TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 320, Number 1, July 1990

L^p INEQUALITIES FOR ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

QAZI I. RAHMAN AND G. SCHMEISSER

ABSTRACT. Let f be an entire function of exponential type τ belonging to L^p on the real line. It has been known since a long time that

$$\int_{-\infty}^{\infty} |f'(x)|^p\,dx \le \tau^p \int_{-\infty}^{\infty} |f(x)|^p\,dx \quad \text{if } p \ge 1.$$

We prove that the same inequality holds also for $0 < p < 1$. Various other estimates of the same kind have been obtained.

1. INTRODUCTION

For a function $g \in L^p(\mathbf{R})$ we write

$$\|g\|_p := \left(\int_{-\infty}^{\infty} |g(x)|^p\,dx\right)^{1/p}, \qquad p > 0.$$

The following theorem contains an L^p analogue of the well-known inequality of Bernstein for entire functions of exponential type.

Theorem A [3, Theorem 11.3.3]. *Let f be an entire function of exponential type τ belonging to $L^p(\mathbf{R})$. Then*

$$\|f'\|_p \le \tau\|f\|_p \tag{1}$$

for $p \ge 1$.

Various extensions and refinements of (1) exist in the literature. We first mention

Theorem B [1, p. 144, Theorem 3, 2^0]. *Let f be as in Theorem A. Then the inequality*

$$\|(\sin\alpha)f' - \tau(\cos\alpha)f\|_p \le \tau\|f\|_p \tag{2}$$

holds for all real α and $p \ge 1$.

Here is another result of the same kind.

Received by the editors September 1, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 30D15, 26Dxx; Secondary 41A17.
Key words and phrases. Bernstein's inequality, L^p inequalities, entire functions of exponential type.

Theorem C [12, Theorem 3]. *If the function f of Theorem* A *assumes real values for real values of x, then for $p \geq 1$ we have*

$$\|((f')^2 + \tau^2 f^2)^{1/2}\|_p \leq 2\tau C_p \|f\|_p, \tag{3}$$

where

$$C_p := \left(\frac{1}{2\pi}\int_0^{2\pi} |1 + e^{i\alpha}|^p \, d\alpha\right)^{-1/p} < 1. \tag{4}$$

Note that

$$((f'(x))^2 + \tau^2 (f(x))^2)^{1/2} = \max_{\alpha \in \mathbf{R}} \{(\sin\alpha) f'(x) - \tau(\cos\alpha) f(x)\}.$$

Next, we mention a refinement of Theorem A under a side condition whose pertinence is explained in [5]. The function h_f appearing in the statement is the Phragmén-Lindelöf indicator of f defined as usual by

$$h_f(\theta) := \limsup_{r\to\infty} \frac{\log|f(re^{i\theta})|}{r}, \qquad \theta \text{ real}.$$

Theorem D [12, Theorem 2]. *If the function f of Theorem* A *is such that $h_f(-\pi/2) = 0$ and $f(z) \neq 0$ for $\operatorname{Im} z < 0$, then for $p \geq 1$*

$$\|f'\|_p \leq \tau C_p \|f\|_p \tag{5}$$

where C_p is the constant defined in (4).

While the results stated above are concerning L^p estimates for the derivative of f on $\mathbf{R}$ it is also of interest to know L^p estimates of f itself on lines parallel to the real axis. In order to state the most important result of this kind in its full generality we need to introduce the

Notation. Let Ω^+ be the set of all functions ϕ given by $\phi(t) := \psi(\log t)$, where ψ is a nonnegative nondecreasing convex function defined on $\mathbf{R}$.

As examples of functions $\phi \in \Omega^+$ we mention $\log^+ t := \max\{0, \log t\}$, $\log(1 + t^p)$ and t^p for any $p \in (0, \infty)$.

Theorem E [3, Theorem 6.7.4]. *Let $\phi \in \Omega^+$ and let f be an entire function of exponential type τ such that*

$$\int_{-\infty}^{\infty} \phi(|f(x)|)\, dx < \infty. \tag{6}$$

Then

$$\int_{-\infty}^{\infty} \phi(e^{-\tau|y|}|f(x+iy)|)\, dx \leq \int_{-\infty}^{\infty} \phi(|f(x)|)\, dx; \tag{7}$$

in particular, if $f \in L^p(\mathbf{R})$ where $p \in (0, \infty)$, then

$$\|f(\cdot + iy)\|_p \leq e^{\tau|y|}\|f\|_p. \tag{8}$$

Variations of Theorem E in the spirit of Theorems B–D are known for $p \geq 1$ only.

Theorem F [4, see (3.3), (3.5)]. *If f satisfies the conditions of Theorem* A, *then for $p \geq 1$ and $\omega \in \mathbf{R}$ we have*

$$(9) \qquad \|f(\cdot + iy)e^{-i\omega} + f(\cdot - iy)e^{i\omega}\|_p \leq 2(\cosh^2 \tau y - \sin^2 \omega)^{1/2}\|f\|_p .$$

If in addition $f(x) \in \mathbf{R}$ for $x \in \mathbf{R}$, then

$$(10) \qquad \| \operatorname{Re}\{f(\cdot + iy)e^{-i\omega}\}\|_p \leq (\cosh^2 \tau y - \sin^2 \omega)^{1/2}\|f\|_p .$$

Theorem G [6, Theorem 1]. *If the function f of Theorem* A *is such that $|f(x+iy)| \leq |f(x-iy)|$ for $y > 0$, then for $y \geq 0$ and $p \geq 1$ we have*

$$(11) \qquad \|f(\cdot + iy)\|_p \leq S_p(\tau, y)\|f\|_p$$

where

$$(12) \qquad S_p(\tau, y) := \left(\int_0^{2\pi} |e^{-\tau y} + e^{i\alpha+\tau y}|^p \, d\alpha \Big/ \int_0^{2\pi} |1 + e^{i\alpha}|^p \, d\alpha\right)^{1/p} .$$

Finally, as a refinement of Theorem G in the situation of Theorem D we mention [6, see (3.1) wherein the factor $e^{\tau y/2}$ is missing due to an obvious oversight]:

Theorem H. *Under the assumptions of Theorem* D *we have*

$$(13) \qquad \|f(\cdot + iy)\|_p \leq e^{\tau y/2} S_p(\tau/2, y)\|f\|_p$$

for $y \geq 0$ and $p \geq 1$.

It should be mentioned that all the theorems stated so far remain true for $p = \infty$ with

$$\|g\|_\infty := \sup_{x \in \mathbf{R}} |g(x)|$$

if the corresponding constants (4) and (12) are defined by

$$C_\infty := 1/2 \quad \text{and} \quad S_\infty(\tau, y) := \cosh \tau y .$$

In fact, it was the case $p = \infty$ which was considered first ([1, p. 140, see Theorem 1; 7, p. 239, see the Corollary; 5, see Theorems 1, 2 and 4; 3, Theorem 6.2.4; 4, see (3.1), (3.4)]).

In view of Theorem E the question arises if the other theorems also hold for $p \in (0, 1)$ or more generally with a function $\phi \in \Omega^+$ instead of $|\cdot|^p$. The known proofs are of no help in finding the answer since they all make essential use of the fact that the function $\phi \colon x \mapsto |x|^p$ is convex if $p \geq 1$. The purpose of this paper is to present an alternative approach to Theorems A–H which leads to an affirmative answer to the question just raised.

2. Statement of results

We shall prove two theorems from which the desired extensions of Theorems A–D and Theorems F–H (respectively) can be deduced as corollaries.

Theorem 1. *Let ϕ and f be as in Theorem* E. *Then for all complex numbers A and B not both zero such that* $\operatorname{Im}(A/B) \geq 0$ *in case $B \neq 0$, we have*

$$\int_{-\infty}^{\infty} \phi\left(\left|\frac{Af(x)+Bf'(x)}{A+i\tau B}\right|\right) dx \leq \int_{-\infty}^{\infty} \phi(|f(x)|)\, dx. \tag{14}$$

For $A = 0$, $B \neq 0$ we immediately obtain

Corollary 1. *Under the assumptions of Theorem* E

$$\int_{-\infty}^{\infty} \phi\left(\left|\frac{f'(x)}{\tau}\right|\right) dx \leq \int_{-\infty}^{\infty} \phi(|f(x)|)\, dx; \tag{15}$$

in particular, (1) *holds for all $p > 0$.*

Setting $A := \tau\cos\alpha$, $B := -\sin\alpha$ in (14) we obtain

Corollary 2. *Let ϕ and f be as in Theorem* E. *Then for all $\alpha \in \mathbf{R}$ we have*

$$\int_{-\infty}^{\infty} \phi\left(\left|\frac{\sin\alpha}{\tau} f'(x) - (\cos\alpha) f(x)\right|\right) dx \leq \int_{-\infty}^{\infty} \phi(|f(x)|)\, dx;$$

in particular, (2) *holds for all $p > 0$.*

The next two corollaries which extend Theorems D and C respectively are not so obvious consequences of Theorem 1. We shall therefore prove them in §4.

Corollary 3. *Let $\phi \in \Omega^+$. If, in addition to satisfying the conditions of Theorem* E, *f is such that $h_f(-\pi/2) = 0$ and $f(z) \neq 0$ for* $\operatorname{Im} z < 0$, *then*

$$\int_{-\infty}^{\infty} \frac{1}{2\pi}\int_{0}^{2\pi} \phi\left(\frac{|f'(x)|}{\tau}\cdot|1+e^{i\alpha}|\right) d\alpha\, dx \leq \int_{-\infty}^{\infty} \phi(|f(x)|)\, dx; \tag{16}$$

in particular, (5) *holds for all $p > 0$.*

Remark 1. For $\phi \in \Omega^+$ the function

$$g\colon w \mapsto \phi\left(\frac{|f'(x)|}{\tau}\cdot|1+w|\right) \quad (x \text{ fixed})$$

is subharmonic [10, p. 46, see Theorem 2.2] and so

$$g(0) \leq \frac{1}{2\pi}\int_{0}^{2\pi} g(e^{i\alpha})\, d\alpha$$

which shows that (16) improves upon (15).

Corollary 4. *Let $\phi \in \Omega^+$. If f satisfies the conditions of Theorem* E *and in addition $f(x) \in \mathbf{R}$ for $x \in \mathbf{R}$, then*

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{2\pi}\int_{0}^{2\pi} \phi\left(\frac{((f'(x))^2+\tau^2(f(x))^2)^{1/2}}{2\tau}|1+e^{i\alpha}|\right) d\alpha\, dx \\ \leq \int_{-\infty}^{\infty} \phi(|f(x)|)\, dx;\end{aligned} \tag{17}$$

in particular, (3) *holds for all $p > 0$.*

The next theorem generalizes Theorem E and is the key to extensions of Theorems F–H.

Theorem 2. *Let* ϕ *and* f *be as in Theorem* E. *Then for all* $\lambda \in \mathbf{C}$ *with* $|\lambda| \leq 1$ *and all* $y \geq 0$, *we have*

$$\int_{-\infty}^{\infty} \phi\left(\left|\frac{f(x-iy)+\lambda f(x+iy)}{e^{\tau y}+\lambda e^{-\tau y}}\right|\right) dx \leq \int_{-\infty}^{\infty} \phi(|f(x)|)\,dx. \tag{18}$$

In order to deduce Theorem E from Theorem 2 we may apply the special case $\lambda = 0$ to f and to the function $z \mapsto f(-z)$.

Setting $\lambda = e^{-2i\omega}$ where $\omega \in \mathbf{R}$ we obtain

$$\left|\frac{f(x-iy)+\lambda f(x+iy)}{e^{\tau y}+\lambda e^{-\tau y}}\right| = \frac{|e^{i\omega} f(x-iy) + e^{-i\omega} f(x+iy)|}{2(\cosh^2 \tau y - \sin^2 \omega)^{1/2}}$$

and so as another consequence of Theorem 2 we may mention

Corollary 5. *Let* ϕ *and* f *be as in Theorem* E. *Then for all real* ω *we have*

$$\int_{-\infty}^{\infty} \phi\left(\frac{|e^{i\omega} f(x-iy) + e^{-i\omega} f(x+iy)|}{2(\cosh^2 \tau y - \sin^2 \omega)^{1/2}}\right) dx \leq \int_{-\infty}^{\infty} \phi(|f(x)|)\,dx. \tag{19}$$

If in addition $f(x) \in \mathbf{R}$ *for* $x \in \mathbf{R}$, *then*

$$\int_{-\infty}^{\infty} \phi\left(\frac{|\operatorname{Re}\{f(x+iy)e^{-i\omega}\}|}{2(\cosh^2 \tau y - \sin^2 \omega)^{1/2}}\right) dx \leq \int_{-\infty}^{\infty} \phi(|f(x)|)\,dx; \tag{20}$$

in particular, Theorem F *holds for all* $p > 0$.

The proofs of Corollaries 6 and 7 given in §4 will show that Theorems G and H can also be extended to arbitrary $\phi \in \Omega^+$ but we state the results only in the case $\phi(x) = |x|^p$ with $p > 0$ since the general form looks somewhat artificial to us (see (35) below).

Corollary 6. *Theorem* G *holds for all* $p > 0$.

Corollary 7. *Theorem* H *holds for all* $p > 0$.

Remark 2. By a result of Hardy [9], if $p > 0$ and $y \geq 0$, then

$$\begin{aligned}\int_0^{2\pi} |e^{-\tau y} + e^{\tau y} e^{i\alpha}|^p \,d\alpha &= e^{\tau y p} \int_0^{2\pi} |1 + e^{-2\tau y} e^{-i\alpha}|^p \,d\alpha \\ &\leq e^{\tau y p} \int_0^{2\pi} |1 + e^{i\alpha}|^p \,d\alpha.\end{aligned}$$

Hence the constant $S_p(\tau, y)$ defined in (12) satisfies $S_p(\tau, y) \leq e^{\tau y}$ for all $p > 0$ and $y \geq 0$. This implies that Corollaries 6 and 7 improve upon (8) for $y \geq 0$.

3. Lemmas

Our approach is based on two powerful tools; namely, a result of Arestov and an approximation method of Hörmander. We first describe the result of Arestov.

Denote by $\mathscr{P}_n$ the set of all polynomials

$$P(z) := \sum_{\nu=0}^{n} a_\nu z^\nu \tag{21}$$

of degree at most n with complex coefficients. For $\gamma := (\gamma_0, \dots, \gamma_n) \in \mathbf{C}^{n+1}$ we define

$$\Lambda_\gamma p(z) := \sum_{\nu=0}^{n} \gamma_\nu a_\nu z^\nu .$$

The linear operator $\Lambda_\gamma : \mathscr{P}_n \to \mathscr{P}_n$ is said to be *admissible* if it preserves one of the following properties:

(i) $P(z)$ has all its zeros in $\{z \in \mathbf{C}: |z| \le 1\}$,
(ii) $P(z)$ has all its zeros in $\{z \in \mathbf{C}: |z| \ge 1\}$.

Next we denote by Ω the set of all functions ϕ given by $\phi(t) = \psi(\log t)$ where ψ is a nondecreasing convex function defined on $\mathbf{R}$. The set Ω^+ introduced in §1 is obviously a subset of Ω. The result of Arestov used in our approach may now be stated as follows.

Lemma 1 [1, Theorem 4]. *For $p \in \mathscr{P}_n$, $\phi \in \Omega$ and every admissible operator Λ_γ we have*

$$\int_{-\pi}^{\pi} \phi(|\Lambda_\gamma P(e^{i\theta})|)\, d\theta \le \int_{-\pi}^{\pi} \phi(c(\gamma, n)|P(e^{i\theta})|)\, d\theta \tag{22}$$

where

$$c(\gamma, n) := \max\{|\gamma_0|, |\gamma_n|\} .$$

We turn now to the method of Hörmander. Let

$$\varphi(x) := \left(\frac{\sin \pi x}{\pi x}\right)^2 .$$

Given $f: \mathbf{R} \to \mathbf{C}$ such that

$$\|f\|_\infty := \sup_{x\in\mathbf{R}} |f(x)| < \infty \tag{23}$$

and $h > 0$, we define

$$f_h(x) := \sum_{\nu=-\infty}^{\infty} \varphi(hx + \nu) f(x + \nu/h) . \tag{24}$$

The function f_h always exists since (23) implies the uniform convergence of the series in (24). The properties of f_h which we shall need are summarized in the following lemma.

Lemma 2. *Let f be an entire function of exponential type τ such $\|f\|_\infty \le M$. Then the following statements hold:*

(i) *f_h may be represented as $f_h(z) = \sum_{\nu=-N}^{N} a_\nu e^{2\pi i\nu hz}$, $z \in \mathbf{C}$, with $N = [\tau/2\pi h] + 1$ and complex coefficients a_ν $(\nu = -N, -N+1, \ldots, N)$.*

(ii) $\|f_h\|_\infty \le M$.

(iii) *$\lim_{h\to 0+} f_h(z) = f(z)$ uniformly on all compact subsets of $\mathbf{C}$.*

(iv) *For $\phi \in \Omega^+$*

$$\int_{-1/2h}^{1/2h} \phi(|f_h(x)|)\,dx \le \int_{-\infty}^{\infty} \phi(|f(x)|)\,dx. \tag{25}$$

Proof. Statements (i)–(iii) are contained in [11, pp. 22–24]. The additional assumption "$f(x) \in \mathbf{R}$ for $x \in \mathbf{R}$" made in [11] is needed for other purposes (also see [8]). As regards (iv), a proof is needed only if $\int_{-\infty}^{\infty} \phi(|f(x)|)dx < \infty$. But then we have

$$\begin{aligned}\int_{-\infty}^{\infty} \phi(|f(x)|)\,dx &= \sum_{\nu=-\infty}^{\infty} \int_{-1/2h}^{1/2h} \phi(|f(x+\nu/h)|)\,dx \\ &\ge \int_{-1/2h}^{1/2h} \phi\left(\sup_{-\infty\le\nu\le\infty} |f(x+\nu/h)|\right) dx \\ &\ge \int_{-1/2h}^{1/2h} \phi\left(\sum_{\nu=-\infty}^{\infty} \varphi(hx+\nu)|f(x+\nu/h)|\right) dx \\ &\ge \int_{-1/2h}^{1/2h} \phi(|f_h(x)|)\,dx.\end{aligned}$$

The above inequalities are all trivial. It is sufficient to keep in mind that ϕ is a continuous nonnegative function (defined on $(0, \infty)$) and that

$$\sum_{\nu=-\infty}^{\infty} |\varphi(hx+\nu)| = \sum_{\nu=-\infty}^{\infty} \varphi(hx+\nu) = 1.$$

Lemma 3. *Let $\phi \in \Omega^+$ be not identically zero. If f is an entire function of exponential type such that* (6) *holds, then f is bounded on $\mathbf{R}$.*

Proof. The lemma holds for f if and only if it holds for $g(z) := f(cz)$ where c is any real number $\ne 0$. As such, we may assume f to be of exponential type less than π. Since

$$\begin{aligned}\int_{-\infty}^{\infty} \phi(|f(x)|)\,dx &= \sum_{n=-\infty}^{\infty} \int_{n/2}^{(n+1)/2} \phi(|f(x)|)\,dx \\ &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \phi(|f(x_n)|)\end{aligned}$$

with appropriate points $x_n \in [n/2, (n+1)/2]$ we see that the sequence $\{\phi(|f(x_n)|)\}_{n\in\mathbf{Z}}$ is bounded. Since $\phi(t) = \psi(\log t)$, where ψ is a nondecreasing convex function not identically zero, we must have $\phi(x) \to \infty$ as $x \to \infty$

and so we conclude that the sequence $\{f(x_n)\}_{n\in\mathbf{Z}}$ itself is bounded. In particular, f is bounded on the sequence of points $\lambda_n := x_{2n}$ $(n \in \mathbf{Z})$ which satisfy $\lambda_{n+1} - \lambda_n \geq \frac{1}{2}$ and $|\lambda_n - n| \leq \frac{1}{2}$. Hence by a theorem of Duffin and Schaeffer [3, Theorem 10.5.1], f is bounded on the whole real line.

We shall also need two lemmas on the location of zeros of polynomials.

Lemma 4. *Let a and b be complex numbers not both zero such that* $\operatorname{Im}(a/b) \geq 0$ *if $b \neq 0$. If $P \in \mathscr{P}_n$ has all its zeros in $\{z \in \mathbf{C}: |z| \leq 1\}$, then so does*

$$Q(z) := \left(a - \frac{i}{2}nb\right) P(z) + ibzP'(z).$$

Proof. Assuming that $P(z)$ is given by (21) we may write

$$Q(z) = \sum_{\nu=0}^{n} \gamma_\nu a_\nu z^\nu =: \Lambda_\gamma P(z) \tag{26}$$

where

$$\gamma = a - ib(n/2 - \nu). \tag{27}$$

The polynomial

$$\sum_{\nu=0}^{n} \binom{n}{\nu} \gamma_\nu z^\nu = \left(a - \frac{i}{2}nb\right)(1+z)^n + ibnz(1+z)^{n-1}$$

has a zero of multiplicity $n-1$ at -1 and a simple zero at

$$\zeta = -\left(a - \frac{i}{2}nb\right) \Big/ \left(a + \frac{i}{2}nb\right).$$

It is easily seen that $|\zeta| \leq 1$ under our assumptions on a and b, i.e., $\sum_{\nu=0}^{n} \binom{n}{\nu}\gamma_\nu z^\nu$ has all its zeros in $\{z \in \mathbf{C}: |z| \leq 1\}$. Hence the desired result may be obtained by applying Szegö's convolution theorem [14].

Lemma 5. *Let $\rho > 1$ and let λ be a complex number such that $|\lambda| \leq 1$. If $P \in \mathscr{P}_n$ has all its zeros in $\{z \in \mathbf{C}: |z| \leq 1\}$, then so does*

$$R(z) := P(\rho z) + \lambda\rho^n P(z/\rho).$$

Proof. Assuming that $P(z)$ is given by (21) we may write

$$R(z) = \sum_{\nu=0}^{n} \delta_\nu a_\nu z^\nu =: \Lambda_\delta P(z) \tag{28}$$

where

$$\delta_\nu = \rho^\nu + \lambda\rho^{n-\nu}. \tag{29}$$

The polynomial

$$\sum_{\nu=0}^{n} \binom{n}{\nu} \delta_\nu z^\nu = (1+\rho z)^n + \rho^n\lambda(1 + z/\rho)^n$$

vanishes at the points

$$\zeta_\nu = \frac{\omega_\nu \rho - 1}{\rho - \omega_\nu} \qquad (\nu = 1, \dots, n)$$

where ω_ν $(\nu = 1, \dots, n)$ are the nth roots of $-\lambda$. Since $|\omega_\nu| = |\lambda|^{1/n} \le 1$ for $1 \le \nu \le n$ and

$$z \mapsto \frac{z\rho - 1}{\rho - z} \qquad (\rho > 1)$$

maps the unit disk onto itself we see that $|\zeta_\nu| \le 1$ for $\nu = 1, \dots, n$. Hence we may again apply Szegö's convolution theorem to obtain the desired conclusion.

4. Proofs of statements in §2

Proof of Theorem 1. Lemma 3 implies that f is bounded on $\mathbf{R}$ and so the corresponding function f_h defined in (24) exists. Now set $N = [\tau/2\pi h] + 1$. Then according to statement (i) of Lemma 2

$$P(e^{iz}) := e^{iNz} f_h(z/2\pi h) \tag{30}$$

defines a polynomial $P \in \mathscr{P}_{2N}$. If a and $b \ne 0$ are complex numbers such that $\operatorname{Im}(a/b) \ge 0$, then by Lemma 4 the operator Λ_γ given by (26) and (27) with $n = 2N$ is admissible. Further,

$$\max\{|\gamma_0|, |\gamma_n|\} = |a + ibN|.$$

Applying Lemma 1 to $P/|a + ibN|$ where P is the polynomial defined in (30) we obtain

$$\int_{-\pi}^{\pi} \phi\left(\left|\frac{af_h(\theta/2\pi h) + (b/2\pi h)f_h'(\theta/2\pi h)}{a + ibN}\right|\right) d\theta \le \int_{-\pi}^{\pi} \phi\left(\left|f_h\left(\frac{\theta}{2\pi h}\right)\right|\right) d\theta$$

or equivalently

$$\int_{-1/2h}^{1/2h} \phi\left(\left|\frac{af_h(x) + (b/2\pi h)f_h'(x)}{a + ibN}\right|\right) dx \le \int_{-1/2h}^{1/2h} \phi(|f_h(x)|)\, dx.$$

Using statement (iv) of Lemma 2 we can replace the right-hand side of the preceding inequality by $\int_{-\infty}^{\infty} \phi(|f(x)|)\, dx$. Setting $a = A$ and $b = 2\pi hB$ we thus obtain

$$\int_{x_0}^{x_1} \phi\left(\left|\frac{Af_h(x) + Bf_h'(x)}{A + i2\pi hNB}\right|\right) dx \le \int_{-\infty}^{\infty} \phi(|f(x)|)\, dx$$

where x_0, x_1 are any two real numbers such that $-1/2h \le x_0 < x_1 \le 1/2h$. Now we let $h \to 0+$. Since f_h is holomorphic the uniform convergence in statement (iii) of Lemma 2 extends to the derivative and so

$$\int_{x_0}^{x_1} \phi\left(\left|\frac{Af(x) + Bf'(x)}{A + i\tau B}\right|\right) dx \le \int_{-\infty}^{\infty} \phi(|f(x)|)\, dx.$$

Finally, letting $x_0 \to -\infty$ and $x_1 \to \infty$ we arrive at (14).

Proof of Corollary 3. Let α be an arbitrary real number and apply Theorem 1 to $g(z) := e^{i\tau z/2} f(z)$ with

$$A = \frac{\tau}{2}(1 - e^{-i\alpha}), \qquad B = \frac{1 + e^{-i\alpha}}{i},$$

which is permissible since $A/B \in \mathbf{R}$ if $B \neq 0$. Noting that g is an entire function of exponential type $\tau/2$ we obtain

$$\int_{-\infty}^{\infty} \phi\left(\left|f(x) + \frac{1 + e^{-i\alpha}}{i\tau} f'(x)\right|\right) dx \\ \leq \int_{-\infty}^{\infty} \phi(|g(x)|)\, dx = \int_{-\infty}^{\infty} \phi(|f(x)|)\, dx.$$

In particular, for any pair of real numbers x_0, x_1 we have

$$\int_{x_0}^{x_1} \phi\left(\left|\frac{f'(x)}{\tau}\right| \cdot \left|1 + e^{i\alpha} \frac{f'(x) + i\tau f(x)}{f'(x)}\right|\right) dx \leq \int_{-\infty}^{\infty} \phi(|f(x)|)\, dx.$$

Integrating both sides with respect to α from 0 to 2π we get

$$\int_{x_0}^{x_1} \frac{1}{2\pi} \int_0^{2\pi} \phi\left(\left|\frac{f'(x)}{\tau}\right| \cdot \left|1 + e^{i\alpha}\sigma(x)\right|\right) d\alpha\, dx \leq \int_{-\infty}^{\infty} \phi(|f(x)|)\, dx$$

where

$$\sigma(x) := \left|\frac{f'(x) + i\tau f(x)}{f'(x)}\right|.$$

By an extension of Laguerre's theorem to entire functions of exponential type [13, Theorem 1] applied to $f(-z)$, it follows that $\sigma(x) \geq 1$ for $x \in \mathbf{R}$. Hence for every fixed $x \in \mathbf{R}$ the operator

$$\Lambda\colon P(w) \mapsto P\left(\frac{w}{\sigma(x)}\right)$$

is admissible. Applying Lemma 1 to the first degree polynomial

$$P(w) := \frac{f'(x)}{\tau}(1 + w\sigma(x))$$

we find that

$$\int_{x_0}^{x_1} \frac{1}{2\pi} \int_0^{2\pi} \phi\left(\left|\frac{f'(x)}{\tau}\right| \cdot |1 + e^{i\alpha}|\right) d\alpha\, dx \leq \int_{-\infty}^{\infty} \phi(|f(x)|)\, dx.$$

Now the proof is completed by letting $x_0 \to -\infty$ and $x_1 \to \infty$.

Proof of Corollary 4. For

$$A := i\tau(1 - e^{i\alpha}), \qquad B := 1 + e^{i\alpha}$$

we obtain from Theorem 1 that for every pair of real numbers x_0, x_1

$$\int_{x_0}^{x_1} \phi\left(\frac{|f'(x) + i\tau f(x) + e^{i\alpha}(f'(x) - i\tau f(x))|}{2\tau}\right) dx \leq \int_{-\infty}^{\infty} \phi(|f(x)|)\, dx.$$

Hence if $f(x) \in \mathbf{R}$ for $x \in \mathbf{R}$ then

$$\int_{x_0}^{x_1} \phi\left(\frac{\sqrt{(f'(x))^2 + \tau^2(f(x))^2}}{2\tau}|1 + e^{i\alpha}\eta(x)|\right) dx \le \int_{-\infty}^{\infty} \phi(|f(x)|)\, dx \tag{31}$$

where

$$\eta(x) := \frac{f'(x) - i\tau f(x)}{f'(x) + i\tau f(x)}.$$

Now the proof is readily completed by integrating both sides of (31) with respect to α from 0 to 2π and taking into account the fact that $|\eta(x)| = 1$ for $x \in \mathbf{R}$.

Proof of Theorem 2. Let f_h, N, and P be as in the proof of Theorem 1. If $\rho > 1$, then by Lemma 5 the operator Λ_δ given by (28) and (29) is admissible and for $n = 2N$

$$\max\{|\delta_0|, |\delta_n|\} = |\rho^{2N} + \lambda|. \tag{32}$$

We choose this Λ_δ as the (admissible) operator in Lemma 1 and apply that lemma to $P/|\rho^{2N} + \lambda|$. Recalling that P is defined by (30) we thus obtain

$$\int_{-\pi}^{\pi} \phi\left(\left|\frac{\rho^N f_h((\theta - i\log\rho)/2\pi h) + \lambda\rho^N f_h((\theta + i\log\rho)/2\pi h)}{\rho^{2N} + \lambda}\right|\right) d\theta$$
$$\le \int_{-\pi}^{\pi} \phi\left(\left|f_h\left(\frac{\theta}{2\pi h}\right)\right|\right) d\theta.$$

Now we set $\theta/2\pi h = x$ and $\rho = e^{2\pi h y}$ with $y > 0$. Using statement (iv) of Lemma 2 we obtain

$$\int_{-1/2h}^{1/2h} \phi\left(\left|\frac{f_h(x - iy) + \lambda f_h(x + iy)}{e^{2\pi hNy} + \lambda e^{-2\pi hNy}}\right|\right) dx \le \int_{-\infty}^{\infty} \phi(|f(x)|)\, dx.$$

The proof may now be completed by letting $h \to 0+$ and arguing as in the proof of Theorem 1.

Proof of Corollary 6. For $\lambda = e^{-i\alpha}$ ($\alpha \in \mathbf{R}$) and fixed $y > 0$ which is not necessarily the imaginary part of z we apply Theorem 2 to $f(z)(e^{\tau y} + e^{-(\tau y + i\alpha)})$ and obtain for any pair of real numbers x_0, x_1

$$\begin{aligned} &\int_{x_0}^{x_1} \phi\left(|f(x + iy)| \cdot \left|1 + e^{i\alpha}\frac{f(x - iy)}{f(x + iy)}\right|\right) dx \\ &\quad \le \int_{-\infty}^{\infty} \phi(|f(x)| \cdot |e^{\tau y} + e^{-(\tau y + i\alpha)}|)\, dx \end{aligned} \tag{33}$$

provided the integral on the right exists. Integrating both sides with respect to α from 0 to 2π we get

$$\begin{aligned} &\int_{x_0}^{x_1} \int_0^{2\pi} \phi\left(|f(x + iy)| \cdot \left|1 + e^{i\alpha}\frac{f(x - iy)}{f(x + iy)}\right|\right) d\alpha\, dx \\ &\quad \le \int_0^{2\pi} \int_{-\infty}^{\infty} \phi(|f(x)| \cdot |e^{\tau y} + e^{-(\tau y + i\alpha)}|)\, dx\, d\alpha. \end{aligned} \tag{34}$$

Since by assumption

$$\left|\frac{f(x-iy)}{f(x+iy)}\right| \geq 1$$

we may apply, to the left-hand side of (34), the reasoning used in the proof of Corollary 3 and obtain (on letting $x_0 \to -\infty$, $x_1 \to \infty$)

$$\begin{aligned}(35)\qquad &\int_{-\infty}^{\infty}\int_0^{2\pi} \phi(|f(x+iy)|\cdot|1+e^{i\alpha}|)\,d\alpha\,dx \\ &\quad\leq \int_0^{2\pi}\int_{-\infty}^{\infty} \phi(|f(x)|\cdot|e^{\tau y}+e^{-(\tau y+i\alpha)}|)\,dx\,d\alpha.\end{aligned}$$

In the case $\phi(x) = |x|^p$ with $p \in (0, \infty)$ the existence of the integral on the right-hand side of (33) is guaranteed by " $f \in L^p(\mathbf{R})$ " and the double integrals in (35) decompose into products of single integrals. This shows that Corollary 6 holds.

Proof of Corollary 7. The function $g(z) := e^{i\tau z/2} f(z)$ is entire and of exponential type $\tau/2$ such that $g(z) \neq 0$ for $\operatorname{Im} z < 0$ and $h_g(\pi/2) \leq \tau/2$, $h_g(-\pi/2) = \tau/2$. In this situation it is known [3, Theorem 7.8.1, Definition 7.8.2] that $|g(x-iy)| \geq |g(x+iy)|$ for $y \geq 0$ and so Corollary 6 applies. Thus we obtain

$$\|e^{-\tau y/2} f(\cdot + iy)\|_p \leq S_p(\tau/2, y)\|f\|_p$$

for $y \geq 0$ and $p > 0$ which is equivalent to the result we were looking for.

References

1. N. I. Achieser, *Theorem of approximation*, Ungar, New York, 1956.
2. V. V. Arestov, *On integral inequalities for trigonometric polynomials and their derivatives*, Math. USSR-Izv. **18** (1982), 1–17.
3. R. P. Boas, Jr., *Entire functions*, Academic Press, New York, 1954.
4. ——, *Inequalities for functions of exponential type*, Math. Scand. 4 (1956), 29–32.
5. ——, *Inequalities for asymmetric entire functions*, Illinois J. Math. **1** (1957), 94–97.
6. R. P. Boas, Jr. and Q. I. Rahman, *L^p inequalities for polynomials and entire functions*, Arch. Rational Mech. Anal. **11** (1962), 34–39.
7. R. J. Duffin and A. C. Schaeffer, *Some properties of functions of exponential type*, Bull. Amer. Math. Soc. **44** (1938), 236–240.
8. T. G. Genčev, *Inequalities for asymmetric entire functions of exponential type*, Soviet Math. Dokl. **19** (1978), 981–985.
9. G. H. Hardy, *The mean value of the modulus of an analytic function*, Proc. London Math. Soc. (2) **14** (1915), 269–277.
10. W. K. Hayman and P. B. Kennedy, *Subharmonic functions*, Vol. 1, Academic Press, 1976.
11. L. Hörmander, *Some inequalities for functions of exponential type*, Math. Scand. **3** (1955), 21–27.
12. Q. I. Rahman, *Functions of exponential type*, Trans. Amer. Math. Soc. **135** (1969), 295–309.

13. Q. I. Rahman and G. Schmeisser, *Extension of a theorem of Laguerre to entire functions of exponential type*, J. Math. Anal. Appl. **122** (1987), 463–468.

14. G. Szegö, *Bemerkungen zu einem Satz von J. H. Grace über die Wurzeln algebraischer Gleichungen*, Math. Z. **13** (1922), 28–55.

Département de Mathématiques et de Statistique, Université de Montréal, C. P. 6128, succursale A, Montréal, H3C 3J7, Canada

Mathematisches Institut, Universität Erlangen-Nürnberg, Bismarckstrasse 1 $\frac{1}{2}$, D-8520 Erlangen, Federal Republic of Germany