TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 320, Number 1, July 1990

THE SCHRÖDINGER EQUATION WITH A QUASI-PERIODIC POTENTIAL

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ABSTRACT. We consider the Schrödinger equation

$$-\frac{d^2}{dx^2}\psi + \varepsilon(\cos x + \cos(\alpha x + \vartheta))\psi = E\psi$$

where ε is small and α satisfies the Diophantine inequality

$$|p + q\alpha| \geq C/q^2 \quad \text{for } p, q \in \mathbf{Z},\ q \neq 0.$$

We look for solutions of the form

$$\psi(x) = e^{iKx}q(x) = e^{iKx}\sum \psi_{mn}e^{inx}e^{im(\alpha x+\vartheta)}.$$

If we try to solve for $\psi = \psi_{mn}$ we are led to the Schrödinger equation on the lattice $\mathbf{Z}^2$

$$H(K)\psi = (\varepsilon\Delta + V(K))\psi = E\psi$$

where Δ is the discrete Laplacian (without diagonal terms) and $V(K)$ is some potential on $\mathbf{Z}^2$. We have two main results:

(1) For ε sufficiently small, $H(K)$ has pure point spectrum for almost every K.

(2) For ε sufficiently small, the operator

$$-d^2/dx^2 + \varepsilon(\cos x + \cos(\alpha x + \vartheta))$$

has no point spectrum.

To prove our results, we must get decay estimates on the Green's function $(E - H)^{-1}$. The decay of the eigenfunction follows from this. In general, we must keep track of small divisors which can make the Green's function large. This is accomplished by a KAM (Kolmogorov, Arnold, Moser) type of multiscale perturbation analysis.

INTRODUCTION

We consider the Schrödinger equation

$$-\frac{d^2}{dx^2}\psi + \varepsilon(\cos x + \cos(\alpha x + \vartheta))\psi = E^*\psi \tag{0.1}$$

where ε is small and α satisfies the Diophantine inequality

$$|p + q\alpha| \geq C/q^2 \quad \text{for } p, q \in \mathbf{Z},\ q \neq 0.$$

Received by the editors September 30, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 34B25; Secondary 34D10.

The set of irrationals α satisfying the above condition for some constant C has full measure.

If the term $\cos(\alpha x + \vartheta)$ is absent from (0.1), then the potential is periodic and the spectrum is known to be purely absolutely continuous. Moreover, any polynomially bounded solution can be expressed as a linear combination of functions of the form

$$\psi(x) = e^{iKx} \sum \psi_n e^{inx}$$

where the coefficients ψ_n decay exponentially fast as $n \to \infty$ [2, 17]. For a general quasi-periodic potential, Dinaburg and Sinai [4] proved the existence of "Bloch type" eigenfunctions $\psi(x) = e^{iKx} q(x)$ where $q(x)$ is quasi-periodic. By definition, a quasi-periodic function is a function $q(x)$ which can be expressed as

$$q(x) = F(\omega_1 x, \dots, \omega_d x) = F(\omega x)$$

where F is continuous and periodic in all d variables. To prove our results we usually assume that the frequencies $\omega = (\omega_1, \dots, \omega_d)$ satisfy some Diophantine inequality such as $|\langle j, \omega\rangle|^{-1} \le C|j|^s$ for $j \in \mathbf{Z}^d$, $j \neq 0$.

With any quasi-periodic potential $v(x)$ we can associate a rotation number

$$\beta(E^*) = \lim_{x\to\infty} \frac{1}{x} \arg(\psi' + i\psi)$$

where ψ is a solution of the differential equation $(-\Delta + v)\psi = E^*\psi$. In [9] it was shown that $\beta(E^*) = \frac{1}{2}\langle j, \omega\rangle$ for E^* in the resolvent set. Dinaburg and Sinai proved their result for a set of rotation numbers "not too close" to $\frac{1}{2}\langle j, \omega\rangle$. Their proof uses KAM (Kolmogorov, Arnold, Moser) type methods. They also established the existence of some absolutely continuous spectrum but did not exclude the presence of point spectrum or singular continuous spectrum.

In our case (0.1) it is natural to try to write

$$\psi(x) = e^{iK^* x} \sum \psi_{mn} e^{inx} e^{im(\alpha x + \vartheta)}.$$

If we assume a solution of this form, we get a recursion formula for the coefficients ψ_{mn}

$$\varepsilon[\psi_{m+1,n} + \psi_{m-1,n} + \psi_{m,n+1} + \psi_{m,n-1}] + (n + m\alpha + K^*)^2 \psi_{mn} = E^* \psi_{mn}.$$

Remark. This equation is independent of ϑ.

We now think of $\psi = \psi(m, n)$ as a function on $\mathbf{Z}^2$ and write a matrix version of the recursion formula

$$H\psi \equiv (\varepsilon\Delta + V)\psi = E^*\psi$$

where Δ is the finite difference Laplacian (without diagonal terms) and V is our lattice potential. The matrix elements are given by

$$\Delta_{ij} = \begin{cases} 1 & \text{if } |i - j| = 1, \\ 0 & \text{otherwise}, \end{cases}$$

$$V_{ij} = \delta_{ij} v_j \quad \text{where } v_j = (j_2 + j_1\alpha + K^*)^2,$$

and $j = (j_1, j_2) \in \mathbf{Z}^2$. For a more convenient notation we define

$$[j] \equiv j_2 + j_1\alpha .$$

With this notation we have

$$v_j = ([j] + K^*)^2$$

and by the Diophantine condition on α we have

$$|[j]| \geq C/|j|^2 \quad \text{for } j \neq 0 . \tag{0.2}$$

To prove our results we must examine the eigenvalues and eigenfunctions of the lattice operator $H(K) = \varepsilon\Delta + V(K)$. We now list our main results.

Theorem 1. *For ε sufficiently small, $H(K)$ has pure point spectrum for almost every K.*

Theorem 2. *For ε sufficiently small, the operator*

$$-d^2/dx^2 + \varepsilon(\cos x + \cos(\alpha x + \vartheta))$$

has no point spectrum.

Remark. For ε large it has been shown that there is pure point spectrum at low energy for almost every ϑ [16].

Conjecture. We believe that the spectrum of the operator

$$-d^2/dx^2 + \varepsilon(\cos x + \cos(\alpha x + \vartheta))$$

is purely absolutely continuous when ε is small.

Most of this paper is devoted to the proof of the following lemma, from which the proof of Theorem 1 follows. See [1].

Main Lemma. *For ε sufficiently small and for almost every K, every polynomially bounded eigenfunction of the operator $H(K)$ decays exponentially fast.*

Other problems of this type on the lattice have recently been studied. In the case where $\{v_j\}$ are independent random variables it was shown that with probability one the spectrum of H is pure point with eigenfunctions which decay exponentially fast [3, 5, 6, 8, 13]. When $v_j(\vartheta) = \cos 2\pi(\alpha j + \vartheta)$ the same result was shown to be true for almost every ϑ [14, 16]. Our methods are closely related to those of [16]. In general, we must deal with the appearance of small divisors in the Green's function $(H - E^*)^{-1}$. To overcome this problem, we will use a multiscale perturbation analysis similar to the one used in the random variable case. Our methods also apply if the potential $v_j = ([j] + K)^2$ is replaced by any other symmetric C^2 potential with a nondegenerate critical point.

The main idea of the proof of the main lemma is to keep track of the sites in $\mathbf{Z}^2$ where ψ may be large. To do this, we will define a sequence of singular sets S_n (for $n \geq 0$). We list some important properties that S_n will have:

(1) ψ decays exponentially fast outside S_n.

(2) The sites in S_n become increasingly sparse as n gets larger.

Remark. As $n \to \infty$ we are left with a set S_∞ which contains all the sites where ψ may be large; moreover, ψ decays exponentially fast outside S_∞.

We now give a brief discussion of S_0. If we examine the eigenvalue equation $H(K^*)\psi = (\varepsilon\Delta + V(K^*))\psi = E^*\psi$, we see that $\psi(c_0^i)$ may be large at the sites $c_0^i \in \mathbf{Z}^2$ where the potential $v(c_0^i)$ is near E^*. For convenience we define

$$E_0^i(K^*) = ([c_0^i] + K^*)^2 = v(c_0^i, K^*) \tag{0.3}$$

to be the value of the potential at c_0^i. Then we define the 0th order singular set to be

$$S_0(K^*, E^*) = \{c_0^i \in \mathbf{Z}^2 : |E_0^i(K^*) - E^*| \le \delta_0\sqrt{E^* + 1}\}$$

where $\delta_0 \cong \exp(-l_0^{2/3})$ is some small number to be chosen later by making l_0 as large as we want. The following theorem is the key estimate needed to prove our results. We will state it here for $S_0(K^*, E^*)$ and later we will prove that it holds true (with the appropriate change of constants) for every singular set $S_n(K^*, E^*)$. (See §3 for the general proof.) In §4 we will show how the Center Theorem is used to prove our main theorems.

Center Theorem. *If* $c_0^r \in S_0(K^*, E^*)$ *for* $r = i, j$ *then*

$$m(c_0^i, c_0^j) \le 2\delta_0^{1/2}$$

where

$$m(c_0^i, c_0^j) \equiv \min(|[c_0^i] - [c_0^j]|, |[c_0^i] + [c_0^j] + 2K^*|).$$

We can see the importance of $m(c_0^i, c_0^j)$ *if we note that*

$$\begin{aligned}|E_0^i(K) - E_0^j(K)| &= |v(c_0^i, K) - v(c_0^j, K)| \\ &= |[c_0^i] - [c_0^j]||[c_0^i] + [c_0^j] + 2K|.\end{aligned}$$

The Center Theorem has two important corollaries which will give us information about the structure of $S_0(K^*, E^*)$.

Corollary 1. *Let* $s_0 = \min_{S_0}|c_0^i - c_0^j| \equiv |c_0^I - c_0^J|$ *and suppose that* $s_0 \le 6l_0^2$. *Then every point* $c_0^i \in S_0(K^*, E^*)$ *has a mirror image*

$$\tilde{c}_0^i = c_0^i \pm (c_0^J - c_0^I)$$

whose sign is uniquely determined by the equation

$$|[c_0^i] + [\tilde{c}_0^i] + 2K^*| \le 6\delta_0^{1/2}. \tag{0.4}$$

Proof. Since $s_0 \le 6l_0^2$, by (0.2) and the Center Theorem we must have $|[c_0^I] + [c_0^J] + 2K^*| \le 2\delta_0^{1/2}$. If $|[c_0^i] - [c_n^I]| \le 2\delta_0^{1/2}$, we define $\tilde{c}_0^i = c_0^i + (c_0^J - c_0^I)$ and it is easy to check that (0.4) holds. Therefore by the Center Theorem we may assume $|[c_0^i]+[c_0^I]+2K^*| \le 2\delta_0^{1/2}$. In this case $\tilde{c}_0^i = c_0^i - (c_0^J - c_0^I)$ is the required mirror image. □

Remark. Equation (0.4) implies

$$|v(\tilde{c}_0^i) - E^*| \le \mathrm{cst}\, \delta_0^{1/2} \sqrt{E^* + 1}\,.$$

Therefore by Corollary 1, if there is one "close" pair of points in S_0 then every point in S_0 has a mirror image that "almost belongs" to S_0.

Corollary 2 (Spacing Lemma). *If* $c_0^m \in S_0(K^*, E^*)$ *for* $m = i, j, p$, *then two of the three points* c_0^m *must be separated by at least* $\delta_0^{-1/6}$.

Proof. If $|[c_0^i] - [c_0^j]| \le 2\delta_0^{1/2}$, the proof follows from (0.2). Therefore by the Center Theorem we may assume

$$|[c_0^i] + [c_0^j] + 2K^*| \le 2\delta_0^{1/2} \quad \text{and} \quad |[c_0^i] + [c_0^p] + 2K^*| \le 2\delta_0^{1/2}.$$

Now we subtract the two inequalities to obtain the result. □

Remark. The two corollaries give us a good description of S_0. There are two possibilities. See Figures 1 and 2.

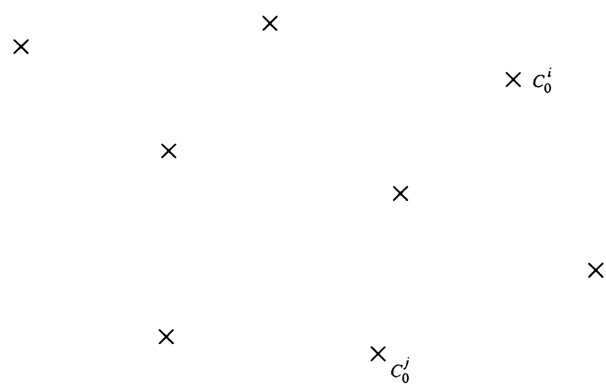


FIGURE 1. $s_0 \ge 6l_0^2$

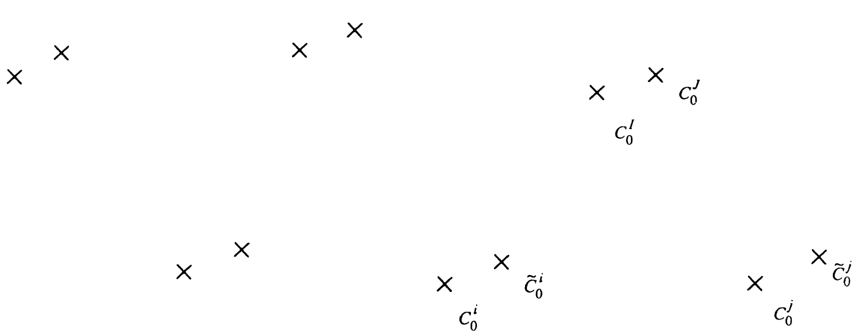


FIGURE 2. $s_0 \le 6l_0^2$

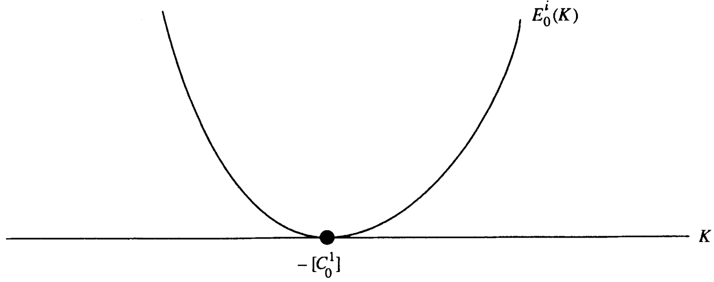


FIGURE 3

We now sketch the proof of the Center Theorem. Since $c_0^r \in S_0(K^*, E^*)$ for $r = i, j$, by definition of S_0 and equation (0.3) we must have

$$2\delta_0\sqrt{E^* + 1} \geq |E_0^i(K^*) - E_0^j(K^*)| = |E_0^j(K^* + \Delta K) - E_0^j(K^*)|$$

where $|\Delta K| = m(c_0^i, c_0^j)$. If we can establish the bound

$$|E_0^j(K^* + \Delta K) - E_0^j(K^*)| \geq \tfrac{1}{2}\sqrt{E^* + 1}|\Delta K|^2,$$

it follows that $m(c_0^i, c_0^j) = |\Delta K| \leq 2\delta_0^{1/2}$. In §1 we will study the structure of $E_0^j(K)$ and establish this bound. Since $E_0^j(K)$ is a simple quadratic function, it will be easy to understand its structure completely and therefore easy to study the set $S_0(K^*, E^*)$. See Figure 3.

We end our discussion of $S_0(K^*, E^*)$ with the following theorem.

Decay Theorem. *If $S_0(K^*, E^*) = \varnothing$ then the matrix elements of $G(K^*) \equiv (H(K^*) - E^*)^{-1}$ decay exponentially fast, i.e.,*

$$|(H(K^*) - E^*)^{-1}(x, y)| \leq \text{cst}\, e^{-\gamma_0|x-y|}.$$

Proof. Since $\|\Delta\| \leq 4$ we have

$$(0.5) \qquad \begin{aligned} (H - E^*)^{-1} &= (\varepsilon\Delta + V - E^*)^{-1} \\ &= \sum_{n=0}^{\infty}(-1)^n\varepsilon^n[(V - E^*)^{-1}\Delta]^n(V - E^*)^{-1} \end{aligned}$$

converges (if ε is small compared to δ_0) and

$$\begin{aligned} |(H - E^*)^{-1}(x, y)| &\leq \frac{\text{cst}}{\delta_0\sqrt{E^* + 1}}\left(\frac{4\varepsilon}{\delta_0\sqrt{E^* + 1}}\right)^{|x-y|} \\ &\equiv \frac{\text{cst}}{\delta_0\sqrt{E^* + 1}}e^{-\gamma_0|x-y|}. \quad \square \end{aligned}$$

Remark. If $E^* \leq -\frac{1}{2}$ then $(H - E^*)^{-1}$ converges. This follows from (0.5) and the fact that $|v_j - E^*| \geq \frac{1}{2}$ for every $j \in \mathbf{Z}^2$. Therefore the only interesting case is when $E^* \geq -\frac{1}{2}$. From now on we will assume that E^* is restricted to this set.

Before we define $S_n(K^*, E^*)$ for every n, we shall discuss some perturbation methods that we will frequently use. We will need to restrict our Hamiltonian $H = (\varepsilon\Delta + V)$ to boxes (certain subsets of $\mathbf{Z}^2$) of increasing size and inductively use information from small boxes to gain information in larger boxes. The smallest boxes will be single sites in $\mathbf{Z}^2$.

If B is a region in $\mathbf{Z}^2$ we define $H(B)$ to be the operator H restricted to B by defining the matrix elements as follows:

$$H(B)(i, j) = H(i, j) \quad \text{for } i, j \in B.$$

We now define the boundary operator Γ_B by

$$\Gamma_B(i, j) = \begin{cases} \varepsilon & \text{if } |i-j| = 1 \text{ and } (i \in B, j \notin B) \text{ or } (i \notin B, j \in B), \\ 0 & \text{otherwise.} \end{cases}$$

Note:

$$H(\Gamma_B) \equiv H - \Gamma_B = H(B) \oplus H(B^c). \tag{0.6}$$

We will need to estimate the Green's function $G = (H - E)^{-1}$. To do this we define

$$G_\Gamma \equiv (H(\Gamma) - E)^{-1} = G(B) \oplus G(B^c)$$

where $G(B) \equiv (H(B) - E)^{-1}$, $G(B^c) \equiv (H(B^c) - E)^{-1}$, and $\Gamma = \Gamma_B$. The following resolvent identity will be used later:

$$G = G_\Gamma - G_\Gamma \Gamma G. \tag{0.7}$$

We can now define $S_{n+1}(K^*, E^*)$ inductively. We assume that $S_n(K^*, E^*)$ is defined and consists of either

(1) "distant" points c_n^i for which we define $c_{n+1}^i \equiv c_n^i$ or

(2) "distant" pairs of points c_n^i, $\tilde{c}_n^i$ for which we define $c_{n+1}^i \equiv \frac{1}{2}(c_n^i + \tilde{c}_n^i)$.

We then put square boxes I_{n+1}^i of length l_{n+1} centered at c_{n+1}^i where $l_{n+1} \equiv l_n^2$ for case (1) and $l_{n+1} = l_n^4$ for case (2). See Figures 4 and 5.

Remark. We must choose our boxes I_{n+1}^i so that the boundary of I_{n+1}^i does not intersect the boundary of any previous box I_m^i $(m \leq n)$. An intersection like this would break up the box I_m^i and cause us to lose information about $\sigma(H(I_m^i))$.

Definition. We call a box Λ n *regular* if $\partial\Lambda \cap \partial I_m^i = \varnothing$ and $\partial\Lambda \cap \partial\tilde{I}_m^i = \varnothing$ for every $M \leq n$.

Remark. By Appendix D we can always deform our boxes so that they are regular when required. From now on we will assume that all boxes are chosen to be regular.

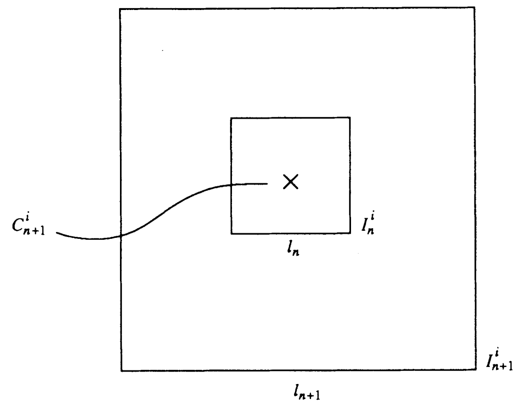


FIGURE 4. $s_n \geq 6l_n^2$

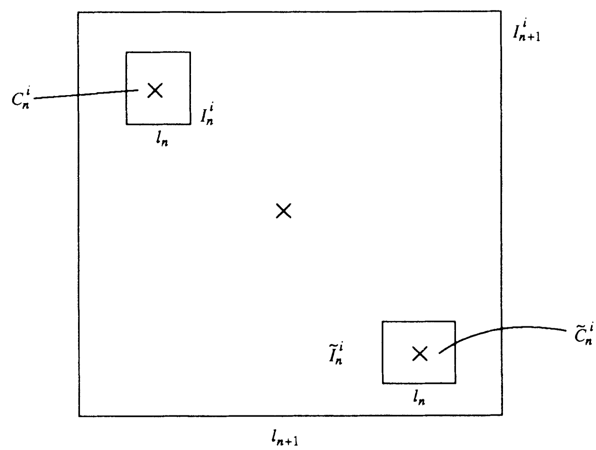


FIGURE 5. $s_n \leq 6l_n^2$

We say that c^i_{n+1} belongs to $S_{n+1}(K^*, E^*)$ if $H(K^*)$ restricted to I^i_{n+1} has an eigenvalue $E^i_{n+1}(K^*)$ such that

$$|E^i_{n+1}(K^*) - E^*| \leq \delta_{n+1}\sqrt{E^* + 1}$$

where $\delta_{n+1} \cong \exp(-\gamma_0 l_{n+1}^{2/3})$ and γ_0 is the constant defined in the Decay Theorem.

Remark 1. We will show that there are at most two eigenvalues in $\sigma(H(I^i_{n+1}))$ that are "near" E^*. See Lemmas 3.8 and 3.11.

Remark 2. In one-dimensional problems it is known that there are no level crossings; i.e., there is a lower bound on the separation of the eigenvalues in $\sigma(H(I^i_{n+1}))$. This is not true in higher dimensions. This will complicate our analysis since we must consider the case when the eigenvalues cross.

§3 is devoted to the study of $S_n(K^*, E^*)$. Here we will list the important results and give a brief sketch of their proofs.

Center Theorem. *If c_n^i and c_n^j belong to S_n, then*

$$m(c_n^i, c_n^j) \le 2\delta_n^{1/2}/\delta_{n-1}$$

where

$$m(c_n^i, c_n^j) \equiv \min(|[c_n^i] - [c_n^j]|, |[c_n^i] + [c_n^j] + 2K^*|).$$

Corollary 1. *Let $s_n = \min_{S_n} |c_n^i - c_n^j| \equiv |c_n^I - c_n^J|$ and suppose that $s_n \le 6l_n^2$. Then every $c_n^i \in S_n$ has a mirror image*

$$\tilde{c}_n^i = c_n^i \pm (c_n^J - c_n^I)$$

whose sign is uniquely determined by the equation

$$|[c_n^i] + [\tilde{c}_n^i] + 2K^*| \le 6\delta_n^{1/2}/\delta_{n-1}. \tag{0.8}$$

Corollary 2 (Spacing Lemma). *If $c_n^m \in S_n$ for $m = i, j, p$ then two of the three points must be separated by at least $\delta_n^{-1/6}$.*

Remark. The proof of these two corollaries goes exactly as it did in the $S_0(K^*, E^*)$ case after we establish the Center Theorem.

Now we will state a generalized form of the Decay Theorem.

Decay Theorem. *If $S_n \cap \Lambda = \varnothing$ then*

$$|(H(\Lambda) - E^*)^{-1}(x, y)| \le (\mathrm{cst}/\delta_0\sqrt{E^* + 1})e^{-\gamma_n|x-y|}$$

provided $|x - y| \ge l_n^{5/6}$ where $(\gamma_0/2) \le \gamma_n \le \gamma_0$.

Proof. See Appendix C. □

Remark. The Decay Theorem will be the main tool used to prove that eigenfunctions decay exponentially fast.

We now sketch the proof of the Center Theorem. If $c_n^r \in S_n$ for $r = i, j$, there are eigenvalues $E_n^r(K^*) \in \sigma(H(I_n^r), K^*)$ such that $|E_n^r(K^*) - E^*| \le \delta_n\sqrt{E^* + 1}$; hence it suffices to show that

$$|E_n^i(K^*) - E_n^j(K^*)| \ge \tfrac{1}{2}\delta_{n-1}^2\sqrt{E^* + 1}\; m(c_n^i, c_n^j)^2. \tag{0.9}$$

To prove (0.9), we will show that

$$\sigma(H(I_n^j, K^*)) = \sigma(H(I_n^i, K^* + \Delta K)) \tag{0.10}$$

where $|\Delta K| = m(c_n^i, c_n^j)$ and

$$\sigma(H(I_n^i, K)) = \sigma(H(I_n^i, -2[c_n^i] - K)) \tag{0.11}$$

for K near K^*. Equation (0.10) follows from the fact that the box I_n^j can be thought of as the box I_n^i translated by the amount $c_n^j - c_n^i$. To prove

(0.11) we note that $v_y(K_s + \delta K) = v_x(K_s - \delta K)$ for every $x \in I_n^i$ where $y = 2c_n^i - x$. Equation (0.11) implies that there is a point $K_s = -[c_n^i]$ about which the eigenvalues are symmetric.

If we are in the simple case where there is a unique eigenvalue $E_n^i \in \sigma(H(I_n^i))$ close to E^*, then (0.10) implies that

$$E_n^j(K^*) = E_n^i(K^* + \Delta K).$$

Therefore the proof of (0.9) reduces to a problem of investigating the derivatives of $E_n^i(K)$ for K near K^*. We will use first and second order perturbation theory to calculate the derivatives. The main result is that the first and second derivatives cannot be small simultaneously. This together with the fact that $E_n^i(K)$ is symmetric yields (0.9). See Figure 6.

If the boxes I_n^i have two eigenvalues near E^*, then the calculations described above become more complex. We must analyze both eigenvalues simultaneously in order to calculate the derivatives. (See Appendix B.)

Now we shall explain why $|E_n'|$ being small forces $|E_n''|$ to be large. This is the key estimate of this paper. The facts we need about the derivatives of the eigenvalue curves will be proved by induction on n as follows. Suppose that $c_n^i \in S_n(K^*, E^*)$. Then by the Spacing Lemma there are at most two points $c_{n-1}^i, \tilde{c}_{n-1}^i \in S_{n-1}(K^*, E^*) \cap I_n^i$ and therefore at most two eigenvalues $E_n^i(K^*)$, $\mathscr{E}_n^i(K^*)$ in $\sigma(H(I_n^i, K^*))$ that are near E^*.

The Decay Theorem will be used to prove that the corresponding eigenfunctions ψ_n^i and Ψ_n^i decay exponentially fast outside $I_{n-1}^i \cup \widetilde{I}_{n-1}^i$. Then using the

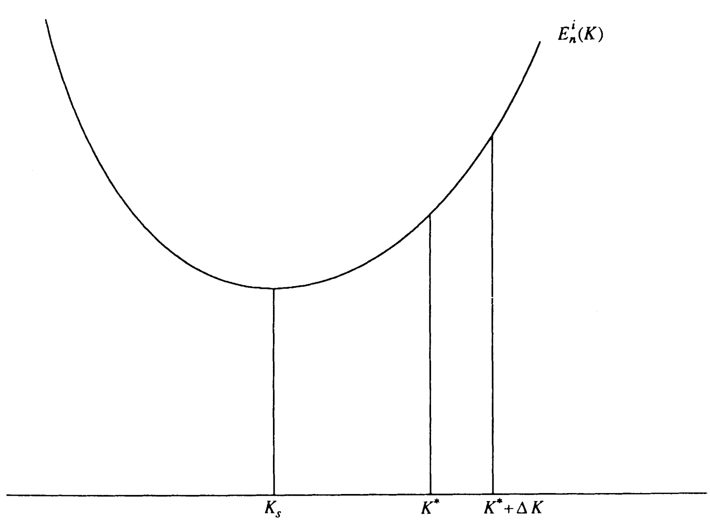


FIGURE 6

fact that these two eigenfunctions are orthonormal we can show that

$$\psi_n^i \cong A\psi_{n-1}^i + B\tilde{\psi}_{n-1}^i,$$
$$\Psi_n^i \cong B\psi_{n-1}^i - A\tilde{\psi}_{n-1}^i$$

where $A^2+B^2 \cong 1$ and ψ_{n-1}^i, $\tilde{\psi}_{n-1}^i$ are eigenfunctions in the boxes I_{n-1}^i, $\tilde{I}_{n-1}^i$ that have eigenvalues E_{n-1}^i, $\tilde{E}_{n-1}^i$ near E^*.

Now we will calculate the derivatives of E_n^i and relate them to the derivatives of E_{n-1}^i and $\tilde{E}_{n-1}^i$. Then by induction we will gain information about E_n^i. By Appendix B we have

$$\begin{aligned}\frac{d}{dK} E_n^i &= \langle \psi_n^i, V'\psi_n^i\rangle \\ &\cong A^2\langle \psi_{n-1}^i, V'\psi_{n-1}^i\rangle + B^2\langle \tilde{\psi}_{n-1}^i, V'\tilde{\psi}_{n-1}^i\rangle \\ &= A^2 \frac{d}{dK} E_{n-1}^i + B^2 \frac{d}{dK} \tilde{E}_{n-1}^i.\end{aligned}$$

Then we can use the symmetry of the potential to show that

$$\frac{d}{dK} \tilde{E}_{n-1}^i \cong -\frac{d}{dK} E_{n-1}^i;$$

therefore

$$\frac{d}{dK} E_n^i \cong (A^2 - B^2) \frac{d}{dK} E_{n-1}^i. \tag{0.12}$$

Again by Appendix B we can show that

$$\frac{d^2}{dK^2} E_n^i = \frac{2\langle \psi_n^i, V'\Psi_n^i\rangle^2}{E_n^i - \mathscr{E}_n^i} + \text{smaller terms}$$

where

$$\begin{aligned}\langle \psi_n^i, V'\Psi_n^i\rangle &\cong AB\langle \psi_{n-1}^i, V'\psi_{n-1}^i\rangle - AB\langle \tilde{\psi}_{n-1}^i, V'\tilde{\psi}_{n-1}^i\rangle \\ &\cong AB \frac{d}{dK} E_{n-1}^i - AB \frac{d}{dK} \tilde{E}_{n-1}^i \\ &\cong 2AB \frac{d}{dK} E_{n-1}^i.\end{aligned}$$

Therefore

$$\frac{d^2}{dK^2} E_n^i \cong 2\left(2AB \frac{d}{dK} E_{n-1}^i\right)^2 (E_n^i - \mathscr{E}_n^i)^{-1} \tag{0.13}$$

Since $A^2 + B^2 \cong 1$, it is impossible for $A^2 - B^2$ and AB to be small at the same time. By induction we will know that dE_{n-1}^i/dK is bounded away from zero; thus by (0.12) and (0.13) it is easy to see that dE_n^i/dK and $d^2E_n^i/dK^2$ cannot be small simultaneously. In particular, we will prove that if

$$\left|\frac{d}{dK} E_n^i\right| \le \delta_{n-1}^2 \sqrt{E^* + 1}$$

then

$$\left|\frac{d^2}{dK^2} E_n^i\right| \geq \frac{1}{2}\sqrt{E^* + 1}.$$

Now equation (0.9) follows immediately from this estimate. See Appendix A for more details.

We end our introduction with a brief discussion of the proof of Theorem 2. To prove the absence of point spectrum we assume that there is a solution to the equation

$$-\frac{d^2}{dx^2}\psi + \varepsilon(\cos x + \cos(\alpha x + \vartheta))\psi = E^*\psi.$$

Let $\phi_{mn}^K = e^{im\vartheta}\hat{\psi}(K + n + m\alpha)$ where $\hat{\psi}$ is the Fourier transform of ψ. Then it is easy to show that ϕ^K is a solution to our lattice equation

$$H(K)\phi^K = (\varepsilon\Delta + V(K))\phi^K = E^*\phi^K. \tag{0.14}$$

In §4 we will prove the following facts:

(1) For n large enough we must have

$$S_n(K, E^*) \cap \Lambda_n = \phi$$

for almost every $K \in \mathbf{R}$, where Λ_n are boxes around the origin of length l_n.

(2) ϕ^K is polynomially bounded for almost every K.

If we assume these facts, we can restrict (0.14) to the boxes Λ_n and write

$$H(\Lambda_n)\phi^K = E^*\phi^K + \Gamma(\Lambda_n)\phi^K,$$

which implies

$$\phi^K = (H(\Lambda_n) - E^*)^{-1}\Gamma(\Lambda_n)\phi^K \equiv G(\Lambda_n)\Gamma(\Lambda_n)\phi^K. \tag{0.15}$$

The Decay Theorem together with fact (1) implies that the matrix elements G_{xy} decay exponentially fast. Since ϕ^K is polynomially bounded, it follows that $\phi^K(x) \equiv 0$ for every x and almost every K. Therefore $\hat{\psi}(K) = 0$ for almost every K, which implies that $\psi = 0$ and therefore not an eigenfunction. This is a contradiction to our assumption.

1. Definition and properties of S_0

We begin by defining the 0th singular set

$$S_0(K^*, E^*) = \{c_0^i \in \mathbf{Z}^2 : |E_0^i(K^*) - E^*| \leq \delta_0\sqrt{E^* + 1}\}.$$

For convenience we have defined

$$E_0^i(K) = ([c_0^i] + K)^2 \tag{1.1}$$

to be the value of the potential at c_0^i. In this section we will study the structure of S_0 and prove the Center Theorem. As stated in the introduction, we will always assume that $E^* \geq -\frac{1}{2}$.

Lemma 1.1. *Let* $c_0^r \in S_0$ *for* $r = i, j$; *then*

(a) $m(c_0^i, c_0^j) \le \sqrt{2\delta_0 (E^* + 1)^{1/2}}$.

(b) $E_0^i(K^*) = E_0^j(K^* + \Delta K)$ *where* $\Delta K = [c_0^i] - [c_0^j]$ *or* $-([c_0^i] + [c_0^j] + 2K^*)$ *and* $|\Delta K| = m(c_0^i, c_0^j)$.

(c) $dE_0^i/dK| \le 3\sqrt{E^* + 1}$ *for* $|K - K^*| \le 12\sqrt{2\delta_0(E^* + 1)^{1/2}}$.

Proof. By definition of S_0, we have

$$2\delta_0\sqrt{E^* + 1} \ge |E_0^i(K^*) - E_0^j(K^*)| = |([c_0^i] + K^*)^2 - ([c_0^j] + K^*)^2| = |[c_0^i] - [c_0^j]||[c_0^i] + [c_0^j] + 2K^*|,$$

which establishes part (a). The proof of part (b) is obvious from (1.1).

We will now prove part (c). Since $c_0^i \in S_0$, we have

$$|E_0^i(K^*) - E^*| = |([c_0^i] + K^*)^2 - E^*| \le \delta_0\sqrt{E^* + 1},$$

which implies $|[c_0^i] + K^*| \le \frac{5}{4}\sqrt{E^* + 1}$. Now we can estimate

$$\left|\frac{d}{dK} E_0^i\right| = 2|[c_0^i] + K| = 2|[c_0^i] + K^* + K - K^*| \le 3\sqrt{E^* + 1}. \quad \square$$

Remark. Lemma 1.1(a) is only a weak version of the Center Theorem since E^* may be very large. We will remove the E^* dependence and strengthen the lemma.

We note that $E_0^i(K) = ([c_0^i] + K)^2$ is symmetric about $K_s = -[c_0^i]$. The next lemma tells us that if the first derivative is small, then K is near the symmetry point and the second derivative is big.

Lemma 1.2. *If* $|dE_0^i/dK| \le \sqrt{E^* + 1}$ *for some* K *in the interval* $|K - K^*| \le 12\sqrt{2\delta_0(E^* + 1)^{1/2}}$, *then*

(a) $|d^2E_0^i/dK^2| = 2 \ge \sqrt{E^* + 1}$ *for all* K.

(b) $|dE_0^i/dK| \ge \sqrt{E^* + 1}|K + [c_0^i]|$.

Proof. The equality in part (a) follows immediately from (1.1). Since $|E_0^i(K^*) - E^*| \le \delta_0\sqrt{E^* + 1}$, then

$$([c_0^i] + K^*)^2 \ge E^* - \delta_0\sqrt{E^* + 1}.$$

By assumption we have $2|[c_0^i] + K| = |dE_0^i/dK| \le \sqrt{E^* + 1}$ for some K in the interval $|K - K^*| \le 12\sqrt{2\delta_0(E^* + 1)^{1/2}}$; therefore

$$|[c_0^i] + K^*| = |[c_0^i] + K + K^* - K| \le \tfrac{1}{2}\sqrt{E^* + 1} + 12\sqrt{2\delta_0(E^* + 1)^{1/2}} \le \tfrac{2}{3}\sqrt{E^* + 1}.$$

If we combine this inequality with the previous inequality, we get

$$E^* - \delta_0\sqrt{E^* + 1} \le ([c_0^i] + K^*)^2 \le \tfrac{4}{9}(E^* + 1),$$

which implies $E^* \le 1$.

We now prove part (b). By part (a) we can assume $|d^2E_0^i/dK^2| \geq \sqrt{E^*+1}$ for all K. Therefore,

$$\left(\frac{d}{dK}E_0^i\right)(K) = \left(\frac{d}{dK}E_0^i\right)(-[c_0^i]) + \left(\frac{d^2}{dK^2}E_0^i\right)(K+[c_0^i])$$

so that

$$\left|\frac{d}{dK}E_0^i\right| \geq \sqrt{E^*+1}\,|K+[c_0^i]|. \quad \square$$

Lemma 1.3. *If $c_0^r \in S_0$ for $r=i, j$ then*

(a) $2\delta_0\sqrt{E^*+1} \geq |E_0^i(K^*) - E_0^j(K^*)| \geq \frac{1}{2}\sqrt{E^*+1}\; m(c_0^i, c_0^j)^2$.

(b) (*Center Theorem*) $m(c_0^i, c_0^j) \leq 2\delta_0^{1/2}$.

Proof. The first inequality is obvious since $c_0^r \in S_0$. By Lemma 1.1(b) we can write $E_0^i(K^*) = E_0^j(K^*+\Delta K)$ where $|\Delta K| = m(c_0^i, c_0^j)$; hence it suffices to estimate

$$|E_0^j(K^*+\Delta K) - E_0^j(K^*)|.$$

If $|dE_0^j/dK| \geq \sqrt{E^*+1}$ for all K in the interval $|K-K^*| \leq 12\sqrt{2\delta_0(E^*+1)^{1/2}}$, we are done. If $|dE_0^j/dK| \leq \sqrt{E^*+1}$ for some K in the interval $|K-K^*| \leq 12\sqrt{2\delta_0(E^*+1)^{1/2}}$, Lemma 1.2(a) implies that $|d^2E_0^j/dK^2| \geq \sqrt{E^*+1}$ for all K. Since $E_0^j(K)$ is symmetric about $K_s = -[c_0^j]$, we may assume $K_s \leq K^* \leq K^*+\Delta K$. Therefore

$$\begin{aligned}|E_0^j(K^*+\Delta K) - E_0^j(K^*)| &= E_0^j(K^*+\Delta K) - E_0^j(K^*)\\ &= \frac{d}{dK}E_0^j(K^*)\Delta K + \frac{1}{2}\left(\frac{d^2}{dK^2}E_0^j\right)(\Delta K)^2\\ &\geq \frac{1}{2}\sqrt{E^*+1}\; m(c_0^i, c_0^j)^2.\end{aligned}$$

The Center Theorem follows immediately from part (a). $\square$

Remark. We have removed the E^* dependence from Lemma 1.1(a).

Lemma 1.4 (Decay Theorem). *If $S_0 \cap \Lambda = \varnothing$ then*

$$|(H(\Lambda)-E)^{-1}(x, y)| \leq \frac{\mathrm{cst}}{\delta_0\sqrt{E^*+1}} e^{-\gamma_0|x-y|}$$

for $|E-E^*| \leq (\delta_0/5)\sqrt{E^*+1}$ and $|K-K^*| \leq 4\delta_0^{3/2}$ where

$$\gamma_0 = \ln((\delta_0\sqrt{E^*+1})/4\varepsilon).$$

Proof. See (0.5). $\square$

Before we consider the next singular set S_1, we group together nearby elements of S_0. There are two cases to consider, depending on the size of $s_0 = \min_{S_0}|c_0^i - c_0^j|$. If $s_0 \geq 6l_0^2$, we construct boxes I_1^i centered at $c_1^i \equiv c_0^i$. We choose the length of the box to be $l_1 \equiv l_0^2$. If $s_0 \leq 6l_0^2$, by (0.4) each $c_0^i \in S_0$ has a mirror image $\tilde{c}_0^i$ which satisfies

$$|[\tilde{c}_0^i] + [c_0^i] + 2K^*| \leq 6\delta_0^{1/2}. \tag{1.2}$$

In this case we construct boxes I_1^i centered at $c_1^i \equiv \frac{1}{2}(c_0^i + \tilde{c}_0^i)$. We choose the length of the box to be $l_1 \equiv l_0^4$. Finally we define $\overline{S}_0 = \{c_1^i\}$. (See Figures 4 and 5 of the introduction.)

2. Definition and properties of S_1

In this section we shall define S_1 and prove the Center Theorem for S_1.

$$S_1(K^*, E^*) \equiv \{c_1^i \in \overline{S}_0 : \operatorname{dist}(\sigma(H(I_1^i)), E^*) \leq \delta_1\sqrt{E^* + 1}\}.$$

We will consider the two cases described at the end of §1 separately.

Case 1. $s_0 \geq 6l_0^2$.

Lemma 2.1. *If $c_1^r \in S_1$ for $r = i, j$, then $m(c_1^i, c_1^j) \leq \delta_0^{3/2}$.*

Proof. Since $c_1^r = c_0^r$, by Lemma 1.3(a) it suffices to show that $|E_0^i(K^*) - E_0^j(K^*)| = O(\delta_0^4\sqrt{E^* + 1})$. By definition of S_1, there are eigenvalues $E_1^r(K^*) \in \sigma(H(I_1^r))$ such that $|E_1^r(K^*) - E^*| \leq \delta_1\sqrt{E^* + 1}$. Let ψ_1^r be the corresponding wave functions. By choosing $\varepsilon \leq \delta_0^4$ we have

$$\|(V(I_1^r) - E^*)\psi_1^r\| = \|(-\varepsilon\Delta + E_1^r(K^*) - E^*)\psi_1^r\| \leq \text{cst}\,\delta_0^4\sqrt{E^* + 1};$$

hence ψ_1^r serves as a good trial wave function and we conclude that

$$|E_0^r(K^*) - E^*| \leq \text{cst}\,\delta_0^4\sqrt{E^* + 1}. \quad \square \tag{2.1}$$

Remark. With a finer analysis we will show that Lemma 2.1 continues to hold if $\delta_0^{3/2}$ is replaced by $2\delta_1^{1/2}/\delta_0$.

By definition of S_1, there exists an eigenvalue $E_1^i(K^*) \in \sigma(H(I_1^i, K^*))$ which is near E^*. We will show that we can extend E_1^i as a function of K for K near K^*.

Lemma 2.2. *If $c_1^i \in S_1$ then*

(a) *There exists a unique eigenvalue $E_1^i(K) \in \sigma(H(I_1^i))$ such that*

$$|E_1^i(K) - E^*| \leq \text{cst}\,\delta_0^{3/2}\sqrt{E^* + 1}$$

for every K in the interval $|K - K^*| \leq 2\delta_0^{3/2}$.

(b) *The corresponding wave function ψ_1^i decays rapidly away from c_1^i; i.e., $|\psi_1^i(x)| \leq \text{cst}\,e^{-\gamma_0|x - c_1^i|}$.*

(c) *Any other eigenvalue $\mathscr{E}_1^i \in \sigma(H(I_1^i))$ must obey $|\mathscr{E}_1^i - E^*| \geq (\delta_0/5)\sqrt{E^* + 1}$.*

(d) $E_1^j(K^*) = E_1^i(K^* + \Delta K)$ *where* $\Delta K = [c_1^j] - [c_1^i]$ *or* $-([c_1^i] + [c_1^j] + 2K^*)$ *and* $|\Delta K| = m(c_1^i, c_1^j)$.

Proof. By (2.1) and Lemma 1.1(c) we have

$$|E_0^i(K) - E^*| \leq \text{cst}\,\delta_0^{3/2}\sqrt{E^* + 1}$$

for $|K - K^*| \leq 2\delta_0^{3/2}$. Now,

$$\begin{aligned}\|(H(I_1^i) - E^*)\delta(x - c_1^i)\| &= \|(\varepsilon\Delta + V(I_1^i) - E^*)\delta(x - c_1^i)\| \\ &\leq \text{cst}\,\delta_0^{3/2}\sqrt{E^* + 1},\end{aligned}$$

which implies the existence of $E_1^i(K) \in \sigma(H(I_1^i))$ such that

$$|E_1^i(K) - E^*| \leq \text{cst}\,\delta_0^{3/2}\sqrt{E^* + 1} \quad \text{for } |K - K^*| \leq 2\delta_0^{3/2}.$$

To prove part (b) we use the defining equation $H(I_1^i)\psi_1^i = E_1^i\psi_1^i$ and rewrite it as $(\overline{H} - E_1^i)\psi_1^i = R\psi_1^i$. The matrix $\overline{H}$ is the same as $H(I_1^i)$ except that the potential at c_1^i is changed to make S_0 empty and $(\overline{H} - E_1^i)^{-1}$ decay (see Lemma 1.4). We note that $R = \overline{H} - H(I_1^i)$ has one nonzero element in the place corresponding to c_1^i. Thus we can express $\psi_1^i = (\overline{H} - E_1^i)^{-1}R\psi_1^i$, which implies $|\psi_1^i(x)| \leq \text{cst}\,e^{-\gamma_0|x - c_1^i|}$.

To establish part (c) we assume that there is another eigenvalue $\mathscr{E}_1^i$ that obeys $|\mathscr{E}_1^i - E^*| \leq (\delta_0/5)\sqrt{E^* + 1}$. Then the same argument shows that its wave function Ψ_1^i decays away from c_1^i. Since the wave functions ψ_1^i and Ψ_1^i are orthogonal, we have a contradiction.

The proof of part (d) follows from (0.10) and the uniqueness of E_1^j. □

Lemma 2.3. *If* K *belongs to the interval* $|K - K^*| \leq 2\delta_0^{3/2}$, *then* $dE_1^i/dK = dE_0^i/dK + O(\delta_0^2\sqrt{E^* + 1})$.

Proof. By Appendix B, $dE_1^i/dK = \langle\psi_1^i, V'\psi_1^i\rangle$ and $dE_0^i/dK = \langle\psi_0^i, V'\psi_0^i\rangle$ where $\psi_0^i(x) = \delta(x - c_1^i)$. Thus to prove the lemma we need only show that

$$\|\psi_1^i - \psi_0^i\| \leq \text{cst}\,\delta_0^3 \quad \text{and} \quad \|V'\| \leq \delta_0^{-1}\sqrt{E^* + 1}. \tag{2.2}$$

The first inequality is a consequence of Lemma 2.2(b), so it remains to estimate

$$\begin{aligned}\|V'\| &\leq \max_{I_1} |v_x'| = \max 2|[x] + K| \\ &\leq \max 2|[x] - [c_1^i]| + \max 2|[c_1^i] + K| \\ &\leq \text{cst}\,l_1 + \max\left|\frac{d}{dK}E_0^i\right|,\end{aligned}$$

which by Lemma 1.1(c) is bounded by

$$\text{cst}\,l_1 + 3\sqrt{E^* + 1} \leq \text{cst}\,l_1\sqrt{E^* + 1} \leq \delta_0^{-1}\sqrt{E^* + 1}. \quad \square$$

Lemma 2.4. *If* $c_1^i \in S_1$ *then*

(a) $|dE_1^i/dK| \le (3+\delta_0)\sqrt{E^*+1}$ *for every* K *in the interval* $|K-K^*| \le 2\delta_0^{3/2}$.

(b) *If* $|dE_1^i/dK| \le \delta_0^2\sqrt{E^*+1}$ *for some* K *in the interval* $|K-K^*| \le 2\delta_0^{3/2}$ *then*

$$\left|\frac{d^2}{dK^2}E_1^i\right| \ge (1-\delta_0)\sqrt{E^*+1};$$

moreover $d^2E_1^i/dK^2$ *has a unique sign.*

Proof. We use Lemmas 2.3 and 1.1(c) to establish part (a).

We now prove part (b). If the hypothesis holds, then by Lemma 2.3 we must have

$$\left|\frac{d}{dK}E_0^i\right| \le \text{cst}\,\delta_0^2\sqrt{E^*+1}. \tag{2.3}$$

Lemma 1.2(a) implies that $2 \ge \sqrt{E^*+1}$; therefore we can use Appendix B to write

$$\frac{d^2}{dK^2}E_1^i = 2 + 2\langle V'\psi_1^i, (E_1^i - H)_\perp^{-1}V'\psi_1^i\rangle.$$

We will show that the remainder term is $O(\delta_0^3\sqrt{E^*+1})$. Taking norms, the remainder term is bounded by $2\|V'\psi_1^i\|^2\|(E_1^i-H)_\perp^{-1}\|$. By Lemma 2.2(c) we have $\|(E_1^i-H)_\perp^{-1}\| = O(\delta_0\sqrt{E^*+1})^{-1}$, so it remains to estimate $\|V'\psi_1^i\|^2$. Using (2.2) we get

$$\begin{aligned}\|V'\psi_1^i\|^2 &\le (\|V'(\psi_1^i-\psi_0^i)\| + \|V'\psi_0^i\|)^2\\ &\le \left(\|V'\|\delta_0^3 + \left|\frac{d}{dK}E_0^i\right|\right)^2\\ &\le \left(\delta_0^{-1}\sqrt{E^*+1}\,\delta_0^3 + \left|\frac{d}{dK}E_0^i\right|\right)^2\end{aligned}$$

which by (2.3) is bounded by $\text{cst}(\delta_0^2\sqrt{E^*+1})^2$. □

By (0.11) we know that there is a point $K_s = -[c_1^i]$ about which $E_1^i(K)$ is symmetric. It is not clear that the symmetry point belongs to the interval of definition of $E_1^i(K)$. The next lemma tells us that if dE_1^i/dK is small then K must be near the symmetry point.

Lemma 2.5. *If* $|dE_1^i/dK| \le \delta_0^2\sqrt{E^*+1}$ *for some* K *in the interval* $|K-K^*| \le \delta_0^{3/2}$, *then* $|dE_1^i/dK| \ge \frac{1}{2}\sqrt{E^*+1}|K+[c_1^i]|$.

Proof. If the hypotheses holds then by Lemma 2.3 we have

$$|dE_0^i/dK| \le \text{cst}\,\delta_0^2\sqrt{E^*+1}$$

and by Lemma 1.2(b) we get $|K+[c_1^i]| = O(\delta_0^2)$, which implies $|K^*+[c_1^i]| \le 2\delta_0^{3/2}$. Therefore the symmetry point $K_s = -[c_1^i]$ lies in the interval $|K-K^*| \le$

$2\delta_0^{3/2}$, which is the interval of definition of $E_1^i(K)$. We now use Lemma 2.4(b) and Appendix A to complete the proof. □

Lemma 2.6. *If* $c_1^r \in S_1$ *for* $r = i, j$ *then*

(a) $2\delta_1\sqrt{E^*+1} \geq |E_1^i(K^*) - E_1^j(K^*)| \geq \frac{1}{2}\delta_0^2\sqrt{E^*+1}\, m(c_1^i, c_1^j)^2$.

(b) (*Center Theorem*) $m(c_1^i, c_1^j) \leq 2\delta_1^{1/2}/\delta_0$.

Proof. The first inequality in part (a) holds since $c_1^r \in S_1$. By Lemma 2.2(d) we have

$$E_1^i(K^*) - E_1^j(K^*) = E_1^i(K^*) - E_1^i(K^* + \Delta K).$$

If $|dE_1^i/dK| \geq \delta_0^2\sqrt{E^*+1}$ for $|K - K^*| \leq \delta_0^{3/2}$ then part (a) follows immediately. Therefore we may assume $|dE_1^i/dK| \leq \delta_0^2\sqrt{E^*+1}$ for some $|K - K^*| \leq \delta_0^{3/2}$. Then by Lemma 2.5, the symmetry point belongs to the interval of definition of $E_1^i(K)$. With the definition of ΔK from Lemma 2.2(d) we use Lemma 2.4(b) and Appendix A to establish part (a). The Center Theorem follows immediately from part (a). □

Remark 1. Lemma 2.6(b) strengthens Lemma 2.1.

Remark 2. We are now finished with case 1 ($s_0 \geq 6l_0^2$).

Case 2. $s_0 \leq 6l_0^2$.

We would like to include the symmetry point $K_s = -[c_1^i]$ in our interval of K's. The next lemma tells us how wide an interval we need to take in order to do this.

Lemma 2.7. *If* $c_1^r \in S_1$ *for* $r = i, j$ *then*

(a) $|K^* + [c_1^r]| \leq 3\delta_0^{1/2}$.

(b) $\max(|[c_1^i] - [c_1^j]|, |[c_1^i] + [c_1^j] + 2K^*|) \leq 6\delta_0^{1/2}$.

Proof. We use (1.2) and the definition of c_1^i to establish part (a). Part (b) follows immediately from part (a). □

Since $c_0^i \in S_0$, we have $|E_0^i(K^*) - E^*| \leq \delta_0\sqrt{E^*+1}$. For convenience we define $\tilde{E}_0^i(K)$ to be the value of the potential at $\tilde{c}_0^i$; therefore

$$\tilde{E}_0^i(K) = ([\tilde{c}_0^i] + K)^2 = E_0^i(-K - [\tilde{c}_0^i] - [c_0^i]). \tag{2.4}$$

With equation (2.4) we can transfer information from E_0^i to $\tilde{E}_0^i$; in particular, Lemma 1.1(c) implies that $|\tilde{E}_0^i(K^*) - E^*| \leq \text{cst}\,\delta_0^{1/2}\sqrt{E^*+1}$. Thus in each box I_1^i we have two values of the potential near E^* which will be used to generate two eigenvalues in $\sigma(H(I_1^i))$ near E^*.

Lemma 2.8. *If* $c_1^r \in S_1$ *for* $r = i, j$ *then*

(a) *There exist two eigenvalues* $E_1^i(K)$ *and* $\mathcal{E}_1^i(K) \in \sigma(H(I_1^i))$ *such that*

$$|E_1^i(K) - E^*| \leq \text{cst}\,\delta_0^{1/2}\sqrt{E^*+1}$$

for every K *in the interval* $|K - K^*| \leq 6\delta_0^{1/2}$. *The same holds for* $\mathcal{E}_1^i(K)$.

(b) *The corresponding wave functions* ψ_1^i *and* Ψ_1^i *decay rapidly away from* c_0^i *and* $\tilde{c}_0^i$, *i.e.,*

$$|\psi_1^i(x)| \leq \mathrm{cst}(e^{-\gamma_0|x-c_0^i|} + e^{-\gamma_0|x-\tilde{c}_0^i|}.$$

The same holds for Ψ_1^i.

(c) *If* $\widehat{E}$ *is any other eigenvalue in* $\sigma(H(I_1^i))$ *then*

$$|\widehat{E} - E^*| \geq \delta_0^{1/8}\sqrt{E^* + 1}.$$

(d) $E_1^j(K^*) = E_1^i(K^*+\Delta K)$ *or* $E_1^j(K^*) = \mathscr{E}_1^i(K^*+\Delta K)$ *where* $\Delta K = [c_1^j]-[c_1^i]$ *and* $|\Delta K| \leq 6\delta_0^{1/2}$.

Remark. We assume that the eigenvalues $E_1^i(K)$ and $\mathscr{E}_1^i(K)$ are labeled so that they are differentiable functions of K. See Appendix B for details.

Proof. We already know that

$$|E_0^i(K^*) - E^*| \leq \mathrm{cst}\,\delta_0^{1/2}\sqrt{E^* + 1}.$$

By Lemma 1.1(c) we can extend this for K near K^*, i.e.,

$$|E_0^i(K) - E^*| \leq \mathrm{cst}\,\delta_0^{1/2}\sqrt{E^* + 1}$$

for $|K-K^*| \leq 6\delta_0^{1/2}$. We now use $\delta(x-c_0^i)$ as a trial wave function for $H(I_1^i)$, giving us

$$\|(H(I_1^i) - E^*)\delta(x - c_0^i)\| \leq \mathrm{cst}\,\delta_0^{1/2}\sqrt{E^* + 1},$$

which implies the existence of $E_1^i(K) \in \sigma(H(I_1^i))$ such that

$$|E_1^i(K) - E^*| \leq \mathrm{cst}\,\delta_0^{1/2}\sqrt{E^* + 1}.$$

The same analysis with $E_0^i(K)$ replaced by $\widetilde{E}_0^i(K)$ gives us the second eigenvalue $\mathscr{E}_1^i(K) \in \sigma(H(I_1^i))$.

To establish the decay of the wave function we note that $|v_j(K) - E^*| \geq 2\delta_0^{1/8}\sqrt{E^* + 1}$ for every $j \in I_1^i\backslash\{c_0^i, \tilde{c}_0^i\}$ and every $|K - K^*| \leq 6\delta_0^{1/2}$, or else there would be three points j, c_0^i, $\tilde{c}_0^i$ belonging to $S_0(K, E^*) \cap I_1^i$ (with δ_0 replaced by $2\delta_0^{1/8}$). This is impossible by the Spacing Lemma. Therefore

$$|v_j(K) - E_1^i(K)| \geq \delta_0^{1/8}\sqrt{E^* + 1} \quad \text{for } j \in I_1^i\backslash\{c_0^i, \tilde{c}_0^i\}.$$

We write the defining equation $H(I_1^i)\psi_1^i = E_1^i\psi_1^i$ as $(\overline{H} - E_1^i)\psi_1^i = R\psi_1^i$ where $\overline{H}$ is the same as $H(I_1^i)$ except the potential is changed at c_0^i and $\tilde{c}_0^i$ to make $(\overline{H} - E_1^i)^{-1}$ decay. Now we can express $\psi_1^i = (\overline{H} - E_1^i)^{-1}R\psi_1^i$, which gives us

$$|\psi_1^i(x)| \leq \mathrm{cst}(e^{-\gamma_0|x-c_0^i|} + e^{-\gamma_0|x-\tilde{c}_0^i|}).$$

To prove part (c) we assume that there is another eigenvalue $\widehat{E} \in \sigma(H(I_1^i))$ such that $|\widehat{E} - E^*| \leq \delta_0^{1/8}\sqrt{E^* + 1}$. Then the same argument shows that its wave function decays away from c_0^i and $\tilde{c}_0^i$, thus violating orthogonality. The proof of part (d) follows from (0.10) and Lemma 2.7(b). □

Since we have two eigenvalues in $\sigma(H(I_1^i))$ which are close to E^*, the 0th order eigenvalues E_0^i and $\widetilde{E}_0^i$ have some special properties.

Lemma 2.9. *If* K *is in the interval* $|K - K^*| \leq 6\delta_0^{1/2}$, *then*

(a) $|dE_0^i/dK + d\widetilde{E}_0^i/dK| \leq \text{cst}\,\delta_0^{1/2}$.

(b) $|dE_0^i/dK| \geq \delta_0^{1/8}\sqrt{E^* + 1}$.

Proof. By (2.4) we have

$$\begin{aligned}\left|\frac{d}{dK}E_0^i + \frac{d}{dK}\widetilde{E}_0^i\right| &= \left|\left(\frac{d}{dK}E_0^i\right)(K) - \left(\frac{d}{dK}E_0^i\right)(-K - [c_0^i] - [\tilde{c}_0^i])\right| \\ &\leq \max\left|\frac{d^2}{dK^2}E_0^i\right| |2K + [c_0^i] + [\tilde{c}_0^i]| \\ &= 2|2K + [c_0^i] + [\tilde{c}_0^i]|,\end{aligned}$$

which by (1.2) is bounded by $\text{cst}\,\delta_0^{1/2}$.

We now prove part (b). If $|dE_0^i/dK| \leq \delta_0^{1/8}\sqrt{E^* + 1}$ for some K, then by Lemma 1.2(b) we have $|dE_0^i/dK| \geq \sqrt{E^* + 1}|K + [c_0^i]|$. Now

$$|K + [c_0^i]| \geq |[c_0^i] - [c_1^i]| - |K - K^*| - |K^* + [c_1^i]|,$$

which by Lemma 2.7(a) is greater than

$$|[c_0^i] - [c_1^i]| - \text{cst}\,\delta_0^{1/2}.$$

We note that

$$\left|[c_0^i] - [c_1^i]\right| = \left|[c_0^i] - \frac{[c_0^i] + [\tilde{c}_0^i]}{2}\right| = \tfrac{1}{2}|[c_0^i] - [\tilde{c}_0^i]|,$$

which by (0.2) is bounded below by $\text{cst}/|c_0^i - \tilde{c}_0^i|^2 \geq \text{cst}/l_0^4$. Therefore

$$|dE_0^i/dK| \geq (\text{cst}/l_0^4 - \text{cst}\,\delta_0^{1/2})\sqrt{E^* + 1} \geq \delta_0^{1/8}\sqrt{E^* + 1},$$

giving us a contradiction. □

Lemma 2.10. *If* $c_1^i \in S_1$ *then*

$$\begin{aligned}\psi_1^i(x) &= A\delta(x - c_0^i) + B\delta(x - \tilde{c}_0^i) + R(x), \\ \Psi_1^i(x) &= B\delta(x - c_0^i) - A\delta(x - \tilde{c}_0^i) + S(x),\end{aligned}$$

where $\|R\| \leq \delta_0$, $\|S\| \leq \delta_0$, *and* $1 \geq A^2 + B^2 \geq 1 - \delta_0$.

Proof. It is obvious that we can express

$$\psi_1^i(x) = A\delta(x - c_0^i) + B\delta(x - \tilde{c}_0^i) + R(x).$$

Lemma 2.8(b) gives us $\|R\| \le \delta_0^2$, and since ψ is normalized we have $1 \ge A^2 + B^2 \ge 1 - O(\delta_0^2)$. We can express Ψ_1^i in a similar manner. Since ψ_1^i and Ψ_1^i are orthogonal, we must have

$$\Psi_1^i(x) = B\delta(x - c_0^i) - A\delta(x - \tilde{c}_0^i) + S(x). \quad \square$$

Lemma 2.11. *If $c_1^i \in S_1$ then*

(a)

$$\frac{d}{dK} E_1^i = (A^2 - B^2)\frac{d}{dK} E_0^i + O(\delta_0^{1/2}\sqrt{E^* + 1}),$$

$$\frac{d}{dK} \mathscr{E}_1^i = (B^2 - A^2)\frac{d}{dK} E_0^i + O(\delta_0^{1/2}\sqrt{E^* + 1}).$$

(b)

$$\frac{d^2}{dK^2} E_1^i = \frac{2\langle \psi_1^i, V'\Psi_1^i\rangle^2}{E_1^i - \mathscr{E}_1^i} + O(\delta_0^{-1/8}\sqrt{E^* + 1}),$$

$$\frac{d^2}{dK^2} \mathscr{E}_1^i = \frac{2\langle \psi_1^i, V'\Psi_1^i\rangle^2}{\mathscr{E}_1^i - E_1^i} + O(\delta_0^{-1/8}\sqrt{E^* + 1})$$

(*both hold where* $E_1^i \ne \mathscr{E}_1^i$).

(c)

$$\langle \psi_1^i, V'\Psi_1^i\rangle = 2AB\frac{d}{dK}E_0^i + O(\delta_0^{1/2}\sqrt{E^* + 1}).$$

(d)

$$\left|\frac{d}{dK} E_1^i + \frac{d}{dK}\mathscr{E}_1^i\right| \le \mathrm{cst}\,\delta_0^{1/2}\sqrt{E^* + 1}.$$

Proof. By Lemma 2.8(b) we have

$$\begin{aligned}\frac{d}{dK} E_1^i = \langle \psi_1^i, V'\psi_1^i\rangle &= A^2\frac{d}{dK} E_0^i + B^2\frac{d}{dK}\tilde{E}_0^i + O(\delta_0^2\sqrt{E^* + 1}) \\ &= (A^2 - B^2)\frac{d}{dK} E_0^i + B^2\left(\frac{d}{dK} E_0^i + \frac{d}{dK}\tilde{E}_0^i\right) + O(\delta_0^2\sqrt{E^* + 1}).\end{aligned}$$

We use Lemma 2.9(a) to bound the second term of the last line and complete the proof of part (a).

To prove part (b) we use Appendix B to write

$$\frac{d^2}{dK^2} E_1^i = 2 + 2\frac{\langle \psi_1^i, V'\Psi_1^i\rangle^2}{E_1^i - \mathscr{E}_1^i} + 2\langle V'\psi_1^i, (E_1^i - H)_{\perp\perp}^{-1} V'\psi_1^i\rangle.$$

The remainder term is bounded by $2\|V'\psi_1^i\|^2\|(E_1^i - H)_{\perp\perp}^{-1}\|$, and by Lemma 2.8(c) we have $\|(E_1^i - H)_{\perp\perp}^{-1}\| \le (\delta_0^{1/8}\sqrt{E^* + 1})^{-1}$. It remains to estimate

$$\|V'\psi_1^i\|^2 = A^2\left(\frac{d}{dK} E_0^i\right)^2 + B^2\left(\frac{d}{dK}\tilde{E}_0^i\right)^2 + O(\delta_0^2\sqrt{E^* + 1})^2.$$

We now use Lemma 1.1(c) to bound the first two terms of the last equation. Therefore $\|V'\psi_1^i\|^2 \le \mathrm{cst}(E^* + 1)$, and part (b) is established.

We now prove part (c). By Lemma 2.9(a) we can estimate

$$\begin{aligned}\langle \psi_1^i, V'\Psi_1^i\rangle &= AB\frac{d}{dK}E_0^i - AB\frac{d}{dK}\tilde{E}_0^i + O(\delta_0^2\sqrt{E^*+1})\\ &= 2AB\frac{d}{dK}E_0^i + O(\delta_0^{1/2}\sqrt{E^*+1}).\end{aligned}$$

Part (d) follows immediately from parts (a) and (b). □

Lemma 2.12. *If* $|dE_1^i/dK| \le \delta_0^{1/4}\sqrt{E^*+1}$ *for some* K *in the interval* $|K-K^*| \le 6\delta_0^{1/2}$ *then*

(a) $|\langle \psi_1^i, V'\Psi_1^i\rangle| \ge \frac{1}{3}\delta_0^{1/8}\sqrt{E^*+1}$.

(b) $|d^2E_1^i/dK^2| \ge \sqrt{E^*+1}$ *at all points where* $E_1^i \ne \mathscr{E}_1^i$; *moreover* $d^2E_1^i/dK^2$ *has a unique sign.*

Remark 1. Lemma 2.12 holds if E_1^i is replaced by $\mathscr{E}_1^i$.

Remark 2. We will show later that under the hypothesis of Lemma 2.12, $E_1^i \ne \mathscr{E}_1^i$ for $|K-K^*| \le 6\delta_0^{1/2}$.

Proof. If $|dE_1^i/dK| \le \delta_0^{1/4}\sqrt{E^*+1}$ then by Lemma 2.11(a) we have

$$|A^2 - B^2||dE_0^i/dK| \le \text{cst}\,\delta_0^{1/4}\sqrt{E^*+1},$$

which by Lemma 2.9(b) implies $|A^2-B^2| = O(\delta_0^{1/8})$. Since $1 \ge A^2+B^2 \ge 1-\delta_0$ we must have $|AB| \ge \frac{1}{4}$. Lemmas 2.11(c) and 2.9(b) give us

$$|\langle \psi_1^i, V'\Psi_1^i\rangle \ge \tfrac{1}{3}\delta_0^{1/8}\sqrt{E^*+1}.$$

To prove part (b) we use Lemmas 2.11(b), 2.8(a) and part (a). □

By (0.11) we have

$$E_1^i(K_s+\delta K) = E_1^i(K_s-\delta K) \quad \text{or} \quad E_1^i(K_s+\delta K) = \mathscr{E}_1^i(K_s-\delta K). \tag{2.5}$$

Each eigenvalue curve itself might not be symmetric about K_s, but the union of both curves is symmetric. (See Figure 2.)

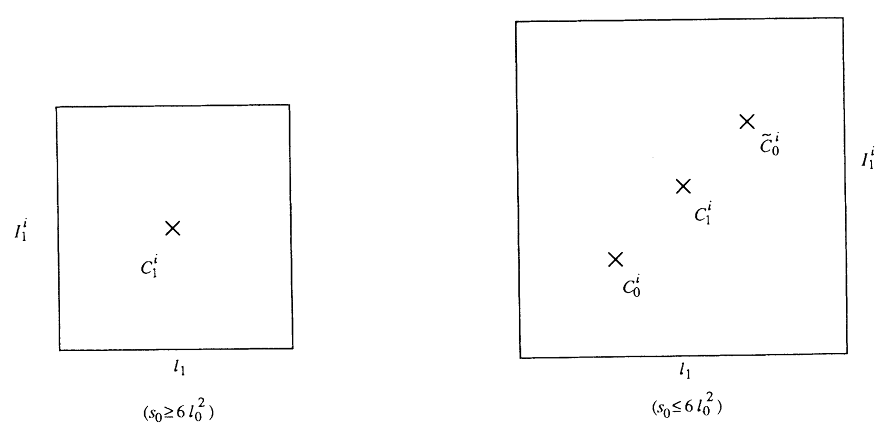


FIGURE 1

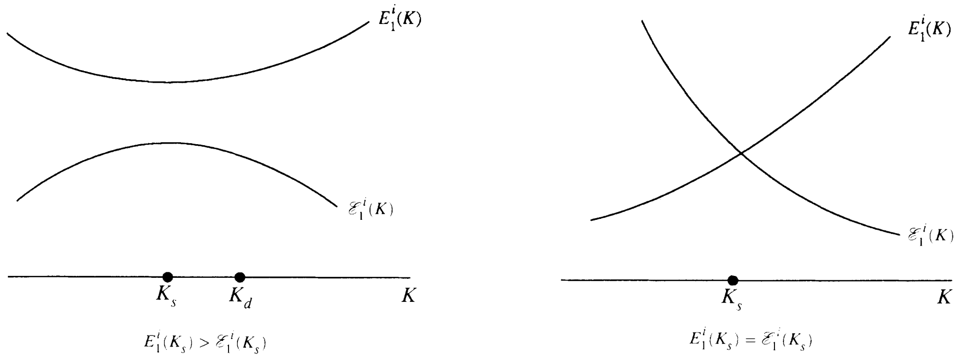


FIGURE 2

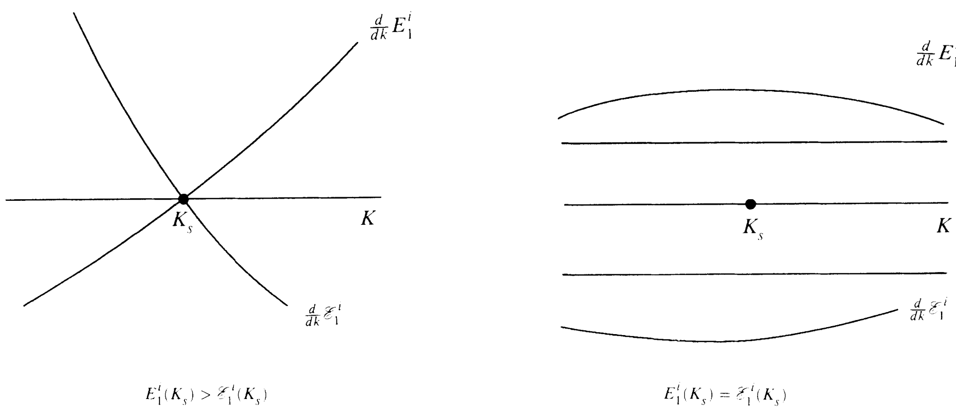


FIGURE 3

Lemma 2.13. *If* $c_1^i \in S_1$ *then*

$$|E_1^i(K_2) - E_1^i(K_1)| \geq \delta_0^2 \sqrt{E^* + 1} \min \begin{cases} |K_2 - K_1|^2, \\ |K_2 + K_1 + 2[c_1^i]|^2 \end{cases}$$

for any points K_1, K_2 *belonging to the interval* $|K - K^*| \leq 6\delta_0^{1/2}$. *The same is true for* $\mathscr{E}_1^i$.

Proof. There are two cases to consider:

Case I. $E_1^i(K_s) \neq \mathscr{E}_1^i(K_s)$ (see Figures 2, 3).

Without loss of generality we may assume $E_1^i(K_s) > \mathscr{E}_1^i(K_s)$. By (2.5) we must have

$$E_1^i(K_s + \delta K) = E_1^i(K_s - \delta K) \quad \text{and} \quad \mathscr{E}_1^i(K_s + \delta K) = \mathscr{E}_1^i(K_s - \delta K)$$

for δK small; therefore

$$(dE_1^i/dK)(K_s) = (d\mathscr{E}_1^i/dK)(K_s) = 0. \tag{2.6}$$

By Lemmas 2.12(a) and 2.11(b) we see that $d^2E_1^i/dK^2$ and $d^2\mathscr{E}_1^i/dK^2$ are large with opposite signs. Appendix A and Lemma 2.12(b) give us

$$|E_1^i(K_2) - E_1^i(K_1)| \geq \delta_0^2\sqrt{E^* + 1}\min\begin{cases} |K_2 - K_1|^2, \\ |K_2 + K_1 + 2[c_1^i]|^2. \end{cases}$$

Case II. $E_1^i(K_s) = \mathscr{E}_1^i(K_s)$ (see Figures 2, 3).

In this case we will show that

$$\left|\frac{d}{dK}E_1^i\right| \geq \delta_0^{1/4}\sqrt{E^* + 1} \quad \text{and} \quad \left|\frac{d}{dK}\mathscr{E}_1^i\right| \geq \delta_0^{1/4}\sqrt{E^* + 1} \tag{2.7}$$

holds for every K in the interval $|K - K^*| \leq 6\delta_0^{1/2}$; moreover the derivatives have opposite signs.

If (2.7) holds, the lemma follows immediately. We must now prove (2.7). We start by showing that (2.7) is true *at* the symmetry point K_s. Differentiating the equation $H(I_1^i)\psi_1^i = E_1^i\psi_1^i$ yields

$$H'\psi_1^i + H\left(\frac{d}{dK}\psi_1^i\right) = \left(\frac{d}{dK}E_1^i\right)\psi_1^i + E_1^i\left(\frac{d}{dK}\psi_1^i\right).$$

Note that $H' = V'$ and then set $K = K_s$. Let P be the projection onto the nullspace of $(H - E)$ where $E = E_1^i(K_s) = \mathscr{E}_1^i(K_s)$. Now multiply the last equation by P. We have

$$PV'\psi_1^i + PH\left(\frac{d}{dK}\psi_1^i\right) = \left(\frac{d}{dK}E_1^i\right)P\psi_1^i + E_1^iP\left(\frac{d}{dK}\psi_1^i\right).$$

Notice that the second term on the left, $PH(d\psi_1^i/dK) = HP(d\psi_1^i/dK) = E_1^iP(d\psi_1^i/dK)$, is canceled by the same term on the right; therefore $PV'\psi_1^i = (dE_1^i/dK)P\psi_1^i$. Using the fact that $P\psi_1^i = \psi_1^i$, we rewrite this equation as $(PV'P)\psi_1^i = (dE_1^i/dK)(K_s)\psi_1^i$, which says that $(dE_1^i/dK)(K_s)$ is an eigenvalue of the 2×2 matrix $PV'P$. The same argument with E_1^i replaced by $\mathscr{E}_1^i$ implies that $(d\mathscr{E}_1^i/dK)(K_s)$ is the other eigenvalue.

To calculate these eigenvalues we will represent $PV'P$ in a special basis. To do this we define the operator S by

$$(S\phi)(x) = \phi(2c_1^i - x).$$

The eigenfunctions of S are ψ_s and ψ_a, symmetric and antisymmetric wave functions, i.e.,

$$\psi_s(2c_1^i - x) = \psi_s(x) \quad \text{and} \quad \psi_a(2c_1^i - x) = -\psi_a(x).$$

Note that $H(I_1^i, K_s)$ commutes with S, which allows us to express $PV'P$ in the basis $\{\psi_s, \psi_a\}$. In this basis we have

$$PV'P = \begin{pmatrix} \langle\psi_s, V'\psi_s\rangle & \langle\psi_s, V'\psi_a\rangle \\ \langle\psi_s, V'\psi_a\rangle & \langle\psi_a, V'\psi_a\rangle \end{pmatrix} \quad (\text{at } K = K_s).$$

Now by the symmetry properties of ψ_s, ψ_a, and $V'(K_s)$ we have $\langle \psi_s, V'\psi_s\rangle = \langle \psi_a, V'\psi_a\rangle = 0$, which gives us

$$PV'P = \begin{pmatrix} 0 & \langle \psi_s, V'\psi_a\rangle \\ \langle \psi_s, V'\psi_a\rangle & 0 \end{pmatrix}$$

and therefore

$$\left(\frac{d}{dK} E_1^i\right)(K_s) = -\left(\frac{d}{dK} \mathscr{E}_1^i\right)(K_s) = \langle \psi_s, V'\psi_a\rangle .$$

We must show that $(dE_1^i/dK)(K_s)$ is not too small and then extend this for $|K - K^*| \le 6\delta_0^{1/2}$. Using the symmetry properties and the decay of the wave function, we have

$$\left(\frac{d}{dK} E_1^i\right)(K_s) = 2\psi_s(c_0^i)\psi_a(c_0^i)\left(\frac{d}{dK} E_0^i\right)(K_s) + O(\delta_0^2\sqrt{E^*+1}) \tag{2.8}$$

where $|\psi_s(c_0^i)| \cong 1/\sqrt{2}$ and $|\psi_a(c_0^i)| \cong 1/\sqrt{2}$. Lemma 2.9(b) together with (2.8) gives us

$$|(dE_1^i/dK)(K_s)| \ge \delta_0^{1/4}\sqrt{E^*+1}.$$

We must show that this continues to hold for all K in the interval $|K - K^*| \le 6\delta_0^{1/2}$. To see this, we plot the derivatives of $E_1^i(K)$ and $\mathscr{E}_1^i(K)$. We claim that these curves cannot enter the dashed region. (See Figure 3.) If $|dE_1^i/dK| \le \delta_0^{1/4}\sqrt{E^*+1}$ for some K, by Lemmas 2.12(b) and 2.11(b) we have $d^2E_1^i/dK^2 > 0$. This contradicts Figure 3. □

Lemma 2.14. *Let* $c_1^i \in S_1$; *then*

(a) $|dE_1^i/dK| \le (3 + \delta_0^{1/4})\sqrt{E^*+1}$.

(b) *If* $|dE_1^i/dK| \le \delta_0^2\sqrt{E^*+1}$ *for some* K *in the interval* $|K - K^*| \le 6\delta_0^{1/2}$ *then* $|dE_1^i/dK| \ge \frac{1}{2}\sqrt{E^*+1}|K + [c_1^i]|$.

Remark. The same is true for $\mathscr{E}_1^i$.

Proof. We use Lemmas 2.11(a) and 1.1(c) to prove part (a). To establish part (b) we use Appendix A and Lemma 2.12(b). □

Lemma 2.15. *If* $c_1^r \in S_1$ *for* $r = i, j$ *then*

(a) $|E_1^i(K^*) - E_1^j(K^*)| \ge \delta_0^2\sqrt{E^*+1}\, m(c_1^i, c_1^j)^2$. *The same is true for* $|\mathscr{E}_1^i(K^*) - \mathscr{E}_1^j(K^*)|$.

(b) $|E_1^i(K^*) - \mathscr{E}_1^j(K^*)| \ge \delta_0^2\sqrt{E^*+1}\, m(c_1^i, c_1^j)^2$.

(c) (*Center Theorem*) $m(c_1^i, c_1^j) \le 2\delta_1^{1/2}/\delta_0$.

Proof. If $E_1^j(K^*) = E_1^i(K^* + \Delta K)$, we must estimate $|E_1^i(K^*) - E_1^i(K^* + \Delta K)|$. By Lemma 2.13, this difference is bounded below by

$$\delta_0^2\sqrt{E^*+1}\,\min\begin{cases} |\Delta K|^2, \\ |2K^* + \Delta K + 2[c_1^i]|^2. \end{cases}$$

We now use the definition of ΔK in Lemma 2.8(d) to obtain part (a).

We must consider the case where $E_1^j(K^*) = \mathscr{E}_1^i(K^* + \Delta K)$ and estimate $|E_1^i(K^*) - \mathscr{E}_1^i(K^* + \Delta K)|$. A careful inspection of the geometry of the eigenvalue curves gives us

$$|E_1^i(K^*) - \mathscr{E}_1^i(K^* + \Delta K)| \geq \min \begin{cases} |E_1^i(K^*) - E_1^i(K^* + \Delta K)|, \\ |\mathscr{E}_1^i(K^*) - \mathscr{E}_1^i(K^* + \Delta K)|. \end{cases}$$

We now use the result in the first part of the proof to establish parts (a) and (b).

To prove the Center Theorem, we note that by definition of S_1, one of the eigenvalue differences in parts (a) and (b) must be bounded above by $2\delta_1\sqrt{E^* + 1}$. □

It is important to know that the eigenvalues $E_1^i(K)$ and $\mathscr{E}_1^i(K)$ may cross only at the symmetry point $K_s = -[c_1^i]$, and their separation grows as K moves away from K_s.

Lemma 2.16. *If* $c_1^i \in S_1$ *then*

$$|E_1^i(K) - \mathscr{E}_1^i(K)| \geq \delta_0^2\sqrt{E^* + 1}|K + [c_1^i]|$$

for all K *in the interval* $|K - K^*| \leq 6\delta_0^{1/2}$.

Proof. We must consider two cases:

Case I. $E_1^i(K_s) = \mathscr{E}_1^i(K_s)$ (see Figures 2, 3).

In this case, by (2.7) we have

$$\begin{aligned} |(E_1^i - \mathscr{E}_1^i)(K)| &= \left|(E_1^i - \mathscr{E}_1^i)(K_s) + \frac{d}{dK}(E_1^i - \mathscr{E}_1^i)(\widehat{K})(K - K_s)\right| \\ &\geq 2\delta_0^{1/4}\sqrt{E^* + 1}|K - K_s| \\ &= 2\delta_0^{1/4}\sqrt{E^* + 1}|K + [c_1^i]|. \end{aligned}$$

Case II. $E_1^i(K_s) > \mathscr{E}_1^i(K_s)$ (see Figures 2, 3).

By (2.6) and Lemma 2.12(a) we have

$$|\langle \psi_1^i, V'\Psi_1^i\rangle(K_s)| \geq \tfrac{1}{3}\delta_0^{1/8}\sqrt{E^* + 1};$$

therefore there must be an interval $K_s \leq K \leq K_d$ where $|\langle \psi_1^i, V'\Psi_1^i\rangle(K)| \geq \frac{1}{3}\delta_0^{1/8}\sqrt{E^* + 1}$. If K is in this interval then

$$\begin{aligned} (E_1^i - \mathscr{E}_1^i)(K) = (E_1^i - \mathscr{E}_1^i)(K_s) &+ \frac{d}{dK}(E_1^i - \mathscr{E}_1^i)(K_s)(K - K_s) \\ &+ \frac{1}{2}\frac{d^2}{dK^2}(E_1^i - \mathscr{E}_1^i)(\widehat{K})(K - K_s)^2. \end{aligned}$$

By Lemma 2.11(b) we have

$$\begin{aligned}\frac{d^2}{dK^2}(E_1^i - \mathscr{E}_1^i)(\hat{K}) &= \frac{4\langle \psi_1^i, V'\Psi_1^i\rangle^2}{(E_1^i - \mathscr{E}_1^i)(\hat{K})} + O(\delta_0^{-1/8}\sqrt{E^* + 1}) \\ &\geq \mathrm{cst}\frac{(\delta_0^{1/8}\sqrt{E^*+1})^2}{(E_1^i - \mathscr{E}_1^i)(K)},\end{aligned}$$

which implies

$$(E_1^i - \mathscr{E}_1^i)(K) \geq \mathrm{cst}\frac{(\delta_0^{1/8}\sqrt{E^*+1})^2}{(E_1^i - \mathscr{E}_1^i)(K)}(K - K_s)^2$$

and proves the lemma.

We must now consider the case when $K \geq K_d$. By definition of K_d we have

$$|\langle \psi_1^i, V'\Psi_1^i\rangle| \leq \tfrac{1}{3}\delta_0^{1/8}\sqrt{E^*+1} \quad \text{for } K \geq K_d;$$

therefore by Lemma 2.12(a) we must have

$$\frac{d}{dK}E_1^i \geq \delta_0^{1/4}\sqrt{E^*+1} \quad \text{and} \quad \frac{d}{dK}\mathscr{E}_1^i \leq -\delta_0^{1/4}\sqrt{E^*+1}$$

giving us

$$\begin{aligned}(E_1^i - \mathscr{E}_1^i)(K) &= (E_1^i - \mathscr{E}_1^i)(K_d) + \frac{d}{dK}(E_1^i - \mathscr{E}_1^i)(\hat{K})(K - K_d) \\ &\geq (E_1^i - \mathscr{E}_1^i)(K_d) + 2\delta_0^{1/4}\sqrt{E^*+1}\,(K - K_d).\end{aligned}$$

Now by the first part of the proof we can bound

$$(E_1^i - \mathscr{E}_1^i)(K_d) \geq \mathrm{cst}\,\delta_0^{1/8}\sqrt{E^*+1}\,(K_d - K_s)$$

which gives us

$$(E_1^i - \mathscr{E}_1^i)(K) \geq \delta_0^2\sqrt{E^*+1}\,(K - K_s) = \delta_0^2\sqrt{E^*+1}\,|K + [c_1^i]|. \quad \square$$

Remark. We are done with Case 2 ($s_0 \leq 6l_0^2$). We will list the most important properties of S_1 in our induction hypothesis.

Induction hypothesis. For $n \geq 1$ we define

$$S_n(K^*, E^*) \equiv \{c_n^i \in \overline{S}_{n-1} : \mathrm{dist}(\sigma(H(I_n^i, K^*)), E^*) \leq \delta_n\sqrt{E^*+1}\}.$$

We assume that every point $c_n^i \in S_n$ belongs to either class A or class B. We also assume that the Decay Theorem holds for S_n. (See Appendix C.)

Class A. For every K in the interval $|K - K^*| \leq 2\delta_{n-1}^{3/2}$, there exists a unique eigenvalue $E_n^i(K) \in \sigma(H(I_n^i))$ such that

(H1) $|E_n^i(K) - E^*| \leq \mathrm{cst}\,\delta_{n-1}^{3/2}\sqrt{E^*+1}$.

(H2) $|dE_n^i/dK| \leq (3 + \mu_n)\sqrt{E^*+1}$ where $\mu_0 = 0$ and $|\mu_n - \mu_{n-1}| \leq 1/4^n$.

(H3) If $|dE_n^i/dK| \leq \delta_{n-1}^2 \sqrt{E^* + 1}$ for some K in the interval $|K - K^*| \leq \delta_{n-1}^{3/2}$ then

$$\left| \frac{d}{dK} E_n^i \right| \geq \tfrac{1}{2} \sqrt{E^* + 1} |K + [c_n^i]|$$

and

$$\left| \frac{d^2}{dK^2} E_n^i \right| \geq (1 - \nu_n) \sqrt{E^* + 1}$$

where $\nu_0 = 0$ and $|\nu_n - \nu_{n-1}| \leq 1/4^n$; moreover $d^2 E_n^i / dK^2$ has a unique sign.

(H4) $|\psi_n^i(x)| \leq \mathrm{cst}\, e^{-(\gamma_0/4) l_n}$ for $x \in \partial I_n^i$.

(H5) $|E_n^i(K^*)| - E_n^j(K^*)| \geq \frac{1}{2} \delta_{n-1}^2 \sqrt{E^* + 1} m(c_n^i, c_n^j)^2$.

Class B. For every K in the interval $|K - K^*| \leq 6\delta_{n-1}^{1/2} / \delta_{n-2}$, there exist exactly two eigenvalues $E_n^i(K)$, $\mathcal{E}_n^i(K) \in \sigma(H(I_n^i))$ such that

(H6) $|E_n^i(K) - E^*| \leq \mathrm{cst}\, \delta_{n-1}^{1/2} / \delta_{n-2}$. The same holds for $\mathcal{E}_n^i$.

(H7) $|K^* + [c_n^i]| \leq 3\delta_{n-1}^{1/2} / \delta_{n-2}$.

Note: $\delta_{-1} \equiv 1$.

(H8) $|E_n^i(K) - \mathcal{E}_n^i(K)| \geq \delta_{n-1}^2 \sqrt{E^* + 1} |K + [c_n^i]|$.

(H9) $|dE_n^i/dK| \leq (3 + \mu_n) \sqrt{E^* + 1}$ where $\mu_0 = 0$ and $|\mu_n - \mu_{n-1}| \leq 1/4^n$. The same holds for $\mathcal{E}_n^i$.

(H10) If $|dE_n^i/dK| \leq \delta_{n-1}^2 \sqrt{E^* + 1}$ for some K in the interval $|K - K^*| \leq 6\delta_{n-1}^{1/2} / \delta_{n-2}$ then

$$\left| \frac{d}{dK} E_n^i \right| \geq \tfrac{1}{2} \sqrt{E^* + 1} |K + [c_n^i]|$$

and

$$\left| \frac{d^2}{dK^2} E_n^i \right| \geq (1 - \nu_n) \sqrt{E^* + 1}$$

where $\nu_0 = 0$ and $|\nu_n - \nu_{n-1}| \leq 1/4^n$; moreover $d^2 E_n^i / dK^2$ has a unique sign. The same holds for $\mathcal{E}_n^i$.

(H11) $|\psi_n^i(x)| \leq \mathrm{cst}\, e^{-(\gamma_0/4) l_n}$ for $x \in \partial I_n^i$. The same holds for Ψ_n^i.

(H12) $|E_n^i(K^*) - E_n^j(K^*)| \geq \frac{1}{2} \delta_{n-1}^2 \sqrt{E^* + 1}\, m(c_n^i, c_n^j)^2$. The same holds for $|\mathcal{E}_n^i(K^*) - \mathcal{E}_n^j(K^*)|$ and $|E_n^i(K^*) - \mathcal{E}_n^j(K^*)|$.

3. Definition and properties of S_{n+1}

In this section, we will assume that the induction hypothesis is true for S_n and then prove that it holds for S_{n+1}. Many of the proofs run along the same lines as their counterparts in §2 and will be omitted.

We construct $\overline{S}_n$ from S_n in the usual way by pairing elements together if $s_n \leq 6 l_n^2$. Then we define

$$S_{n+1}(K^*, E^*) \equiv \{c_{n+1}^i \in \overline{S}_n : \mathrm{dist}(\sigma(H(I_{n+1}^i)), E^*) \leq \delta_{n+1} \sqrt{E^* + 1}\}.$$

Case 1. $s_n \geq 6l_n^2$.

This case will be broken up into two subcases, according to the number of eigenvalues in $\sigma(H(I_n^i))$ that are near E^*. Before we consider the two subcases, we will prove some general facts.

Lemma 3.1. *If* $c_{n+1}^r \in S_{n+1}$ *for* $r = i, j$ *then*

$$m(c_{n+1}^i, c_{n+1}^j) \leq \delta_n^{3/2}.$$

Proof. Since $c_{n+1}^r = c_n^r$, by (H5), (H12) it suffices to show that

$$|E_n^i(K^*) - E_n^j(K^*)| = O(\delta_n^4\sqrt{E^* + 1})$$

where $E_n^r \in \sigma(H(I_n^r))$.

By definition of S_{n+1}, we have eigenvalues $E_{n+1}^r(K^*) \in \sigma(H(I_{n+1}^r))$ such that $|E_{n+1}^r(K^*) - E^*| \leq \delta_{n+1}\sqrt{E^* + 1}$. We define $A = I_{n+1}^r \backslash \widehat{I}_n^r$ where $\widehat{I}_n^r$ is a box with center c_{n+1}^r whose length is chosen to be $O(l_n^{1/2})$ so that $S_n \cap A = \varnothing$. By induction we have

$$|(H(A) - E_{n+1}^r)^{-1}(x, y)| \leq \frac{\text{cst}}{\delta_0\sqrt{E^* + 1}} e^{-(\gamma_0/2)|x-y|}$$

provided $|x - y| \geq l_n^{5/6}$. This will allow us to prove that the wave function ψ_{n+1}^r decays outside I_n^r. To see this we restrict the eigenvalue equation

$$H(I_{n+1}^r)\psi_{n+1}^r = E_{n+1}^r\psi_{n+1}^r \tag{3.1}$$

to the set A,

$$\psi_{n+1}^r = (H(A) - E_{n+1}^r)^{-1}\,\Gamma(\widehat{I}_n^r)\psi_{n+1}^r.$$

Suppose $x \notin I_n^r$. Since $\text{dist}(x, \partial\widehat{I}_n^r) \geq l_n^{5/6}$, we have

$$|\psi_{n+1}^r(x)| \leq \sum_{\partial \hat{I}_n^r} \frac{\text{cst}\,\varepsilon}{\delta_0\sqrt{E^* + 1}} e^{-(\gamma_0/2)|x-y|}$$

and since $|x - y| \geq |x - c_{n+1}^r| - l_n^{1/2}$, we get

$$|\psi_{n+1}^r(x)| \leq \text{cst}\, e^{-(\gamma_0/4)|x-c_{n+1}^r|} \quad \text{for } x \notin I_n^r. \tag{3.2}$$

We will use ψ_{n+1}^r as a trial wave function for $H(I_n^r)$. To do this, we restrict (3.1) to I_n^r, yielding

$$(H(I_n^r) - E_{n+1}^r)\psi_{n+1}^r = \Gamma(I_n^r)\psi_{n+1}^r;$$

therefore by (3.2) we get

$$\|(H(I_n^r) - E_{n+1}^r)\psi_{n+1}^r\| \leq \delta_n^4\sqrt{E^* + 1}.$$

Thus by trial wave function we have

$$|E_n^r(K^*) - E_{n+1}^r(K^*)| \leq \delta_n^4\sqrt{E^* + 1};$$

therefore

$$|E_n^r(K^*) - E^*| \leq 2\delta_n^4\sqrt{E^* + 1} \quad \text{for } r = i, j. \quad \square \tag{3.3}$$

Remark 1. This lemma will be strengthened later by replacing $\delta_n^{3/2}$ with $2\delta_{n+1}^{1/2}/\delta_n$.

Remark 2. By (H2), (H9), and (3.3) we have

$$|E_n^i(K) - E^*| \leq \text{cst}\,\delta_n^{3/2}\sqrt{E^* + 1} \quad \text{for } |K - K^*| \leq 4\delta_n^{3/2}. \tag{3.4}$$

We are now ready to break up case 1 into two subcases.

Subcase 1A.

$$\text{dist}(\sigma(H(I_n^i)) - E_n^i, E^*) \geq \delta_{n-1}\sqrt{\delta_n}\sqrt{E^* + 1} \tag{3.5}$$

for every K in the interval $|K - K^*| \leq 4\delta_n^{3/2}$. In this case we will show how to get back to class A of the induction hypothesis.

Lemma 3.2. *If* $c_{n+1}^i \in S_{n+1}$ *then*

(a) *There exists a unique eigenvalue* $E_{n+1}^i(K) \in \sigma(H(I_{n+1}^i))$ *such that*

$$|E_{n+1}^i(K) - E^*| \leq \text{cst}\,\delta_n^{3/2}\sqrt{E^* + 1}$$

for every K *in the interval* $|K - K^*| \leq 4\delta_n^{3/2}$.

(b) $|\psi_{n+1}^i(x)| \leq \text{cst}\,e^{-(\gamma_0/4)|x - c_{n+1}^i|}$ *for* $x \notin I_n^i$.

(c) $\|\psi_{n+1}^i - \psi_n^i\| \leq \text{cst}\,\delta_n^3$.

(d) *Any other eigenvalue* $\mathscr{E}_{n+1}^i \in \sigma(H(I_{n+1}^i))$ *must obey*

$$|\mathscr{E}_{n+1}^i - E^*| \geq (\delta_n/5)\sqrt{E^* + 1}.$$

(e) $E_{n+1}^j(K^*) = E_{n+1}^i(K^* + \Delta K)$ *where* $\Delta K = [c_{n+1}^j] - [c_{n+1}^i]$ *or* $-([c_{n+1}^j] + [c_{n+1}^i] + 2K^*)$ *and* $|\Delta K| = m(c_{n+1}^i, c_{n+1}^j)$.

Proof. We use (H4), (H11) to get

$$\|(H(I_{n+1}^i) - E_n^i)\psi_n^i\| = \|\Gamma(I_n^i)\psi_n^i\| \leq \delta_n^4\sqrt{E^* + 1}.$$

Hence ψ_n^i serves as a good trial wave function and we conclude that there exists an eigenvalue $E_{n+1}^i(K) \in \sigma(H(I_{n+1}^i))$ such that

$$|E_{n+1}^i(K) - E_n^i(K)| \leq \delta_n^4\sqrt{E^* + 1}. \tag{3.6}$$

Therefore by (3.4) we have

$$|E_{n+1}^i(K) - E^*| \leq \text{cst}\,\delta_n^{3/2}\sqrt{E^* + 1}.$$

We now examine the wave function ψ_{n+1}^i. Part (b) follows from the fact that $S_n \cap (I_{n+1}^i \backslash \widehat{I}_n^i) = \varnothing$. (See (3.2).) To prove part (c) we must show that ψ_{n+1}^i

is close to ψ_n^i inside I_n^i. To see this, we restrict (3.1) to I_n^i and arrange the terms as follows:

$$(H(I_n^i) - E_{n+1}^i)\psi_{n+1}^i = \Gamma(I_n^i)\psi_{n+1}^i.$$

By Lemma 3.2(b) we can bound the right-hand side so that

$$\psi_{n+1}^i = A\psi_n^i + (H(I_n^i) - E_{n+1}^i)_\perp^{-1}(O(\delta_n^4\sqrt{E^*+1})).$$

Using (3.5) we get $\|\psi_{n+1}^i - A\psi_n^i\| \le \mathrm{cst}\,\delta_n^3$, and since ψ_{n+1}^i is normalized, we must have $|A-1| = O(\delta_n^3)$.

To prove part (d) we assume that there is another eigenvalue $\mathscr{E}_{n+1}^i \in \sigma(H(I_{n+1}^i))$ such that $|\mathscr{E}_{n+1}^i - E^*| \le (\delta_n/5)\sqrt{E^*+1}$. The same argument shows that its wave function decays outside I_n^i and is close to ψ_n^i inside I_n^i. This violates orthogonality. □

The next lemma tells us that the derivatives of E_{n+1}^i closely approximate those of E_n^i. The proof uses Cauchy's theorem, so we must state the lemma in its complex form.

Lemma 3.3. *If* $c_{n+1}^i \in S_{n+1}$ *then*

(a) $|dE_{n+1}^i/dK - dE_n^i/dK| \le \mathrm{cst}\,\delta_n^2\sqrt{E^*+1}$ *for every* K *in the complex disk* $|K-K^*| \le 4\delta_n^{3/2}$.

(b) $|d^2E_{n+1}^i/dK^2 - d^2E_n^i/dK^2| \le \mathrm{cst}\,\delta_n^{1/2}\sqrt{E^*+1}$ *for every* K *in the complex disk* $|K-K^*| \le 2\delta_n^{3/2}$.

Proof. Since $H(K) = \varepsilon\Delta + V(K)$ is analytic, we can extend the eigenvalues and eigenfunctions to be complex analytic functions of K for K in the disk $|K-K^*| \le 4\delta_n^{3/2}$. For complex K we have $dE_{n+1}^i/dK = \langle\psi_{n+1}^i, V'\psi_{n+1}^i\rangle_{\mathbf{R}}$ and $dE_n^i/dK = \langle\psi_n^i, V'\psi_n^i\rangle_{\mathbf{R}}$ where $\langle\phi, \psi\rangle_{\mathbf{R}} = \sum\phi(i)\psi(i)$. To prove part (a) it suffices to show that

$$\text{(3.7)} \qquad \|\psi_{n+1}^i - \psi_n^i\| \le \mathrm{cst}\,\delta_n^3 \quad\text{and}\quad \|V'\| \le \delta_n^{-1}\sqrt{E^*+1}.$$

We note that

$$\begin{aligned}\|V'\| &\le \max_{I_{n+1}}|v_x'| = \max 2|[x]+K| \le \max 2|[x] - [c_{n+1}^i]| \\ &\quad + \max 2|[c_{n+1}^i] + K^*| + \max 2|K-K^*| \le \mathrm{cst}\, l_{n+1}\sqrt{E^*+1} \\ &\le \delta_n^{-1}\sqrt{E^*+1}.\end{aligned}$$

To bound the difference of the eigenfunctions we need to establish decay estimates on the Green's function for complex K. We begin by taking a contour $C\colon |z-E^*| = \frac{1}{2}\delta_{n-1}\sqrt{\delta_n}\sqrt{E^*+1}$. By (3.4) and (3.5) we see that $E_n^i(K^*) \in \sigma(H(I_n^i))$ is an isolated eigenvalue inside C. When $K = K^*$ the operator H is selfadjoint; therefore

$$\|G_n(K^*, z)\| = \|(H(I_n^i, K^*) - z)^{-1}\| \le \mathrm{cst}(\delta_{n-1}\sqrt{\delta_n}\sqrt{E^*+1})^{-1}$$

for $z \in C$. For K in the complex disk we use the resolvent identity

$$G_n(K, z) = G_n(K^*, z) + G_n(K^*, z)(V(K^*) - V(K))G_n(K, z)$$

to bound

$$\|G_n(K, z)\| \le \mathrm{cst}(\delta_{n-1}\sqrt{\delta_n}\sqrt{E^* + 1})^{-1}$$

for $z \in C$ and $|K - K^*| \le 4\delta_n^{3/2}$. Since

$$G_n^{\perp}(K, E) = \frac{1}{2\pi i}\oint_C \frac{G_n(K, z)}{z - E}\, dz$$

we have

$$\|G_n^{\perp}(K, E)\| \le \mathrm{cst}(\delta_{n-1}\sqrt{\delta_n}\sqrt{E^* + 1})^{-1}$$

provided $|E - E^*| \le \mathrm{cst}\,\delta_n\sqrt{E^* + 1}$.

If we restrict the eigenvalue equation $H(I_{n+1}^i)\psi_{n+1}^i = E_{n+1}^i\psi_{n+1}^i$ to I_n^i we get

$$\psi_{n+1}^i = A\psi_n^i + G_n^{\perp}(K, E_{n+1}^i)\Gamma(I_n^i)\psi_{n+1}^i .$$

Since (3.2) holds for complex K we have

$$\|\psi_{n+1}^i - \psi_n^i\| \le \mathrm{cst}\,\delta_n^3$$

provided $|E_{n+1}^i - E^*| \le \mathrm{cst}\,\delta_n\sqrt{E^* + 1}$ for $|K - K^*| \le 4\delta_n^{3/2}$. To show that E_{n+1}^i remains close to E^* we use two contours

$$C_1\colon |z - E^*| = (\delta_n/10)\sqrt{E^* + 1},$$
$$C_2\colon |z - E^*| = (\delta_n/8)\sqrt{E^* + 1}.$$

By Lemma 3.2(d) we see that $E_{n+1}^i(K^*) \in \sigma(H(I_{n+1}^i))$ is an isolated eigenvalue inside C_2. Using the same argument as before, we can bound

$$\|G_{n+1}(K, z)\| \le \mathrm{cst}(\delta_n\sqrt{E^* + 1})^{-1}$$

for $z \in C_2$, $|K - K^*| \le 4\delta_n^{3/2}$ and

$$\|G_{n+1}^{\perp}(K, E)\| \le \mathrm{cst}(\delta_n\sqrt{E^* + 1})^{-1}$$

for $|E - E^*| \le (\delta_n/10)\sqrt{E^* + 1}$. But

$$G_{n+1}^{\perp}(K, E_{n+1}^i) = \frac{1}{2\pi i}\oint_{C_1} \frac{G_{n+1}(K, Z)}{z - E_{n+1}}\, dz$$

is uniformly bounded when E_{n+1}^i is inside C_1; therefore by continuity E_{n+1}^i must stay inside C_1 for every K in the disk $|K - K^*| \le 4\delta_n^{3/2}$.

To prove part (b) we apply Cauchy's theorem to part (a). □

Lemma 3.4. *If* $c_{n+1}^i \in S_{n+1}$ *then*

(a) $|dE_{n+1}^i/dK| \le (3 + \mu_{n+1})\sqrt{E^* + 1}$.

(b) *If* $|dE^i_{n+1}/dK| \le \delta_n^2\sqrt{E^*+1}$ *for some* K *in the interval* $|K-K^*| \le 2\delta_n^{3/2}$ *then*

$$\left|\frac{d^2}{dK^2}E^i_{n+1}\right| \ge (1-\nu_{n+1})\sqrt{E^*+1};$$

moreover $d^2E^i_{n+1}/dK^2$ *has a unique sign.*

Proof. We use Lemma 3.3(a) and (H2), (H9) to establish part (a). We now prove part (b). If the hypothesis holds, then by Lemma 3.3(a),

$$\left|\frac{d}{dK}E^i_n\right| \le \operatorname{cst}\delta_n^2\sqrt{E^*+1}.$$

By Lemma 3.3(b) and (H3), (H10) we get

$$\left|\frac{d^2}{dK^2}E^i_{n+1}\right| \ge (1-\nu_n)\sqrt{E^*+1} - O(\delta_n^{1/2}\sqrt{E^*+1}) \equiv (1-\nu_{n+1})\sqrt{E^*+1}.$$

□

Lemma 3.5. *If* $|dE^i_{n+1}/dK| \le \delta_n^2\sqrt{E^*+1}$ *for some* K *in the interval* $|K-K^*| \le \delta_n^{3/2}$, *then*

$$\left|\frac{d}{dK}E^i_{n+1}\right| \ge \tfrac{1}{2}\sqrt{E^*+1}|K+[c^i_{n+1}]|.$$

Proof. If the hypothesis holds, then by Lemma 3.3(a) we have $|dE^i_n/dK| \le \operatorname{cst}\delta_n^2\sqrt{E^*+1}$. Therefore by (H3), (H10) it follows that $|K+[c^i_{n+1}]| = O(\delta_n^2)$. This implies that the symmetry point $K_s = -[c^i_{n+1}]$ belongs to the interval $|K-K^*| \le 2\delta_n^{3/2}$ which is contained in the interval of definition of $E^i_{n+1}(K)$. We can now use Lemma 3.4(b) and Appendix A to complete the proof. □

Lemma 3.6. *If* $c^r_{n+1} \in S_{n+1}$ *for* $r = i, j$ *then*

(a) $2\delta_{n+1}\sqrt{E^*+1} \ge |E^i_{n+1}(K^*)-E^j_{n+1}(K^*)| \ge \frac{1}{2}\delta_n^2\sqrt{E^*+1}m\,(c^i_{n+1}, c^j_{n+1})^2$.

(b) (*Center Theorem*) $m(c^i_{n+1}, c^j_{n+1}) \le 2\delta_{n+1}^{1/2}/\delta_n$.

Proof. The proof goes word for word the same as the proof of Lemma 2.6 if we replace the indices 0, 1 with n, $n+1$. □

Remark. We are now back to class A of the induction hypothesis.

Subcase 1B.

$$\operatorname{dist}(\sigma(H(I^i_n)) - E^i_n, E^*) \le \delta_{n-1}\sqrt{\delta_n}\sqrt{E^*+1} \tag{3.8}$$

for some $\widehat{K}$ in the interval $|K-K^*| \le 4\delta_n^{3/2}$. In this case we will show how to get back to class B of the induction hypothesis.

Let $m \le n-1$ be the first scale back where $s_m \le 6l_m^2$. Thus we have two boxes I^i_m, $\widetilde{I}^i_m$ such that

$$I^i_m, \widetilde{I}^i_m \subset I^i_{m+1} \subset I^i_{m+2} \subset \cdots \subset I^i_n \subset I^i_{n+1}.$$

Note that

$$c_{n+1}^{i}=c_{m+1}^{i}=\tfrac{1}{2}(c_m^i+\tilde{c}_m^i) \tag{3.9}$$

and by (0.8) we have

$$|[c_m^i]+[\tilde{c}_m^i]+2K^*|\le 6\delta_m^{1/2}/\delta_{m-1}. \tag{3.10}$$

The next lemma tells us how wide an interval we need to take in order to include the symmetry point $K_s=-[c_{n+1}^i]$.

Lemma 3.7. *If* $c_{n+1}^r\in S_{n+1}$ *for* $r=i,j$ *then*

(a) $|K^*+[c_{n+1}^i]|\le 3\delta_n^{1/2}/\delta_{n-1}$.

(b) $\max(|[c_{n+1}^i]-[c_{n+1}^j]|, |[c_{n+1}^i]+[c_{n+1}^j]+2K^*|)\le 6\delta_n^{1/2}/\delta_{n-1}$.

Proof. By (3.4) we have $|E_n^i(K)-E^*|\le \mathrm{cst}\,\delta_n^{3/2}\sqrt{E^*+1}$ for every K in the interval $|K-K^*|\le 4\delta_n^{3/2}$. Equation (3.8) gives us another eigenvalue $\mathscr{E}_n^i(\widehat{K})\in\sigma(H(I_n^i))$ such that

$$|\mathscr{E}_n^i(\widehat{K})-E^*|\le \delta_{n-1}\sqrt{\delta_n}\sqrt{E^*+1}. \tag{3.11}$$

Thus we have two eigenvalues in $\sigma(H(I_n^i))$ close enough to E^* so that our boxes I_n^i belong to class B of the induction hypothesis. By (3.11), (3.8), and (H9) we have $|\mathscr{E}_n^i(K^*)-E^*|\le 2\delta_{n-1}\sqrt{\delta_n}\sqrt{E^*+1}$; therefore

$$|\mathscr{E}_n^i(K^*)-E_n^i(K^*)|\le 3\delta_{n-1}\sqrt{\delta_n}\sqrt{E^*+1}.$$

Since $c_{n+1}^i=c_n^i$, we can use (H8) to establish part (a) of the lemma. Part (b) follows immediately from part (a). □

Lemma 3.8. *If* $c_{n+1}^r\in S_{n+1}$ *for* $r=i,j$ *then*

(a) *There exist two eigenvalues* $E_{n+1}^i(K)$ *and* $\mathscr{E}_{n+1}^i(K)\in\sigma(H(I_{n+1}^i))$ *such that*

$$|E_{n+1}^i(K)-E^*|\le \mathrm{cst}(\delta_n^{1/2}/\delta_{n-1})\sqrt{E^*+1}$$

for every K *in the interval* $|K-K^*|\le 6\delta_n^{1/2}/\delta_{n-1}$. *The same holds for* $\mathscr{E}_{n+1}^i(K)$.

(b) $|\psi_{n+1}^i(x)|\le \mathrm{cst}(e^{-(\gamma_0/4)|x-c_m^i|}+e^{-(\gamma_0/4)|x-\tilde{c}_m^i|})$ *for* $x\notin I_m^i\cup\widetilde{I}_m^i$. *The same holds for* Ψ_{n+1}^i.

(c) $E_{n+1}^j(K^*)=E_{n+1}^i(K^*+\Delta K)$ *or* $E_{n+1}^j(K^*)=\mathscr{E}_{n+1}^i(K^*+\Delta K)$ *where* $\Delta K=[c_{n+1}^j]-[c_{n+1}^i]$ *and* $|\Delta K|\le 6\delta_n^{1/2}/\delta_{n-1}$.

Proof. From (3.4) and (H9) it follows that

$$|E_n^i(K)-E^*|\le \mathrm{cst}(\delta_n^{1/2}/\delta_{n-1})\sqrt{E^*+1} \tag{3.12}$$

for every K in the interval $|K-K^*|\le 6\delta_n^{1/2}/\delta_{n-1}$. By (H11) we have

$$\|(H(I_{n+1}^i)-E_n^i)\psi_n^i\|=\|\Gamma(I_n^i)\psi_n^i\|\le \mathrm{cst}\,\delta_n^4\sqrt{E^*+1};$$

therefore (3.12) implies

$$\|(H(I^i_{n+1}) - E^*)\psi^i_n\| \le \mathrm{cst}(\delta_n^{1/2}/\delta_{n-1})\sqrt{E^*+1}\,.$$

The same equation holds if we replace ψ^i_n with Ψ^i_n. Thus by trial wave function we establish part (a).

We now prove part (b). Fix $x \notin I^i_m \cup \widetilde{I}^i_m$. Suppose $x \in I^i_n \backslash I^i_{m+1}$. Then $x \in I^i_{r+1}\backslash I^i_r$ for some $m+1 \le r \le n-1$. We choose an annulus A around $I^i_{r+1}\backslash I^i_r$ such that $\mathrm{dist}(x, \partial A) \ge l_r^{5/6}$. By Lemma 3.7(a) and the Center Theorem for S_r we see that $S_r \cap A = \varnothing$. Now we restrict the eigenvalue equation $H(I^i_{n+1})\psi^i_{n+1} = E^i_{n+1}\psi^i_{n+1}$ to the annulus A and then use the Decay Theorem for S_r to prove part (b). The case where $x \notin I^i_n\backslash I^i_{m+1}$ is treated the same way.

To establish part (c) we use (0.10) and Lemma 3.7(b). □

Remark. By Appendix B, we can always assume that the eigenvalues and eigenfunctions are labeled so that they are differentiable functions of K.

Lemma 3.9. *For every K in the interval $|K-K^*| \le 6\delta_n^{1/2}/\delta_{n-1}$ there are eigenvalues $E^i_m(K) \in \sigma(H(I^i_m))$ and $\widetilde{E}^i_m(K) \in \sigma(H(\widetilde{I}^i_m))$ with the following properties:*

(a) $|E^i_m(K) - E^i_{n+1}(K)| \le 2\delta_m^4\sqrt{E^*+1}$.
(b) $|\widetilde{E}^i_m(K) - E^i_{n+1}(K)| \le 2\delta_m^4\sqrt{E^*+1}$.
(c) $|E^i_m(K) - \mathscr{E}^i_{n+1}(K)| \le 2\delta_m^4\sqrt{E^*+1}$.
(d) $|\widetilde{E}^i_m(K) - \mathscr{E}^i_{n+1}(K)| \le 2\delta_m^4\sqrt{E^*+1}$.
(e) $\|(H(I^i_m) - E^i_m)^{-1}_\perp\| \le (\delta^3_{m-1}\sqrt{E^*+1})^{-1}$.
(f) $\|(H(\widetilde{I}^i_m) - \widetilde{E}^i_m)^{-1}_\perp\| \le (\delta^3_{m-1}\sqrt{E^*+1})^{-1}$.

Proof. One of the wave functions, say ψ^i_{n+1}, must have significant amplitude in I^i_m. This follows from Lemma 3.8(b) and the fact that ψ^i_{n+1} and Ψ^i_{n+1} are orthogonal. We use ψ^i_{n+1} as a trial wave function for $H(I^i_m)$, yielding

$$\|(H(I^i_m) - E^i_{n+1})\psi^i_{n+1}\| = \|\Gamma(I^i_m)\psi^i_{n+1}\|,$$

which by Lemma 3.8(b) is bounded by $\delta_m^4\sqrt{E^*+1}$. This implies

$$|E^i_m(K) - E^i_{n+1}(K)| \le \delta_m^4\sqrt{E^*+1} \tag{3.13}$$

which establishes part (a). Lemma 3.8(a) together with (3.13) proves part (c). We will now prove the uniqueness of $E^i_m(K)$. By (H8) we have

$$|\mathscr{E}^i_m(K) - E^i_m(K)| \ge \delta^2_{m-1}\sqrt{E^*+1}\,|K+[c^i_m]|$$

where

$$\begin{aligned}|K+[c^i_m]| &= |K-K^*+K^*+[c^i_{n+1}]+[c^i_m]-[c^i_{n+1}]| \\ &\ge |[c^i_m]-[c^i_{n+1}]| - O(\delta_n^{1/2}/\delta_{n-1})\,.\end{aligned}$$

By (3.9) and (0.2) we have

$$|[c_m^i] - [c_{n+1}^i]| = \tfrac{1}{2}|[c_m^i] - [\tilde{c}_m^i]| \geq \mathrm{cst}/|c_m^i - \tilde{c}_m^i|^2 \geq \mathrm{cst}/l_m^4\,;$$

therefore

$$|K + [c_m^i]| \geq \mathrm{cst}\, l_m^4 - O(\delta_n^{1/2}/\delta_{n-1}) \geq \delta_{m-1}\,, \tag{3.14}$$

which implies

$$|\mathscr{E}_m^i(K) - E_m^i(K)| \geq \delta_{m-1}^3\sqrt{E^* + 1}\,.$$

This shows that $E_m^i(K)$ is the unique eigenvalue in $\sigma(H(I_m^i))$ which is near E^*, thus proving part (e) of the lemma.

A similar argument in $\widetilde{I}_m^i$ shows that there exists an eigenvalue $\widetilde{E}_m^i(K) \in \sigma(H(\widetilde{I}_m^i))$ such that parts (b), (d), and (f) hold. If we transform the equation $H(\widetilde{I}_m^i)\tilde{\psi}_m^i = \widetilde{E}_m^i\tilde{\psi}_m^i$ to the box I_m^i by changing variables $x \to -x + c_m^i + \tilde{c}_m^i$, we get

$$\widetilde{E}_m^i(K) = E_m^i(-K - [c_m^i] - [\tilde{c}_m^i])\,. \tag{3.15}$$

By induction we know the structure of $E_m^i(K)$; therefore equation (3.15) gives us a full description of $\widetilde{E}_m^i(K)$.

Lemma 3.10. *If $c_{n+1}^i \in S_{n+1}$ then*

(a) $\|\psi_{n+1}^i - A\psi_m^i\| \leq \mathrm{cst}\,\delta_m^3$,
(b) $\|\psi_{n+1}^i - B\tilde{\psi}_m^i\| \leq \mathrm{cst}\,\delta_m^3$,
(c) $\|\Psi_{n+1}^i - B\psi_m^i\| \leq \mathrm{cst}\,\delta_m^3$,
(d) $\|\Psi_{n+1}^i + A\tilde{\psi}_m^i\| \leq \mathrm{cst}\,\delta_m^3$,

where $1 \geq A^2 + B^2 \geq 1 - \delta_m$.

Proof. We restrict the equation $H(I_{n+1}^i)\psi_{n+1}^i = E_{n+1}^i\psi_{n+1}^i$ to I_m^i and write it as

$$(H(I_m^i) - E_m^i)\psi_{n+1}^i = (E_{n+1}^i - E_m^i)\psi_{n+1}^i + \Gamma(I_m^i)\psi_{n+1}^i\,.$$

By Lemmas 3.9(a) and 3.8(b), the right-hand side is bounded by $\mathrm{cst}\,\delta_m^4\sqrt{E^*+1}$; therefore

$$\psi_{n+1}^i = A\psi_m^i + (H(I_m^i) - E_m^i)_{\perp}^{-1}(O(\delta_m^4\sqrt{E^*+1}))\,.$$

By Lemma 3.9(e) we get $\|\psi_{n+1}^i - A\psi_m^i\| \leq \mathrm{cst}\,\delta_m^3$. We can do the same thing in $\widetilde{I}_m^i$ to get $\|\psi_{n+1}^i - B\tilde{\psi}_m^i\| \leq \mathrm{cst}\,\delta_m^3$. Orthogonality and normalization give us parts (c) and (d) and the relationship between A and B. □

We have two eigenvalues E_{n+1}^i and $\mathscr{E}_{n+1}^i$ in $\sigma(H(I_{n+1}^i))$ that are close enough to E^* to make their wave functions decay exponentially fast outside $I_m^i \cup \widetilde{I}_m^i$. Any other eigenvalue must be far enough away from E^* to make its wavefunction orthogonal to ψ_{n+1}^i and Ψ_{n+1}^i.

Lemma 3.11. *Any other eigenvalue* $\widehat{E} \in \sigma(H(I^i_{n+1}))$ *obeys*

$$|\widehat{E} - E^*| \geq \delta_m^4\sqrt{E^*+1} \geq \delta_{n-1}^4\sqrt{E^*+1}.$$

Proof. Suppose that there is a third eigenvalue $e^i_{n+1} \in \sigma(H(I^i_{n+1}))$ such that $|e^i_{n+1} - E^*| \leq \delta_m^4\sqrt{E^*+1}$. Then we can show that its wave function ϕ^i_{n+1} decays exponentially fast outside $I^i_m \cup \widetilde{I}^i_m$ and also satisfies

$$\|\phi^i_{n+1} - A\psi^i_m\| \leq \text{cst}\,\delta_m^3 \quad \text{and} \quad \|\phi^i_{n+1} - B\tilde{\psi}^i_m\| \leq \text{cst}\,\delta_m^3.$$

(See Lemmas 3.8(b) and 3.10.) Therefore it is impossible for ψ^i_{n+1}, Ψ^i_{n+1}, and ϕ^i_{n+1} to be orthogonal. □

In Lemma 3.10 we expressed ψ^i_{n+1} and Ψ^i_{n+1} in terms of ψ^i_m and $\tilde{\psi}^i_m$. This will allow us to relate the derivatives of E^i_{n+1} and $\mathscr{E}^i_{n+1}$ to those of E^i_m and $\widetilde{E}^i_m$. To do this we need to prove some technical lemmas about E^i_m and $\widetilde{E}^i_m$.

Lemma 3.12. $|dE^i_m/dK + d\widetilde{E}^i_m/dK| \leq \delta_m^{1/4}\sqrt{E^*+1}$.

Proof. By (3.10) and (3.15) we have

$$\begin{aligned}
&\left|\left(\frac{d}{dK}E^i_m\right)(K) + \left(\frac{d}{dK}\widetilde{E}^i_m\right)(K)\right| \\
&\quad = \left|\left(\frac{d}{dK}E^i_m\right)(K) - \left(\frac{d}{dK}E^i_m\right)(-K - [c^i_m] - [\tilde{c}^i_m])\right| \\
&\quad \leq \max\left|\frac{d^2}{dK^2}E^i_m\right| \times O((\delta_m^{1/2})/\delta_{m-1}).
\end{aligned}$$

We must estimate

$$\frac{d^2}{dK^2}E^i_m = 2 + 2\langle V'\psi^i_m, (E^i_m - H)^{-1}_{\perp}V'\psi^i_m\rangle \leq 2 + 2\|V'\psi^i_m\|^2\|(E^i_m - H)^{-1}_{\perp}\|.$$

By Lemma 3.9(e) we have $\|(E^i_m - H)^{-1}_{\perp}\| \leq (\delta_{m-1}^3\sqrt{E^*+1})^{-1}$, so it remains to estimate

$$\|V'\psi^i_m\|^2 \leq \|V'(I^i_m)\|^2 \leq \left(\max_{I_m}|v'_x|\right)^2 \leq (l_m\sqrt{E^*+1})^2.$$

Thus

$$\left|\frac{d}{dK}E^i_m + \frac{d}{dK}\widetilde{E}^i_m\right| \leq \text{cst}\left(\frac{l_m^2\sqrt{E^*+1}}{\delta_{m-1}^3}\frac{\delta_m^{1/2}}{\delta_{m-1}}\right) \leq \delta_m^{1/4}\sqrt{E^*+1}. \quad □$$

Lemma 3.13. *If* K *is in the interval* $|K - K^*| \leq 6\delta_n^{1/2}/\delta_{n-1}$, *then* $|dE^i_m/dK| \geq \delta_{m-1}^2\sqrt{E^*+1}$.

Proof. If $|dE^i_m/dK| \leq \delta_{m-1}^2\sqrt{E^*+1}$ then by (H3), (H10) we have $|dE^i_m/dK| \geq \frac{1}{2}\sqrt{E^*+1}|K + [c^i_m]|$, contradicting (3.14). □

Now we can prove the lemma which relates the derivatives of E^i_{n+1}, $\mathscr{E}^i_{n+1}$ to those of E^i_m, $\widetilde{E}^i_m$.

Lemma 3.14. *If* $c^i_{n+1} \in S_{n+1}$ *then*

(a)

$$\frac{d}{dK} E^i_{n+1} = (A^2 - B^2)\frac{d}{dK} E^i_m + O(\delta_m^{1/4}\sqrt{E^* + 1}),$$
$$\frac{d}{dK} \mathscr{E}^i_{n+1} = (B^2 - A^2)\frac{d}{dK} E^i_m + O(\delta_m^{1/4}\sqrt{E^* + 1}).$$

(b)

$$\frac{d^2}{dK^2} E^i_{n+1} = 2\frac{\langle \psi^i_{n+1}, V'\Psi^i_{n+1}\rangle^2}{E^i_{n+1} - \mathscr{E}^i_{n+1}} + O\left(\frac{l^2_{n-1}\sqrt{E^* + 1}}{\delta^4_{n-1}}\right),$$
$$\frac{d^2}{dK^2} \mathscr{E}^i_{n+1} = 2\frac{\langle \psi^i_{n+1}, V'\Psi^i_{n+1}\rangle^2}{\mathscr{E}^i_{n+1} - E^i_{n+1}} + O\left(\frac{l^2_{n-1}\sqrt{E^* + 1}}{\delta^4_{n-1}}\right),$$

(*both hold where* $E^i_{n+1} \neq \mathscr{E}^i_{n+1}$).

(c) $\langle \psi^i_{n+1}, V'\Psi^i_{n+1}\rangle = 2AB\, dE^i_m/dK + O(\delta_m^{1/4}\sqrt{E^* + 1})$.

(d) $|dE^i_{n+1}/dK + d\mathscr{E}^i_{n+1}/dK| \leq \mathrm{cst}\,\delta_m^{1/4}\sqrt{E^* + 1}$.

Proof. Since $dE^i_{n+1}/dK = \langle \psi^i_{n+1}, V'\psi^i_{n+1}\rangle$, $dE^i_m/dK = \langle \psi^i_m, V'\psi^i_m\rangle$, and $d\widetilde{E}^i_m/dK = \langle \widetilde{\psi}^i_m, V'\widetilde{\psi}^i_m\rangle$, we use Lemmas 3.10 and 3.8(b) and the fact that $\|V'(I^i_m)\| \leq \mathrm{cst}\, l_m\sqrt{E^* + 1}$ to get

$$\begin{aligned}\frac{d}{dK} E^i_{n+1} &= A^2 \frac{d}{dK}E^i_m + B^2 \frac{d}{dK} \widetilde{E}^i_m + O(\delta^2_m\sqrt{E^* + 1}) \\ &= (A^2 - B^2)\frac{d}{dK} E^i_m + B^2 \left(\frac{d}{dK} E^i_m + \frac{d}{dK} \widetilde{E}^i_m\right) + O(\delta^2_m\sqrt{E^* + 1}).\end{aligned}$$

Now we can use Lemma 3.12 to establish part (a). The same argument is used to prove part (c).

We now prove part (b). By Appendix B we have

$$\frac{d^2}{dK^2} E^i_{n+1} = 2 + 2\frac{\langle \psi^i_{n+1}, V'\psi^i_{n+1}\rangle^2}{E^i_{n+1} - \mathscr{E}^i_{n+1}} + 2\langle V'\psi^i_{n+1}, (E^i_{n+1} - H)^{-1}_{\perp\perp} V'\psi^i_{n+1}\rangle,$$

and the remainder term is bounded by

$$\mathrm{cst}\|V'\psi^i_{n+1}\|^2\|(E^i_{n+1} - H)^{-1}_{\perp\perp}\|.$$

Lemma 3.11 implies $\|(E^i_{n+1} - H)^{-1}_{\perp\perp}\| \leq (\delta^4_{n-1}\sqrt{E^* + 1})^{-1}$, so it remains to estimate $\|V'\psi^i_{n+1}\|^2$. By Lemma 3.8(b) we have

$$\begin{aligned}\|V'\psi^i_{n+1}\|^2 &\leq \|V'(I^i_m)\|^2 + \|V'(\widetilde{I}^i_m)\|^2 + O(\delta^2_m\sqrt{E^* + 1}) \\ &\leq \mathrm{cst}(l_m\sqrt{E^* + 1})^2 + O(\delta^2_m\sqrt{E^* + 1}) \\ &\leq \mathrm{cst}(l_{n-1}\sqrt{E^* + 1})^2 \quad (\text{since } m \leq n - 1).\end{aligned}$$

Therefore the remainder term is bounded by $\mathrm{cst}(l^2_{n-1}\sqrt{E^* + 1}/\delta^4_{n-1})$ and part (b) is established. The proof of part (d) follows immediately from part (a). □

Lemma 3.15. *If* $|dE^i_{n+1}/dK| \le \delta^3_{m-1}\sqrt{E^*+1}$ *for some* K *in the interval* $|K - K^*| \le 6\delta_n^{1/2}/\delta_{n-1}$ *then*

(a) $|\langle \psi^i_{n+1}, V'\Psi^i_{n+1}\rangle| \ge \frac{1}{3}\delta^2_{m-1}\sqrt{E^*+1}$.

(b) $|d^2E^i_{n+1}/dK^2| \ge \sqrt{E^*+1}$ *at all points where* $E^i_{n+1} \ne \mathscr{E}^i_{n+1}$; *moreover* $d^2E^i_{n+1}/dK^2$ *has a unique sign.*

Remark 1. Lemma 3.15 holds if E^i_{n+1} is replaced by $\mathscr{E}^i_{n+1}$.

Remark 2. We will show later that under the hypothesis of Lemma 3.15, $E^i_{n+1} \ne \mathscr{E}^i_{n+1}$ for every K in the interval $|K - K^*| \le 6\delta_n^{1/2}/\delta_{n-1}$.

Proof. If $|dE^i_{n+1}/dK| \le \delta^3_{m-1}\sqrt{E^*+1}$ for some K, then by Lemma 3.14(a) we have $|A^2 - B^2||dE^i_m/dK| \le \mathrm{cst}\,\delta^3_{m-1}\sqrt{E^*+1}$. By Lemma 3.13 we get $|A^2 - B^2| = O(\delta_{m-1})$, and therefore by Lemma 3.10 we must have $|AB| \ge \frac{1}{4}$. Lemmas 3.14(c) and 3.13 give us

$$|\langle \psi^i_{n+1}, V'\Psi^i_{n+1}\rangle| \ge \tfrac{1}{3}\delta^2_{m-1}\sqrt{E^*+1}.$$

To prove part (b) we use Lemmas 3.14(b) and 3.8(a) and part (a). □

Lemma 3.16. *Let* $c^i_{n+1} \in S_{n+1}$; *then*

(a) $|dE^i_{n+1}/dK| \le (3+\mu_{n+1})\sqrt{E^*+1}$.

(b) *If* $|dE^i_{n+1}/dK| \le \delta_n^2\sqrt{E^*+1}$ *for some* K *in the interval* $|K - K^*| \le 6\delta_n^{1/2}/\delta_{n-1}$ *then*

$$\left|\frac{d}{dK}E^i_{n+1}\right| \ge \tfrac{1}{2}\sqrt{E^*+1}|K + [c^i_{n+1}]|.$$

Remark. The same holds for $\mathscr{E}^i_{n+1}$.

Proof. The proof of part (a) follows immediately from (H9) and Lemma 3.14(a). To establish part (b) we use Appendix A and Lemma 3.15. □

Lemma 3.17. *Let* $c^i_{n+1} \in S_{n+1}$; *then*

$$|E^i_{n+1}(K_2) - E^i_{n+1}(K_1)| \ge \delta_n^2\sqrt{E^*+1}\,\min\begin{cases} |K_2 - K_1|^2, \\ |K_2 + K_1 + 2[c^i_{n+1}]|^2 \end{cases}$$

for any two points K_1, K_2 *in the interval* $|K - K^*| \le 6\delta_n^{1/2}/\delta_{n-1}$. *The same holds for* $\mathscr{E}^i_{n+1}$.

Proof. The proof goes the same as the proof of Lemma 2.13. There will be two cases to consider:

Case I. $E^i_{n+1}(K_s) > \mathscr{E}^i_{n+1}(K_s)$ where $K_s = -[c^i_{n+1}]$.

In this case, the analog of (2.6) holds; i.e.,

$$\left(\frac{d}{dK}E^i_{n+1}\right)(K_s) = \left(\frac{d}{dK}\mathscr{E}^i_{n+1}\right)(K_s) = 0. \tag{3.16}$$

By Lemmas 3.15(a) and 3.14(b) we see that $d^2E^i_{n+1}/dK^2$ and $d^2\mathscr{E}^i_{n+1}/dK^2$ are large with opposite signs so that $E^i_{n+1}(K)$ and $\mathscr{E}^i_{n+1}(K)$ never cross. Appendix A and Lemma 3.15(b) give us

$$|E^i_{n+1}(K_2) - E^i_{n+1}(K_1)| \geq \delta_n^2\sqrt{E^* + 1}\ \min\begin{cases} |K_2 - K_1|^2, \\ |K_2 + K_1 + 2[c^i_{n+1}]|^2. \end{cases}$$

Case II. $E^i_{n+1}(K_s) = \mathscr{E}^i_{n+1}(K_s)$.

In this case we will show that

$$\left|\frac{d}{dK}E^i_{n+1}\right| \geq \delta^3_{m-1}\sqrt{E^* + 1} \quad \text{and} \quad \left|\frac{d}{dK}\mathscr{E}^i_{n+1}\right| \geq \delta^3_{m-1}\sqrt{E^* + 1} \tag{3.17}$$

holds for $|K - K^*| \leq 6\delta_n^{1/2}/\delta_{n-1}$; moreover, the derivatives have opposite signs. To see this we calculate the derivatives at the symmetry point K_s using the special basis $\{\psi_s, \psi_a\}$ of symmetric and antisymmetric wave functions (see the proof of Lemma 2.13 for details). This calculation yields

$$\left(\frac{d}{dK}E^i_{n+1}\right)(K_s) = -\left(\frac{d}{dK}\mathscr{E}^i_{n+1}\right)(K_s) = \langle\psi_s, V'\psi_a\rangle.$$

By symmetry and the decay of the wave function, we may restrict this inner product to I^i_m. Inside I^i_m we can express $\psi_s \cong A\psi^i_m$ and $\psi_a \cong C\psi^i_m$ where $|A| \cong 1/\sqrt{2}$ and $|C| \cong 1/\sqrt{2}$; therefore the analog of (2.8) holds:

$$\left(\frac{d}{dK}E^i_{n+1}\right)(K_s) \cong 2\langle A\psi^i_m, V'C\psi^i_m\rangle \cong \frac{d}{dK}E^i_m.$$

Now we use Lemma 3.13 to prove (3.17). □

Lemma 3.18. *Let* $c^r_{n+1} \in S_{n+1}$ *for* $r = i, j$; *then*

(a) $|E^i_{n+1}(K^*) - E^j_{n+1}(K^*)| \geq \delta_n^2\sqrt{E^* + 1}\ m(c^i_{n+1}, c^j_{n+1})^2$; *the same holds for* $|\mathscr{E}^i_{n+1}(K^*) - \mathscr{E}^j_{n+1}(K^*)|$.

(b) $|E^i_{n+1}(K^*) - \mathscr{E}^j_{n+1}(K^*)| \geq \delta_n^2\sqrt{E^* + 1}\ m(c^i_{n+1}, c^j_{n+1})^2$.

(c) (*Center Theorem*) $m(c^i_{n+1}, c^j_{n+1}) \leq 2\delta^{1/2}_{n+1}/\delta_n$.

Proof. See the proof of Lemma 2.15 and replace the indices 0, 1 with n, $n+1$. □

Lemma 3.19. *Let* $c^i_{n+1} \in S_{n+1}$; *then*

$$|E^i_{n+1}(K) - \mathscr{E}^i_{n+1}(K)| \geq \delta_n^2\sqrt{E^* + 1}|K + [c^i_{n+1}]|$$

for every K *in the interval* $|K - K^*| \leq 6\delta_n^{1/2}/\delta_{n-1}$.

Proof. The proof goes like the proof of Lemma 2.16. There are two cases to consider:

Case I. $E^i_{n+1}(K_s) = \mathscr{E}^i_{n+1}(K_s)$ where $K_s = -[c^i_{n+1}]$.

In this case we use (3.17) and follow the proof of Lemma 2.16 (Case I).

Case II. $E^i_{n+1}(K_s) > \mathscr{E}^i_{n+1}(K_s)$.

By (3.16) and Lemma 3.15(a) we have

$$|\langle \psi^i_{n+1}, V'\Psi^i_{n+1}\rangle(K_s)| \geq \tfrac{1}{3}\delta^2_{m-1}\sqrt{E^* + 1}.$$

Now we follow the proof of Lemma 2.16 (Case II). □

Remark. We are back to class B of the induction hypothesis.

Case 2. $s_n \leq 6l^2_n$.

After we make a few remarks, the proof of case 2 goes exactly as the proof of Case 1B. In Case 2 there are two boxes I^i_n, $\widetilde{I}^i_n \subset I^i_{n+1}$, and the Center Theorem for S_{n-1} implies that $s_{n-1} \geq l^2_{n-1}$.

Lemma 3.20. *If* $c^r_{n+1} \in S_{n+1}$ *for* $r = i, j$ *then*

(a) $|K^* + [c^i_{n+1}]| \leq 3\delta^{1/2}_n/\delta_{n-1}$.

(b) $\max(|[c^i_{n+1}] - [c^j_{n+1}]|, |[c^i_{n+1}] + [c^j_{n+1}] + 2K^*|) \leq 6\delta^{1/2}_n/\delta_{n-1}$.

Proof. We use (0.8) and the definition of c^i_{n+1} to prove part (a). Part (b) follows immediately from part (a). □

Since $c^i_n \in S_n$, we have $|E^i_n(K^*) - E^*| \leq \delta_n\sqrt{E^* + 1}$. By (H2), (H9) we get $|E^i_n(K) - E^*| \leq \text{cst}(\delta^{1/2}_n/\delta_{n-1})\sqrt{E^* + 1}$ for every K in the interval $|K - K^*| \leq 6\delta^{1/2}_n/\delta_{n-1}$. If we transform the equation $H(I^i_n)\psi^i_n = E^i_n\psi^i_n$ from I^i_n to $\widetilde{I}^i_n$ we see that $\widetilde{E}^i_n(K) = E^i_n(-K - [c^i_n] - [\tilde{c}^i_n])$ is an eigenvalue in $\sigma(H(\widetilde{I}^i_n))$ which is near E^*. We use ψ^i_n and $\tilde{\psi}^i_n$ as trial wave functions for $H(I^i_{n+1})$ to get two eigenvalues $E^i_{n+1}(K)$ and $\mathscr{E}^i_{n+1}(K) \in \sigma(H(I^i_{n+1}))$ which obey

$$|E^i_{n+1}(K) - E^*| \leq \text{cst}(\delta^{1/2}_n/\delta_{n-1})\sqrt{E^* + 1}$$

and

$$|\mathscr{E}^i_{n+1}(K) - E^*| \leq \text{cst}(\delta^{1/2}_n/\delta_{n-1})\sqrt{E^* + 1}.$$

We are now in the same setup as subcase 1B and the rest of the proof will be omitted.

4. Proof of the main results

In this section we will prove our main results.

Main Lemma. *For* ε *sufficiently small and for almost every* K, *every polynomially bounded eigenfunction of the operator* $H(K) = \varepsilon\Delta + V(K)$ *decays exponentially fast.*

To establish the Main Lemma we will follow the proof given in [6]. First we will show that inside large enough boxes there must be points of S_n. Then the Center Theorem will give us annuli around these boxes which do not intersect S_n. Finally, the Decay Theorem will force the Green's function and the wave function to decay at points in the annulus. We will piece together annuli to prove that ψ decays for all $|x|$ large. To accomplish this we need to set some notation and prove two lemmas.

Definition 1. We call E a *generalized eigenvalue* if E satisfies the equation $H\psi = E\psi$ where ψ is polynomially bounded.

Definition 2. We define $B(l)$ to be a square box of length l centered at the origin of $\mathbf{Z}^2$. Let $\Lambda_n \equiv B(\frac{1}{2}l_{n+1})$ and $A_n \equiv B(2l_{n+2})\backslash B(\frac{1}{2}l_{n+1})$ where l_n is the length scale defined in the introduction.

Lemma 4.1. *Let E^* be a generalized eigenvalue for $H(K^*)$; then there exists an integer $N(K^*, E^*)$ such that*

$$S_n(K^*, E^*) \cap \Lambda_n \neq \varnothing \quad \textit{for } n \geq N(K^*, E^*).$$

Proof. Assume not. Then there exists a sequence $n_i \to \infty$ such that

$$S_{n_i}(K^*, E^*) \cap \Lambda_{n_i} = \varnothing.$$

Fix any $x \in \mathbf{Z}^2$ and choose i large enough so that $x \in \Lambda_{n_i}$ and $\operatorname{dist}(x, \partial\Lambda_{n_i}) \geq l_{n_i}^{5/6}$. We will show that $\psi \equiv 0$ and therefore not an eigenfunction.

To see this we restrict the equation $H\psi = E^*\psi$ to the sets Λ_{n_i}:

$$\psi = (H(\Lambda_{n_i}) - E^*)^{-1}(\Gamma_{n_i}\psi).$$

This gives us

$$\psi(x) = \sum_{y \in \partial\Lambda_{n_i}} (H(\Lambda_{n_i} - E^*)^{-1}(x, y)(\Gamma_{n_i}\psi)(y).$$

Since $|x - y| \geq l_{n_i}^{5/6}$, the Decay Theorem and the polynomial boundedness of ψ imply

$$|\psi(x)| \leq \text{cst}\, l_{n_i}^p \exp(-(\gamma_0/2)l_{n_i}^{5/6}) \to 0 \quad \text{as } i \to \infty.$$

Therefore $\psi(x) \equiv 0$ for all x. □

Lemma 4.2. *There exists a set $\mathscr{K}$ of Lebesgue measure zero with the following property:*

If E^* is a generalized eigenvalue for $H(K^*)$ where $K^* \notin \mathscr{K}$, then there exists an integer $N(K^*, E^*)$ such that $S_n(K^*, E^*) \cap A_n = \varnothing$ for $n \geq N(K^*, E^*)$.

Proof. For any value of E, we define

$$B_n(E) \equiv \{K \in \mathbf{R} : S_n(K, E) \cap A_n \neq \varnothing,\ S_n(K, E) \cap \Lambda_n \neq \varnothing\}.$$

Let $C_n \equiv \bigcup_E (B_n(E))$. If we show that $\sum_{n=0}^{\infty} \mu(C_n)$ converges then it is easy to see that

$$\mu\left(\bigcap_{m=0}^{\infty} \bigcup_{n \geq m} C_n\right) = 0.$$

Define $\mathscr{K} = \bigcap_{m=0}^{\infty} \bigcup_{n \geq m} C_n$. Let E^* be any generalized eigenvalue for $H(K^*)$ and suppose that $K^* \notin \mathscr{K}$. Since $K^* \notin \mathscr{K}$ there must be an integer $N(K^*, E^*)$ such that $K^* \notin B_n(E^*)$ for $n \geq N(K^*, E^*)$. By Lemma 4.1

we may assume that $S_n(K^*, E^*) \cap \Lambda_n \neq \emptyset$ for $n \geq N(K^*, E^*)$; therefore by definition of $B_n(E^*)$ we must have $S_n(K^*, E^*) \cap A_n = \emptyset$ for $n \geq N(K^*, E^*)$.

It remains to show that $\sum \mu(C_n) < \infty$. In particular, we will prove that $\mu(C_n) \leq l_{n+2}^4 \delta_n^{1/3}$. To see this let $K \in C_n$. Then $K \in B_n(E)$ for some E; therefore we have points $a \in S_n \cap A_n$, $b \in S_n \cap \Lambda_n$ where $|a - b| \leq 2l_{n+2}$. By the Center Theorem for S_n and (0.2) it follows that

$$|[a] + [b] + 2K| \leq 2\delta_n^{1/2}/\delta_{n-1}.$$

Therefore K belongs to an interval of measure $O(\delta_n^{1/2}/\delta_{n-1})$, but this interval depends on E and K since a, b depend on E and K. However, there are at most $O(l_{n+2}^4)$ points $a, b \in B(2l_{n+2})$, thus

$$\mu(C_n) \leq \text{cst}\, l_{n+2}^4 \delta_n^{1/2}/\delta_{n-1}. \quad \square$$

We are now ready to prove the Main Lemma.

Proof (*of Main Lemma*). Choose $x \in B(l_{n+2}) \backslash B(l_{n+1}) \subset A_n$ where n is large enough so that Lemma 4.2 applies. If we restrict the equation $H\psi = E^*\psi$ to A_n we get

$$\psi = (H(A_n) - E^*)^{-1})(\Gamma(A_n)\psi).$$

The Decay Theorem, Lemma 4.2, and the polynomial boundedness of ψ give us

$$|\psi(x)| \leq \text{cst}\, e^{-(\gamma_0/16)|x|}. \quad \square$$

With the help of the Main Lemma we can now go on to the proof of Theorems 1 and 2. For the proof of Theorem 1 see [1].

Theorem 2. *For ε sufficiently small, the operator*

$$-d^2/dx^2 + \varepsilon(\cos x + \cos(\alpha x + \vartheta))$$

has no point spectrum.

To prove Theorem 2 we will need the following lemma.

Lemma 4.3. *Fix E^*. Then for almost every $K \in \mathbf{R}$, there exists an integer $N(K)$ such that $S_n(K, E^*) \cap \Lambda_n = \emptyset$ for $n \geq N(K)$.*

Proof. If we define

$$B_n = \{K \in \mathbf{R} : S_n(K, E^*) \cap \Lambda_n \neq \emptyset\},$$

then the set $\mathscr{K} = \bigcap_{m=0}^{\infty} \bigcup_{n \geq m} B_n$ will be the set of measure zero we need to prove the lemma. To prove that this set has measure zero we define

$$B_n^i = \{K \in \mathbf{R} : c_n^i \in S_n(K, E^*) \cap \Lambda_n\}$$

which decomposes B_n; i.e., $B_n = \bigcup_{\Lambda_n} B_n^i$. If we can show that $\sum \mu(B_n \cap I)$ converges (for intervals I), then by the Borel-Cantelli theorem it follows that

$\mu(\mathscr{K} \cap I) = 0$ and therefore $\mu(\mathscr{K}) = 0$. To finish the proof we need to establish the estimate

$$\mu(B_n^i \cap I) \leq \mathrm{cst}(\delta_n^{1/4})/(\delta_{n-1}^{3/2}). \tag{4.1}$$

Let $K^* \in B_n^i$. Then $c_n^i \in S_n(K^*, E^*) \cap \Lambda_n$; therefore there exists an eigenvalue $E_n^i(K^*) \in \sigma(H(I_n^i))$ such that

$$|E_n^i(K^*) - E^*| \leq \delta_n \sqrt{E^* + 1}.$$

By induction we know that $E_n^i(K)$ is defined in the interval $|K - K^*| \leq \mathrm{cst}\,\delta_{n-1}^{3/2}$, and by induction hypotheses (H3) and (H10) we can show that $|E_n^i(K \pm \delta_n^{1/4}) - E^*| \geq \delta_n \sqrt{E^* + 1}$. From this it follows that every interval of length $O(\delta_{n-1}^{3/2})$ can intersect B_n^i in a set of measure at most $O(\delta_n^{1/4})$. □

Now we can prove Theorem 2.

Proof (*of Theorem* 2). Suppose that we have a solution to the equation

$$-d^2\psi/dx^2 + \varepsilon(\cos x + \cos(\alpha x + \vartheta))\psi = E^*\psi.$$

Let $\phi_{mn}^K = e^{im\vartheta}\hat{\psi}(K + m + n\alpha)$ where $\hat{\psi}$ is the Fourier transform of ψ. Then ϕ^K satisfies the eigenvalue equation

$$H(K)\phi^K = (\varepsilon\Delta + V(K))\phi^K = E^*\phi^K$$

on the lattice $\mathbf{Z}^2$. Later we will prove that ϕ^K is polynomially bounded for almost all K. If we use this fact with Lemma 4.3 and an argument similar to the one used in the proof of lemma 4.1, we can show that $\phi^K \equiv 0$ for almost every K. Thus $\hat{\psi}(K) = 0$ for almost every K, which implies that $\psi = 0$ and not an eigenfunction.

Now we need only show that ϕ^K is polynomially bounded for almost every K. To see this we note that since $\psi \in L_2$ we have

$$\int |\phi_{mn}^K|^2\, dK = \int |\psi(K)|^2\, dK = 1;$$

therefore

$$\int \sum_{m,n} \frac{|\phi_{mn}^K|^2}{1 + |m|^4 + |n|^4}\, dK = \sum_{m,n} \frac{1}{1 + |m|^4 + |n|^4} < \infty.$$

This implies that $\sum_{m,n}(|\phi_{mn}^K|^2/(1 + |m|^4 + |n|^4))$ is bounded by some constant C_K for almost every K; therefore ϕ^K is polynomially bounded for almost every K. □

APPENDIX A

Lemma. *Let $E(K)$ be defined and twice differentiable for every K in the interval $|K - K^*| \leq \eta \leq \frac{1}{2}$. Suppose that there is a point K_s in the interval such that*

$E(K_s - \delta K) = E(K_s + \delta K)$. *We also assume that* $|E'| \le s$ *implies* $|E''| \ge t$ *and* E'' *has a unique sign. Then*

(a)

$$|E(K_2) - E(K_1)| \ge \min(s/2, t/4) \min \begin{cases} |K_2 - K_1|^2, \\ |K_2 + K_1 - 2K_s|^2 \end{cases}$$

for any points K_1, K_2 *belonging to the interval* $|K - K^*| \le \eta$.

(b)

$$|E'(K)| \ge \min \begin{cases} s, \\ t|K - K_s|. \end{cases}$$

Proof. Without loss of generality, we may consider the case $|E'| \le s$ implies $E'' \ge t$. By symmetry we must have $E'(K_s) = 0$; therefore $E''(K_s) \ge t$. Let K_d be the largest point with the following property:

$$E''(K) \ge t \quad \text{for } K_s \le K \le K_d.$$

This implies that $E(K)$ is an increasing function to the right of the symmetry point. By definition of K_d we have $E''(K_d + \delta K) < t$ for δK small; therefore $E'(K_d + \delta K) \ge s$. This inequality must hold for every $K > K_d$ or else we would have a point K where $E''(K) \ge t > 0$. This is impossible. (See Figure A.) Therefore

$$E'(K) \ge s \quad \text{for } K > K_d.$$

We now split up the proof into cases.

Case 1. $K_s \le K_1 \le K_2 \le K_d$.

$$\begin{aligned} |E(K_2) - E(K_1)| &= E(K_2) - E(K_1) \\ &= E'(K_1)(K_2 - K_1) + \tfrac{1}{2}E''(\widehat{K})(K_2 - K_1)^2 \\ &\ge (t/2)|K_2 - K_1|^2. \end{aligned}$$

Case 2. $K_d \le K_1 \le K_2$.

$$\begin{aligned} |E(K_2) - E(K_1)| &= E(K_2) - E(K_1) = E'(\widehat{K})(K_2 - K_1) \\ &\ge s|K_2 - K_1| \ge s|K_2 - K_1|^2. \end{aligned}$$

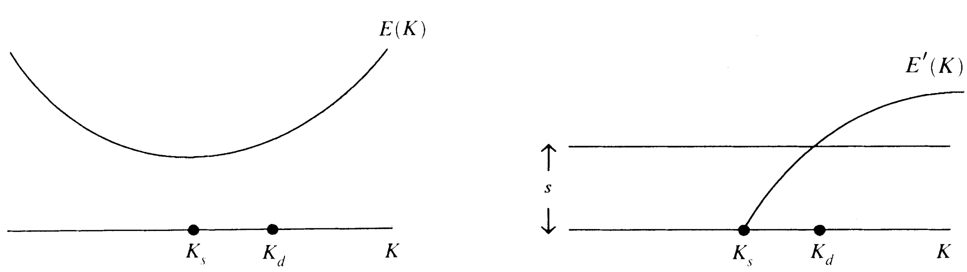


FIGURE A

Case 3. $K_s \le K_1 \le K_d \le K_2$.

$$|E(K_2) - E(K_1)| = E(K_2) - E(K_d) + E(K_d) - E(K_1).$$

By Cases 1 and 2 we get

$$\begin{aligned}|E(K_2) - E(K_1)| &\ge s|K_2 - K_d|^2 + (t/2)|K_d - K_1|^2 \\ &\ge \tfrac{1}{2}\min(s, t/2)|K_2 - K_1|^2 \\ &= \min(s/2, t/4)|K_2 - K_1|^2.\end{aligned}$$

Case 4. $K_1 \le K_s \le K_2$. We use the symmetry of $E(K)$ to write

$$|E(K_2) - E(K_1)| = |E(K_2) - E(2K_s - K_1)|,$$

which by Cases 1–3 gives us

$$|E(K_2) - E(K_1)| \ge \min(s/2, t/4)|K_2 + K_1 - 2K_s|^2.$$

To prove part (b), we need to consider two cases.

Case 1. $K_s \le K \le K_d$.

$$E'(K) = E'(K_s) + E''(\widehat{K})(K - K_s) \ge t|K - K_s|.$$

Case 2. $K_d \le K$. In this case we have $|E'(K)| \ge s$. □

APPENDIX B

Lemma. *Suppose* $H\psi = E\psi$ *and* $H\Psi = \mathscr{E}\Psi$, *where* $\mathscr{E}$ *is the closest eigenvalue to* E. *We assume that the eigenfunctions are normalized and everything is defined for* K *in the interval* $|K - K^*| \le \eta$. *Then*

(a) *The eigenvalues and eigenfunctions can be chosen to be analytic functions of* K.

(b) $E'(K) = \langle \psi, V'\psi \rangle$.

(c)

$$\begin{aligned}E''(K) &= 2 + 2\langle V'\psi, (E - H)_{\perp}^{-1} V'\psi \rangle \\ &= 2 + \frac{2\langle \psi, V'\Psi \rangle^2}{E - \mathscr{E}} + 2\langle V'\psi, (E - H)_{\perp\perp}^{-1} V'\psi \rangle\end{aligned}$$

at all points where $E(K) \ne \mathscr{E}(K)$.

Remark. From the Spectral Theorem we have

$$G_{\perp} \equiv (E - H)_{\perp}^{-1} = \sum_{E_\alpha \ne E} (E - E_\alpha)^{-1} P(E_\alpha)$$

where $P(E_\alpha)$ are projections. Similarly, we write

$$G_{\perp\perp} \equiv (E - H)_{\perp\perp}^{-1} = \sum_{E_\alpha \ne E, \mathscr{E}} (E - E_\alpha)^{-1} P(E_\alpha).$$

Proof. If the eigenvalues never cross, part (a) obviously holds. When there are level crossings, we can always find a way to label our eigenvalues so that they are analytic. This follows from the fact that our operator $H(K)$ is analytic and selfadjoint. See [10].

To prove part (b) we differentiate the equation $H\psi = E\psi$ to get $H'\psi + H\psi' = E'\psi + E\psi'$. Note that $H' = V'$, and then take the inner product with ψ yielding

$$\langle\psi, V'\psi\rangle + \langle\psi, H\psi'\rangle = E'\langle\psi, \psi\rangle + E\langle\psi, \psi'\rangle. \tag{B1}$$

Since the second term on the left $\langle\psi, H\psi'\rangle = \langle H\psi, \psi'\rangle = E\langle\psi, \psi'\rangle$ is canceled by the same term on the right, we get $E' = \langle\psi, V'\psi\rangle$.

To establish part (c) we differentiate the result in part (b) to get

$$E'' = \langle\psi', V'\psi\rangle + \langle\psi, V''\psi\rangle + \langle\psi, V'\psi'\rangle = \langle\psi, V''\psi\rangle + 2\langle\psi, V'\psi'\rangle.$$

By definition of V we have $V'' = 2I$ (where I is the identity matrix); therefore

$$E'' = 2 + 2\langle\psi, V'\psi'\rangle.$$

To complete the proof we must calculate ψ'. We rewrite (B1) as $(E-H)\psi' = V'\psi - E'\psi$, thus

$$\psi' = (E-H)_\perp^{-1}(V'\psi - E'\psi) = (E-H)_\perp^{-1}V'\psi. \tag{B2}$$

If we put (B2) back into our last equation for E'' we get

$$E''(K) = 2 + 2\langle V'\psi, (E-H)_\perp^{-1}V'\psi\rangle. \tag{B3}$$

We can further express

$$(E-H)_\perp^{-1} = (E-H)_{\perp\perp}^{-1} + P(\mathscr{E})/(E-\mathscr{E}), \tag{B4}$$

and if we note that $P(\mathscr{E})V'\psi = \langle\psi, V'\Psi\rangle\Psi$, we substitute (B4) into (B3) to get

$$E'' = 2 + 2\langle\psi, V'(E-H)_{\perp\perp}^{-1}V'\psi\rangle + 2\langle\psi, V'\Psi\rangle^2/(E-\mathscr{E}). \quad \square$$

APPENDIX C

Decay Theorem. *If Λ is $n+1$ regular and $S_{n+1}(K^*, E^*) \cap \Lambda = \varnothing$ then*

$$|(H(\Lambda) - E)^{-1}(x, y)| \le \frac{\text{cst}}{\delta_0\sqrt{E^* + 1}} e^{-\gamma_{n+1}|x-y|}$$

provided $|x-y| \ge l_{n+1}^{5/6}$, $|K - K^*| \le 4\delta_{n+1}^{3/2}$, and $|E - E^*| \le (\delta_{n+1}/5)\sqrt{E^* + 1}$, where $\gamma_0 \ge \gamma_{n+1} \ge \gamma_0/2$.

Proof. Let Λ be $n+1$ regular and suppose $\overline{S}_n \cap \Lambda$ consists of just one point c_{n+1}^i. Then there exists a box $\widehat{I}_{n+1}^i \subset I_{n+1}^i$ with the following properties:

(a) $S_n \cap (\Lambda \backslash \widehat{I}_{n+1}^i) = \varnothing$.
(b) $\Lambda \backslash \widehat{I}_{n+1}^i$ is n regular.
(c) $\operatorname{dist}(\partial \widehat{I}_{n+1}^i, \{x, y\}) \ge l_n^{5/6}$.
(d) length of $\widehat{I}_{n+1}^i = O(l_{n+1}^{1/2})$.

By (0.7) we have

$$\text{(C1)}\qquad G(x, y) \equiv (H(\Lambda) - E)^{-1}(x, y) = [G_\Gamma + G_\Gamma \Gamma G](x, y)$$

$$\text{(C2)}\qquad = [G_\Gamma + G_\Gamma \Gamma G_\Gamma + G_\Gamma \Gamma G \Gamma G_\Gamma](x, y)$$

where $\Gamma = \partial \widehat{I}^i_{n+1}$. Notice that the second term on the right in (C2) vanishes unless Γ separates x and y. By property (d), we see that it is impossible for x and y simultaneously to belong to $\widehat{I}^i_{n+1}$. If we assume that $x, y \in \Lambda \backslash \widehat{I}^i_{n+1}$, then $G_\Gamma = G(\Lambda \backslash \widehat{I}^i_{n+1})$ decays by induction. To estimate G we assume

$$\text{(C3)}\qquad |G(u', v')| \leq 4(\delta_{n+1}\sqrt{E^* + 1})^{-1} \quad \text{for } u', v' \in \widehat{I}^i_{n+1}.$$

Therefore by (C2) we get

$$\begin{aligned} |G(x, y)| &\leq \frac{\text{cst}}{\delta_0\sqrt{E^*+1}} e^{-\gamma_n|x-y|} \\ &\quad + \frac{\text{cst}}{\delta_0\sqrt{E^*+1}} e^{-\gamma_n|x-y|} e^{\text{cst}\,\gamma_n(l^{1/2}_{n+1} + l^{2/3}_{n+1})} \\ &\leq \frac{\text{cst}}{\delta_0\sqrt{E^*+1}} e^{-\gamma_{n+1}|x-y|} \end{aligned}$$

where

$$\gamma_{n+1} = \gamma_n \left(1 - \text{cst} \frac{l^{1/2}_{n+1} + l^{2/3}_{n+1}}{l^{5/6}_{n+1}} \right).$$

If $x \in \widehat{I}^i_{n+1}$ and $y \notin \widehat{I}^i_{n+1}$, we use the resolvent identity $G = G_\Gamma + G \Gamma G_\Gamma$ to obtain a similar result.

It remains to prove (C3). By alternate application of the two resolvent identities

$$G = G_{\overline{\Gamma}} + G_{\overline{\Gamma}} \overline{\Gamma}_G \quad \text{and} \quad G = G_\Gamma + G_\Gamma \Gamma G$$

(where $\overline{\Gamma} = \partial I^i_{n+1}$) we get

$$\text{(C4)}\qquad G(u', v') = [G_{\overline{\Gamma}} + G_{\overline{\Gamma}} \overline{\Gamma} G_\Gamma + G_{\overline{\Gamma}} \overline{\Gamma} G_\Gamma \Gamma G_{\overline{\Gamma}} + \cdots](u', v').$$

Since $G_{\overline{\Gamma}} = G(I^i_{n+1})$ and by assumption $\text{dist}(\sigma(H(I^i_{n+1}, K^*)), E^*) \geq \delta_{n+1}\sqrt{E^*+1}$, we get $|G_{\overline{\Gamma}}(u', v')| \leq 2(\delta_{n+1}\sqrt{E^*+1})^{-1}$ for K near K^* and E near E^*. Note that in all nonvanishing terms of (C4) we have $G_\Gamma = G(\Lambda \backslash \widehat{I}^i_{n+1})$, and since $\text{dist}(\Gamma, \overline{\Gamma}) \geq l_n$ we have $|\overline{\Gamma} G(\Lambda \backslash \widehat{I}^i_{n+1}) \Gamma| \leq e^{-\gamma_n l_n}$. Therefore

$$|G(u', v')| \leq 4(\delta_{n+1}\sqrt{E^*+1})^{-1}.$$

The general case where Λ contains many points $c^i_{n+1} \in \overline{S}_n$ is treated using the block resolvent expansion as explained in [15]. □

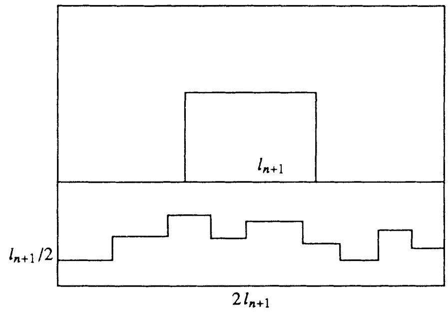
Appendix D

Lemma. *Let $B(l)$ be a square box of length l. Then there exists an n regular box Λ such that*

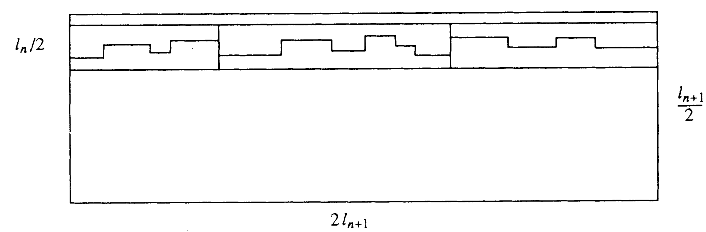
$$B(l_{n+1}) \subset \Lambda \subset B(2l_{n+1})$$

whose perimeter is bounded by l^2_{n+1}.

Proof. It suffices to show that we can pass a broken line across a rectangle of dimension $\frac{1}{2}l_{n+1} \times 2l_{n+1}$ which misses every box I^i_m for $m \leq n$.



The proof will be by induction on n. Assume we are given a rectangle with dimensions $\frac{1}{2}l_{n+1} \times 2l_{n+1}$. By the Center Theorem for S_n, it is possible to put a strip of width $l_n/2$ across our rectangle so that it misses every box I^i_n. We now break up the strip into rectangles which by induction have paths that avoid I^i_m for $m \leq n-1$. The length of the path is bounded by induction. □



Remark. It is possible to choose the box Λ to be symmetric about its center.

Acknowledgment. I would like to thank Professor T. Spencer for introducing me to the subject of this paper and for the many discussions which led to the final result.

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