

CURVATURES AND SIMILARITY OF OPERATORS WITH HOLOMORPHIC EIGENVECTORS

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ABSTRACT. The curvature of the holomorphic vector bundle generated by eigenvectors of operators is estimated, and the necessary and sufficient conditions for contractions to be similar or quasi-similar with unilateral shifts are given.

1. INTRODUCTION

Let H be a separable complex Hilbert space, $g_r(n, H)$ the set of all n -dimensional subspaces of H , and γ a mapping from an open connected set Ω in the complex plane $\mathbb{C}$ to $g_r(n, H)$. Then γ is called a holomorphic curve over Ω , if for each w_0 in Ω , there is a nbhd Δ of w_0 and vector valued holomorphic functions γ_{iw} on Δ ($i = 1, \dots, n$) satisfying $\gamma_w = \bigvee \{\gamma_{iw} : i = 1, \dots, n\}$ for w in Δ . In this case, the Hermitian holomorphic vector bundle (E_γ, Ω, π) is defined as

$$E_\gamma = \{(x, w) \in H \times \Omega : x \text{ in } \gamma_w\}, \quad \pi(x, w) = w,$$

and hence for this bundle, the canonical connection and curvature $\mathcal{K}_\gamma$ are well defined [19]. We call $\gamma_{1w}, \dots, \gamma_{nw}$ a frame for E_γ on Δ . The matrix form of $\mathcal{K}_\gamma(w)$ with respect to the above frame is

$$(1.1) \quad -\frac{\partial}{\partial \bar{w}} \left(G\gamma^{-1} \frac{\partial G\gamma}{\partial w} \right),$$

where $G_\gamma(w)$ is the Gram matrix whose (i, j) component is $(\gamma_j(w), \gamma_i(w))$ (cf. [4]).

In case of $n = 1$, we have especially

$$\mathcal{K}_\gamma(w) = -\frac{\partial^2}{\partial w \partial \bar{w}} \log \|\gamma_{1w}\|^2.$$

We explain some notations about relations between given bounded operators T_1, T_2 . Suppose there is an intertwining bounded operator X such that $XT_1 = T_2X$, then we denote by $T_1 \overset{d}{\prec} T_2$, $T_1 \overset{i}{\prec} T_2$, $T_1 \prec T_2$, $T_1 \approx T_2$, and $T_1 \cong T_2$,

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X with dense range, X injective, quasi-affinity (that is, X is injective and has dense range), invertible, and unitary, respectively. Moreover we write $T_1 \sim T_2$ and say that T_1 and T_2 are quasi-similar, if $T_1 \prec T_2$ and $T_2 \prec T_1$. In [4], Cowen-Douglas defined the class $B_n(\Omega)$ consisting of bounded operator T satisfying

- (a) $\Omega \subset \sigma(T)$,
- (b) $\text{range}(T - w) = H$ for each w in Ω ,
- (c) $\bigvee_{w \in \Omega} \ker(T - w) = H$,
- (d) $\dim \ker(T - w) = n$ for w in Ω .

Now we introduce the class $B_n^h(\Omega)$ as

Definition. T belongs to $B_n^h(\Omega)$ if there is a holomorphic curve $\gamma: \Omega \rightarrow g_r(n, H)$ such that $\gamma(w) \subset \ker(T - w)$, and $\bigvee_{w \in \Omega} \gamma(w) = H$. It is known that $B_n(\Omega) \subset B_n^h(\Omega)$. If T is in $B_n^h(\Omega)$, then the bundle is well defined by the curve $\gamma(w)$. We denote it and its curvature by E_T and $\mathcal{K}_T$.

The purpose of this paper is to estimate $\mathcal{K}_T$ of T in $B_n^h(\Omega)$ and to research what kind of operator is similar or quasi-similar to the shifts.

Now we show some examples. Let $\{e_n\}_{n=0}^\infty$ be a C.O.N.B. of H and A a weighted shift with positive weight $\{a_n\}_{n=1}^\infty$, that is $Ae_n = a_{n+1}e_{n+1}$. Set $b_n = a_1 \cdots a_n$ and $r_1(A) = \lim_{n \rightarrow \infty} (\inf_k b_{k+n}/b_k)^{1/n}$. Then we have $A^* \in B_1(\{w: |w| < r_1(A)\})$, (see [13 or 12]). Especially, the adjoint of unilateral shift S corresponding to $a_n = 1$ for all n and the adjoint of the Bergman shift B corresponding to $a_n = \sqrt{n/(n+1)}$ for all n are both in $B_1(D)$, where D is the open unit disk. And $\mathcal{K}_{S^*}(w) = -1/(1 - |w|^2)^2$ and $\mathcal{K}_{B^*}(w) = -2/(1 - |w|^2)^2$.

In [17, 18] we studied a contraction T with $I - T^*T$ in the trace class, and showed that $S_n^* \prec T^*$ if and only if T is in C_{10} (that is, $T^n x \nrightarrow 0$, $T^{*n} x \rightarrow 0$ as $n \rightarrow \infty$ for every $x \neq 0$) [17], and that these are equivalent with $T^* \in B_n^h(D)$ [18]. We should notice that $B_n^h(\Omega) \subset B_n^h(\Delta)$ for $\Delta \subset \Omega$ (cf. p. 193 of [4]).

2. CURVATURES

It was shown that the curvature of a vector bundle generated by a holomorphic curve was nonpositive, and if T is in $B_1(\Omega)$, then

$$(2.1) \quad \mathcal{K}_T(w)^{-1} = -\text{trace } N_w^* N_w,$$

where $N_w = (T - w)|_{\ker(T - w)^2}$ [4]. Let Ω be a finitely connected Jordan region and $\text{cl}\Omega$ (closure of Ω) is a spectral set for T , that is $\sigma(T) \subset \text{cl}\Omega$ and $\|f(T)\| \leq \|f\|_\infty$ for every rational function f with no poles in $\text{cl}\Omega$. Then the curvature of T in $B_1(\Omega)$ was estimated by Misra [9] as

$$(2.2) \quad \mathcal{K}_T(w) \leq -\hat{K}_\Omega(w, \bar{w})^2,$$

where $\hat{K}_\Omega$ is the Szegő kernel of Ω . His proof is based on (2.1). In this section we will extend (2.2) to the case of the $B_n^h(\Omega)$ by virtue of the canonical model

theory of contraction due to Sz.-Nagy and Foias [14]; let T be a contraction on H in C_0 , that is $T^{**n}x \rightarrow 0$ for x in H . Then there is the characteristic function $\theta(z)$, which is a $B(F_1, F_2)$ -valued holomorphic contractive function defined on D and $\theta(z)$ is isometric from F_1 to F_2 a.e. on the unit circle, where F_1 and F_2 are the subspaces of H called defect spaces of T . And then T on H is unitarily equivalent to $S(\theta)$ on $H(\theta)$ given as the following:

$$(2.3) \quad H(\theta) = H^2(F_2) \ominus \theta H^2(F_1), \quad S(\theta)^* = M_z^*|_{H(\theta)},$$

where M_z is the multiplication by z on $H^2(F_2)$, which is the Hardy class of F_2 -valued holomorphic functions on D . We remark that $S_n := S \oplus \cdots \oplus S \cong M_z$ on $H^2(\mathbb{C}_n)$.

Theorem 2.1. *Let $\gamma: \Omega \rightarrow g_r(n, H)$ be a holomorphic curve such that $\Omega \subset D$, Ω is open, $\bigvee_{w \in \Omega} \gamma(w) = H$. Suppose there is a contraction T such that $\gamma(w) \subset \ker(T^* - w)$ for $w \in \Omega$. Then $\mathcal{K}_\gamma(w) (= \mathcal{K}_{T^*}(w)) \leq -I_n/(1 - |w|^2)^2$ for w in Ω .*

Proof. Since $T^{**k}\gamma(w) = w^k\gamma(w) \rightarrow 0$ ($k \rightarrow \infty$), $\|T^*\| \leq 1$ implies $T \in C_0$. So we may consider $S(\theta)$ of (2.3) instead of T . For any $w_0 \in \Omega$, there is a nbhd Δ of w_0 and a frame $\gamma_{1w}, \dots, \gamma_{nw}$ for γ_w on Δ . Then, since $M_z^*\gamma_{iw} = w\gamma_{iw}$, we can represent γ_{iw} as the function in $H(\theta)$:

$$(2.4) \quad \gamma_{iw}(z) = \frac{\gamma_{iw}(0)}{1 - wz} \quad \text{for } z \in D.$$

Thus we have

$$(2.5) \quad \gamma_{iw}(0) \perp \theta(\bar{w})F_2$$

and

$$(2.6) \quad \begin{aligned} (\gamma_{jw}, \gamma_{iw})_{H(\theta)} &= \frac{1}{2\pi} \int_{\partial D} (\gamma_{jw}(z), \gamma_{iw}(z))_{F_2} |dz| \\ &= \frac{1}{1 - |w|^2} (\gamma_{jw}(0), \gamma_{iw}(0))_{F_2}, \end{aligned}$$

which implies $\gamma_{iw}(0), \dots, \gamma_{nw}(0)$ are linearly independent. Hence, if we set $\gamma_w^0 = \bigvee \{\gamma_{iw}(0) : i = 1, \dots, n\}$ for each $w \in \Delta$, then $\gamma^0: \Delta \rightarrow g_r(n, F_2)$ is a holomorphic curve. From (1.1) and (2.6), it follows that

$$(2.7) \quad \mathcal{K}_\gamma(w) = -\frac{I_n}{(1 - |w|^2)^2} + \mathcal{K}_{\gamma^0}(w) \quad \text{for } w \text{ in } \Delta.$$

Since $\mathcal{K}_{\gamma^0}(w) \leq 0$, we can conclude the proof.

Proposition 2.2. *If T is a contraction in $B_n^h(D)$ and $\mathcal{K}_T(w) = I_n/(1 - |w|^2)^2$ on an open set $\Delta \subset D$, then $T \cong S_n^*$.*

Proof. Since $\mathcal{K}_T(w) = \mathcal{K}_{S_n^*}(w)$ for w in Δ , from Proposition 3.3 of [4], there is a holomorphic isometric bundle map $U(w)$ satisfying $U(w)\ker(T - w) =$

$\ker(S_n^* - w)$ for w in Δ . Since T is in $B_n^h(\Delta)$, by the rigidity theorem (cf. p. 202 of [4]), there is a unitary U on H such that $U \ker(T - w) = \ker(S_n^* - w)$ and hence $UT = S_n^*U$. Thus the proof is complete.

Let Ω_1, Ω_2 be connected open sets, $\gamma: \Omega_2 \rightarrow g_r(n, H)$ a holomorphic curve, and ϕ an injective holomorphic mapping from Ω_1 to Ω_2 . Then by the chain rule and (1.1) we have

$$(2.8) \quad \mathcal{K}_{\gamma \circ \phi}(w) = |\phi'(w)|^2 \mathcal{K}_\gamma(\phi(w)) \quad \text{for } w \text{ in } \Omega_1.$$

Proposition 2.3. *If T is a bounded operator in $B_n(\Omega)$, where Ω is an open connected set, then*

$$\mathcal{K}_T(w) \leq -\frac{I_n}{(\|T\|^2 - |w|^2)^2} \quad \text{for } w \in \Omega.$$

Proof. From (2.8) $\mathcal{K}_{T/\|T\|}(w/\|T\|) = \|T\|^2 \mathcal{K}_T(w)$ follows. Since $\Omega/\|T\| \subset D$, Theorem 2.1 implies the above inequality.

Theorem 2.5. *Let Ω be a p -ply connected Jordan region, and $T \in B_n^h(\Delta)$ for some $\Delta \subset \Omega$. Suppose $\text{cl } \Omega$ is a spectral set of T . Then we have*

$$\mathcal{K}_T(w) \leq -\widehat{K}_\Omega(w, \bar{w})^2 I_n \quad \text{for } w \in \Delta.$$

Proof. For each w_0 in Δ there is a holomorphic function F from Ω to a p -sheeted disc such that $F(w_0) = 0$, $F'(w_0) \neq 0$, and F is continuous on $\text{cl } \Omega$ (cf. [7, 2]). From Mergerlyan's theorem there is a sequence of rational functions with no poles in $\text{cl } \Omega$ which uniformly converges to F on $\text{cl } \Omega$. We denote it by $\{R_n\}$. Then Riesz functional $R_n(T)$ is well defined and $\{R_n(T)\}$ converges uniformly. We represent its limit by $F(T)$. Then for a holomorphic curve $\gamma(w) \subset \ker(T - w)$ on Δ , $\|F(T)\| \leq \|F\| = 1$, and $\{F(T) - F(w)\}\gamma(w) = 0$ follows, because $\{R_n(T) - R_n(w)\}\gamma(w) = 0$. From $F'(w_0) \neq 0$ we can take neighbourhoods Ω_1 of w_0 and Ω_2 of 0 such that $F|_{\Omega_1}: \Omega_1 \rightarrow \Omega_2$ is bijective. Let ϕ be the inverse of $F|_{\Omega_1}$. Then we have $\{F(T) - z\}\gamma(\phi(z)) = 0$ for z in Ω_2 . Since

$$\bigvee \{\gamma(\phi(z)): z \in \Omega_2\} = \bigvee \{\gamma(w): w \in \Omega_1\} = \bigvee \{\gamma(w): w \in \Omega\} = H$$

follows from p. 194 of [4], a contraction $F(T)$ and curve $\gamma \circ \phi$ satisfy the conditions of Theorem 2.1. Thus at the origin $\mathcal{K}_{\gamma \circ \phi}(0) \leq -I_n$, from which, using (2.8), we get

$$\mathcal{K}_\gamma(w_0) \leq -|F'(w_0)|^2 I_n = -\widehat{K}_\Omega(w_0, \bar{w}_0)^2 I_n,$$

because the second equality follows from p. 118 of [2]. Consequently we can conclude the proof.

At the end of this section we consider the question proposed on p. 329 of [5], that is, if T_1 and T_2 are contractions in $B_1(D)$ such that $\mathcal{K}_{T_1} \leq \mathcal{K}_{T_2}$, then does there exist a bounded operator X such that $XT_1 = T_2X$? Corollary 2.2 shows $\mathcal{K}_T \leq \mathcal{K}_{S^*}$ for any contraction T in $B_1(D)$, and the existence of X

with dense range satisfying $XT = S^*X$ is well known (cf. [16], or see the proof of Proposition 3.6). Hence the question is true in the case of $T_2 = S^*$. In [10] Misra showed that a contraction T in $B_1(D)$ is unitarily equivalent to $\phi(T)$ for every Möbius transformation ϕ of D if and only if $\mathcal{K}_T(w) = -\alpha/(1-|w|^2)^2$, where α is a constant and $\alpha \geq 1$.

Proposition 2.6. *Let T_1, T_2 be contractions in $B_1(D)$ with curvature $\mathcal{K}_{T_i}(w) = -\alpha_i/(1-|w|^2)^2$ ($\alpha_i \geq 1$). Then next conditions are equivalent: (i) $\mathcal{K}_{T_2} \leq \mathcal{K}_{T_1}$, (ii) there is a bounded operator X such that $XT_2 = T_1X$, and (iii) $T_2 \prec T_1$.*

Proof. Let A_i be the weighted shift with weight $a_{ni} = \sqrt{n/(\alpha_i + n - 1)}$ for $i = 1, 2$. Then we have $r_1(A_i) = 1$ and hence $A_i^* \in B_1(D)$. Since the square of the norm of a holomorphic eigenvector of $A_i^* - w$ is $(1-|w|^2)^{\alpha_i}$, $\mathcal{K}_{A_i^*}(w) = \mathcal{K}_{T_i}(w)$, and hence $A_i^* \cong T_i$ (see [5]). Thus we may identify A_i^* with T_i . Assume (i). Then diagonal quasi-affinity Y defined by $Ye_n = \{(a_{12} \cdots a_{n2})/(a_{11} \cdots a_{n1})\}e_n$ satisfies $YA_1 = A_2Y$ and hence $Y^*T_2 = T_1Y^*$, which implies (iii). Assume (ii). Since $X^*A_1 = A_2X^*$, setting $b_{mn} = (X^*e_n, e_m)$, we obtain

$$b_{m,n+1}a_{n+1,1} = \begin{cases} 0 & (m=0), \\ b_{m-1,n}a_{m,2} & (m \geq 1). \end{cases}$$

Since there is a nonvanishing b_{ij} ($i \geq j$), boundedness of X implies that $\prod_{k=1}^{\infty} a_{i+k,2}/a_{j+k,1}$ is bounded. To show (i), suppose $\alpha_1 > \alpha_2$, then each term of the infinite product is larger than 1. Hence

$$\sum_{k=1}^{\infty} \left(\left(\frac{\alpha_1 + j + k - 1}{j + k} \right) / \left(\frac{\alpha_2 + i + k - 1}{i + k} \right) - 1 \right)$$

must converge, however this is impossible. Consequently (i) follows. (iii) obviously implies (ii), and the proof is complete.

We can apply the previous result to show that $S \prec B$, where B is the Bergman shift, but there is not a bounded operator X such that $XB = SX$, though it is possible to get them by another simple method.

3. EXACT SEQUENCE AND INTERTWINING OPERATORS

In this section we give the conditions for a contraction T to be $T \prec S_n$ or $T \approx S_n$. At the beginning we will refer to a result about exact sequence of Hardy classes and use it to show that if $T \prec S_n$, then $T^* \in B_n(D)$. A $B(F_1, F_2)$ -valued holomorphic function $\Gamma(z)$ on D is called bounded if $\sup_{z \in D} \|\Gamma(z)\| < \infty$. In this case a bounded operator Γ from $H^2(F_1)$ to $H^2(F_2)$ is determined by $(\Gamma f)(z) = \Gamma(z)f(z)$.

Theorem 3.1. *Let Γ_1, Γ_2 be operator-valued bounded holomorphic functions on D , and suppose*

$$H^2(F_1) \xrightarrow{\Gamma_1} H^2(F_2) \xrightarrow{\Gamma_3} H^2(C_n)$$

is exact and Γ_2 has the dense range. Then the next sequence is exact for every z in D :

$$F_1 \xrightarrow{\Gamma_1(z)} F_2 \xrightarrow{\Gamma_2(z)} C_n \rightarrow 0.$$

Proof. Since $\Gamma_2(z)\Gamma_1(z) = 0$, we have only to show $\ker \Gamma_2(z) \subset \Gamma_1(z)F$. Since Γ_2 has the dense range, from the Cauchy integral formula, the range of $\Gamma_2(z)$ is dense and hence coincident with C_n . Thus $\tilde{\Gamma}_2(z) := \Gamma_2(\bar{z})^*$ is injective with closed range. Fix an arbitrary z_0 in D . There is an isometry V from C_n to F_2 such that $\det V^* \tilde{\Gamma}_2(z_0) \neq 0$. Then $\Omega := \{z \in D : \det V^* \tilde{\Gamma}_2(z) = 0\}$ is a set of isolated points. In the same way as Theorem 1 of [17] or p. 94 of [8] we can obtain a $B(F, F_2)$ -valued bounded holomorphic function $\Phi(z)$ defined on D such that $\tilde{\Gamma}_2(z)C_n \oplus \Phi(\bar{z})F = F_2$ for $z \in D \setminus \Omega$, where F is an auxiliary Hilbert space. This implies $\ker \Gamma_2(\bar{z}) = \Phi(\bar{z})F$ for $z \in D \setminus \Omega$ and hence $\Gamma_2\Phi = 0$. Thus we have $\Phi H^2(F) \subset \ker \Gamma_2 = \Gamma_1 H^2(F_1)$. Taking F -valued constant functions we get $\Phi(z)F \subset \Gamma_1(z)F_1$ for $z \in D$. Thus we have $\ker \Gamma_2(\bar{z}_0) = \Phi(\bar{z}_0)F \subset \Gamma_1(\bar{z}_0)F_1$. The proof is complete.

Remark. The converse assertion of the theorem is false. In fact, set

$$\Gamma_1(z) = \begin{pmatrix} \exp \frac{z+1}{z-1} \\ 0 \end{pmatrix}, \quad \Gamma_2(z) = (0, 1),$$

then

$$C_1 \xrightarrow{\Gamma_1(z)} C_2 \xrightarrow{\Gamma_2(z)} C_1 \rightarrow 0$$

is exact for each z , but

$$\Gamma_1 H^2(C_1) = \exp \frac{z+1}{z-1} H^2(C_1) \oplus 0 \subsetneq H^2(C_1) \oplus 0 = \ker \Gamma_2.$$

Corollary 3.2 (K. Takahashi [16]). *Let T be a contraction with $T \prec S_n$, then $T^* \in B_n(D)$.*

Proof. Since T is in class $C_{0,0}$, we may identify $S(\theta)$ given by (2.3) with T . Let X be a quasi-affinity such that $XS(\theta) = S_n X$. Then, from the lifting theorem (see [14]) there is a $B(F_2, C_n)$ -valued bounded holomorphic function $\Gamma(z)$ defined on D such that $\Gamma\theta = 0$ and $Xh = \Gamma h$ for h in $H(\theta)$. That X is a quasi-affinity implies that

$$H^2(F_1) \xrightarrow{\theta} H^2(F_2) \xrightarrow{\Gamma} H^2(C_n)$$

is exact, and that Γ has the dense range. Thus from the theorem we get $\theta(w)F_1$ is closed and $\dim\{F_2 \ominus \theta(w)F_1\} = n$ for w in D . The next equivalent conditions:

- (1) $\theta(w)F_1$ is closed in F_2 ,
- (2) $\frac{z-w}{1-\bar{w}z} H^2(F_2) \oplus \frac{\theta(w)F_1}{1-\bar{w}z}$ is closed in $H^2(F_2)$,
- (3) $\frac{z-w}{1-\bar{w}z} H^2(F_2) + \theta H^2(F_1)$ is closed in $H^2(F_2)$,
- (4) $P_{H(\theta)} \frac{z-w}{1-\bar{w}z} H(\theta)$ is closed in $H(\theta)$,
- (5) $(S(\theta) - w)(I - \bar{w}S(\theta))^{-1} H(\theta)$ is closed in $H(\theta)$,

show that the range of $(S(\theta) - w)^*$ is closed for w in D . Similarly we have $\dim \ker(S(\theta) - w)^* = n$, hence the proof is complete.

Remark. The latter half in the above proof is trivial if we notice that θ is the characteristic function of $S(\theta)$ [14]. But we showed it directly.

Theorem 3.3. *Let T be a contraction. Then $T \prec S_n$ if and only if $T^* \in B_n^h(D)$ and there is a frame $\{\gamma_{1w}, \dots, \gamma_{nw}\}$ for $\ker(T^* - w)$ on D such that*

$$\sup_{w \in D} (1 - |w|^2) \|\gamma_{iw}\|^2 < \infty \text{ for each } i.$$

Proof. Let $\{e_1, \dots, e_n\}$ be the O.N.B. of $\mathbb{C}_n$. Then eigenvectors of $(S_n^* - w)$ are $e_1/(1 - wz), \dots, e_n/(1 - wz)$. If X is the quasi-affinity such that $XT = S_n X$, then $\gamma_{iw} = X^* e_i/(1 - wz)$ satisfies the norm condition. The rest of “only if” part is clear. In order to show “if” part, we consider $S(\theta)$ instead of T . Then γ_{iw} is given by (2.4). By the norm condition and (2.6), $\|\gamma_{iw}(0)\|$ is uniformly bounded for w in D . For each z in D , we determine the operator $\Gamma(z): F_2 \rightarrow \mathbb{C}_n$ by

$$\Gamma(z)y = \sum_{i=1}^n (y, \gamma_{iz}(0)) e_i.$$

Then from (2.5) we have $\Gamma(z)\theta(z) = 0$, and clearly $\sup_{z \in D} \|\Gamma(z)\| < \infty$. Let us determine the bounded operator $X: H(\theta) \rightarrow H^2(\mathbb{C}_n)$ by $Xh = \Gamma h$ for h in $H(\theta)$. Then it clearly follows that $XS(\theta) = S_n X$. For any i, k , and any ζ, w in D , since z is the variable of a function, we have

$$\begin{aligned} \left(X^* \frac{e_i}{1 - wz}, \frac{\gamma_{k\zeta}(0)}{1 - \zeta z} \right)_{H(\theta)} &= \left(\frac{e_i}{1 - wz}, \sum_j \frac{(\gamma_{k\zeta}(0), \gamma_{jz}(0)) e_j}{1 - \zeta z} \right)_{H^2(\mathbb{C}_n)} \\ &= \left(\frac{\gamma_{iz}(0)}{1 - wz}, \frac{\gamma_{k\zeta}(0)}{1 - \zeta z} \right)_{L^2(F_2)} = \left(P_{H^2(F_2)} \frac{\gamma_{iz}(0)}{1 - wz}, \frac{\gamma_{k\zeta}(0)}{1 - \zeta z} \right)_{H^2(F_2)} \\ &= \left(\frac{\gamma_{iw}(0)}{1 - wz}, \frac{\gamma_{k\zeta}(0)}{1 - \zeta z} \right)_{H(\theta)} = (\gamma_{iw}, \gamma_{k\zeta})_{H(\theta)}, \end{aligned}$$

which shows that $X^* e_i/(1 - wz) = \gamma_{iw}$, because $\bigvee_{k\zeta} \gamma_{k\zeta} = H(\theta)$, and hence that X^* has the dense range. Thus X is injective. Since the rank of $\Gamma(z)$ is n , $S_n|_{\text{cl } XH(\theta)} = S_n|_{\text{cl } \Gamma H^2(F_2)}$ is unitarily equivalent to S_n . To accomplish the proof, it suffices to take PX to be the intertwining quasi-affinity, where P is the projection from $H^2(\mathbb{C}_n)$ to $\text{cl } XH(\theta)$. The proof is complete.

Suppose T be a completely nonunitary (c.n.u.) contraction. In [1], Alexander called vectors $h_1, \dots, h_n$ analytically independent under T if a relation $\phi_1(T)h_1 + \dots + \phi_n(T)h_n = 0$ with $\phi_i \in H^\infty$ implies $\phi_1 = \dots = \phi_n = 0$, and showed that $S_n \prec T$ if and only if T has n cyclic vectors which are analytically independent under T . We remark that a contraction T with the adjoint in $B_n^h(D)$ satisfies $T^{*n} \rightarrow 0$ so that T is c.n.u.

Corollary 3.4. *Let T be a contraction. Then $T \sim S_n$ if and only if T has n -cyclic vectors, $T^* \in B_n^h(D)$ and there is a frame $\{\gamma_{1w}, \dots, \gamma_{nw}\}$ for $\ker(T^* - w)$ on D such that*

$$\sup_{w \in D} (1 - |w|^2) \|\gamma_{iw}\|^2 < \infty \quad \text{for each } i.$$

Proof. We have only to show “if” part. From above theorem $T \prec S_n$ follows. Let X be a quasi-affinity satisfying $XT = S_n X$, and $h_1, \dots, h_n$ cyclic vectors for T . Then $Xh_1, \dots, Xh_n$ are cyclic vectors for S_n . It is trivial to show that for each z in D $(Xh_1)(z), \dots, (Xh_n)(z)$ span $\mathbf{C}_n$ and hence $\det((Xh_1)(z), \dots, (Xh_n)(z)) \neq 0$. Thus, from [1], $Xh_1, \dots, Xh_n$ are analytically independent under S_n . Since $X\phi_i(T)h_i = \phi_i(S_n)(Xh_i)$, $h_1, \dots, h_n$ are analytically independent under T . Thus we obtain $S_n \prec T$ and hence $S_n \sim T$.

In [20], P. Y. Wu gave a necessary and sufficient condition for the characteristic function of T to be $T \sim S_n$. That S_n^* has a cyclic vector was shown by D. Sarason. Now we can extend it as follows:

Theorem 3.5. *If Ω is a connected open set and $T^* \in B_n^h(\Omega)$, then T^* has a cyclic vector. Especially if T is a contraction with $T^* \in B_n^h(D)$, then $S \prec T^*$.*

Proof. Fix an arbitrary w_0 in Ω , then there is a nbhd Δ of w_0 , and a frame $\gamma_{1w}, \dots, \gamma_{nw}$ for $\ker(T^* - w)$ on Δ . Since $B_n^h(\Omega) \subset B_n^h(\Delta)$,

$$\bigvee \{\gamma_{iw} : 1 \leq i \leq n, w \in \Delta\} = H$$

follows. By the Taylor expansion we have $\bigvee \{\gamma_i^{(k)} : 1 \leq i \leq n, 1 \leq k < \infty\} = H$, where $\gamma_i^{(k)} = (d^k \gamma_{iw} / dw^k)_{w=w_0} \in H$. From $(T^* - w)\gamma_{iw} = 0$, it follows that $(T^* - w_0)\gamma_i^{(k)} = k\gamma_i^{(k-1)}$. Setting $a_k = 1/k!$, clearly $\sum_{k=0}^{\infty} \|\gamma_i^{(k)}\| a_k / k! < \infty$. In case of $n = 1$, $x = \sum_{k=0}^{\infty} \gamma_1^{(k)} a_k / k!$ is a cyclic vector. In fact,

$$(T^* - w_0)^m x = \sum_{k=0}^{\infty} \frac{\gamma_1^{(k)}}{k!} a_{m+k}$$

implies that

$$\left\| \frac{(T^* - w_0)^m}{a_m} x - \gamma_1^{(0)} \right\| \leq \frac{a_{m+1}}{a_m} \sum_{k=1}^{\infty} \frac{\|\gamma_1^{(k)}\|}{k!} \frac{a_{m+k}}{a_{m+1}} \leq \frac{a_{m+1}}{a_m} \left(\sum_{k=1}^{\infty} \frac{\|\gamma_1^{(k)}\|}{k!} \frac{a_k}{a_1} \right) \rightarrow 0$$

as $m \rightarrow \infty$. Thus $\gamma_1^{(0)} \in \bigvee_{m=0}^{\infty} (T^* - w_0)^m x$. From

$$\begin{aligned} & \left\| \frac{1}{a_m} ((T^* - w_0)^{m-1} x - a_{m-1} \gamma_1^{(0)}) - \gamma_1^{(1)} \right\| \\ & \leq \frac{a_{m+1}}{a_m} \sum_{k=2}^{\infty} \frac{\|\gamma_1^{(k)}\|}{k!} \frac{a_k}{a_2} \rightarrow 0 \quad (m \rightarrow \infty), \end{aligned}$$

we have $\gamma_1^{(1)} \in \bigvee_{m=0}^{\infty} (T^* - w_0)^m x$. Similarly we get $\gamma_1^{(k)} \in \bigvee_{m=0}^{\infty} (T^* - w_0)^m x$, consequently $\bigvee_{m=0}^{\infty} (T^* - w_0)^m x = H$, and hence $\bigvee_{m=0}^{\infty} T^{*m} x = H$. In case of

$n > 1$

$$x = \gamma_1^{(0)} a_0 + \frac{\gamma_2^{(1)}}{1!} a_1 + \frac{\gamma_3^{(2)}}{2!} a_2 + \cdots + \frac{\gamma_n^{(n-1)}}{(n-1)!} a_{n-1} \\ + \frac{\gamma_1^{(n)}}{n!} a_n + \frac{\gamma_2^{(n+1)}}{(n+1)!} a_{n+1} + \cdots$$

is a cyclic vector for T^* . To show the rest, suppose $\phi(T^*)x = 0$ for $\phi \in H^\infty$. Since $\phi(T^*)T^{*m}x = T^{*m}\phi(T^*)x = 0$, we have $\phi(T^*) = 0$. From $T^*\gamma_{iw} = w\gamma_{iw}$, it follows that $\phi(T^*)\gamma_{iw} = \phi(w)\gamma_{iw}$ for every w in D and hence $\phi(w) = 0$, which implies that x is analytically independent under T^* . Consequently we get $S \prec T^*$.

Proposition 3.6. *If T is a contraction and $T \prec S_n$, then there is an invariant subspace L for T such that $T|_L \sim S_n$.*

Proof. Let us consider $S(\theta)$ instead of T . Then the eigenvector γ_{i0} of T^* is given by (2.4). Since it is constant vector valued, we can determine a bounded operator Y from $H^2(\mathbf{C}_n) = H^2(\mathbf{C}_1) \oplus \cdots \oplus H^2(\mathbf{C}_1)$ to $H(\theta)$ by

$$Y(h_1 \oplus \cdots \oplus h_n) = P_{H(\theta)}(h_1\gamma_{10} + \cdots + h_n\gamma_{n0}).$$

Suppose $Y(h_1 \oplus \cdots \oplus h_n) = 0$. Then $\sum h_i\gamma_{i0} \in \theta H^2(F_1)$ so that there is f in $H^2(F_1)$ such that $\sum h_i\gamma_{i0} = \theta f$. By (2.5) and linear independence of $\gamma_{10}(0), \dots, \gamma_{n0}(0)$, we have $h_i(0) = 0$ and $f(0) = 0$. Since

$$\sum h'_i(0)\gamma_{i0}(0) = \theta'(0)f(0) + \theta(0)f'(0) = \theta(0)f'(0),$$

we have $h'_i(0) = 0$ and $f'(0) = 0$ too. Thus to show $h_i = 0$ it suffices to continue this process. Set $L = \text{cl } YH^2(\mathbf{C}_n)$. Then $TL \subset L$ and $S_n \prec T|_L$. Let X be a quasi-affinity satisfying $XT = S_nX$. Then XY is injective and commutes with S_n . From the characterizations of invariant subspaces for S_n , it follows that $S_n|_{\text{cl } XL} = S_n|_{\text{cl } XYH^2(\mathbf{C}_n)} \cong S_n$, and hence $T|_L \prec S_n$. Thus we have $T|_L \sim S_n$ and the proof is complete.

Next we will give the conditions for contractions to be similar to S_n by using the Rosenblum's infinite corona theorem [11]. Suppose

$$\sup_{z \in D} \sum_{j=1}^n \sum_{i=1}^{\infty} |h_{ij}(z)|^2 < \infty, \quad \text{where } h_{ij} \in H^\infty.$$

Then a $B(\mathbf{C}_n, l^2)$ -valued holomorphic function $A(z) = (h_{ij}(z))$ is bounded on D . Under this setting we have

Proposition 3.7. *There is a $B(l^2, \mathbf{C}_n)$ -valued bounded holomorphic function $B(z)$ such that $B(z)A(z) = I$ for z in D , if and only if there is a positive constant δ such that $\|A(z)x\| \geq \delta\|x\|$ for every x in $\mathbf{C}_n$ and every z in D .*

Proof. Suppose $\|A(z)x\| \geq \delta\|x\|$. Then $A(z)^*A(z) \geq \delta^2$ and hence

$$\delta^{2n} \leq \det(A(z)^*A(z)) = \sum_{i_1 < \cdots < i_n} |\det A_{i_1 \dots i_n}(z)|^2,$$

where $A_{i_1 \dots i_n}$ is the $n \times n$ submatrix of A . Since $\det(A(z)^* A(z))$ is upper bounded, by the infinite corona theorem, there are $b_{i_1 \dots i_n} \in H^\infty$ such that

$$\sup_{z \in D} \sum_{i_1 < \dots < i_n} |b_{i_1 \dots i_n}(z)|^2 < \infty, \quad \sum b_{i_1 \dots i_n} \det A_{i_1 \dots i_n} = 1 \quad \text{on } D.$$

Thus we can construct a bounded holomorphic function $B(z)$ such that $B(z)A(z) = I$ in the same way as Fuhrmann [6]. The converse is trivial, so we can conclude the proof.

Theorem 3.8. *Let T be a contraction. Then T is similar to S_n if and only if $T^* \in B_n^h(D)$, and there is a holomorphic frame $\gamma_{1w}, \dots, \gamma_{nw}$ for $\ker(T^* - w)$ and positive constants M, δ such that for any $x_i \in \mathbb{C}$ and $w \in D$*

$$(3.1) \quad M \sum_{i=1}^n |x_i|^2 \geq (1 - |w|^2) \left\| \sum_{i=1}^n x_i \gamma_{iw} \right\|^2 \geq \delta \sum_{i=1}^n |x_i|^2.$$

Proof. We use the notations in the proof of Theorem 3.3. Let Y be an invertible operator satisfying $YT = S_n Y$. Then $\gamma_{iw} = Y^* e_i / (1 - wz)$ satisfies (3.1). It is clear that T^* is in $B_n^h(D)$. Thus we must only show “if” part. We represent γ_{iw} as (2.4), and determine $\Gamma(z): F_2 \rightarrow C_n$ by $\Gamma(z)y = \sum_{i=1}^n (y, \gamma_{iz}(0)) e_i$. Then we have $\Gamma^\sim(z)x = \sum_{i=1}^n (x, e_i) \gamma_{iz}(0)$ for $x \in C_n, z \in D$. Thus, since

$$\begin{aligned} \|\Gamma^\sim(z)x\|^2 &= \left\| \sum (x, e_i) \gamma_{iz}(0) \right\|^2 \\ &= (1 - |z|^2) \left\| \sum (x, e_i) \gamma_{iz} \right\|^2 \quad \text{for every } z \in D, \end{aligned}$$

applying Proposition 3.7, $\Gamma(z)$ has the bounded right inverse. Therefore we have $H^2(C_n) = \Gamma H^2(F_2) = \Gamma H(\theta)$, because $\Gamma\theta = 0$. Consequently X given by $Xh = \Gamma h$ is an invertible operator from $H(\theta)$ to $H^2(C_n)$ satisfying $XT = S_n X$ (see the proof of Theorem 3.3). Hence the proof is complete.

We observe that we can substitute $(1 - |w|^2)G(w)$ for the middle term of (3.1), where $G(w)$ is the Gram matrix of $\gamma_{1w}, \dots, \gamma_{nw}$.

Proposition 3.9. *The contraction T is similar to the isometry if and only if T satisfies one of the following equivalent conditions:*

- (a) *there is a positive constant δ such that $\|T^n x\| \geq \delta \|x\|$ for x in H .*
- (b) *There is a power-bounded operator B satisfying $BT = I$.*
- (c) *There is a bounded operator B such that $BT = I$ and for any w in D $(I - wB^*)^{-1}$ exists and $\sup_{w \in D} (1 - |w|) \|(I - wB^*)^{-1}\| < \infty$*

Proof. In [15], Sz.-Nagy and Foias showed that T satisfies (a) if and only if T is similar to isometry. (a) $\Leftrightarrow$ (b) is trivial. Moreover it is clear that (c) follows from similarity of T and isometry, and its converse is able to be shown in the same way as Castern [3], by considering

$$\sum_{n=1}^{\infty} r^n e^{int} B^{*n} + \sum_{n=1}^{\infty} r^n e^{-int} T^{*n}$$

instead of $\sum_{n=-\infty}^{\infty} r^n e^{int} S^n$ on p. 191 of [3].

At the end of this section we remark that from the above proposition we can get conditions for T to be similar to S_n . For instance it suffices to add $T \in C_0$ and $\dim \ker T^* = n$ to each condition of the above.

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