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WEIGHT STRINGS IN NONSTANDARD REPRESENTATIONS OF KAC-MOODY ALGEBRAS

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ABSTRACT. We consider the weights which occur in arbitrary irreducible highest weight representations of Kac-Moody algebras and determine conditions under which certain weights may or may not occur.

INTRODUCTION

Much has been written in recent years about the so-called standard or integrable highest weight representations of affine Lie algebras, the attention due in part to the intimate connections between these representations and the physicists string theories. (See [1–3, 6, 8, 10] for a sampling of the literature.) In this work we offer the slightly broader view of Lie algebra representations afforded by considering the class of irreducible highest weight representations of arbitrary Kac-Moody algebras, a class which includes the standard representations of affine Lie algebras and the irreducible finite-dimensional representations of finite-dimensional simple Lie algebras over $\mathbf{C}$.

The most fundamental object of our scrutiny is a Kac-Moody algebra $\mathfrak{g}$. Generally infinite-dimensional, $\mathfrak{g}$ is a complex Lie algebra defined by generators which satisfy relations as in §1. We point out here that underlying the relations satisfied by $\mathfrak{g}$ is an indecomposable $n \times n$ generalized Cartan matrix (GCM) A. (See [6, 7, 12–14].) If there is a column vector u with components greater than zero ($u > 0$) such that $Au > 0$, then $\mathfrak{g}$ is simple and finite dimensional. If there is $u > 0$ such that $Au = 0$ then $\mathfrak{g}$ is neither simple nor finite dimensional but is said to be *affine*. (See [4, 5] for the theory of finite-dimensional Lie algebras over $\mathbf{C}$; for the theory of affine Lie algebras, see [6, 7, 11–14].) If $\mathfrak{g}$ is neither finite dimensional nor affine it is *indefinite*.

Our objective here is to describe the weights and the nature (finite or infinite) of the weight strings which occur in irreducible highest weight representations of $\mathfrak{g}$, an arbitrary Kac-Moody algebra. We give background on $\mathfrak{g}$ and its representations in §1. (All the material there may be found in [6].)

We start §2 with a technical result concerning the roots of $\mathfrak{g}$. We then fix a

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set of weight vectors in $\mathfrak{g}$ and go on to modify some of the terminology and results from §1 to fit the context of nonstandard representations of $\mathfrak{g}$.

In §3, we consider (nonstandard) irreducible highest weight representations of $\mathfrak{g}$ coming finally to our main result, Theorem 3.8, where we describe the weight strings which occur in these modules.

1. Background

Let $\mathfrak{g}$ be a Kac-Moody algebra, e_i, f_i $(i = 1, \dots, n)$ its *Chevalley generators*, $\Pi^\vee = \{\alpha_q^\vee, \dots, \alpha_1^\vee\}$ its *simple coroots*, $\Pi = \{\alpha_1, \dots, \alpha_n\}$ its *simple roots*. Recall that $\Pi^\vee$ is linearly independent in the complex vector space $\mathfrak{h}^*$, the dual of $\mathfrak{h}$, the *Cartan subalgebra* of $\mathfrak{g}$. Letting $\langle \cdot, \cdot \rangle$ denote the pairing of $\mathfrak{h}$ and $\mathfrak{h}^*$, we have that $a_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle$ is a nonpositive integer when $i \neq j$ and that $a_{ii} = 2$ for $i = 1, \dots, n$. Recall that $\mathfrak{g}$ satisfies

$$[e_i\ f_j] = \delta_{ij}\alpha_i^\vee, \qquad [h\ e_i] = \langle \alpha_i, h \rangle e_i,$$

$$[h\ f_i] = -\langle \alpha_i, h \rangle f_i, \qquad [h\ h'] = 0,$$

$\operatorname{ad} e_i^{-a_{ij}+1} e_j = 0 = \operatorname{ad} f_i^{-a_{ij}+1} f_j$ for h, h' in $\mathfrak{h}$ and $i, j = 1, \dots, n$. If we let $\mathfrak{g}_{(\alpha_i)} := \mathbf{C}e_i + \mathbf{C}\alpha_i^\vee + \mathbf{C}f_i$ we see that $\mathfrak{g}_{(\alpha_i)} \cong \mathrm{sl}_2(\mathbf{C})$ for $i = 1, \dots, n$.

Denote by ω the involutive automorphism of $\mathfrak{g}$ determined by $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$, $\omega(h) = -h$ for h in $\mathfrak{h}$, $i = 1, \dots, n$. We call ω the *Cartan involution* of $\mathfrak{g}$.

If $[\cdots[e_{i_1}\ e_{i_2}]\cdots e_{i_m}] = [e_{i_1} \cdots e_{i_m}]$ is nonzero, then using the Cartan involution we see that $[f_{i_1} \cdots f_{i_m}]$ is also nonzero. In this case, $\alpha = \sum_{j=1}^m \alpha_{i_j}$ is called a *positive root* of $\mathfrak{g}$, $-\alpha$, a *negative root*. We denote the set of roots of $\mathfrak{g}$ by Δ, the positive roots by Δ_+, the negative roots by Δ_-. Recall that $\mathfrak{g}$ has the root space decomposition $\mathfrak{g} = \bigoplus_{\alpha \in \Delta \cup \{0\}} \mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [hx] = \langle \alpha, h \rangle x$ for all $h \in \mathfrak{h}\}$ and $\mathfrak{g}_0 = \mathfrak{h}$.

Let $\mathfrak{n}_+ = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$, $\mathfrak{n}_- = \bigoplus_{\alpha \in \Delta_-} \mathfrak{g}_\alpha$. If $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$, the PBW theorem (see [4, 5]) give us that $U(\mathfrak{g}) = U(\mathfrak{n}_-) \otimes U(\mathfrak{g}) \otimes U(\mathfrak{g}_+)$.

If α is any element in the $\mathbf{Z}_+$ span of Π, then we define the *height* of α, denoted ht α, by $\operatorname{ht} \alpha = \sum_i^n k_i$ where $\alpha = \sum_1^n k_i \alpha_i$. The *support* of α, denoted $\operatorname{supp} \alpha$, is the set of α_i in Π such that $k_i \neq 0$ again when $\alpha = \sum_1^n k_i \alpha_i$.

Each α_i in Π determines r_i, the *fundamental reflection* on $\mathfrak{h}^*$, given by $r_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$ for λ in $\mathfrak{h}^*$. The group W which is generated by these reflections is called the *Weyl group*. We use W to classify the roots of $\mathfrak{g}$ as follows.

If α in Δ has the property that there is w in W such that $w(\alpha)$ belongs to Π, then α is a *real root*, otherwise, α is an *imaginary root*. We denote the real and imaginary roots of $\mathfrak{g}$ by Δ^{re} and Δ^{im} respectively. Note that if α belongs to Δ^{re}, then $\dim \mathfrak{g}_\alpha = 1$; if α belongs to Δ^{im}, then $\infty > \dim \mathfrak{g}_\alpha \geq 1$.

Just as r_i is defined on $\mathfrak{h}^*$ for $i = 1, \ldots, n$ we may define the reflection determined by $\alpha_i^\vee$ on $\mathfrak{h}$ via $r_i(h) = h - \langle h, \alpha_i \rangle \alpha_i^\vee$. The group generated by these reflections may be identified with W. If α belongs to Δ^{re} and $w(\alpha_i) = \alpha$ for some w in W, α_i in Π, we define $\alpha^\vee$ in $\mathfrak{h}$ via $w(\alpha_i^\vee) = \alpha^\vee$. We then have r_α in W defined for all α in Δ^{re} via $r_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ for λ in $\mathfrak{h}^*$.

Suppose now that $\pi: \mathfrak{g} \to \mathrm{End}_c(V)$ is a representation of $\mathfrak{g}$ such that $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$, where $V_\lambda = \{v \in V : h \cdot v = \langle \lambda, h \rangle v \text{ for all } h \in \mathfrak{h}\}$. V is then $\mathfrak{h}$-*diagonalizable* and the set of all λ in $\mathfrak{h}^*$ such that $V_\lambda \neq 0$ is the set of *weights* of V. (Note that under the adjoint action, $\mathfrak{g}$ is itself an $\mathfrak{h}$-diagonalizable $\mathfrak{g}$ module.) The *multiplicity* of a weight λ, denoted $\mathrm{mult}\,\lambda$, is the dimension of V_λ.

If x in $\mathfrak{g}$ has the property that for each v in V there is N in $\mathbf{Z}_+$ such that $x^N \cdot v = 0$ then x is said to be *locally nilpotent* on V. If all e_i and f_i $(i = 1, \ldots, n)$ are locally nilpotent on an $\mathfrak{h}$-diagonalizable $\mathfrak{g}$ module then the representation is said to be *integrable* or *standard*. (Note that $U(\mathfrak{g})$ is an integrable $\mathfrak{g}$ module under the adjoint action of $\mathfrak{h}$.)

If V is an integrable $\mathfrak{g}$ module we define automorphisms on V by $r_i^\pi = \exp \pi(e_i) \exp \pi(-f_i) \exp \pi(e_i)$, $i = 1, \ldots, n$. Then for $x_1, \ldots, x_m$ in $\mathfrak{g}$, v in V (see [16, Lemma 14])

$$r_i^\pi(x_1 \cdots x_m \cdot v) = r_i^{\mathrm{ad}}(x_1) \cdots r_i^{\mathrm{ad}}(x_m) \cdot r_i^\pi(v). \tag{1.1}$$

Applying the representation theory of $\mathrm{sl}_2(\mathbf{C}) \cong \mathfrak{g}_{(\alpha_i)}$ to V we get (see [6, Proposition 3.6])

Proposition 1.2 (Kac). *If V is an integrable $\mathfrak{g}$ module then*

(a) *V decomposes into a direct sum of finite-dimensional $\mathfrak{g}_{(\alpha_i)}$ modules;*

(b) *if λ is a weight of V, then*

(i) $r_i^\pi(V_\lambda) = V_{r_i(\lambda)}$;

(ii) *$\lambda + t\alpha_i$ is a weight of V if and only if t is an integer lying between $-p$ and q where $p = q = \infty$ or p and q are fixed nonnegative integers which satisfy $\langle \lambda, \alpha_i^\vee \rangle = p - q$;*

(iii) *$e_i \cdot V_\lambda \neq 0$ if $\lambda + \alpha_i$ is a weight;*

(iv) *$\langle \lambda, \alpha_i^\vee \rangle \geq 0$ (respectively $\langle \lambda, \alpha_i^\vee \rangle \leq 0$) if $\lambda + \alpha_i$ (respectively $\lambda - \alpha_i$) is not a weight.* □

If λ is a weight of V and α is in Δ then the *α-string through λ* is the set of all weights of the form $\lambda + t\alpha$ where t is an integer.

Suppose now that V is a *highest weight module*, i.e., that there is v^+ in V such that $\mathfrak{n}_+ \cdot v^+ = 0$, $h \cdot v^+ = \langle \Lambda, h \rangle v^+$ for some Λ in $\mathfrak{h}^*$ for all h in $\mathfrak{h}$, and that $V = U(\mathfrak{n}_-) \cdot v^+$. We call v^+ a *highest weight vector* and Λ the *highest weight* of V. Note that V is $\mathfrak{h}$-diagonalizable and that for all weights λ of V, $\mathrm{mult}\,\lambda < \infty$. If V is, in addition, freely generated as a $U(\mathfrak{n}_-)$ module, then V has a unique maximal proper submodule V'. Letting $M(\Lambda) = V/V'$, we get the (unique) irreducible $\mathfrak{g}$ module with highest weight Λ. $M = M(\Lambda)$ will be the focus of our attention in the sequel.

2. Preliminaries

We need a technical result before undertaking a study of the representations of $\mathfrak{g}$.

Lemma 2.1. *If α is in Δ_+ and α_k belongs to* $\operatorname{supp}\alpha$, *then α may be written in the form $\alpha = \alpha_k + \sum_{j=1}^{m} \alpha_{i_j}$ where α_{i_j} belongs to Π and for all t in $\{1, \dots, m\}$, $\alpha_k + \sum_{j=1}^{t} \alpha_{i_j}$ belongs to Δ_+.*

Proof. We proceed by induction on $\operatorname{ht}\alpha$. If α is in Π the result is trivial so assume that $\operatorname{ht}\alpha > 1$. Pick α_1 in $\operatorname{supp}\alpha$ so that

$$\beta = \alpha - \alpha_1 \in \Delta_+. \tag{2.1.1}$$

Since $\operatorname{ht}\beta < \operatorname{ht}\alpha$, it follows by induction on $\operatorname{ht}\alpha$ that the result is true for β, hence for α in the event that $\operatorname{supp}\alpha = \operatorname{supp}\beta$. Suppose then that $\operatorname{supp}\alpha \neq \operatorname{supp}\beta$, i.e., that α_1 is not in $\operatorname{supp}\beta$. Since $\alpha = \beta + \alpha_1$ it suffices to show that $\alpha = \alpha_1 + \sum_{j=1}^{m} \alpha_{i_j}$ where $\alpha_1 + \sum_{j=1}^{t} \alpha_{i_j}$ is in Δ_+ for $t = 1, \dots, m$, for some sequence of elements α_{i_j} in Π. Proposition 1.2 gives us that $\langle \beta, \alpha_1^\vee \rangle \leq 0$ since $\beta + \alpha_1$ is in Δ_+. Since $\beta - \alpha_1$ is not in Δ_+, Proposition 1.2 also gives us that $\langle \beta, \alpha_1^\vee \rangle < 0$; in particular, there is α_{i_1} in $\operatorname{supp}\beta$ such that $\langle \alpha_{i_1}, \alpha_1^\vee \rangle < 0$ implying that $\alpha_{i_1} + \alpha_1$ belongs to Δ_+. Again since $\operatorname{ht}\beta < \operatorname{ht}\alpha$ we have by induction that $\beta = \alpha_{i_1} + \sum_{k=2}^{m} \alpha_{i_j}$ where α_{i_j} is in Π for $j = 2, \dots, m$ and $\sum_{j=1}^{t} \alpha_{i_j}$ is in Δ_+ for $t = 2, \dots, m$. Since α_1 is not in $\operatorname{supp}\beta$ we have for $j = 2, \dots, m$ $\langle \alpha_{i_j}, \alpha_1^\vee \rangle \leq 0$ thus, that $\langle \sum_{j=1}^{t} \alpha_{i_j}, \alpha_1^\vee \rangle < 0$ for $t = 2, \dots, m$. We conclude that $\alpha_1 + \sum_{j=1}^{t} \alpha_{i_j}$ belongs to Δ_+ for $t = 2, \dots, m$ as desired. □

For convenience, we shall fix a set of root vectors of $\mathfrak{g}$. It turns out that if α is real, $[\mathfrak{g}_\alpha\ \mathfrak{g}_{-\alpha}] = \mathbf{C}\alpha^\vee$. Given any positive real root α, we may thus pick e_α in $\mathfrak{g}_\alpha$ and f_α in $\mathfrak{g}_{-\alpha}$ such that $[e_\alpha\ f_\alpha] = \alpha^\vee$. Since $\langle \alpha, \alpha^\vee \rangle = 2$ we have that $\mathfrak{g}_{(\alpha)} := \mathbf{C}e_\alpha + \mathbf{C}\alpha^\vee + \mathbf{C}f_\alpha$ is isomorphic to $\mathrm{sl}_2(\mathbf{C})$ for all α in Δ_+^{re}. We shall exploit this extensively below. If α is an arbitrary positive root, we shall use x_α and $x_{-\alpha}$ to designate nonzero elements of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ respectively.

Let $\pi: \mathfrak{g} \to \operatorname{End}_{\mathbf{C}}(V)$ be an $\mathfrak{h}$-diagonalizable representation of $\mathfrak{g}$. If V is not necessarily integrable we say that the representation is *nonstandard* or *nonintegrable.* If α is a positive root with the property that all x_α and $x_{-\alpha}$ are locally nilpotent on V then α is an *integrable root* relative to V. If α is real and integrable we define an automorphism on V $r_\alpha^\pi = \exp \pi(e_\alpha) \exp \pi(-f_\alpha) \exp \pi(e_\alpha)$. As in (1.1) $r_\alpha^\pi(x_1 \cdots x_m \cdot v) = r_\alpha^{\mathrm{ad}}(x_1) \cdots r_\alpha^{\mathrm{ad}}(x_m) \cdot r_\alpha^\pi(v)$ when $x_1, \dots, x_m$ belong to $\mathfrak{g}$, v to V. If a positive root α is not integrable, we say that α is a *nonintegrable root.* Using the representation theory of $\mathrm{sl}_2(\mathbf{C}) \cong \mathfrak{g}_{(\alpha)}$ for α real integrable, we generalize Proposition 1.2 as follows.

Proposition 2.2. *Let V be an $\mathfrak{h}$-diagonalizable $\mathfrak{g}$ module and α a real integrable root relative to V. Then*

(a) *V decomposes into a direct sum of finite-dimensional irreducible $\mathfrak{g}_{(\alpha)}$ modules;*

(b) *if λ is a weight of V, then*

(i) $r_\alpha^\pi(V_\lambda) = V_{r_\alpha(\lambda)}$;

(ii) *$\lambda + t\alpha$ is a weight of V if and only if t is an integer lying between $-p$ and q where $p = q = \infty$ or p and q are fixed nonnegative integers which satisfy $\langle \lambda, \alpha^\vee \rangle = p - q$;*

(iii) *$e_\alpha \cdot V_\lambda \neq 0$ if $\lambda + \alpha$ is a weight;*

(iv) *$\langle \lambda, \alpha^\vee \rangle \geq 0$ (respectively $\langle \lambda, \alpha^\vee \rangle \leq 0$) if $\lambda + \alpha$ (respectively $\lambda - \alpha$) is not a weight.* □

We close this section with a technical lemma.

Lemma 2.3. *Let V be an $\mathfrak{h}$-diagonalizable $\mathfrak{g}$ module and λ a weight of V. If v is an element of V_λ then for all α in Δ_+^{re} and N in $\mathbf{Z}_+$,*

$$e_\alpha \cdot f_\alpha^N \cdot v = N(\langle \lambda, \alpha^\vee \rangle - (N-1)) f_\alpha^N \cdot v + f_\alpha^N \cdot e_\alpha \cdot v. \quad \square$$

Proof. See [7, §7.2].

3. Weights in nonstandard modules

Let M be an irreducible highest weight module over $\mathfrak{g}$ with highest weight Λ in $\mathfrak{h}^*$. Let v^+ be a highest weight vector of M and $M = \bigoplus_{\lambda \leq \Lambda} M_\lambda$ its weight space decomposition. We make no assumptions as to the integrability of M but do assume that the representation is faithful. Our first goal is to determine which positive roots of $\mathfrak{g}$ are integrable.

Let $\Pi(\Lambda) = \{\alpha_i \in \Pi : \langle \Lambda, \alpha_i^\vee \rangle \in \mathbf{Z}_+\}$.

Lemma 3.1. *f_i is locally nilpotent on M if and only if α_i belongs to $\Pi(\Lambda)$.*

Proof. By Lemma 2.3,

$$e_i \cdot f_i^N \cdot v^+ = N(\langle \Lambda, \alpha_i^\vee \rangle - (N-1)) f_i^{n-1} \cdot v^+$$

for all N in $\mathbf{Z}_+$. If α_i is not in $\Pi(\Lambda)$, it follows that $e_i \cdot f_i^N \cdot v^+$ is nonzero for all positive integers N, thus, that $f_i^N \cdot v^+$ is nonzero.

If α_i is in $\Pi(\Lambda)$, let $N = \langle \Lambda, \alpha_i^\vee \rangle + 1$. Then $e_j \cdot f_i^N \cdot v^+ = 0$ for $j = 1, \ldots, n$ implying by [6, §9.3] that $f_i^N \cdot v^+ = 0$. Since v^+ generates M, the result follows by [6, Lemma 3.4]. □

If α belongs to Δ^{re}, let W_α be the subgroup of W generated by the set $\{r_i : \alpha_i \in \operatorname{supp} \alpha\}$.

Lemma 3.2. *If α is in Δ^{re} there is w in W_α and α_j in $\operatorname{supp} \alpha$ such that $\alpha = w(\alpha_j)$.*

Proof. If $\alpha > 0$ and ht $\alpha > 1$ then by [6, Lemma 5.4] there is α_j in Π such that $\langle\alpha, \alpha_j^\vee\rangle > 0$. We then know that $r_j(\alpha) = \alpha - \langle\alpha, \alpha_j^\vee\rangle\alpha_j$ is a positive real root. Since $\mathrm{ht}(r_j(\alpha)) < \mathrm{ht}(\alpha)$ the result follows in this case by induction on $\mathrm{ht}\,\alpha$.

If $\alpha < 0$, $-\alpha$ is in Δ_+^{re} so there is w in $W_{-\alpha} = W_\alpha$ and α_j in $\mathrm{supp}(-\alpha) = \mathrm{supp}\,\alpha$ such that $-\alpha = w(\alpha_j)$. We conclude that $\alpha = w(-\alpha_j) = w(r_j(\alpha_j))$ as desired. □

Let $\Delta_+(\Lambda) = \{\alpha \in \Delta_+ : \mathrm{supp}\,\alpha \subset \Pi(\Lambda)\}$.

Lemma 3.3. *If α belongs to $\Delta^{\mathrm{re}} \cap \Delta_+(\Lambda)$, then f_α is locally nilpotent on M.*

Proof. Let λ be a weight of M and α an element of $\Delta^{\mathrm{re}} \cap \Delta_+(\Lambda)$. For all α_i in $\mathrm{supp}\,\alpha$, $r_i^\pi(M_\lambda) = M_{r_i(\lambda)}$ so that if $w = r_{i_k} \cdots r_{i_1}$ belongs to W_α where α_{i_j} is in $\mathrm{supp}\,\alpha$, $j = 1, \ldots, k$, then $w^\pi(M_\lambda) = M_{w(\lambda)}$.

Invoking Lemma 3.2, we choose w in W_α so that $w(\alpha_i) = \alpha$ for some α_i in $\mathrm{supp}\,\alpha$. Since α_i is integrable, there is N in $\mathbf{Z}_+$ such that $M_{\Lambda - N\alpha_i} = 0$. Let $\lambda = w^{-1}(\Lambda)$. Then $w^\pi(M_{\lambda - N\alpha_i}) = M_{w(\lambda - N\alpha_i)} = M_{\Lambda - N\alpha} = 0$ implying in particular that $f_\alpha^N \cdot M_\Lambda = 0$. By [6, Lemma 3.4], we conclude that f_α is locally nilpotent on M. □

The following lemma [1] will facilitate our study of nonintegrable roots.

Lemma 3.4. *If α is in Δ_+^{re} and $f_\alpha^N \cdot v = 0$ for some nonzero v in M, N in $\mathbf{Z}_+$, then f_α is locally nilpotent on M.*

Proof. Without loss of generality we may assume that $f_\alpha^{N-1} \cdot v \neq 0$. The irreducibility of M then implies that $M = U(\mathfrak{g}) \cdot f_\alpha^{N-1} \cdot v$. Given u in M, there is then g in $U(\mathfrak{g})$ so that $u = g \cdot f_\alpha^{N-1} \cdot v$. Noting that

$$f_\alpha \cdot u = f_\alpha \cdot g \cdot f_\alpha^{N-1} \cdot v = [f_\alpha g] \cdot f_\alpha^{N-1} \cdot v + g \cdot f_\alpha^N \cdot v = \mathrm{ad}\, f_\alpha \cdot g \cdot f_\alpha^{N-1} \cdot v$$

we see that $f_\alpha^r \cdot u = (\mathrm{ad}\, f_\alpha)^r \cdot g \cdot f^{N-1} \cdot v$. As f_α is locally nilpotent on $U(\mathfrak{g})$ under the adjoint action, we may choose r large enough so that $(\mathrm{ad}\, f_\alpha)^r \cdot g = 0$, hence, $f_\alpha^r \cdot u = 0$ proving the lemma. □

Our first result concerning nonintegrable roots is now obvious.

Lemma 3.5. *If α_j belongs to $\Pi \backslash \Pi(\Lambda)$, then $f_j \cdot v$ is nonzero for all nonzero v in M.*

Proof. The result is an immediate consequence of Lemmas 3.1 and 3.4. □

Recall that if $\mathfrak{g}$ is affine it has a minimal positive imaginary root δ with the property that $\langle\delta, \alpha_i^\vee\rangle = 0$ for all $\alpha_i^\vee$ in $\Pi^\vee$. If β belongs to Δ^{im}, then $\beta = r\delta$ for some integer r. The imaginary roots of an affine Lie algebra in an irreducible highest weight representation behave in a singular fashion as the next result shows.

[1] I would like to thank the referee who pointed this out to me.

Lemma 3.6. *Suppose* $\mathfrak{g}$ *is affine,* β *is in* Δ_+^{im}*, and* $x_{-\beta}$ *is a nonzero element of* $\mathfrak{g}_{-\beta}$. *For all weights* λ *of* M, $\lambda - \beta$ *is also a weight; in fact,* $x_{-\beta} \cdot v$ *is nonzero for all nonzero* v *in* M.

Proof. See [9, Proposition 2.11]. □

Remark. Lemma 3.6 is not true for arbitrary Kac-Moody algebras as the following example shows.

Let

$$A = \begin{bmatrix} 2 & -4 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

If $u^t = [7\ 8\ 3]$ it is easily checked that $Au < 0$ implying by [6, Corollary 4.3] that A is a GCM of indefinite type. Since DA is a symmetric matrix where $D = \operatorname{diag}(\frac{1}{2}\ 1\ 1)$, by [6, Theorem 9.11] the Kac-Moody algebra $\mathfrak{g}$ associated to A is given by generators $\{e_i, f_i, i = 1, 2, 3;$ with basis $\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee\}$ and relations as given in §1, where $A = [a_{i_j}]$.

As A is symmetrizable we have the standard invariant form on $\mathfrak{g}$ satisfying $(\alpha_1^\vee, \alpha_1^\vee) = 4$, $(\alpha_1^\vee, \alpha_2^\vee) = -4$, $(\alpha_1^\vee, \alpha_3^\vee) = 0$, $(\alpha_2^\vee, \alpha_3^\vee) = -1$, $(\alpha_3^\vee, \alpha_3^\vee) = 2$, $(\alpha_2^\vee, \alpha_2^\vee) = 2$. The canonical isomorphism $\nu: \mathfrak{h} \to \mathfrak{h}^*$ gives us $(\ ,\ )$ on $\mathfrak{h}^*$ where $(\alpha_1, \alpha_1) = 1$, $(\alpha_1, \alpha_2) = -2$, $(\alpha_1, \alpha_3) = 0$, $(\alpha_2, \alpha_2) = 2$, $(\alpha_2, \alpha_3) = -1$, $(\alpha_3, \alpha_3) = 2$, where $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ is the set of simple roots of $\mathfrak{g}$. We see that $\alpha_1 + \alpha_2$ belongs to Δ_+ and that $(\alpha_1 + \alpha_2, \alpha_1 + \alpha_2) = -1$. By [6, Proposition 5.2], $\alpha_1 + \alpha_2$ is thus imaginary.

Now let M be the irreducible highest weight module over $\mathfrak{g}$ with highest weight Λ in $\mathfrak{h}^*$ given by $\langle \Lambda, \alpha_i^\vee \rangle = \delta_{i3}$. It is easy to check that M is faithful. Notice that since $\langle \Lambda, \alpha_1^\vee \rangle = \langle \Lambda, \alpha_2^\vee \rangle = 0$, $f_1 \cdot v^+ = f_2 \cdot v^+ = 0$ for any highest weight vector v^+ in M. Thus $x_{-(\alpha_1+\alpha_2)} \cdot v^+ = 0$ for any nonzero $x_{-(\alpha_1+\alpha_2)}$ in $\mathfrak{g}_{-(\alpha_1+\alpha_2)}$. □

Our next result, along with Lemma 3.3, gives us a criterion for determining the integrability or nonintegrability of a real root.

Lemma 3.7. *If* α *belonging to* Δ_+ *has* α_k *in its support such that* α_k *belongs to* $\Pi \backslash \Pi(\Lambda)$ *and if* λ *is a weight of* M, *then* $\lambda - \alpha$ *is also a weight. Thus if* α *is real it is nonintegrable.*

Proof. Using Lemma 2.1 we write $\alpha = \alpha_k + \sum_{j=1}^r \alpha_{i_j}$ where α_{i_j} is simple for $j = 1, \dots, r$ and $\alpha_k + \sum_{j=1}^t \alpha_{i_j}$ is in Δ_+ for $t = 1, \dots, r$. Lemma 3.5 gives us the result when $\operatorname{ht} \alpha = 1$ so we assume that $\operatorname{ht} \alpha > 1$ and proceed by induction on $\operatorname{ht} \alpha$. Assume that for any weight λ, $\lambda - \alpha_k - \sum_{j=1}^{r-1} \alpha_{i_j}$ is a weight of M.

If $\lambda - \alpha_{i_r}$ is a weight, the result is immediate so we consider the case when $\lambda - \alpha_{i_r}$ is not a weight of M. Notice then, that by Lemmas 3.5 and 3.1, α_{i_r} is integrable. It follows by Proposition 2.2 that $\lambda + t\alpha_{i_r}$ is a weight for all integers t lying between 0 and N where N is a nonnegative integer satisfying

$\langle \lambda, \alpha_{i_r}^{\vee} \rangle = -N$. Proposition 2.2 also gives us nonnegative integers p and q such that $\alpha_k + \sum_{j=1}^{r-1} \alpha_{i_j} + s\alpha_{i_r}$ is a root for all integers s between $-p$ and q (note that q is not 0), where

$$p - q = \left\langle \alpha_k + \sum_{j=1}^{r-1} \alpha_{i_j}, \alpha_{i_r}^{\vee} \right\rangle .$$

Since $\operatorname{ht}(\alpha_k + \sum_{j=1}^{r-1} \alpha_{i_j} - s\alpha_{i_r}) \leq \operatorname{ht}(\alpha_k + \sum_{j=1}^{r-1} \alpha_{i_j})$ for all s between 0 and p it follows that

$$\lambda + t\alpha_{i_r} - \left(\alpha_k + \sum_{j=1}^{r-1} \alpha_{i_j} - s\alpha_{i_r} \right)$$

is a weight when $s + t$ is between 0 and $N + p$. Since

$$\left\langle \lambda - \left(\alpha_k + \sum_{j=1}^{r-1} \alpha_{i_j} \right), \alpha_{i_r} \right\rangle = -(N + p) + q ,$$

Proposition 2.2 again applies to give us that $\lambda - (\alpha_k + \sum_{j=1}^{r} \alpha_{i_j})$ is a weight, as desired. □

We see that in the event that $\mathfrak{g}$ is affine or finite dimensional and simple, we may readily classify a given positive root as integrable or nonintegrable. In general, however, the behavior of the imaginary roots under a given representation is not clearly understood. The following summarizes our results in terms of root strings.

Theorem 3.8. *Let $\mathfrak{g}$ be a Kac-Moody algebra, M a faithful irreducible highest weight representation of $\mathfrak{g}$, α a positive root of $\mathfrak{g}$, and λ a weight of M.*

(a) *If $\mathfrak{g}$ is not indefinite, the following statements are equivalent:*
  (i) *α is integrable;*
  (ii) *the α-string through λ is finite;*
  (iii) *α belongs to $\Delta_+(\Lambda) \cap \Delta^{\mathrm{re}}$.*

(b) *Suppose $\mathfrak{g}$ is indefinite.*
  (i) *If α is real, it is integrable if and only if it belongs to $\Delta_+(\Lambda)$.*
  (ii) *If α does not belong to $\Delta_+(\Lambda)$, then the α-string through λ is infinite.* □

Bibliography

1. I. B. Frenkel, *Representations of Kac-Moody algebras and dual resonance models*, Lectures in Appl. Math., vol. 21, Amer. Math. Soc., Providence, R.I., 1985, pp. 325–353.
2. ——, *Two constructions of affine Lie algebra representations and boson-fermion correspondence in quantum field theory*, J. Funct. Anal. **44** (1981), 259–327.
3. I. B. Frenkel and V. G. Kac, *Basic representations of affine Lie algebras and dual resonance models*, Invent. Math. **62** (1980), 23–66.

4. J. Humphreys, *Introduction to Lie algebras and representation theory*, 2nd ed., Springer-Verlag, Berlin, Heidelberg and New York, 1972.

5. N. Jacobson, *Lie algebras*, Interscience, New York, 1962.

6. V. G. Kac, *Infinite dimensional Lie algebras*, Birkhäuser, Boston, Mass., 1983.

7. ——, *Simple irreducible graded Lie algebras of finite growth*, Math. USSR-Izv. **2** (1968), 1271–1311.

8. V. G. Kac, D. A. Kazhdan, J. Lepowsky and R. L. Wilson, *Realizations of the basic representations of the Euclidean Lie algebras*, Adv. in Math. **42** (1981), 83–112.

9. V. G. Kac and D. H. Peterson, *Infinite dimensional Lie algebras, theta functions and modular forms*, Adv. in Math. **53** (1984), 125–264.

10. J. Lepowsky, *Some constructions of the affine Lie algebra* $A_1^{(1)}$, Lectures in Appl. Math., vol. 21, Amer. Math. Soc., Providence, R. I., 1985, pp. 375–397.

11. ——, *Lectures on Kac-Moody algebras*, Université Paris VI, spring 1978.

12. D. Mitzman, *Integral bases for affine Lie algebras and their universal enveloping algebras*, Contemp. Math., vol. 40, Amer. Math. Soc., Providence, R.I., 1985.

13. R. V. Moody, *Euclidean Lie algebras*, Canad. J. Math. **21** (1969), 1432–1454.

14. ——, *Lie algebras associated with generalized Cartan matrices*, Bull. Amer. Math. Soc. **73** (1967), 217–221.

15. ——, *A new class of Lie algebras*, J. Algebra **10** (1968), 211–230.

16. R. Steinberg, *Lectures on Chevalley groups*, Yale Univ. lecture notes, 1967.

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