TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 320, Number 1, July 1990

MINIMAL IDENTITIES OF SYMMETRIC MATRICES

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ABSTRACT. Let $H_n(F)$ denote the subspace of symmetric matrices of $M_n(F)$, the full matrix algebra with coefficients in a field F. The subspace $H_n(F) \subset M_n(F)$ does not have any polynomial identity of degree less than $2n$. Let

$$T_k^i(x_1, \dots, x_k) = \sum_{\substack{\sigma \in \mathscr{S}_k \\ 1 \le i \le k, \sigma^{-1}(i) \equiv 1, 2 \bmod 4}} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(k)},$$

and $e(n) = n$ if n is even, $n+1$ if n is odd. For all $n \ge 1$, T_{2n}^i is an identity of $H_n(F)$. If the characteristic of F does not divide $e(n)!$ and if $n \ne 3$, then any homogeneous polynomial identity of $H_n(F)$ of degree $2n$ is a consequence of T_{2n}^i. The case $n = 3$ is also dealt with. The proofs are algebraic, but an equivalent formulation of the first result in graph-theoretical terms is given.

1. INTRODUCTION

Razmyslov [5] introduced the concept of *weak identities*, namely, polynomials which evaluate to zero on some fixed subspace W of an algebra A. We wish to determine the weak identities of minimal degree for the subspace $H_n(F)$ of symmetric matrices of the full matrix algebra $M_n(F)$. We refer to these as identities of H_n. Amitsur and Levitzki [1, Theorems 1 and 3] have shown that $M_n(F)$ has no polynomial identity of degree less than $2n$ and that if $n > 2$ or $|F| > 2$ then any polynomial identity of M_n of degree $2n$ is a scalar multiple of the *standard polynomial* S_{2n}, where

$$(1) \qquad S_k(x_1, x_2, \dots, x_k) = \sum_{\sigma \in \mathscr{S}_k} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(k)},$$

$\mathscr{S}_k$ the symmetric group on k objects and $(-1)^\sigma$ the sign of the permutation σ. In particular S_{2n} is an identity of H_n. An easy substitution argument [9, Proposition 2] shows there are no identities of H_n of degree less than $2n$.

Received by the editors September 15, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 16A38; Secondary 05C50, 17C05.
The research of the second author was supported in part by an NSERC grant and by an SSHN fellowship of the French Foreign Ministry.

To state our main result we must introduce other polynomials. Let

$$T_k^i(x_1, \ldots, x_k) = \sum_{\substack{\sigma \in \mathcal{S}_k \\ 1 \leq i \leq k, \sigma^{-1}(i) \equiv 1, 2 \bmod 4}} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(k)}, \tag{2}$$

Let $[a, b]$ denote the commutator $ab - ba$ and $\{a\,b\,c\}$ the triple product $abc + cba$. Let

$$Q(x_1, x_2, x_3, x_4, x_5, x_6) = \sum_{\substack{(1\,2\,3) \\ (4\,5\,6)}} \{[x_1, x_2][x_3, x_4][x_5, x_6]\}, \tag{3}$$

where the commutators are the arguments of the triple product and the sum is taken over cyclic permutations of 1 2 3 and 4 5 6, so that Q is the sum of nine triple products. One checks easily that T_k and Q are not the zero polynomial irrespective of the characteristic of the base field. We will prove

Theorem 1. *Let F be a field of arbitrary characteristic. For all $n \geq 1$ polynomials $T_{2n}^i(x_1, x_2, \ldots, x_{2n})$ are identities of $H_n(F)$ and $Q(x_1, x_2, x_3, x_4, x_5, x_6)$ is an identity of H_3.*

Let $e(n) = 2[(n+1)/2]$.

Theorem 2. *If characteristic $F \nmid e(n)!$ and $|F| > 2n$, then for $n \neq 3$ all identities of $H_n(F)$ of degree $2n$ are consequences of T_{2n}^1. If $n = 3$, then all identities of degree 6 of $H_3(F)$ are consequences of T_6^1 and Q.*

2. The identities

In this section we prove Theorem 1.1. Razmyslov [6, p. 732] gave a proof of the Amitsur-Levitzki theorem, which shows that it follows from the fact that a matrix satisfies its characteristic polynomial. Rowen [8, Theorem 1] used a variation of this proof to prove Kostant's theorem that for even n's S_{2n-2} is an identity for skew-symmetric matrices. We in turn use a variation of Rowen's proof to prove Theorem 1.1 for n even.

We start by recalling a few useful facts. Let A be a central simple associative algebra over the field F, * an involution of A of the first kind, i.e., $F \subset H(A, {}^*) = \{a \in A | a^* = a\}$. The involution * is said to be *orthogonal* if $(A \otimes \overline{F}, {}^*) \cong (M_n(\overline{F}), t)$, where $\overline{F}$ is the algebraic closure of F and t the transpose; it is said to be *symplectic* if $(A \otimes \overline{F}, {}^*) \cong (M_n(\overline{F}), s)$, s the symplectic involution. In the last case n must be even, say $n = 2m$. If * is a symplectic involution of A, then the elements of $H(A, {}^*)$ satisfy a polynomial of degree m which is analogous to the characteristic polynomial but obtained using the Pfaffian instead of the determinant (for details see [3, p. 231]).

$$x^m - \mu_1 x^{m-1} + \mu_2 x^{m-2} - \cdots + (-1)^i \mu_i x^{m-i} + \cdots + (-1)^m \mu_m. \tag{1}$$

Just as in the case of the characteristic polynomial the coefficients μ_i are polynomials in traces of powers of x. If char $F = 0$ they are given inductively by

$$(2) \qquad \mu_0 = 1, \qquad 2k\mu_k = \sum_{i=1}^{k}(-1)^{i-1}\mu_{k-i}\operatorname{tr}(x^i).$$

These are the Newton identities with an extra 2, which comes in because the generic trace of $H(A, {}^*)$, * symplectic, is half the reduced trace of A.

Recall

$$(3) \qquad \begin{aligned} S_{k+1}(x_1, \dots, x_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i-1}x_i S_k(x_1, \dots, \hat{x}_i, \dots, x_{k+1}) \\ &= \sum_{i=1}^{k+1}(-1)^{k+1-i}S_k(x_1, \dots, \hat{x}_i, \dots, x_{k+1})x_i. \end{aligned}$$

If n is even, two monomials which are cyclic permutations of one another have opposite signs in S_n. Since they have the same trace we have

$$(4) \qquad \operatorname{tr}(S_{2k}(x_1, \dots, x_{2k})) = 0.$$

Similarly,

$$(5) \qquad \sum_{\sigma\in\mathcal{S}_{2k+1}} (-1)^{\sigma}\operatorname{tr}([y, x_{\sigma(1)}][x_{\sigma(2)}, x_{\sigma(3)}]\cdots[x_{\sigma(2k)}, x_{\sigma(2k+1)}]) = 0,$$

or

$$(5') \qquad \sum_{\sigma\in\mathcal{S}_{2k+1}} (-1)^{\sigma}\operatorname{tr}([y, x_{\sigma(1)}]x_{\sigma(2)}x_{\sigma(3)}\cdots x_{\sigma(2k)}x_{\sigma(2k+1)}) = 0.$$

It is sometimes helpful to set apart the variable of T_k which is not alternating, so we let

$$(6) \qquad T_k(x_1, x_2, \dots, x_{k-1}; y) = T_k^1(y, x_1, x_2, \dots, x_{k-1});$$

or

$$(6') \qquad T_k^i(x_1, \dots, x_k) = (-1)^{i-1}T_k(x_1, \dots, \hat{x}_i, \dots, x_k; x_i).$$

Denote by $I(H_n(F))$ or $I(H_n)$ the ideal of identities of $H_n(F)$ in a free associative algebra on an at most countable number of variables (drawn from the end of the alphabet). Let $K(A, {}^*) = \{a - a^* | a \in A\}$. These are the *-skew elements of A if $\operatorname{char} F \neq 2$. Let $K_n(F) = K(M_n(F), t)$. If e_{ij} is a set of matrix units of M_n let $e[ij] = e_{ij} + e_{ji}$, $i \neq j$, $e[ii] = e_{ii}$. The set $\{e[ij] | 1 \le i \le j \le n\}$ is a basis of H_n. Since T_{2n} is multilinear, it is an identity of H_n if and only if it is zero for every substitution from a given basis of H_n. Since the above basis belongs to $H_n(\mathbf{Z})$ it suffices to prove $T_{2n} \in I(H_n(F))$ for F of characteristic 0.

Proposition 1. *If n is even, $T_{2n}(x_1, \dots, x_{2n-1}; y) \in I(H_n)$.*

Proof. We may assume that F is of characteristic 0 and write $n = 2m$. If c is an invertible element of K_n, then the map $a^* = c^{-1}a^t c$, $a \in M_n$, defines an involution of symplectic type on M_n whose symmetric elements are of the form bc, $b \in K_n$. (Were $c \in H_n$, the involution * would be of orthogonal type.) By (1) they satisfy

$$(7)\qquad (bc)^m - \mu_1(bc)^{m-1} + \cdots + (-1)^i \mu_i (bc)^{m-i} + \cdots + (-1)^m \mu_m = 0.$$

By a Zariski density argument, (7) holds for all $b, c \in K_n$, not only for invertible c's. Each μ_i is a polynomial in traces of powers of bc with rational coefficients. Linearizing (7) completely we get

$$0 = \sum_{\sigma, \pi \in \mathscr{S}_m} b_{\sigma(1)} c_{\pi(1)} b_{\sigma(2)} c_{\pi(2)} \cdots b_{\sigma(m)} c_{\pi(m)}$$
$$+ \text{ a linear combination of products of the form } b_{\sigma(i_1)} c_{\pi(i_1)} \cdots b_{\sigma(i_l)} c_{\pi(i_l)},$$

$1 \le i_1 \le i_2 < \cdots < i_l \le m$, $l < m$, whose coefficients are polynomials in traces of products of similar form, i.e., $b_{\sigma(j_1)} c_{\pi(j_1)} \cdots b_{\sigma(j_r)} c_{\pi(j_r)}$.

Substitute $b_1 = [d, a_1]$, $c_1 = [a_2, a_3]$, $b_2 = [a_4, a_5], \ldots, c_m = [a_{2n-2}, a_{2n-1}]$, d, a_i's $\in H_n$, so that the commutators are elements of K_n. Then sum over all permutation in S_{2n-1} multiplying each summand by the sign of the permutation. The first part of this sum is $2^{m-1} T_{2n}(a_1, \ldots, a_{2n-1}; d)$. The remaining terms have as coefficients polynomials in traces. The traces in which d does not appear have the form $\operatorname{tr}(S_{2k}(a_{i_1}, \ldots, a_{i_{2k}}))$ which is zero by (4). If d is present, since we may permute the commutators cyclically without changing the value of the trace, we may assume that d is in the first commutator and we have

$$\sum_{\sigma \in \mathscr{S}_{2k+1}} (-1)^\sigma \operatorname{tr}([d, a_{\sigma(i_1)}][a_{\sigma(i_2)}, a_{\sigma(i_3)}] \cdots [a_{\sigma(2k)}, a_{\sigma(2k+1)}]),$$

which is zero by (5). Thus the sum of all remaining terms is zero and

$$T_{2n}(x_1, \ldots, x_{2n-1}; y) \in I(H_n)$$

when n is even.

The following lemma will allow us to pass from n even to n odd.

Lemma 2. *If $P(x_1, \ldots, x_k)$ is a multilinear identity of H_n, then*

$$P(x_1, \ldots, x_k) = P'(x_1, \ldots, x_{k-2}) x_{k-1} x_k$$
$$+ \text{ terms which do not end in } x_{k-1} x_k$$

and $P'(x_1, \ldots, x_{k-2})$ is an identity of H_{n-1}.

Proof. The first half of the statement is obvious. Consider substitutions $x_l \in \{e[ij] \| 1 \le i, j \le n-1\}$, $1 \le l < k-1$ and $x_{k-1} = e[jn]$, $x_k = e[nn]$. Terms in e_{in} must come from $P'(x_1, \ldots, x_{k-2}) x_{k-1} x_k$. Since j is allowed to vary, $P'(x_1, \ldots, x_{k-2})$ must be zero for every substitution from H_{n-1} viewed as embedded in the upper left corner of M_n.

The same result holds for terms which begin in $x_k x_{k-1}$.

If n is odd, $n+1$ is even and $T_{2n+2} \in I(H_{n+1})$. Since

$$T_{2n+2}(x_1, \dots, x_{2n+1}; y) = T_{2n}(x_1, \dots, x_{2n-1}; y)x_{2n}x_{2n+1} + \text{terms which do not end in } x_{2n}x_{2n+1},$$

$T_{2n}(x_1, \dots, x_{2n-1}; y) \in I(H_n)$ by Lemma 2 and we have proved the first part of Theorem 1.1.

Since the polynomial $Q(x_1, x_2, x_3, x_4, x_5, x_6)$ is multilinear, we may assume that F is of characteristic 0. If $a, b, c \in K_3$, a simple computation shows that

$$\{abc\} = \tfrac{1}{2}(\mathrm{tr}(ab)c + \mathrm{tr}(cb)a), \qquad a, b, c \in K_3. \tag{8}$$

If $a_i \in H_3$, $1 \le i \le 6$, then $[a_i, a_j] \in K_3$, and (8) may be used to express $Q(a_1, a_2, a_3, a_4, a_5, a_6)$ in terms of traces. Collecting terms and using (5′) with $k = 1$, this is seen to be zero. Thus we have $Q(x_1, x_2, x_3, x_4, x_5, x_6) \in I(H_3)$, which completes the proof of Theorem 1.1.

Standard arguments yield

Corollary 3. *If R is a commutative ring with unit element* 1, *then*

$$T_{2n}(x_1, \dots, x_{2n-1}; y) \in I(H_n(R)).$$

If A is a central simple associative algebra of degree n over its centre and * an orthogonal involution of A, then $T_{2n}(x_1, \dots, x_{2n-1}; y) \in I(H(A, ^*))$. If $n = 3$, then $Q(x_1, x_2, x_3, x_4, x_5, x_6) \in I(H(A, ^*))$.

We will need a few properties of T_n. By definition $T_k(x_1, \dots, x_{k-1}; y)$ is multilinear and it is alternating in the x's. From the definition, if $k \equiv 2, 3$ mod 4,

$$T_{k+1}(x_1, \dots, x_k; y) = \sum_{i=1}^{k} (-1)^{k-i} T_k(x_1, \dots, \hat{x}_i, \dots, x_k; y)x_i, \qquad k \equiv 2, 3 \mod 4. \tag{9}$$

While if $k \equiv 0, 1$ mod 4,

$$\begin{aligned} T_{k+1}(x_1, \dots, x_k; y) &= (-1)^k S_k(x_1, \dots, x_k)y \\ &\quad + \sum_{i=1}^{k} (-1)^{k-i} T_k(x_1, \dots, \hat{x}_i, \dots, x_k; y)x_i, \end{aligned} \qquad k \equiv 0, 1 \mod 4. \tag{9′}$$

Lemma 4. (i) *For all $j \ge 0$, $S_{2n+j}, T_{2n+j} \in I(H_n)$.*

(ii) *If any one of the variables is set equal to* 1, *then S_{2n} and T_{2n} vanish.*

(iii) $\sum_{i=1}^{2n} T_{2n}^i(x_1, \dots, x_{2n}) = e(n)S_{2n}(x_1, \dots, x_{2n})$.

(iv) *If $n \ge 2$ and* $\mathrm{char}\, F$ *does not divide $e(n)!$, then $\{T_{2n}^i \mid 1 \le i \le 2n\}$ are linearly independent.*

Proof. Since S_{2n} and $T_{2n} \in I(H_n)$, (i) follows from equations (3), (9), and (9′) by induction on j. Part (ii) follows from the alternating nature of S_{2n} and T_{2n}.

Consider $\sum_{j=1}^{2n} T_{2n}^j$. We claim that this polynomial is alternating. If we transpose x_i and x_{i+1}, then every summand changes sign except possibly T_{2n}^i and T_{2n}^{i+1}. Using (6′) these are

$$(-1)^{i-1}T_{2n}(x_1, \dots, \hat{x}_i, \dots, x_{2n}; x_i) + (-1)^i T_{2n}(x_1, \dots, \hat{x}_{i+1}, \dots, x_{2n}; x_{i+1}),$$

which become

$$(-1)^{i-1}T_{2n}(x_1, \dots, x_{i-1}, x_i, \hat{x}_{i+1}, x_{i+2}, \dots, x_{2n}; x_{i+1}) + (-1)^i T_{2n}(x_1, \dots, x_{i-1}, \hat{x}_i, x_{i+1}, \dots, x_{2n}; x_i)$$

or $-(T_{2n}^{i+1} + T_{2n}^i)$. So the above sum is alternating and hence a multiple of S_{2n}. Considering the monomial $x_1x_2\cdots x_{2n}$, we see that its coefficient is $e(n)$ and we have (iii).

If char F divides $e(n)$ then (iii) provides a linear dependence relation among the T_{2n}^i. We prove (iv) by induction on n. If $n = 2$ there are four T_4^i, and one checks that if $\operatorname{char} F \neq 2$ they are linearly independent. Assume that $\{T_{2n}^i | 1 \leq i \leq 2n\}$ are linearly independent when $\operatorname{char} F \nmid e(n)!$. If $\operatorname{char} F \nmid e(n+1)!$, then a fortiori $\operatorname{char} F \nmid e(n)!$. By (9) and (9′), for $1 \leq i \leq 2n$

$$T_{2n+2}^i(x_1, \dots, x_{2n+2}) = T_{2n}^i(x_1, \dots, x_{2n})[x_{2n+1}, x_{2n+2}] + \text{terms which do not end in } x_{2n+1}x_{2n+2} \text{ or } x_{2n+2}x_{2n+1}. \tag{10}$$

Therefore $\{T_{2n+2}^i | 1 \leq i \leq 2n\}$ are linearly independent. If n is odd then by (9), T_{2n+2}^{2n+1} and T_{2n+2}^{2n+2} have no terms ending in $x_{2n+1}x_{2n+2}$ or $x_{2n+2}x_{2n+1}$. In this case we need only to check the linear independence of T_{2n+1}^{2n+1} and T_{2n+2}^{2n+2}. The induction hypothesis will do it if we consider the terms ending in x_1x_2 for these two polynomials.

If n is even

$$T_{2n+2}^{2n+1} = S_{2n}(x_1, \dots, x_{2n})[x_{2n+1}, x_{2n+2}] + \text{terms which do not end in } x_{2n+1}x_{2n+2} \text{ or } x_{2n+2}x_{2n+1}, \tag{11}$$

and

$$T_{2n+2}^{2n+2} = S_{2n}(x_1, \dots, x_{2n})[x_{2n+1}, x_{2n+2}] + \text{terms which do not end in } x_{2n+1}x_{2n+2} \text{ or } x_{2n+2}x_{2n+1}. \tag{12}$$

Subtracting

$$\frac{1}{e(n)} \sum_{i=1}^{2n} T_{2n}^i[x_{2n+1}, x_{2n+2}]$$

from T_{2n+2}^{2n+2} reduces the proof to checking that these two polynomials are independent. Again considering the terms ending in x_1x_2 does it and we have (iv).

The F-span of $\{T_{2n}^i | 1 \le i \le 2n\}$ is an S_{2n} module under the action which permutes the variables. One can check that if $\operatorname{char} F \nmid e(n)!$ the character of this representation is $[2, 1^{2n-2}] + [1^{2n}]$ in the usual notation for Young diagrams.

Since $S_{2n} \in I(H_n)$ irrespective of the characteristic

$$S_{2n}(x_1, \dots, x_{2n}) - T_{2n}^i(x_1, \dots, x_{2n}) = \sum_{\substack{\sigma \in S_{2n} \\ \sigma^{-1}(i) \equiv 0, 3 \mod 4}} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(2n)} \in I(H_n). \tag{13}$$

Just as the Amitsur-Levitzki theorem lends itself to a graph-theoretic interpretation and is equivalent to a result on unicursal (or Eulerian) paths on a directed graph [2, p. 232; 11, Theorem 2], Theorem 1.1 is equivalent to a result on unicursal paths on undirected graphs. We still state this for the first part of the theorem.

Let (V, E) be a finite graph with vertices $V = \{v_1, \dots, v_n\}$ and (undirected) edges $E = \{w_1, \dots, w_r\}$ in an arbitrary but fixed ordering. Then unicursal paths (if they exist) correspond to permutations $\sigma \in \mathscr{S}_r$, where the path is $w_{\sigma(1)} w_{\sigma(2)} \cdots w_{\sigma(r)} = p_\sigma$. Define $\varepsilon(p_\sigma)$ to be $(-1)^\sigma$.

Theorem 5. *Let (V, E) be an undirected graph with $|E| \ge 2|V|$. Choose a distinguished edge w_i. Then for any two fixed vertices v_j, v_k (not necessarily distinct) the number of unicursal paths p_σ from v_j to v_k with w_i in positions which are congruent to* 1 *or* 2 *modulo* 4 *and with $\varepsilon(p_\sigma) = 1$ is equal to the number of such paths with $\varepsilon(p_\sigma) = -1$. The same holds for unicursal paths with w_i in positions which are congruent to* 0 *or* 3 *modulo* 4.

It is easy to see the equivalence if we identify an edge joining v_i and v_j to $e[ij] = e[ji]$ (i may equal j) and note that unicursal paths correspond to nonzero monomials.

Since T_{2n} is not alternating, putting two properly chosen variables equal will yield a polynomial of degree 2 in that variable. This polynomial, as expected, will play a role in the proof of Theorem 2. For i, j, k with $1 \le i \le j-1 \le k-2$ define

$$G_k^{ij}(x_1, \dots, x_{k-2}; y) = \sum_{\sigma \in \mathscr{S}_{k-2}} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(i-1)} [y, x_{\sigma(i)}] x_{\sigma(i+1)}, \dots, x_{\sigma(j-2)} [y, x_{\sigma(j-1)}] x_{\sigma(j)} \cdots x_{\sigma(k-2)}, \tag{14}$$

$$G_k(x_1, \dots, x_{k-2}; y) = \sum_{j=0}^{[k/4-1]} \sum_{i=1}^{[k/2-2j-1]} (-1)^{i-1} G_k^{2i-1\, 2i-1+4j+2}(x_1, \dots, x_{k-2}; y). \tag{15}$$

Proposition 6. *$T_{2n}(x_1, \ldots, x_{2n-2}, y; y) = -G_{2n}(x_1, \ldots, x_{2n-2}; y)$ and therefore G_{2n} belongs to $I(H_n)$.*

Proof. A typical expression of $T_{2n}(x_1, \ldots, x_{2n-2}, y; y)$ is of the form

$$x_{\sigma(1)} \cdots x_{\sigma(i-1)}[y, x_{\sigma(i)}]x_{\sigma(i+1)} \cdots x_{\sigma(j-2)}[y, x_{\sigma(j-1)}]x_{\sigma(j)} \cdots x_{\sigma(2n-2)}. \tag{16}$$

To avoid counting the same monomial twice, we may assume that i and j are odd. Such an expression may arise from

$$x_{\sigma(1)} \cdots x_{\sigma(i-1)}[y, x_{\sigma(i)}]x_{\sigma(i+1)} \cdots x_{\sigma(j-2)}[x_{\sigma(2n-1)}, x_{\sigma(j-1)}]x_{\sigma(j)} \cdots x_{\sigma(2n-2)}$$

if $i \equiv 1 \bmod 4$; in this case its sign is $(-1)^j(-1)^\sigma = -(-1)^\sigma$ since j is odd. Or it may arise from

$$x_{\sigma(1)} \cdots x_{\sigma(i-1)}[x_{\sigma(2n-1)}, x_{\sigma(i)}]x_{\sigma(i+1)} \cdots x_{\sigma(j-2)}[y, x_{\sigma(j-1)}]x_{\sigma(j)} \cdots x_{\sigma(2n-2)}$$

if $j \equiv 1 \bmod 4$; in this case the sign is $(-1)^{i-1}(-1)^\sigma = (-1)^\sigma$ since i is odd. Both will occur if $i \equiv 1 \bmod 4$ and $j \equiv 1 \bmod 4$ in which case they cancel since they have opposite signs. So we are left with (16) with i, j odd and exactly one index is congruent to $1 \bmod 4$. If $i \equiv 1$ and $j \equiv 3$ or $i \equiv 3$ and $j \equiv 1 \pmod 4$ their difference is congruent to $2 \bmod 4$. So we get exactly the same terms as are present in G_{2n} and we must check that they have opposite signs. If $i \equiv 1$ and $j \equiv 3$ the sign is $-(-1)^\sigma$ while when $i = 3$ the sign is $(-1)^\sigma$. Writing $i = 2i' - 1$ we get $-(-1)^\sigma$ when i' is odd and $(-1)^\sigma$ when i' is even. Comparing with (15) we see that $T_{2n}(x_1, \ldots, x_{2n-2}; y; y)$ and $G_{2n}(x_1, \ldots, x_{2n-2}; y)$ have opposite signs.

The polynomial $Q(x_1, x_2, x_3, x_4, x_5, x_6)$ is skew in x_1, x_2, x_3 and x_4, x_5, x_6. Moreover

$$Q(x_4, x_5, x_6, x_1, x_2, x_3) = -Q(x_1, x_2, x_3, x_4, x_5, x_6). \tag{17}$$

This can be used to show that if $\operatorname{char} F \neq 2$ then the F-span of

$$Q(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(6)}),$$

$\sigma \in \mathscr{S}_6$, has dimension 10 and its character as an $\mathscr{S}_6$-module is $[2^2, 1^2]+[1^6]$. Let

$$Q(x_1, x_2, x_3, x_4; y) = Q(x_1, x_2, y, x_3, x_4, y), \tag{18}$$

$$Q(x_1, x_2; y, z) = Q(x_1, z, x_2, z; y). \tag{19}$$

The polynomial $Q(x_1, x_2, x_3, x_4; y)$ is x_1, x_2 skew, x_3, x_4 skew and

$$Q(x_3, x_4, x_1, x_2; y) = -Q(x_1, x_2, x_3, x_4; y). \tag{17$'$}$$

This is enough to prove that the F-span of $Q(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}; y)$, $\sigma \in \mathscr{S}_4$, is the F-span of

$$Q(x_1, x_2, x_3, x_4; y), \quad Q(x_1, x_3, x_4, x_2; y), \quad \text{and} \quad Q(x_1, x_4, x_2, x_3; y).$$

In fact one can check that these three polynomials and $G_6(x_1, x_2, x_3, x_4; y)$ are linearly independent.

3. Uniqueness

Essentially Theorem 1.2 says that, under some restrictions on the base field and if $n \neq 3$, any homogeneous polynomial identity of H_n of degree $2n$ is obtained from T_{2n}. To prove that, we must substitute values of H_n for the variables and conclude that some polynomials are not identities. While we try to be as systematic as possible, the need to include Q when $n = 3$ forces us to consider low n's very carefully.

We recall a few standard facts concerning polynomial identities stating them for H_n. If $H_n(F)$ satisfies a polynomial identity, then it satisfies a multilinear identity of the same degree. If $P(x_1, \dots, x_m)$ is a homogeneous identity of $H_n(F)$ then its linearization in all but one variable is an identity of the same total degree and of the same degree as P in the nonlinearized variable. If $P(x_1, \dots, x_m, y)$ is a homogeneous polynomial, linear in the x's and of degree k in y, then the polynomial obtained by first linearizing P completely then replacing the new variables by y is $k!P(x_1, \dots, x_m, y)$. Therefore if $\operatorname{char} F \nmid k!$, $P(x_1, \dots, x_m, y) \in I(H_n)$ if and only if the total linearization of $P \in I(H_n)$. One advantage in dealing with multilinear identities is that they remain identities when the base field is extended. Finally if the degree of each x_i in an identity $P(x_1, \dots, x_m)$ is less than $|F|$ then each homogeneous component of P is also an identity.

Following Osborn [4, p. 78] we introduce a partial ordering on the set of homogeneous polynomials in noncommuting indeterminates $x_1, \dots, x_m$. If $P(x_1, \dots, x_m)$ is a homogeneous polynomial of degree n, we say that it is of *type* $[n_1, n_2, \dots, n_m]$ if n_j is the degree of x_j in P and $n_1 \geq n_2 \geq \cdots \geq n_m$. If P and P' are homogeneous of degree n and n' and of type $[n_1, n_2, \dots, n_m]$ and $[n'_1, n'_2, \dots, n'_m]$ respectively, then P is lower than P' in the *partial ordering* if and only if either (i) $n < n'$ or (ii) $n = n'$ and $n_j > n'_j$ for the first integer j such that $n_j \neq n'_j$. Otherwise two polynomials are not comparable. If an integer is repeated k times, we will denote this by an exponent. For example $[3, 2^2, 1^3]$ means $[3, 2, 2, 1, 1, 1]$.

An identity is *absolutely irreducible* if it does not imply an identity lower than itself in the partial ordering. While the following theorem is proved in [4, Theorem 3] for elements of the free nonassociative algebra, the author notes that essentially the same proof will work in the free associative algebra.

Theorem (Osborn). *Let n be a positive integer and $P(x_1, \dots, x_m)$ an absolutely irreducible homogeneous polynomial with coefficients in a field F of characteristic not dividing $n!$. Then P is either symmetric or skew-symmetric in its arguments of degree n, depending on whether n is even or odd.*

Let * denote the unique involution on a free associative algebra which restricts to the identity on the generators. A polynomial P is said to be *symmetric* if $P^* = P$, *skew-symmetric* if $P^* = -P$. If P is multilinear of degree m,

$$P(x_1, x_2, \dots, x_m) = \sum_{\sigma \in \mathcal{S}_m} \alpha_\sigma x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(m)},$$

then

$$P(x_1, x_2, \dots, x_m)^* = \sum_{\sigma \in \mathcal{S}_m} \alpha_{\rho\sigma} x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(m)},$$

where ρ is the reversal permutation

$$(1m)(2m-1)\cdots\left(\left[\frac{m}{2}\right]\left[\frac{m+1}{2}\right]+1\right).$$

We multiply permutations from right to left. The sign of ρ is 1 if m is congruent to 0 or 1 mod 4, -1 if m is congruent to 2 or 3 mod 4. Taking the transpose of any evaluation we see that if $P \in I(M_n)$ or $I(H_n)$ then so does P^*. If the characteristic is not 2 then any identity is the sum of its symmetric and skew-symmetric parts and these are also identities.

The uniqueness part of the Amitsur-Levitzki theorem is proved using two types of substitutions. The first corresponds to

Lemma 1. *If* $P(x_1, \dots, x_{2n}) = \sum_{\sigma \in \mathcal{S}_{2n}} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(2n)}$ *is a multilinear identity of* H_n, *then*

$$\alpha_{\sigma(2i-1\,2i)} = -\alpha_\sigma, \qquad 1 \le i \le n.$$

If P is a homogeneous identity of H_n of degree $2n$, then the words which differ only by a transposition of the indeterminates in positions $2i-1$ and $2i$ have coefficients which differ only in sign. In particular if $\operatorname{char} F \neq 2$ *the words having the same indeterminate in positions $2i-1$ and $2i$, have coefficient 0 in P.*

Proof. Substituting $e[11], e[12], e[22], e[23], \dots, e[i-1\,i-1], e[i-1\,i]$, $e[ii]$, $e[ii]$, $e[i\,i{+}1]$, $e[i{+}1\,i{+}1]$, $\dots$, $e[n{-}1\,n]$, $e[nn]$ for $x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}$, $x_{\sigma(4)}, \dots, x_{\sigma(2i-3)}, x_{\sigma(2i-2)}, x_{\sigma(2i-1)}, x_{\sigma(2i)}, x_{\sigma(2i+1)}, x_{\sigma(2i+2)}, \dots, x_{\sigma(2n)}$ in P and considering the coefficient of e_{1n} yields the first result. The second part is obtained by linearizing and applying the first result.

From now on we assume that F is a field of characteristic not 2.

Corollary 2. *Any homogeneous identity P of H_n of degree $2n$ is a linear combination of associative products of commutators. In particular no variable has degree greater than n.*

Since Jordan polynomials are symmetric, identities which are symmetric are of particular interest.

Lemma 3. *If* $P(x_1, \dots, x_{2n}) = \sum_{\sigma \in \mathcal{S}_{2n}} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(2n)}$ *is a multilinear identity of* H_n, *then*

$$\alpha_\sigma + \alpha_{\rho\sigma} + \alpha_{\sigma(2i\,2i+1)} + \alpha_{\rho\sigma(2i\,2i+1)} = 0, \qquad 1 \le i \le n-1.$$

In particular if P *is symmetric then*

$$\alpha_{\sigma(2i\,2i+1)} = -\alpha_\sigma\,, \qquad 1 \le i \le n-1\,.$$

Proof. Substituting

$$e[12]\,, e[22]\,, \ldots\,, e[i-1\,i]\,, e[ii]\,, e[ii]\,, e[i\,i+1]\,,$$
$$e[i+1\,i+1]\,, e[i+1\,i+2]\,, \ldots\,, e[nn]e[1n]$$

for $x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(2i-1)}, x_{\sigma(2i)}, \; x_{\sigma(2i+1)}, x_{\sigma(2i+2)}, x_{\sigma(2i+3)}, x_{\sigma(2i+4)}, \ldots,$ $x_{\sigma(2n-1)}, x_{\sigma(2n)}$ in P and considering the coefficient of e_{11} yields the first result. If P is symmetric, then $\alpha_\sigma = \alpha_{\rho\sigma}$ for all σ and since $\operatorname{char} F \neq 2$ the second equality holds.

If P is a symmetry identity of H_n homogeneous of degree $2n$, then, by Lemmas 1 and 3, its complete linearization is a multiple of S_{2n}. Hence P was multilinear to start with. Considering the sign of ρ, S_{2n} is symmetric if n is even, skew-symmetric if n is odd and we have

Corollary 4. *If* n *is odd then* H_n *has no symmetric homogeneous identity of degree* $2n$. *If* n *is even, the only symmetric homogeneous identities of* H_n *of degree* $2n$ *are multiples of* S_{2n}. *Thus any homogeneous identity of* H_n *of degree* $2n$ *of lower type than* $[1^{2n}]$ *must be skew-symmetric.*

Since Smith [10, Corollary 2] has shown that the standard polynomial is not Jordan if $\operatorname{char} F \nmid (2n)!$, in this case H_n does not satisfy any Jordan polynomial of degree $2n$.

Lemma 5. *Let* $P(x_1, \ldots, x_{2n-1}; y)$ *be an identity of* H_n *of type* $[2, 1^{2n-2}]$. *Then we may write*

$$\begin{aligned} &P(x_1, \ldots, x_{2n-2}; y) \\ &\quad = \sum_{1\le i\le j\le n-1} \sum_{\sigma\in\mathscr{S}_{2n-2}} \alpha_\sigma^{ij}[x_{\sigma(1)}, x_{\sigma(2)}]\cdots[y, x_{\sigma(2i-1)}][x_{\sigma(2i)}, x_{\sigma(2i+1)}] \\ &\qquad\qquad \cdots[y, x_{\sigma(2j)}][x_{\sigma(2j+1)}, x_{\sigma(2j+2)}]\cdots[x_{\sigma(2n-3)}, x_{\sigma(2n-2)}]\,, \end{aligned}$$

and

$$\alpha_\sigma^{ii} = -\alpha_{\sigma(2i-1\,2i)}^{ii}\,.$$

Hence a homogeneous identity of H_n *of degree* $2n$ *cannot have terms*

$$\cdots[y, x][y, x]\cdots$$

with nonzero coefficient.

Proof. By Corollary 2 we may write P as above. Consider the coefficient of e_{1n} when $e[11], e[12], \ldots, e[ii], e[i+1\,i+1], e[i+2\,i+2], e[ii+2], \ldots, e[nn],$ $e[n-1\,n]$ are substituted for $x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(2i-1)}, x_{\sigma(2i)}, x_{\sigma(2i+1)}, x_{\sigma(2i+2)},$ $\ldots, x_{\sigma(2n-3)}, x_{\sigma(2n-2)}$ and $e[i\,i+1]$ for y in P. Since it is $(\alpha_\sigma^{ii} - \alpha_{\sigma(2i-1\,2i)}^{ii})$ we have $\alpha_\sigma^{ii} = -\alpha_{\sigma(2i-1\,2i)}^{ii}$. The last part is obtained by partial linearization.

We are now ready to consider the case $n = 2$.

Proposition 6. *If* $\operatorname{char} F \neq 2$ *and* $|F| > 3$, *then the identities of minimal degree of* $H_2(F)$ *are consequences of* $T_4(x_1, x_2, x_3; y)$.

Proof. Since the base field has more than 4 elements we need only consider homogeneous polynomials. By Lemma 2.4(iii), since $\operatorname{char} F \neq 2$, S_4 is a consequence of T_4. Moreover since

$$\sum_{i=1}^{4} T_4(x_1, \ldots, \hat{x}_i, \ldots, x_4; x_i) = 2[[x_2, x_4], [x_1, x_3]],$$

$[[x_1, x_2], [x_3, x_4]]$ and hence $[[y, x_1], [y, x_2]]$ are consequences of T_4.

In the case $n = 2$, Proposition 2.6 becomes

$$T_4(x_1, x_2, y; y) = -[[y, x_1], [y, x_2]].$$

We prefer obtaining $[[y, x_1], [y, x_2]]$ this way because $T_4(x_1, x_2, x_3; y)$ is alternating in the x's.

If $P \in I(H_2)$ is homogeneous of degree 4 then it could be of type $[4]$, $[3, 1]$, $[2^2]$, $[2, 1^2]$ and $[1^4]$. By Corollary 2 the first two types cannot occur and an identity of type $[2^2]$ must be of the form

$$\alpha[x, y]^2, \qquad \alpha \neq 0,$$

which is not an identity either directly or by Lemma 5.

If $P(x_1, x_2; y)$ is an identity of type $[2, 1^2]$ then, by Corollary 2 and Lemma 5, it must be a multiple of $[y, x_1][y, x_2] - [y, x_2][y, x_1] = [[y, x_1], [y, x_2]]$.

If $P(x_1, x_2, x_3, x_4) \in I(H_2)$ is multilinear, then $P(y, y, x_3, x_4)$ must be a multiple of $[[y, x_3], [y, x_4]]$. Subtracting a multiple of $[[x_1, x_3], [x_2, x_4]]$ if necessary, we may assume that P is skew in x_1, x_2. Consider next $P(x_1, y, y, x_4)$. It is a multiple of $[[y, x_1], [y, x_4]]$. Subtracting a multiple of $T_4(x_1, x_2, x_4; x_3)$ if necessary, we may assume that $P(x_1, y, y, x_4)$ is identically zero and hence that $P(x_1, x_2, x_3, x_4)$ is skew in x_2, x_3 without losing the skewness in x_1, x_2. Thus we have P alternating in x_1, x_2, x_3. Finally $P(x_1, x_2, y, y)$ is a multiple of $[[y, x_1], [y, x_2]]$. Subtracting a multiple of $T_4(x_1, x_2, x_3; x_4)$, we may assume that P is skew in x_3, x_4 and still alternating in x_1, x_2, x_3. Hence P is alternating in all x's and a multiple of S_4. Since S_4 is a consequence of T_4 this proves the proposition.

We have given this last argument in detail because analogous ones will be used in the next case.

Remark. If F is an infinite field and $P(x_1, \ldots, x_m; y)$ is a homogeneous polynomial of type $[k, 1^m]$ then to show that P is an identity of H_n it suffices, by a Zariski density argument, to prove that P is zero whenever y is diagonal with distinct eigenvalues and the x's belong to a fixed basis of H_n. If $\operatorname{char} F \nmid k!$ then a field extension argument allows us to drop the hypothesis F infinite.

Lemma 7. *Let* $P(x_1, \dots, x_m; y)$ *be an identity of* $H_n(F)$, $n > 1$, *of type* $[k, 1^m]$ *which is a linear combination of terms ending in* $[y, \ ]$. *Then*

$$P(x_1, \dots, x_m; y) = \sum_{i=1}^{m} P_i(x_1, \dots, \hat{x}_i, \dots, x_m; y)[y, x_i].$$

If F *is infinite or if* $\operatorname{char} F \nmid k!$ *then* $P_i(x_1, \dots, \hat{x}_i, \dots, x_m; y)$ *is an identity of* H_{n-1} *of type* $[k-1, 1^{m-1}]$.

Proof. By the remark we may assume that F is infinite. We need only show that the P_i's are identities of H_{n-1}. Fix i, $1 \le i \le m$. Let $y = \sum_{r=1}^{n} \lambda_r e_r$; λ_r distinct, $x_i = e[jn]$ for an arbitrary but fixed j, $1 \le j < n$, and $x_s \in \{e[pq] \| 1 \le p, q < n\}$, $1 \le s \le m$, $s \neq i$. Since $[y, x_i] = (\lambda_j - \lambda_n)(e_{jn} - e_{nj}) \neq 0$, the coefficient of the matrix units e_{tn}, $1 \le t < n$, in $P(x_1, \dots, x_m; y)$ must come from $P_i(x_1, \dots, \hat{x}_i, \dots, x_m; y)$. Since j is arbitrary, P_i must vanish on H_{n-1} embedded in the upper left-hand corner of H_n. So $P_i \in I(H_{n-1})$.

Corollary 8. *Assume that* $\operatorname{char} F \nmid k!$, $k > 1$. *If* H_{n-1} *has no identity of type* $[k-1, 1^{2n-k-1}]$ *then* H_n *has no identity of type* $[k, 1^{2n-k}]$.

Proof. If $P(x_1, \dots, x_{2n-k}; y) \in I(H_n)$ is of type $[k, 1^{2n-k}]$ then, by Lemma 2.2, P is a sum of terms ending in $[y, \ ]$. Then Lemma 7 yields the result.

Proposition 9. *Assume that* $n > 1$ *and* $\operatorname{char} F \nmid n!$. *Then* $H_n(F)$ *has no identity of degree* $2n$ *of type* $[3, 1^{2n-3}]$ *or lower.*

Proof. The proof is by induction on n. We already have the result for $n = 2$. Assume that for some $n > 2$, H_{n-1} has no identity of type $[3, 1^{2n-5}]$. Then, by Corollary 8, H_n also has no identity of type $[4, 1^{2n-4}]$ or lower.

It remains to show that H_n has no identity of type $[3, 1^{2n-3}]$. Let

$$P(x_1, \dots, x_{2n-3}; y) \in I(H_n)$$

be of type $[3, 1^{2n-3}]$. By Lemma 2.2 used on both sides and the induction hypothesis, none of the summands of P start or end in $x_i x_j$. Therefore every summand starts and ends in $[y, \ ]$. So

$$P(x_1, \dots, x_{2n-3}; y) = \sum_{i=1}^{n-2} \sum_{\sigma \in \mathcal{S}_{2n-3}} \alpha_\sigma^i [y, x_{\sigma(1)}][x_{\sigma(2)}, x_{\sigma(3)}] \cdots [y, x_{\sigma(2i)}][x_{\sigma(2i+1)}, x_{\sigma(2i+2)}] \cdots [y, x_{\sigma(2n-3)}].$$

Since $P^* = -P$,

$$\begin{aligned} P(x_1, \dots, x_{2n-3}; y) &= \sum_{i=1}^{[(n-2)/2]} \sum_{\sigma \in \mathcal{S}_{2n-3}} \alpha_\sigma^i ([y, x_{\sigma(1)}] \cdots [y, x_{\sigma(2i)}] \cdots [y, x_{\sigma(2n-3)}] \\ &\qquad + [y, x_{\sigma(2n-3)}] \cdots [y, x_{\sigma(2i)}] \cdots [y, x_{\sigma(1)}]) \\ &+ \sum_{\sigma \in \mathcal{S}_{2n-3}} \beta_\sigma [y, x_{\sigma(1)}] \cdots [x_{\sigma(n-3)}, x_{\sigma(n-2)}][y, x_{\sigma(n-1)}] \cdots [y, x_{\sigma(2n-3)}], \end{aligned}$$

the last term occurring only if n is odd.

We show first that $\alpha_\sigma^1 = 0$ for all σ, then that $\alpha_\sigma^i = 0$ implies $\alpha_\sigma^{i+1} = 0$ and finally that for n odd, $\beta_\sigma = 0$.

For $\sigma \in \mathcal{S}_{2n-3}$ let $y = e[1n]$, $x_{\sigma(1)} = e[2n]$, $x_{\sigma(2)} = e[22]$, $x_{\sigma(3)} = e[23]$, $x_{\sigma(4)} = e[33]$, $x_{\sigma(5)} = e[34], \dots, x_{\sigma(2n-6)} = e[n-2\, n-2]$, $x_{\sigma(2n-5)} = e[n-2\, n-1]$, $x_{\sigma(2n-4)} = e[1\, n-1]$, $x_{\sigma(2n-3)} = e[12]$. For this substitution, $[y, x_{\sigma(j)}] = 0$ except $[y, x_{\sigma(1)}] = e_{12} - e_{21}$, $[y, x_{\sigma(2n-4)}] = e_{n\, n-1} - e_{n-1\, n}$ and $[y, x_{\sigma(2n-3)}] = e_{n2} - e_{2n}$. For the remaining x's, $x_{\sigma(j)}$ has a nonzero commutator only with $x_{\sigma(j-1)}$ and $x_{\sigma(j+1)}$. Since $x_{\sigma(1)}$, $x_{\sigma(2n-4)}$ and $x_{\sigma(2n-3)}$ are already accounted for, we must have $[x_{\sigma(2)}, x_{\sigma(3)}]$, $[x_{\sigma(4)}, x_{\sigma(5)}], \dots, [x_{\sigma(2n-6)}, x_{\sigma(2n-5)}]$. For $1 \le j \le n-3$, $[x_{\sigma(2j)}, x_{\sigma(2j+1)}] = e_{j+1\, j+2} - e_{j+2\, j+1}$. Therefore the only way to obtain e_{12} is to take the product

$$[y, x_{\sigma(1)}][x_{\sigma(2)}, x_{\sigma(3)}] \cdots [x_{\sigma(2n-6)}, x_{\sigma(2n-5)}][y, x_{\sigma(2n-4)}][y, x_{\sigma(2n-3)}]$$

and $\alpha_\sigma^1 = 0$.

Assume $\alpha_\sigma^i = 0$ for all $\sigma \in \mathcal{S}_{2n-3}$. For a fixed $\sigma \in \mathcal{S}_{2n-3}$, let $x_{\sigma(2i)} = y$ and consider the coefficient of

$$\begin{aligned} &[y, x_{\sigma(1)}][x_{\sigma(2)}, x_{\sigma(3)}] \\ &\qquad \cdots [y, x_{\sigma(2i+1)}][y, x_{\sigma(2i+2)}][x_{\sigma(2i+3)}, x_{\sigma(2i+4)}] \cdots [y, x_{\sigma(2n-3)}]. \end{aligned}$$

It is $\alpha_\sigma^{i+1} + \alpha_{\sigma(2i\, 2i+1)}^i = \alpha_\sigma^{i+1}$ since $\alpha_\sigma^i = 0$. Since H_n has no identity of type $[4, 1^{2n-4}]$, P with $x_{\sigma(2i)}$ replaced by y is the zero polynomial and $\alpha_\sigma^{i+1} = 0$. So all α_σ^j's are 0. The same argument using $x_{\sigma(n-3)} = y$ will yield $\beta_\sigma = 0$ in case n is odd. So H_n has no identity of type $[3, 1^{2n-3}]$.

Proposition 10. *If* $\operatorname{char} F \neq 2, 3$ *then any homogeneous identity of* $H_3(F)$ *of degree* 6 *is a consequence of* T_6 *and* Q.

Proof. By Proposition 9, H_3 has no identity of type $[3, 1^3]$ or lower. By Corollary 2 and Lemma 5, an identity of type $[2^3]$ must be a multiple of $S_3([x, y], [x, z], [y, z])$. This is a central polynomial [7, Remark 2.5.15] but not an identity. So H_3 has no identity of type $[2^3]$.

Let $P(x_1, x_2; y, z)$ be an identity of H_3 linear in x_1, x_2 and of degree 2 in y and z. Since H_3 has no identity of type $[2^3]$, $P(x, x; y, z)$ is the zero polynomial and P is skew in x_1, x_2. By Corollary 2, P is a linear combination of products of commutators. The only commutator which could appear twice in the same products is $[y, z]$ and by Lemma 5 these can not be adjacent. By Corollary 4, $P^* = -P$ so we have a linear combination of terms of the form $\{[\ ,\ ][\ ,\ ][\ ,\ ]\}$. Putting all this together using Lemma 5, we have

$$\begin{aligned} P(x_1, x_2; y, z) = \alpha(&\{[y, z][y, x_1][z, x_2]\} - \{[y, z][y, x_2][z, x_1]\} \\ &- \{[y, x_1][y, z][z, x_2]\} + \{[y, x_2][y, z][z, x_1]\} \\ &+ \{[y, z][z, x_2][y, x_1]\} - \{[y, z][z, x_1][y, x_2]\}) \\ &+ \beta\{[y, z][x_1, x_2][y, z]\}. \end{aligned}$$

Since H_3 has no identity of type $[3, 2, 1]$, $P(x_1, y; y, z)$ is identically zero. This yields $\alpha = \beta$. Thus $P(x_1, x_2; y, z)$ is a multiple of $Q(x_1, x_2; y, z)$.

Let $P(x_1, x_2, x_3, x_4; y) \in I(H_3)$ be of type $[2, 1^4]$. Since

$$P(z, z, x_3, x_4; y) \in I(H_3)$$

is of type $[2^2, 1^2]$ it is a scalar multiple $Q(x_3, x_4; y, z)$. So subtracting a multiple of $Q(x_1, x_2, x_3, x_4; y)$ if necessary, we may assume that P is skew in x_1, x_2. Similarly $P(x_1, z, z, x_4; y)$ is a multiple of $Q(x_1, x_4; y, z)$. Subtracting a multiple of $Q(x_1, x_2, x_3, x_4; y)$ if necessary, we may assume that P is skew in x_2, x_3 without losing the fact that it is skew in x_1, x_2. Finally $P(x_1, x_2, z, z; y)$ is a multiple of $Q(x_1, x_2; y, z)$. The polynomial

$$Q(x_1, x_2, x_3, x_4; y) + Q(x_2, x_3, x_1, x_4; y) + Q(x_3, x_1, x_2, x_4; y)$$

is skew in x_1, x_2, x_3 and if we let $x_3 = z = x_4$ we obtain, using (2.17′),

$$\begin{aligned} &Q(x_1, x_2, z, z; y) - Q(x_1, z, x_2, z; y) - Q(x_1, z, x_2, z; y) \\ &\quad = -2Q(x_1, x_2; y, z). \end{aligned}$$

Since $\operatorname{char} F \neq 2$, correcting P by a multiple of this polynomial, we may assume that P is alternating in x_1, x_2, x_3, x_4. As above P is a linear combination of terms of the form $\{[\ ,\ ][\ ,\ ][\ ,\ ]\}$. If the two y's are in the end commuators, for example $\{[y, x_1][x_2, x_3][y, x_4]\}$, then exchanging the other two elements in these commutors yields the same term but with coefficient of opposite sign since P is alternating in the x's. In the example $\{[y, x_4][x_2, x_3][y, x_1]\} = \{[y, x_1][x_2, x_3][y, x_4]\}$. Hence such terms have coefficient zero in P and we are left with a multiple of

$$\sum_{\sigma \in \mathcal{S}_4} (-1)^\sigma \{[y, x_{\sigma(1)}][y, x_{\sigma(2)}][x_{\sigma(3)}, x_{\sigma(4)}]\} = 2T_6(x_1, x_2, x_3, x_4, y; y).$$

So P is a consequence of Q and T_6.

Finally we must consider multilinear identities of H_3 of degree 6. Let $P(x_1, \dots, x_6) \in I(H_3)$ be multilinear. The aim is to subtract from P linear combinations of Q's and T_6 until P is alternating. If we let two x's equal

y we get a linear combination of these Q's and a G_6 as was noted in the previous section. We may therefore assume that P is skew in x_1, x_2. Then

$$P(x_1, y, y, x_4, x_5, x_6) = \alpha_1 Q(x_1, x_4, x_5, x_6; y) + \alpha_2 Q(x_1, x_5, x_6, x_4; y) + \alpha_3 Q(x_1, x_6, x_4, x_5; y) + \beta T_6(x_1, x_4, x_5, x_6, y; y).$$

By (2.18), subtracting

$$\alpha_1 Q(x_1, x_4, x_2, x_5, x_6, x_3) + \alpha_2 Q(x_1, x_5, x_2, x_6, x_4, x_3) + \alpha_3 Q(x_1, x_6, x_2, x_4, x_5, x_3) + \beta T_6(x_1, x_4, x_5, x_6, x_2; x_3)$$

yields an identity which is x_2, x_3 skew and still x_1, x_2 skew. We may therefore assume that P is x_1, x_2, x_3 skew. So $P(x_1, x_2, y, y, x_5, x_6)$ is x_1, x_2 skew, which forces it to be of the form

$$\alpha_1 Q(x_1, x_2, x_5, x_6; y) + \alpha_2 (Q(x_1, x_5, x_6, x_2; y) - Q(x_1, x_6, x_2, x_5; y)) + \beta T_6(x_1, x_2, x_5, x_6, y; y).$$

Subtracting

$$\begin{aligned}\alpha_1 Q(x_1, x_2, x_3, x_5, x_6, x_4) &+ \frac{\alpha_2}{2}(Q(x_1, x_5, x_3, x_6, x_2, x_4) - Q(x_1, x_6, x_3, x_2, x_5, x_4) \\ &- Q(x_1, x_5, x_2, x_6, x_3, x_4) + Q(x_1, x_6, x_2, x_3, x_5, x_4) \\ &- Q(x_2, x_5, x_3, x_6, x_1, x_4) + Q(x_2, x_6, x_3, x_1, x_5, x_4)) \\ &+ \beta T_6(x_1, x_2, x_5, x_6, x_3; x_4)\end{aligned}$$

yields an identity which is x_1, x_2, x_3, x_4 skew. Assume that P is x_1, x_2, x_3, x_4 skew. We must have

$$P(x_1, x_2, x_3, y, y, x_6) = \alpha(Q(x_1, x_2, x_3, x_6; y) + Q(x_1, x_3, x_6, x_2; y) - Q(x_1, x_6, x_2, x_3; y)) + \beta T_6(x_1, x_2, x_3, x_6, y; y).$$

Subtracting

$$\alpha(Q(x_1, x_2, x_4, x_3, x_6, x_5) + Q(x_1, x_3, x_4, x_6, x_2, x_5) - Q(x_1, x_6, x_5, x_2, x_3, x_4) - Q(x_1, x_2, x_3, x_4, x_6, x_5)) + \beta T_6(x_1, x_2, x_3, x_6, x_4; x_5)$$

yields an identity which is x_1, x_2, x_3, x_4, x_5 skew.

Assume that $P(x_1, x_2, x_3, x_4, x_5, x_6) \in I(H_3)$ is multilinear and skew in $x_1, \ldots, x_5$. Then $P(x_1, x_2, x_3, x_4, y, y)$ is alternating in the x's. Using equation (1.17′) this implies that $P(x_1, x_2, x_3, x_4, y, y)$ is a multiple of $T_6(x_1, x_2, x_3, x_4, y; y)$.

Proposition 11. *If* $\operatorname{char} F \nmid 4!$ *then any homogeneous identity of* $H_4(F)$ *of degree* 8 *is a consequence of* T_8.

Proof. By Proposition 9, H_4 has no identity of type $[3, 1^5]$ or lower. We show next that it has no identity of types $[2^4]$, $[2^3, 1^2]$ and $[2^2, 1^4]$.

If $P(x_1, x_2, x_3, x_4) \in I(H_4)$ is of type $[2^4]$ then, by Osborn's theorem, it is symmetric in all four variables and, by Corollary 4, P is skew under the action of *. P is a linear combination of products of four commutators. We will show that the coefficients of these terms must be zero. We say that two terms are equivalent if they are equivalent up to symmetry. Equivalent terms have the same coefficient. Any term is equivalent to a term which starts with $[x_1, x_2]$. If it also ends in $[x_1, x_2]$ then it must be $[x_1, x_2][x_3, x_4][x_3, x_4][x_1, x_2]$ whose coefficient is zero by Lemma 5. If neither x_1 nor x_2 is present in the last commutator then the term ends in $[x_3, x_4]$. Since $[x_1, x_2][x_1, x_2]$ has coefficient zero by Lemma 5, up to equivalence we have $[x_1, x_2][x_3, x_4][x_1, x_2][x_3, x_4]$ or $[x_1, x_2][x_1, x_3][x_2, x_4][x_3, x_4]$. The first term starred is

$$[x_3, x_4][x_1, x_2][x_3, x_4][x_1, x_2]$$

which is equivalent to $[x_1, x_2][x_3, x_4][x_1, x_2][x_3, x_4]$. So its coefficient is zero. Similarly the coefficient of $[x_1, x_2][x_1, x_3][x_2, x_4][x_3, x_4]$ is zero.

If x_1 or x_2 is present in the last commutator (but not both) the term is equivalent to $[x_1, x_2][\ , x_4][\ , x_4][x_1, x_3]$. By Lemma 5, the coefficient of $[x_1, x_2][x_2, x_4][x_3, x_4][x_1, x_3]$ is minus the coefficient of

$$[x_1, x_2][x_3, x_4][x_2, x_4][x_1, x_3].$$

This term starred is $[x_1, x_3][x_2, x_4][x_3, x_4][x_1, x_2]$ which is equivalent to $[x_1, x_2][x_3, x_4][x_2, x_4][x_1, x_3]$. So its coefficient is zero and P is identically zero. Thus H_4 has no identity of type $[2^4]$.

If $P(x_1, x_2, y_1, y_2, y_3) \in I(H_4)$ is of type $[2^3, 1^2]$, then by Osborn's theorem it is skew in the x's and symmetric in the y's. As usual we write P as a linear combination of commutators and consider terms which could have different coefficients. If both x's are in the same commutator then, by Lemma 2.2, it cannot be at the beginning or at the end since H_3 has no identity of type $[2^3]$. Starring if necessary, we may assume that $[x_1, x_2]$ is in second place. Such a term is equivalent to $[y_1, y_2][x_1, x_2][\ , y_3][\ , y_3]$. By Lemma 5, we need only consider

(1) $$[y_1, y_2][x_1, x_2][y_1, y_3][y_2, y_3].$$

Consider terms with the x's in distinct commutators. We may assume that x_1 comes before x_2. If the x's are in the first and second commutators, since by Lemma 5 $[x_1, y_1][x_2, y_1][y_2, y_3][y_2, y_3]$ has coefficient zero, we need only consider

(2) $$[y_1, x_1][y_2, x_2][y_1, y_3][y_2, y_3],$$

(exchanging y_1 and y_2 in the last two commutators results in a sign change by Lemma 5). Terms with the x's in the last two commutators are obtained by starring (2). For terms of the form $[\ , x_1][\ ,\ ][\ , x_2][\ ,\ ]$ we have

(3) $$[y_1, x_1][y_2, y_3][y_1, x_2][y_2, y_3],$$

if the x's have the same mate, and

$$[y_1, x_1][y_1, y_3][y_2, x_2][y_2, y_3], \tag{4}$$

since the coefficient of $[y_1, x_1][y_2, y_3][y_2, x_2][y_1, y_3]$ is minus the coefficient of $[y_1, x_1][y_2, x_2][y_2, y_3][y_1, y_3]$ by Lemma 5. As usual, starring will give us the coefficients of the terms of the form $[\ ,\ ][\ , x_1][\ ,\ ][\ , x_2]$. Moreover applying Lemma 5 to (4) yields the coefficient of $[y_1, y_3][y_1, x_1][y_2, x_2][y_2, y_3]$. We need however

$$[y_2, y_3][y_1, x_1][y_1, x_2][y_2, y_3] \tag{5}$$

and

$$[y_2, y_3][y_1, x_1][y_2, x_2][y_1, y_3]. \tag{6}$$

Finally for terms of the form $[\ , x_1][\ ,\ ][\ ,\ ][\ , x_2]$, since

$$[y_1, x_1][y_2, y_3][y_2, y_3][y_1, x_2]$$

has coefficient zero, we need only $[y_1, x_1][y_1, y_3][y_2, y_3][y_2, x_2]$ (the middle y_1, y_2 can be interchanged) but this has minus the coefficient of (4). Let $\alpha_1, \dots, \alpha_6$ be the coefficients in P of the above six terms. Since H_4 has no identity of type $[3, 2^2, 1]$, the polynomial $P(x_1, y_1, y_1, y_2, y_3)$ must be identically zero. Considering the coefficient of $[y_1, x_1][y_1, y_2][y_1, y_3][y_2, y_3]$ yields

$$\alpha_4 = \alpha_2. \tag{7}$$

The coefficient of $[y_1, y_3][y_2, x_1][y_1, y_2][y_1, y_3]$ yields

$$\alpha_5 = -2\alpha_2. \tag{8}$$

The coefficient of $[y_2, x_1][y_1, y_3][y_1, y_2][y_1, y_3]$ yields

$$\alpha_2 + \alpha_3 + \alpha_4 = 0. \tag{9}$$

Using (7) we have

$$\alpha_3 = -2\alpha_2. \tag{10}$$

The coefficient of $[y_1, y_2][y_1, x_1][y_1, y_3][y_2, y_3]$ yields

$$\alpha_1 + \alpha_2 + \alpha_4 = 0, \quad \text{or} \quad \alpha_1 = -2\alpha_2. \tag{11}$$

The coefficient of $[y_2, y_3][y_1, x_1][y_1, y_2][y_1, y_3]$ yields

$$\alpha_1 - \alpha_2 + \alpha_6 = 0, \quad \text{or} \quad \alpha_6 = 3\alpha_2. \tag{12}$$

Evaluating P at $y_1 = e[24]$, $y_2 = e[12]$, $y_3 = e[23]$, $x_1 = e[34]$ and $x_2 = e[44]$, and considering the coefficient of e_{34} yields

$$2\alpha_1 + 2\alpha_2 = 0,$$

which with (11) gives $\alpha_2 = 0$ and hence all α's are zero. Therefore H_4 has no identity of type $[2^3, 1^2]$.

If $P(x_1, x_2, x_3, x_4, y_1, y_2)$ is an identity of type $[2^2, 1^4]$, by Osborn's theorem, it is skew in the x's and symmetric in the y's. Write P as

$$R + \sum_{i<j}([x_i, x_j]P_{ij} + P'_{ij}[x_i, x_j]),$$

where R does not have any words starting or ending with two x's. Since $P^* = -P$, $P'_{ij} = P^*_{ij}$. By Lemma 2.2, $P_{ij} \in I(H_3)$ and is of type $[2^2, 1^2]$ so by Corollary 4, $P^*_{ij} = -P_{ij}$. In fact P_{ij} must be a multiple α_{ij} of $Q_{ij} = Q(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_4; y_1, y_2)$. Since P is skew in the x's all α_{ij}'s are the same up to sign. Letting $\alpha_{12} = -\alpha$, we get

$$P = R + \sum_{i<j}(-1)^{i+j}\alpha[[x_i, x_j], Q_{ij}]. \tag{13}$$

R is the alternating sum over all permutations of the x_j's of a linear combination of products of four commutators the first and last containing at least one y. To determine the coefficients, using the symmetry of P in y_1, y_2, it suffices to consider terms starting in $[y_1, \ ]$. If it ends in $[y_1, x]$ then we need only

$$[y_1, y_2][y_2, \ ][\ ,\ ][y_1, \ ], \tag{14}$$

$$[y_1, y_2][\ ,\ ][y_2, \ ][y_1, \ ], \tag{15}$$

$$[y_1, \ ][y_2, \ ][y_2, \ ][y_1, \ ]. \tag{16}$$

If it ends in y_2 then we must consider

$$[y_1, \ ][y_1, \ ][y_2, \ ][y_2, \ ], \tag{17}$$

$$[y_1, \ ][y_2, \ ][y_1, \ ][y_2, \ ], \tag{18}$$

$$[y_1, y_2][\ ,\ ][\ ,\ ][y_1, y_2], \tag{19}$$

since $[y_1, y_2][\ ,\ ][y_1, \ ][y_2, \ ]$ is gotten from permuting y_1 and y_2 in (15), and the coefficient of $[y_1, \ ][y_1, y_2][\ ,\ ][y_2, \ ]$ is minus that of

$$[y_1, y_2][y_1, \ ][\ ,\ ][y_2, \ ],$$

that is, the coefficient of $[y_2, y_1][y_1, \ ][\ ,\ ][y_2, \ ]$ which has the same coefficient as (14) by symmetry. The last four terms must have coefficient zero since $P^* = -P$. For example,

$$[y_1, x_1][y_1, x_2][y_2, x_3][y_2, x_4]^* = [y_2, x_4][y_2, x_3][y_1, x_2][y_1, x_1]$$

which has the same coefficient as $[y_1, x_4][y_1, x_3][y_2, x_2][y_2, x_1]$ by symmetry; this term in turn has the same coefficient as $[y_1, x_1][y_1, x_2][y_2, x_3][y_2, x_4]$ since it is obtained by the product of two transpositions. But it also has minus this coefficient since $P^* = -P$. Since $\operatorname{char} F \neq 2$ this coefficient must be zero.

Let α_1, α_2 be the coefficients of (14) and (15) with the x's in order. Since H_4 has no identity of type $[3, 2, 1^3]$, the polynomial $P(x_1, x_2, x_3, y_1; y_1, y_2)$ is identically zero. Considering the coefficient of $[y_1, x_1][y_1, y_2][y_2, x_2][y_1, x_3]$, we get

$$2\alpha = 0. \tag{20}$$

The coefficient of $[y_1, y_2][y_1, x_1][y_2, x_2][y_1, x_3]$ yields

$$\alpha_2 = -\alpha, \tag{21}$$

and the coefficient of $[y_1, y_2][y_1, x_1][y_1, x_2][y_2, x_3]$,

$$\alpha_1 = \alpha_2. \tag{22}$$

Therefore the coefficients are zero and H_4 has no identity of type $[2^2, 1^4]$.

If $P(x_1, \dots, x_6; y) \in I(H_4)$ is of type $[2, 1^6]$ then either by Osborn's theorem or, if one prefers, since H_4 has no identity of type $[2^2, 1^4]$, P is skew in the x's. So P is the alternating sum over all permutations of the x's of a linear combination of products of four commutators. Only four coefficients are needed, those of

$$[y, \ ][\ , \ ][y, \ ][\ , \ ], \tag{23}$$

$$[y, \ ][y, \ ][\ , \ ][\ , \ ], \tag{24}$$

$$[y, \ ][\ , \ ][\ , \ ][y, \ ], \tag{25}$$

$$[\ , \ ][y, \ ][y, \ ][\ , \ ], \tag{26}$$

starring the first two gives the remaining two possibilities. By Lemma 2.2 and the form of $T_6(x_1, \dots, x_4, y; y)$, terms of the form (23) will not appear. Let $\alpha_1, \alpha_2, \alpha_3$ be the coefficients of (24), (25), and (26) in P, with the x's in order. Since H_4 has no identity of type $[3, 1^5]$, $P(x_1, x_2, x_3, x_4, x_5, y; y)$ is identically zero. The coefficient of $[y, x_1][y, x_2][y, x_3][x_4, x_5]$ yields

$$\alpha_1 + \alpha_3 = 0, \tag{27}$$

and that of $[y, x_1][y, x_2][x_3, x_4][y, x_5]$ yields

$$\alpha_1 - \alpha_2 = 0. \tag{28}$$

Thus P is a multiple of $G_8(x_1, x_2, x_3, x_4, x_5, x_6; y)$.

Finally if $P(x_1, \dots, x_8) \in I(H_4)$ is of type $[1^8]$, arguing as in the case of H_2 using the fact that identities of type $[2, 1^6]$ come from T_8 we get P alternating. This completes the proof of the proposition.

We have given this proof in great detail since it is important to know that H_4 has no identity of type $[2^2, 1^4]$. The following proposition completes the proof of Theorem 1.2.

Proposition 12. *If $|F| \geq 2n$ and $\operatorname{char} F \nmid e(n)!$, then for $n \geq 4$ any identity of $H_n(F)$ of degree $2n$ is a consequence of T_{2n}.*

Proof. We prove the proposition by induction on n. It is true for $n = 4$. By Proposition 9, H_n has no identity of type $[3, 1^{2n-3}]$.

Let $n > 4$ and assume that the proposition is true for $n - 1$. Let

$$P(x_1, \dots, x_{2n-2}; y)$$

be an identity of H_n of type $[2, 1^{2n-2}]$. For a fixed but arbitrary pair i, j, $1 \leq i < j \leq 2n-2$,

$$\begin{aligned} P(x_1, \ldots, x_{2n-2}; y) = \alpha_{ij} T_{2n-2}(\ , \hat{x}_i, \ldots, \hat{x}_j, \ldots, y; y)[x_i, x_j] \\ + \beta_{ij}[x_i, x_j] T_{2n-2}(\ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, y; y) \\ + \text{terms which do not start or end in } [x_i, x_j]. \end{aligned}$$

Starring P shows that $\beta_{ij} = -\alpha_{ij}$ and

$$\begin{aligned} P(x_1, \ldots, x_{2n-2}; y) = \alpha_{ij}[T_{2n-2}(\ , \hat{x}_i, \ldots, \hat{x}_j, \ldots, y; y), [x_i, x_j]] \\ + \text{terms which do not start or end in } [x_i, x_j]. \end{aligned}$$

Doing this for different choices of i, j and comparing the coefficients of common terms shows that the α's are the same up to sign. In this way we get that P is a multiple of $T_{2n}(x_1, \ldots, x_{2n-2}, y; y)$ except possibly for terms starting and ending in $[y, \ ]$. For $\sigma \in \mathscr{S}_{2n-2}$, letting $x_{\sigma(2)} = y$ and using the fact that H_n has no identity of type $[3, 1^{2n-3}]$, the coefficient of

$$[y, x_{\sigma(1)}][y, x_{\sigma(3)}][x_{\sigma(4)}, x_{\sigma(5)}] \cdots [y, x_{\sigma(2n-2)}]$$

will tell us that no terms of the form $[y, x_{\sigma(1)}][x_{\sigma(2)}, x_{\sigma(3)}] \cdots [y, x_{\sigma(2n-2)}]$ have nonzero coefficient in P if n is odd and that they have the right coefficient for P to be a multiple of $T_{2n}(x_1, \ldots, x_{2n-2}, y; y)$ if n is even.

If $P(x_1, \ldots, x_{2n})$ is a multilinear identity of H_n then using the fact that any identity of type $[2, 1^{2n-2}]$ comes from T_{2n} we can modify P by subtracting multiples of T_{2n}. We get P alternating and thus a multiple of S_{2n}. This completes the proof of the proposition and of Theorem 1.2.

Remarks. (1) It should be possible to weaken the assumption on the characteristic of F, by considering the identities which are consequences of T_{2n} and S_{2n} when $\operatorname{char} F | e(n)$.

(2) While $I(H_n)$ is not a T-ideal it is stable under permutation of the variables and substitution of Jordan polynomials.

(3) Amitsur has shown that S_{2n} does not generate $I(M_n(F))$. Using Lemma 2.4, his argument [7, Proposition 2.4.23] will show that T_{2n} does not generate $I(H_n(F))$.

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