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$\Lambda(q)$ PROCESSES

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ABSTRACT. Motivated by some classical notions in harmonic analysis, $\Lambda(q)$ processes are introduced in the context of a study of stochastic interdependencies. An extension of a classical theorem of Salem and Zygmund regarding random Fourier series is obtained. The Littlewood exponent of $\Lambda(q)$ processes is estimated and, in some archetypical cases, computed.

0. INTRODUCTION

In [1], we considered stochastic processes with respect to which every deterministic function on $[0, 1]$ was stochastically integrable. Such processes X were normed by

$$\|X\| = \sup\left\{ \mathbf{E}\left|\sum_{j=1}^{n} \varepsilon_j (X(t_j) - X(t_{j-1}))\right| : N > 0,\ \varepsilon_j = \pm 1,\ j = 1, \dots, N,\right.$$
$$\left. 0 = t_0 < t_1 < \cdots < t_N = 1 \right\},$$

and were said to have *finite expectation* (cf. [1, §1]). One of the basic questions arising in this context is how to determine, in some precise sense, a degree of interdependencies between increments of a process with finite expectation. This problem was the motivation behind α-chaos [3] as well as the subsequent computation of its Littlewood exponent [4]; the present paper is a continuation of that work. A description of the intuition underlying results of [3, 4], as well as the present paper, can be found in [5].

We first set the stage. Throughout, $(\Omega, \mathfrak{A}, \mathbf{P})$ will denote a probability space. Let $E = \{X_j\}_{j\in\mathbf{N}}$ be an orthonormal system of random variables in $L^2(\Omega, \mathbf{P})$, and define

$$\phi_E(x) = \sup\left\{ \mathbf{P}\left(\left|\sum_j \sigma_j X_j\right| \geq x\right) : \sum_j \sigma_j^2 = 1 \right\}, \qquad x > 0.$$

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In [3], we said that E was a sub-α-system if

$$\delta_E(\alpha) \equiv \varliminf_{x\to\infty} (\ln(1/\phi_E(x))/x^{2/\alpha}) > 0,$$

and an α-system if

$$\theta_E \equiv \inf\{\gamma: \delta_E(\gamma) > 0\} = \alpha.$$

For infinite E, $\theta_E \in [1, \infty]$. An archetypical example of an α-system is produced by taking a system of independent symmetric uniformly bounded random variables $\{X_j\}_{j\in\mathbf{N}}$, fixing an α-dimensional lattice set $F \subset N^J$, and defining

$$E_\alpha = \{X_{j_1} \cdots X_{j_J}\}_{(j_1, \dots, j_J)\in F}$$

for which $\theta_{E_\alpha} = \alpha$ (e.g., [2]). In the present paper, we bring the case $\theta_E = \infty$ into a sharper focus.

Defintion 1.1. *An orthonormal system* $\{X_j\}_{j\in\mathbf{N}} = E$ *is a sub-*$\Lambda(q)$ *system if*

$$\lambda_E(q) \equiv \varlimsup_{x\to\infty} (\phi_E(x)/x^q) < \infty,$$

and a $\Lambda(q)$ *system if*

$$\sup\{p: \lambda_E(p) < \infty\} = q.$$

An example of an infinite $\Lambda(q)$ system is produced by taking infinitely many independent copies of a symmetric random variable with finite L^q-norm but infinite $L^{q+\varepsilon}$-norm for all $\varepsilon > 0$.

The notions above are naturally transported to a framework of stochastic processes. Throughout, we shall restrict attention to processes X with orthogonal increments whose variance is given by $E|X(t) - X(s)|^2 = t - s$, $0 \le s < t \le 1$ (note: $\|X\| \le 1$). In [3], we defined X to be an α-chaos when

$$\sup\{\theta_E: E \text{ is a system of normalized increments of } X\} = \alpha.$$

The Wiener process is an archetypical example of a 1-chaos, while Wiener's homogeneous chaos of order k, k a positive integer, is an example of a k-chaos (cf. [12], [9], [3, Remark 4.2(1)]). For noninteger α, examples of α-chaos are produced canonically via the existence of α-dimensional lattice sets [3, Theorem 4.1].

Definition 1.2. *A process* X *with orthogonal increments and* $E|X(t) - X(s)|^2 = t - s$, $0 \le s < t \le 1$, *is a sub-*$\Lambda(q)$ *process if*

$$\beta_X(q) \equiv \sup\{\lambda_E(q): E \textit{ is a system of normalized increments of } X\}$$

is finite, and a $\Lambda(q)$ *process if*

$$\sup\{p: \beta_X(p) < \infty\} = q.$$

Clearly, an α-chaos is a sub-$\Lambda(q)$ process for all $q < \infty$. Indeed, the α-scale of [3] can be viewed as a resolution of the "right end-point $(q = \infty)$" of the q-scale in the framework of the present paper.

Since every sub-$\Lambda(q)$ process is a stochastic integrator in the sense of [1, §2], we easily obtain (and state without proof)

Lemma 1.3. *Suppose X is a process with orthogonal increments and*

$$E|X(t)-X(s)|^2 = t-s, \qquad 0 \le s < t \le 1$$

(*in particular,* $\mathbf{E}|\int_{[0,1]} f\,dX|^2 \le \|f\|_2^2$ *for all* $f \in L^2([0,1], dt)$). *Let*

$$\widehat{X}(n) = \int_{[0,1]} e_n\,dX, \qquad n \in \mathbf{N},$$

be the transform of X relative to $\{e_n\}_{n\in\mathbf{N}}$, a given orthonormal basis of $L^2([0,1], dt)$ (cf. [1, §2]). Then, X is a sub-$\Lambda(q)$ process if and only if $\{\widehat{X}(n)\}_{n\in\mathbf{N}}$ is a sub-$\Lambda(q)$ system.

Corollary 1.4. *$\Lambda(q)$ processes exist for every $2 \le q < \infty$.*

Proof. Let $E = \{X_n\}_{n\in\mathbf{N}}$ be a $\Lambda(q)$ system. Let U be the unitary map from $L^2([0,1], dt)$ onto the $L^2(\Omega, \mathbf{P})$-closure of the linear span of E determined by

$$Ue_n = X_n, \qquad n \in \mathbf{N},$$

where $\{e_n\}_{n\in\mathbf{N}}$ is an orthonormal basis of $L^2([0,1], dt)$. Define

$$X(t) = U\mathbf{1}_{[0,t]}, \qquad 0 \le t \le 1.$$

It is easy to see that X satisfies the hypotheses of Lemma 1.3 and that its transform relative to $\{e_n\}_{n\in\mathbf{N}}$ is given by

$$\widehat{X}(n) = X_n, \qquad n \in \mathbf{N},$$

and so, X is a $\Lambda(q)$ process. □

The main result of the next section (Theorem 2.3) is a sufficient condition for a.s. continuity of a random function represented as a Fourier series randomized by a sub-$\Lambda(q)$ system. This theorem, an analogue of [3, Theorem 2.5], is an extension of a classical theorem due to Salem and Zygmund [11]. As an easy consequence, we obtain that the sample paths of sub-$\Lambda(q)$ processes are almost surely continuous and, when $q > 2$, of unbounded variation (Corollary 2.4).

In §3 we estimate Littlewood exponents of sub-$\Lambda(q)$ and $\Lambda(q)$ processes. We recall some definitions [4]. Let X be any stochastic process on some probability space $(\Omega, \mathscr{A}, \mathbf{P})$. Define its "$p$th variation" by

$$\|X\|_{(p)} = \sup\left\{ \left(\sum_{j,k} |\mathbf{E}\mathbf{1}_{A_j}(X(t_k) - X(t_{k-1}))|^p \right)^{1/p} : \{A_j\}_{j\in N} \subset \mathscr{A}, \right.$$
$$\left. \sum_j \mathbf{1}_{A_j} = 1,\ 0 = t_0 < t_1 < \cdots < t_k < \cdots < 1 \right\},$$

and its Littlewood exponent by

$$l_X = \inf\{p : \|X\|_{(p)} < \infty\}.$$

A straightforward application of Littlewood's classical inequality [8] implies that if X has finite expectation then $l_X \leq 4/3$. An adaptation of Littlewood's example (finite Fourier transform), showing that $4/3$ is best possible in the inequality of [8], establishes that there are processes X for which $\|X\| < \infty$, and $l_X = 4/3$. At the other end of the scale, $l_X = 1$ when X is an α-chaos [3, Theorem 2].[1] The main result of §3 in this paper (Theorem 3.1), filling a gap between 1 and 4/3, is that

(i) $l_X \leq (q+2)/(q+1)$ whenever X is a sub-$\Lambda(q)$ process, and
(ii) for every $q \geq 2$, there are $\Lambda(q)$ processes for which $l_X = (q+2)/(q+1)$.

The proof of part (ii) of Theorem 3.1 is based on Bourgain's recent solution of the $\Lambda(q)$-set problem [6]. The question whether for every $\Lambda(q)$ process X, $l_X = (q+2)/(q+1)$ is an open problem.

2. Sub-$\Lambda(q)$ systems and random Fourier series

Lemma 2.1. *Let* $(\mathcal{T}, \mu)$ *be a probability space, and let* $\mathcal{D}$ *be a linear subspace of* $L^\infty(\mathcal{T}, \mu)$ *so that*

$$\rho(\mathcal{D}) = \rho \equiv \inf\{\mu(|f| \geq \|f\|_\infty/2) : f \in \mathcal{D}\} > 0.$$

Suppose $\{X_j\}_{j\in\mathbf{N}} = E \subset L^2(\Omega, \mathbf{P})$ *is a sub-*$\Lambda(q)$ *system. Let* $\{f_j\}$ *be a finite collection of functions in* $\mathcal{D}$ *so that*

$$\left\| \sum_j |f_j|^2 \right\|_\infty \leq 1, \tag{2.1}$$

and define the random function

$$g = \sum_j f_j \otimes X_j.$$

Then,

$$\mathbf{P}(\|g\|_\infty > x) < (2^q \lambda_E(q)/\rho) x^q, \qquad x > 0.$$

$(\|g\|_\infty \equiv \operatorname{ess\,sup}_{s\in\mathcal{T}} |\sum_j f_j(s) X_j|.)$

Proof. By (2.1), we have for all $s \in \mathcal{T}$ and all $x > 0$,

$$\mathbf{P}(|g(s)| > x) = \mathbf{E}\mathbf{1}_{\{|g(s)|>x\}} \leq \lambda_E(q)/x^q.$$

Integrating the inequality above and applying Fubini's Theorem, we obtain

$$\mathbf{E}\int_{\mathcal{T}} \mathbf{1}_{\{|g(s)|>x\}}\, d\mu(s) \leq \lambda_E(q)/x^q,$$

$$\mathbf{E}\int_{\{s\,:\, |g(s)|\geq\|g\|_\infty/2\}} \mathbf{1}_{\{|g(s)|>x\}}\, d\mu(s) \leq$$

$$\rho\mathbf{E}\mathbf{1}_{\{\|g\|_\infty>2x\}} \leq$$

which implies the conclusion of the lemma. □

[1] When X is a simple process, $l_X = 0$. When $l_X < 1$, stochastic integration with respect to X reduces to usual integration over $\Omega \times [0, 1]$. From our point of view, the interesting range of l_X is $[1, 4/3]$.

The following is an immediate consequence.

Lemma 2.2. *Suppose $\{X_j\}_{j\in\mathbf{N}} = E$ is a sub-$\Lambda(q)$ system. Fix an arbitrary positive integer N, let T_N denote the space of trigonometric polynomials on $[0, 1]$ of degree N, and let $\{f_j\} \subset T_N$ be finite. Then*

$$\mathbf{P}\left(\left\|\sum_j f_j \otimes X_j\right\|_\infty \geq Dx\left(\left\|\sum_j |f_j|^2\right\|_\infty\right)^{1/2}\right) \leq 2\pi N/x^q$$

for all $x > 0$, where $D > 0$ depends only on $\lambda_E(q)$.

Proof. By Bernstein's theorem (e.g. [7, Exercise I.2.12]), $\rho(T_N) \geq 1/2\pi N$. Now apply Lemma 2.1. □

The following is an extension of [3, Theorem 2.5].

Theorem 2.3. *Let $\{X_j\}_{j\in\mathbf{N}}$ be a sub-$\Lambda(q)$ system. Define blocks of integers*

$$B_k = \{\pm[k^{q/2}], \pm([k^{q/2}]+1), \dots, \pm([(k+1)^{q/2}]-1)\}, \qquad k = 1, 2, \dots$$

($[\cdot]$ denotes the "closest integer" function). Let $\{a_n\}_{n=-\infty}^{\infty}$ be a sequence of scalars so that

$$\left(\sum_{n\in B_k} |a_n|^2\right)^{1/2} = s_k, \quad k = 1, 2, \dots, \text{ is a decreasing sequence} \tag{2.2}$$

and

$$\sum_{k=1}^{\infty} (\ln k) s_k < \infty. \tag{2.3}$$

Then, $\sum_{n=-\infty}^{\infty} a_n X_n e^{2\pi i n t}$ is almost surely a Fourier series of a continuous function on $[0, 1]$.

Proof. Define blocks of integers

$$C_k = \{\pm 2^k, \pm(2^k+1), \dots, \pm(2^{k+1}-1)\},$$

define the corresponding random trigonometric polynomials

$$p_k(t) = \sum_{n\in C_k} a_n X_n e^{2\pi i n t},$$

and consider the events

$$E_k = \left\{\|p_k\|_\infty \geq D(2^{k/q})k\left(\sum_{n\in C_k} |a_n|^2\right)^{1/2}\right\}, \qquad k = 1, 2, \dots$$

($D > 0$ is the constant appearing in Lemma 2.2). By Lemma 2.2,

$$\mathbf{P}(E_k) \leq 2\pi/k^q.$$

Therefore, by the Borel-Cantelli lemma, we obtain $\mathbf{P}(\overline{\lim} E_k) = 0$ which implies that

$$(2.4)\qquad (\|p_k\|_\infty)_{k=1}^\infty \text{ is } \mathscr{O}\left((2^{k/q})k\left(\sum_{n\in C_k}|a_n|^2\right)^{1/2}\right) \text{ almost surely.}$$

Observe that $|B_k| \approx k^{(q/2)-1}$ and that, following a partition of each C_k into B_n's, we have

$$\left(\sum_{n\in C_k}|a_n|^2\right)^{1/2} \approx \left(\sum_{n=[4^{k/q}]}^{[4^{(k+1)/q}]}|s_n|^2\right)^{1/2}.$$

Therefore, since $(s_n)_{n=1}^\infty$ is a decreasing sequence, we have

$$(2.5)\qquad \left(\sum_{n\in C_k}|a_n|^2\right)^{1/2} \le K[2^{k/q}]s_{[4^{k/q}]}.$$

And so, following (2.4) and (2.5), to obtain that $\sum_{k=1}^\infty \|p_k\|_\infty$ is almost surely convergent and thus the theorem, we need to verify

$$(2.6)\qquad \sum_{k=1}^\infty [4^{k/q}]s_{[4^{k/q}]}k < \infty.$$

Finally, observe that (2.6) is implied, via a change of index, by the assumption (2.3). □

Corollary 2.4. *The sample paths of every sub-$\Lambda(q)$ process are almost surely continuous and of unbounded variation.*

Proof. The stochastic series of X relative to the usual trigonometric system is given by

$$(2.7)\qquad X(t) - X(0) = \widehat{X}(0)t + \sum_{n\neq 0}\frac{\widehat{X}(n)}{2\pi i n}(e^{2\pi i n t} - 1).$$

By Lemma 1.1, $\{\widehat{X}(n)\}_{n=-\infty}^\infty$ is a sub-$\Lambda(q)$-system. Therefore, since $a_n = 1/n$ satisfies the hypothesis (2.3) in Theorem 2.3, we obtain that the stochastic series (2.7) represents almost surely a continuous function on $[0, 1]$.

Following a computation similar to the one in [3, Remark 3.8], we deduce that every sub-$\Lambda(q)$ process is *chaotic* and, by [3, Proposition 3.9], that the sample paths of X are almost surely of unbounded variation. □

3. The Littlewood exponent of sub-$\Lambda(q)$ processes

Theorem 3.1. (i) *Let X be a sub-$\Lambda(q)$ process. Then*

$$l_X \le (q+2)/(q+1).$$

(ii) *For every* $q \geq 2$, *there exist* $\Lambda(q)$ *processes* X *so that*

$$l_X = (q+2)/(q+1).$$

In what follows below, we fix a sub-$\Lambda(q)$ process X, a measurable partition $\{A_j\}_{j\in\mathbf{N}}$ of Ω, and a subdivision $0 = t_0 < t_1 < \cdots < t_k < \cdots \leq 1$. Denote

$$a_{jk} = \mathbf{E1}_{A_j}(X(t_k) - X(t_{k-1})), \qquad j, k \in N.$$

Throughout, K will denote a numerical constant.

Lemma 3.2. *For all* $p > q/(q-1)$,

$$\sum_j \left(\sum_k |a_{jk}|\right)^p \leq K \tag{3.1}$$

where $K > 0$ *depends only on* p *and* $\beta_X(q)$.

Proof. We verify (3.1) by duality. Fix $p > q/(q-1)$ and $(b_{jk}) \subset \mathbf{C}$ so that

$$\sum_j \left(\sup_k |b_{jk}|\right)^{p'} = 1, \qquad \frac{1}{p} + \frac{1}{p'} = 1.$$

Following a rearrangement of the j's, we can assume that

$$\sup_k |b_{jk}|^{p'} \leq \frac{1}{j}, \qquad j = 1, 2, \ldots. \tag{3.2}$$

Write $Y_k = (X(t_k) - X(t_{k-1}))/\sqrt{t_k - t_{k-1}}$, $d_{jk} = b_{jk}/\sqrt{t_k - t_{k-1}}$, and obtain from (3.2),

$$\left(\sum_k |d_{jk}|^2\right)^{1/2} \leq 1/j^{1/p'}, \qquad j = 1, 2, \ldots. \tag{3.3}$$

Estimate

$$\begin{aligned}
\left|\sum_{j,k} a_{jk} b_{jk}\right| &= \left|\sum_j \mathbf{E1}_{A_j} \sum_k d_{jk} Y_k\right| \leq \mathbf{E} \sum_j \mathbf{1}_{A_j} \left|\sum_k d_{jk} Y_k\right| \\
&\leq \sup_j \left|\sum_k d_{jk} Y_k\right| = \int_0^\infty \mathbf{P}\left(\bigcup_j \left\{\left|\sum_k d_{jk} Y_k\right| > t\right\}\right) dt \\
&\leq 1 + \int_1^\infty \sum_j \mathbf{P}\left(\left|\sum_k d_{jk} Y_k\right| > t\right) dt \\
&\leq 1 + \left(\int_1^\infty t^{-q}\, dt\right) \beta_X(q) \sum_j 1/j^{q/p'} \equiv K < \infty
\end{aligned}$$

(the last line above follows from (3.3) and the assumption $\beta_X(q) < \infty$). □

Lemma 3.3.

$$\sum_j \left(\sum_k |a_{jk}|^2 \right)^{1/2} \leq K. \tag{3.4}$$

Proof. (3.4) is a consequence of Littlewood's inequality for bounded bilinear forms [8]; this argument yields $K \leq \sqrt{2}$ (= Khintchin's constant). We shall give a direct proof in our specific context, bypassing Littlewood's inequality and obtaining $K = 1$. We establish (3.4) by duality: suppose $(b_{jk}) \subset \mathbf{C}$ satisfies

$$\sup_j \left(\sum_k |b_{jk}|^2 \right)^{1/2} = 1, \tag{3.5}$$

and estimate

$$\begin{aligned} \left| \sum_{j,k} a_{jk} b_{jk} \right| &= \left| \sum_{j,k} \mathbf{E} \mathbf{1}_{A_j} (X(t_k) - X(t_{k-1})) b_{jk} \right| \\ &= \left| \sum_k \mathbf{E} \left(\sum_j b_{jk} \mathbf{1}_{A_j} \right) (X(t_k) - X(t_{k-1})) \right| \end{aligned}$$

(without loss of generality, we assume that the sums above are performed over finitely many j's and k's)

$$\begin{aligned} &\leq \sum_k \left(\mathbf{E} \left| \sum_j b_{jk} \mathbf{1}_{A_j} \right|^2 \right)^{1/2} (t_k - t_{k-1})^{1/2} \\ &\qquad\qquad\qquad\qquad \text{(by Schwarz's inequality)} \\ &= \sum_k \left(\sum_j |b_{jk}|^2 \mathbf{P}(A_j) \right)^{1/2} (t_k - t_{k-1})^{1/2} \\ &\leq \left(\sum_k \sum_j |b_{jk}|^2 \mathbf{P}(A_j) \right)^{1/2} \left(\sum_k (t_k - t_{k-1}) \right)^{1/2} \\ &= \left\| \sum_j \left(\sum_k |b_{jk}|^2 \right)^{1/2} \mathbf{1}_{A_j} \right\|_{L^2(\Omega, P)} \\ &\leq \left\| \sum_j \left(\sum_k |b_{jk}|^2 \right)^{1/2} \mathbf{1}_{A_j} \right\|_{L\infty(\Omega, P)} \\ &= \sup_j \left(\sum_k |b_{jk}|^2 \right)^{1/2} = 1 \quad \text{(by (3.5))}. \quad \square \end{aligned}$$

Proof of Theorem 3.1. (i) We need to show that $\|X\|_{(p)} < \infty$ for all $p = (b+2)/(b+1) > (q+2)/(q+1)$. To this end, we will verify

$$\sum_{j,k} |a_{jk}|^{(b+2)/(b+1)} \leq \left(\sum_j \left(\sum_k |a_{jk}|^2\right)^{1/2}\right)^{2/(b+1)} \cdot \left(\sum_j \left(\sum_k |a_{jk}|^{b/(b-1)}\right)\right)^{(b-1)/(b+1)}, \tag{3.6}$$

and then apply Lemmas 3.2 and 3.3. To establish (3.6), first write

$$\sum_{j,k} |a_{jk}|^{(b+2)/(b+1)} = \sum_j \sum_k |a_{jk}|^{2/(b+1)} |a_{jk}|^{b/(b+1)},$$

and apply Hölder's inequality to $\sum_k$ with exponents $b+1$ and $(b+1)/b$ to obtain

$$\sum_{j,k} |a_{jk}|^{(b+2)/(b+1)} \leq \sum_j \left(\sum_k |a_{jk}|^2\right)^{1/(b+1)} \left(\sum_k |a_{jk}|\right)^{b/(b+1)}.$$

Now apply Hölder's inequality to $\sum_j$ above with exponents $(b+1)/2$ and $(b+1)/(b-1)$, and deduce (3.6).

(ii) We consider the discrete abelian group $\Gamma = \bigoplus \mathbf{Z}_{k_j}$, where (k_j) is a sequence of integers increasing to infinity, and view its dual group $\widehat{\Gamma} = \bigotimes \mathbf{Z}_{k_j} = \Omega$ as a probability space with $\mathbf{P} =$ Haar measure. Fix $q > 2$, and a set of characters $E \subset \Gamma$ which is a $\Lambda(q)$ system: such systems E were produced in [6] (in the terminology of [6], E is a $\Lambda(q)$ set but not $\Lambda(q+\varepsilon)$ for any $\varepsilon > 0$). Such $E \subset \bigoplus \mathbf{Z}_{k_j}$, by the productions in [6], can be assumed to satisfy

$E = \bigcup_{j=1}^{\infty} E_j$, $\quad E_j \subset \mathbf{Z}_{k_j}$ (the coordinates of E_j are nonzero only at the k_jth entry; this means that the E_j's are mutually independent systems of random variables on Ω),

$$\begin{cases} \sup_j \lambda_{E_j}(q) < \infty, \\ \sup \lambda_{E_j}(q+\varepsilon) = \infty, \quad \text{for all } \varepsilon > 0, \end{cases} \tag{3.7}$$

and

$$|E_j| \approx [k_j^{2/q}]. \tag{3.8}$$

Let U be a unitary map from $L^2([0,1], dt)$ into $L^2_E(\Omega, \mathbf{P})$, and define

$$X(t) = U\mathbf{1}_{[0,t]}, \qquad 0 \leq t \leq 1.$$

By (3.7) and Lemma 2.3, X is a $\Lambda(q)$ process and therefore, by part (iii), $l_X \leq (q+2)/(q+1)$. By (3.8), following an estimation similar to the one in [4, Theorem 2, part (ii)], we obtain $\|X\|_{(p)} = \infty$ for all $p < (q+2)/(q+1)$, and therefore $l_X = (q+2)/(q+1)$. □

References

1. R. Blei, *Multi-linear measure theory and multiple stochastic integration*, Probability Theory and Related Fields **81** (1989), 569–584.
2. ——, *Combinatorial dimension and certain norms in harmonic analysis*, Amer J. Math. **106** (1984), 847–887.
3. ——, *α-chaos*, J. Functional Anal. **81** (1988), 279–297.
4. R. Blei and J.-P. Kahane, *A computation of the Littlewood exponent of stochastic processes*, Math. Proc. Cambridge Philos. Soc. **103** (1988), 367–370.
5. ——, *Measurements of interdependencies of stochastic increments*, Proceedings of 1987 Summer Conference on Harmonic Analysis, Contemp. Math., vol. 91, Amer. Math. Soc., Providence, R. I., 1989, pp. 29–36.
6. J. Bourgain, *Bounded orthogonal systems and the $\Lambda(p)$-set problem*, Acta Math. **162** (1989), 227–245.
7. Y. Katznelson, *An introduction to harmonic analysis*, Dover, New York, 1976.
8. J. E. Littlewood, *On bounded bilinear forms in an infinite number of variables*, Quart. J. Math Oxford **1** (1930), 164–174.
9. H. P. McKean, *Geometry of differential space*, Ann. Probab. **1** (1973), 197–206.
10. W. Rudin, *Trigonometric series with gaps*, J. Math. Mech. **9** (1960), 203–227.
11. R. Salem and A. Zygmund, *Some properties of trigonometric series whose terms have random signs*, Acta Math. **91** (1954), 245–301.
12. N. Wiener, *The homogeneous chaos*, Amer. J. Math. **60** (1938), 897–936.

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