m-1} f_j^{**}(x)$ and get

$$\|\hat{f}_j\|_{L_{q_1}(\Omega_i)} \le C s_i^{m+1} \sum_{|\alpha|=m+1} \|\hat{F}_{j,\alpha}\|_{L_{q_1}} \le C(s_i/s_j)^{m+1} \|f_j^{**}\|_{L_{p_1}}. \tag{7}$$

Recall that we are assuming $0 < s_i < s_j < \infty$ in this case. By Lemma 1 we have $s_i/s_j \le C\beta^{(i-j)/(n[d-\min(1,p/p_1)])}$ for $j > i$. Combining the inequalities (5) and (7) we get

$$\|\hat{f}_j\|_{L_{q_1}(\Omega_i)} \le C \min(1, \beta^{(m+1)(i-j)/(n[d-\min(1,p/p_1)])}) \|f_j^{**}\|_{L_{p_1}}. \tag{8}$$

In order to estimate the right-hand side of (8) by the L_v^p-norm of f_j^{**} we will use a lemma which concerns weighted norms of sums $\sum_k \lambda_k \chi_{Q_k}$ of characteristic functions of cubes.

Lemma 4. *Let φ be a nonnegative compactly supported function in the Schwartz class with $\varphi_t(x) = \varphi(x/t)t^{-n}$ and let v be a weight on R^n satisfying the doubling condition. Then for $0 < p \le p_1 < \infty$ and for each constant $c > 1$ there is a constant C depending only on c, p and p_1, and a constant C_1 depending only on φ, p and c such that for all $t > 0$*

$$\left(t^{-n}\int_{R^n} |f(x)|^{p_1} dx\right)^{1/p_1} \le C\left(t^{-n}\int_{R^n} |f(x)|^p dx\right)^{1/p}$$

and

$$\int_{R^n} |f(x)|^p \varphi_t * v(x)dx \le C_1 \int_{R^n} |f(x)|^p v(x)dx$$

for all $f = \sum_k \lambda_k \chi_{Q_k}$ with $\lambda_k > 0$ and side of $Q_k > t/c$ for all k.

The assumption that φ has compact support is not really necessary but is just for simplicity of the proof.

Proof of Lemma 4. The lemma is a direct consequence of the following inequality [Str-T, Chapter 8, Lemma 4]:

$$\int_{R^n} \left|\sum_k \lambda_k \chi_{c_1 Q_k}(x)\right|^q v(x)dx \le C \int_{R^n} \left|\sum_k \lambda_k \chi_{Q_k}(x)\right|^q v(x)dx$$

for any constant $c_1 > 1$ with C depending only on c_1, p and v. To see why, we first split R^n into a union of cubes $\tilde{Q}_l = \{x \in R^n : tl_i \le x_i < t(l_i+1), i = 1, \dots, n\}$, $l \in Z^n$.

Since all cubes Q_k have sides larger than t/c, then

$$|f(x)| = \sum_k \lambda_k \chi_{Q_k}(x) \le \sum_k \lambda_k \chi_{c_1 Q_k}(tl) \quad \text{when } x \in \tilde{Q}_l$$

and also

$$\sum_k \lambda_k \chi_{c_1 Q_k}(tl) \le \sum_k \lambda_k \chi_{c_2 Q_k}(x) \quad \text{when } x \in \tilde{Q}_l.$$

(These inequalities hold if we choose $c_1 \ge 1 + 2c$ and $c_2 \ge c_1 + 2c$.)

From these two inequalities we get

$$\begin{aligned}
\left(t^{-n} \int_{R^n} |f(x)|^{p_1} dx\right)^{1/p_1} &\le \left(\sum_{l \in Z^n} \left(\sum_k \lambda_k \chi_{c_1 Q_k}(tl)\right)^{p_1}\right)^{1/p_1} \\
&\le \left(\sum_{l \in Z^n} \left(\sum_k \lambda_k \chi_{c_1 Q_k}(tl)\right)^{p}\right)^{1/p} \\
&\le \left(t^{-n} \int_{R^n} \left(\sum_k \lambda_k \chi_{c_2 Q_k}(x)\right)^p dx\right)^{1/p} \\
&\le C\left(t^{-n} \int_{R^n} |f(x)|^p dx\right)^{1/p},
\end{aligned}$$

where we have used the result of [Str-T] to obtain the last inequality. Since

$$\int_{R^n} |f(x)|^p \varphi_t * v(x) dx = \int_{R^n} \varphi_t(y) \left(\int_{R^n} |f(x+y)|^p v(x) dx\right) dy$$

and

$$|f(x+y)| \le \sum_k \lambda_k \chi_{c_3 Q_k}(x)$$

if $\varphi_t(y) \ne 0$ and we choose c_3 such that $c_3 \ge 1 + 2c_4 c$ where $\operatorname{supp} \varphi \subset \{|y| \le c_4\}$, we also get

$$\begin{aligned}
\int_{R^n} |f(x)|^p \varphi_t * v(x) dx &\le \left(\int \varphi_t dy\right) \left(\int_{R^n} \left(\sum_k \lambda_k \chi_{c_3 Q_k}(x)\right)^p v(x) dx\right) \\
&\le C_1 \int_{R^n} |f(x)|^p v(x) dx.
\end{aligned}$$

This completes the proof of Lemma 4. □

Remark. We do not need the last part of the lemma, which uses the doubling condition of v, if we replace the inequalities in Proposition 1 by

$$\|\hat{f}\|_{L^q_u} \le C\|f^{***}\|_{L^p_v} \quad \text{and} \quad \|\hat{f}\|_{L^q_u} \le C\|f^{*r**}\|_{L^p_v} \quad \text{resp.},$$

where $f^{***} = \sum_k \lambda_k \chi_{c_3 Q_k}$ and $f^{*r**} = (\sum_k \lambda_k^r \chi_{c_3 Q_k})^{1/r}$ As we shall see in §6, these somewhat weaker inequalities will be sufficient to prove Theorem 1.

The function f_j^{**} is the finite sum of characteristic functions of cubes Q_k, $k \in I_j$. We may assume that I_j is not the empty set. Let

$$\bar{s}_j = 1/\min_{k \in I_j} s(Q_k).$$

Thus $s_j \le \bar{s}_j \le s_{j+1}$ and $0 < \bar{s}_j < \infty$. The function f_j^{**} is a sum of characteristic functions of cubes with sides larger than $1/\bar{s}_j$. In the case $p \le p_1$ we get by the definition of s_j the inequality

$$\bar{s}_j^{n(1/p-1/p_1)} \beta^{-j/p} \le C \left(\inf_{x \in R^n} \varphi_{1/\bar{s}_j} * v(x) \right)^{1/p}.$$

By this inequality and the first inequality in Lemma 4 we get in the case $p \le p_1$

$$\|f_j^{**}\|_{p_1} \le C(\bar{s}_j)^{n(\frac{1}{p}-\frac{1}{p_1})} \|f_j^{**}\|_p \le C\beta^{\frac{j}{p}} \left(\int_{R^n} |f_j^{**}(x)|^p \varphi_{1/\bar{s}_j} * v(x) dx \right)^{\frac{1}{p}}.$$

Similary, in the case when $p > p_1$ we get by Hölder's inequality

$$\begin{aligned} \|f_j^{**}\|_{p_1} &\le \left(\int_{R^n} (\varphi_{1/\bar{s}_j} * v)^{-\frac{p_1}{p-p_1}} dx \right)^{\frac{p-p_1}{p_1 p}} \left(\int_{R^n} |f_j^{**}|^p \varphi_{1/\bar{s}_j} * v dx \right)^{\frac{1}{p}} \\ &\le C\beta^{j/p} \left(\int_{R^n} |f_j^{**}(x)|^p \varphi_{1/\bar{s}_j} * v(x) dx \right)^{\frac{1}{p}}. \end{aligned}$$

By the second inequality in Lemma 4, we conclude in both cases that

$$\|f_j^{**}\|_{p_1} \le C\beta^{j/p} \|f_j^{**}\|_{L_v^p}. \tag{9}$$

By (4), (8), and (9) we have

$$\|\hat{f}_j\|_{L_u^q(\Omega_i)} \le C\beta^{(j-i)/p} \min(1, \beta^{(m+1)(i-j)/(n[d-\min(1,p/p_1)])}) \|f_j^{**}\|_{L_v^p}.$$

From this we get (3) for any

$$0 < \delta \le \min\{1/p, (m+1)/(n[d-\min(1,p/p_1)]) - 1/p\}$$

provided $m+1 > n[d-\min(1,p/p_1)]/p$.

5. Summation of all pieces

In order to prove Proposition 1 it remains only to put the pieces together again using estimate (3), which was

$$\|\hat{f}_j\|_{L_u^q(\Omega_i)} \le \beta^{-\delta|i-j|} \|f_j^{**}\|_{L_v^p}$$

for some $\delta > 0$. Let $q_0 = \min(q, 1)$. By Minkowski's inequality or the triangle inequality we have

$$\|\hat{f}\|_{L_u^q(\Omega_i)}^{q_0} \le \sum_j \|\hat{f}_j\|_{L_u^q(\Omega_i)}^{q_0},$$

and since the sets $\Omega_i, i = \ldots, -1, 0, 1, \ldots$, are disjoint we also have

$$\|\hat{f}\|_{L_u^q} \le \left(\sum_i \|\hat{f}\|_{L_u^q(\Omega_i)}^q \right)^{1/q}.$$

Thus by combining inequalities, we get

$$\|\hat{f}\|_{L^q_u} \le C \left(\sum_i \left(\sum_j \beta^{-q_0 \delta |i-j|} \|f_j^{**}\|^{q_0}_{L^p_v} \right)^{q/q_0} \right)^{1/q}. \tag{10}$$

Let $0 < r \le 1$. Then

$$\sum_j (f_j^{**}(x))^r = \sum_j \left(\sum_{k \in I_j} \lambda_k \chi_{Q_k}(x) \right)^r \le \sum_k \lambda_k^r \chi_{Q_k}(x) = \left(f^{*r*}(x) \right)^r.$$

Let $p_0 = \max(p, r)$. We use the pointwise inequality

$$\sum_j (f_j^{**}(x))^p \le \left(\sum_j (f_j^{**}(x))^r \right)^{p/r}$$

when $p \ge r$ and the concavity of the $L_v^{p/r}$-'norm' for positive functions (i.e., the inequality opposite to the usual Minkowski inequality) when $0 < p \le r$ to obtain the inequality

$$\left(\sum_j (\|f_j^{**}\|_{L^p_v})^{p_0} \right)^{1/p_0} \le \left\| \left(\sum_j \left(f_j^{**} \right)^r \right)^{1/r} \right\|_{L^p_v} \le \|f^{*r*}\|_{L^p_v}. \tag{11}$$

To complete the proof of Proposition 1 we need only to show that the right side of (10) is less than the left side of (11). To do this, we will apply the following discrete version of Hardy's inequality with the exponents $r_0 = q$, $r_1 = q_0$ and $r_2 = p_0$.

Lemma 5. *Let* $0 < r_1 < \infty$, $0 < r_2 < \infty$, $r_0 \ge r_2$, $\beta > 1$, *and* $\delta > 0$. *Then*

$$\left(\sum_i \left(\sum_j \beta^{-\delta|i-j|} a_j^{r_1} \right)^{r_0/r_1} \right)^{1/r_0} \le C \left(\sum_j a_j^{r_2} \right)^{1/r_2}$$

for any sequence $\{a_j\}_{-\infty}^{\infty}$ *of nonnegative numbers.*

Proof of Lemma 5. Using Hölder's inequality on the inner sum in the case $r_1 < r_2$ we get

$$\sum_j \beta^{-\delta|i-j|} a_j^{r_1} \le \left(\sum_j \beta^{-\delta|i-j|} a_j^{r_2} \right)^{r_1/r_2} \left(\sum_j \beta^{-\delta|i-j|} \right)^{1-r_1/r_2}$$
$$\le C \left(\sum_j \beta^{-\delta|i-j|} a_j^{r_2} \right)^{r_1/r_2}.$$

For $r_1 \geq r_2$, Hölder's inequality is replaced by the inequality

$$\sum_j b_j^{r_1} \leq \left(\sum_j b_j^{r_2}\right)^{r_1/r_2}$$

for $b_j \geq 0$. Using the inequality

$$\left(\sum_i b_i^{r_0/r_2}\right)^{1/r_0} \leq \left(\sum_i b_i\right)^{1/r_2},$$

for $b_i \geq 0$, we see that there is a $\delta_1 > 0$ such that the left-hand side of the inequality in Lemma 5 is bounded by

$$\leq C\left(\sum_i \sum_j \beta^{-\delta_1|i-j|} a_j^{r_2}\right)^{1/r_2} \leq C\left(\sum_j a_j^{r_2}\right)^{1/r_2}.$$

This completes the proof of Lemma 5. □

Combining Lemma 5 with (10) and (11) we get

$$\|\hat{f}\|_{L^q_u} \leq C\|f^{*r*}\|_{L^p_v}.$$

This completes the proof of Proposition 1. □

6. Limiting arguments

In this section we are going to use the estimate for the Fourier transform of a finite sum of atoms to obtain a similar estimate for functions in $\mathscr{S}_{0,0}$, and then we will extend that estimate to all tempered distributions in H^p_v.

If f is a function in $\mathscr{S}_{0,0}$ then it can be written as an infinite sum of atoms, i.e., $f = \sum_{k=1}^{\infty} \lambda_k a_k$ (with $\lambda_k > 0$) such that the partial sums $f_N = \sum_{k=1}^{N} \lambda_k a_k$ converge pointwise to f almost everywhere . Furthermore, for any $m \geq 0$, $M > 0$ and $r > 0$, the decomposition can be done such that

$$f^{*r*}(x) = \left(\sum_k \lambda_k^r \chi_{B_k}(x)\right)^{1/r} \leq C_{M,m,r} f^{*M}(x) \quad \text{for all } x,$$

where B_k is a supporting ball for the atom a_k and with the atoms a_k satisfying all moment conditions of order less than or equal to m (see [Str-T]). (In fact, by careful use of the Whitney decomposition of the sets $\{f^{*M} > 2^l\}$ when doing the atomic decomposition we may replace χ_{B_k} by χ_{cB_k} for any fixed $c > 0$ in the inequality above.) For any $\lambda > 0$ and any ψ in $\mathscr{S}$ with nonzero integral, the right-hand side is dominated by $CN^\lambda_\psi f(x)$ provided M is large enough depending on λ.

For all $\lambda > 0$ and all $f \in \mathscr{S}_{0,0}$ we have

$$N^\lambda_\psi f(x) \leq C_{\lambda,f,\psi}(1 + |x|)^{-\lambda}, \qquad x \in R^n.$$

If $\lambda > n$ the finite sums of atoms f_N are dominated by $C_{\lambda,f,\psi}(1+|x|)^{-\lambda} \in L^1$, and hence the f_N converge to f in L^1 norm by the dominated convergence theorem. Thus the Fourier transforms $\hat{f}_N(y)$ converge to $\hat{f}(y)$ uniformly for all $y \in R^n$. Since

$$\|\hat{f}_N\|_{L^q_u} \leq C\|f_N^{*r*}\|_{L^p_v} \leq C\|N_\psi^\lambda f\|_{L^p_v},$$

by Proposition 1, provided $0 < r \leq \min(q, 1)$, we conclude that

$$\|\hat{f}\|_{L^q_u} \leq C\|N_\psi^\lambda f\|_{L^p_v} \quad \text{for } f \in \mathscr{S}_{0,0}.$$

Next assume that f is a tempered distribution in H^p_v. Since $\mathscr{S}_{0,0}$ is dense in H^p_v, we get a sequence of functions $f_N \in \mathscr{S}_{0,0}$ which converges to f in H^p_v norm and hence also in the distributional sense. We may assume that $\|f_N - f_{N-1}\|_{H^p_v} \leq 2^{-N}$. We may assume that $\int |\varphi_{1/s} * v|^{-p_1/(p-p_1)}dx < \infty$ if $p > p_1$ (resp., $\inf_{x\in R^n} |\varphi_{1/s} * v| \geq c > 0$ when $p \leq p_1$), since otherwise the $PW(q, p; q_1, p_1)$ condition would imply that u is identically zero. If $u_1(x) = (1 + |x|)^{-l}$, then it follows from Lemma 1 that the pair of weights (u_1, v) satisfies the $PW(q, p; q_1, p_1)$ condition provided l is chosen large enough. By the estimate above of the Fourier transform of a function in $\mathscr{S}_{0,0}$ it follows that $\{\hat{f}_N\}$ is a Cauchy sequence in $L^q_{u_1}$ and hence converges in norm to a function g in $L^q_{u_1}$. Since the weight $u_1 \geq c > 0$ on compact sets, $\hat{f}_N$ converges to g in measure on compact sets. The sequence $\{\hat{f}_N\}$ is also a Cauchy sequence in L^q_u and since $\hat{f}_N$ converges in measure to g on compact sets we conclude that $\hat{f}_N$ converges to g in L^q_u norm and that

$$\|g\|_{L^q_u} = \lim_{N\to\infty} \|\hat{f}_N\|_{L^q_u} \leq C \lim_{N\to\infty} \|f_N\|_{H^p_v} = C\|f\|_{H^p_v}.$$

It remains to show that the tempered distribution $\hat{f}$ is obtained from the function g, i.e., to show that

$$\langle \hat{f}, \phi\rangle = \int_{R^n} g(x)\phi(x)dx \quad \text{for all } \phi \in \mathscr{S}.$$

Since f_N converges to f in the distributional sense we have

$$\langle \hat{f}, \phi\rangle = \langle f, \hat{\phi}\rangle = \lim_{N\to\infty} \langle f_N, \hat{\phi}\rangle = \lim_{N\to\infty} \langle \hat{f}_N, \phi\rangle,$$

and since $\phi \in \mathscr{S}$, which is contained in the dual space of $L^q_{u_1}$, and also $\hat{f}_N$ converges to g in $L^q_{u_1}$, we conclude that

$$\lim_{N\to\infty} \langle \hat{f}_N, \phi\rangle = \int_{R^n} g(x)\phi(x)dx.$$

Thus we have shown that

$$\langle \hat{f}, \phi\rangle = \int_{R^N} g(x)\phi(x)dx$$

with

$$\|g\|_{L^q_u} \leq C\|f\|_{H^p_v}$$

This completes the proof of Theorem 1. $\square$

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