TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 321, Number 1, September 1990

# DICHROMATIC LINK INVARIANTS

JIM HOSTE AND MARK E. KIDWELL

ABSTRACT. We investigate the skein theory of oriented dichromatic links in $S^3$. We define a new chromatic skein invariant for a special class of dichromatic links. This invariant generalizes both the two-variable Alexander polynomial and the twisted Alexander polynomial. Alternatively, one may view this new invariant as an invariant of oriented monochromatic links in $S^1 \times D^2$, and as such it is the exact analog of the twisted Alexander polynomial. We discuss basic properties of this new invariant and applications to link interchangeability. For the full class of dichromatic links we show that there does not exist a chromatic skein invariant which is a mutual extension of both the two-variable Alexander polynomial and the twisted Alexander polynomial.

## INTRODUCTION

In 1984 Vaughan Jones discovered a significant new polynomial invariant of knots and links [J]. Since then others have found similar invariants that are more general than Jones's original polynomial, and collectively these new invariants have generated a tremendous resurgence of interest on John Conway's skein theory. Indeed, Conway's procedure of computing link invariants by changing and smoothing crossings in a link projection (actually discovered by Alexander, but neglected for forty years) has been virtually the only concept from classical link theory to be used successfully in studying the new invariants. Skein theory before Jones had been regarded as a computational tool allowing the recursive computation of known invariants (mainly the Alexander polynomial) of complicated links in terms of simpler links.

Following the discovery of the Jones polynomial, knot theorists dared to use skein theory to *define* new invariants [$P_1$]. In fact, several groups simultaneously discovered the two-variable twisted Alexander polynomial (also known as the HOMFLY, FLYPMOTH, generalized Jones, two-variable Jones, Jones-Conway, or skein polynomial), which can be regarded as satisfying a universal linear skein relation for oriented links in $S^3$ [FYHLMO, PT]. (It thus includes the one-variable Alexander polynomial and the Jones polynomial as special cases.) Other groups soon discovered a similar polynomial invariant of unoriented links

Received by the editors May 11, 1988 and, in revised form, September 29, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 57M25.
*Key words and phrases.* Knot, link, dichromatic link, skein theory, Alexander polynomial, Conway polynomial, Jones polynomial, HOMFLY polynomial, twisted Alexander polynomial.
Work by Jim Hoste supported in part by National Science Foundation grant DMS-87-96209.

[Ho, BLM]. Shortly thereafter, Kauffman discovered yet another new polynomial link invariant which related their invariant of unoriented links to the Jones polynomial [$K_1$]. Surveys of these new invariants are given in [LM, $K_2$, $P_2$].

To date, only the classical multivariable Alexander polynomial takes into account the possibility that components of a link may have different labels or "colors." Two such chromatic links are equivalent if one can be moved to the other by an ambient isotopy so as to preserve colors. The multivariable Alexander polynomial has one variable for each color used. (Several components of a chromatic link may have the same color.) Attempts to compute this polynomial recursively have been made by Hartley [Ha], Kidwell [Ki], Nakanishi [N], and Turaev [T]. The added difficulty in dealing with chromatic links, rather than monochromatic links, is the inability to smooth a crossing between strands of different colors.

Our purpose in this paper is to investigate the skein theory of two-colored, or *dichromatic*, links in $S^3$. We show that there does not exist a chromatic skein invariant which is a mutual extension of both the twisted Alexander polynomial and the two-variable Alexander polynomial. The main obstacle is revealed by the link $7^2_6$ in Rolfsen's tables [R]. This is the first link in the table which has both components unknotted and which is *noninterchangeable*; that is, no homeomorphism of $S^3$ interchanges the components of $7^2_6$. Thus, the two ways of coloring $7^2_6$ are truly distinct. With each coloring one may reduce to simpler links by performing skein operations on only one of the two colors. Attempts to reconcile these four calculations lead to a drastic reduction in the possibilities for an invariant.

In contrast to the general case, we show that for a special class of dichromatic links the most general possible chromatic skein invariant does in fact exist and that the twisted Alexander polynomial and the two-variable Alexander polynomial are special cases of this more general invariant. We call this invariant $\Omega^1$. The subclass of dichromatic links for which it is defined consists of those links colored with the two colors $\{1, 2\}$ where the color 1 is used only to color a single unknotted component. The color 2 is used to color all the remaining components. We call these *1-trivial* dichromatic links. (A similar invariant, $\Omega^2$, is defined for *2-trivial* links.) These links correspond in an obvious way to monochromatic links inside a solid torus $S^1 \times D^2$. Thus we may reinterpret our result by saying that we have found the analog of the twisted Alexander polynomial of links in $S^3$ for links in $S^1 \times D^2$. Or equivalently, our result may be regarded as the computation of the skein module of $S^1 \times D^2$. The skein module is defined for all 3-manifolds $M$ but has previously been computed only for $M = S^3$ [$P_1$].

The paper is organized as follows. In §1 we precisely formulate the problem we wish to investigate. We also list identities satisfied by both the twisted Alexander polynomial and the two-variable Alexander polynomial (as normalized by Conway [C]). The nature of these identities and the similarities between them provide the motivation for much of our work. In §2 we exhibit a dichromatic link invariant obtained by essentially applying monochromatic invariants to each of the two pure colored sublinks of a dichromatic link. We call this the *uncoupled* invariant. Moreover, we show that unless certain restrictions are made, only the uncoupled invariant is possible. However, in §3 we show that these restrictions are unnecessary provided we limit our attention to 1-trivial (or, equivalently, 2-trivial) links. In this case we establish the existence of a general 1-trivial link invariant $\Omega^1$. The proof that $\Omega^1$ exists is similar, yet more complicated than, the existence proof given by the first author in [H] for the twisted Alexander polynomial. Returning to the general case of dichromatic links in §4, we discover additional restrictions that must be satisfied by any dichromatic skein invariant. In particular, we show that there does not exist a dichromatic skein invariant which is a mutual extension of both the twisted Alexander polynomial and the two-variable Alexander polynomial. Finally, in §5, we discuss invariants which may be derived from $\Omega^1$ and their applications to problems in knot theory. We show that $\Omega^1$ is particularly adept at detecting noninterchangeability for two-component links with trivial components.

## 1. Preliminaries

We shall deal exclusively with two-color, or dichromatic, oriented links in $S^3$ and use "1" and "2" as labels representing the two colors. The sublink of a dichromatic link $L$ consisting of the components colored $i$ will be called the *i-sublink* and denoted $L_i$. Unless otherwise stated, we will consider only links where both the 1- and 2-sublinks are nonempty. In other words, the link is truly dichromatic. A dichromatic link is *i-trivial* if $L_i$ is an unknot. In this case we call $L_i$ the *i-component*. If $K$ and $J$ are two dichromatic links, and *i-connected sum* $J \#_i K$ is an ordinary connected sum of $J$ and $K$ where additionally the connection takes place between an $i$-colored component of $J$ and an $i$-colored component of $K$. Note that if $J$ and $K$ are $i$-trivial, so is their $i$-connected sum.

Throughout this paper we will use Conway's normalized versionof the Alexander polynomial [C], with a minor change of variables introduced by the second author. (Our variables $z_1$ and $z_2$ correspond to Conway's $\{r\}$ and $\{s\}$, respectively. See [Ki] for details.) We have included in Table 1.2 a list of identities satisfied by both the two-variable Conway polynomial $\nabla(z_1, z_2)$ and the two-variable twisted Alexander polynomial $P(v, z)$. In listing these properties we have adopted notational conventions now common in the literature. For example, suppose $D_+$, $D_-$, and $D_0$ are three dichromatic diagrams which are identical except near a single right-handed crossing between two $i$-colored strands of $D_+$. Near this crossing the three diagrams appear as shown in Figure 1.1. Then the Conway polynomials of $D_+$, $D_-$, and $D_0$ are related by

$$\nabla_{D_+}(z_1, z_2) - \nabla_{D_-}(z_1, z_2) = z_i \nabla_{D_0}(z_1, z_2),$$

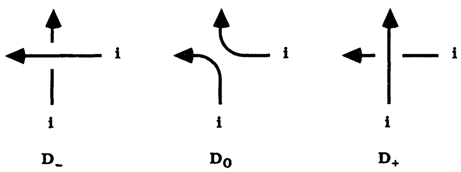



FIGURE 1.1

which we abbreviate as

$$\nabla({}_i\times_i) - \nabla({}_i\times_i) = z_i\nabla({}_i\asymp_i).$$

The reader should note that our choice of parametrization of the twisted Alexander polynomial differs from much of the literature. However, we are consistent with [M] and have adopted this convention because of the ease with which $P$ reduces to the one-variable Conway polynomial, namely by setting $v = 1$. The two-variable Conway polynomial is related to the one-variable Conway polynomial by $\nabla(z) = z\nabla(z, z)$.

We shall distinguish between a *diagram* of a link, in which overcrossings and undercrossings are indicated in the usual way, and a *projection* of a link, in which the actual crossing (intersection) of strands is depicted. In a diagram of a dichromatic link, crossings between strands of the same color are called *pure colored*, while all others are called *mixed*. We can further divide pure colored crossings into *pure* 1*-colored* and *pure* 2*-colored* and mixed crossings into 1*-over-*2 and 2*-over-*1.

TABLE 1.2. Properties satisfied by both $\nabla(z_1, z_2)$ and $P(v, z)$ for dichromatic links colored with "1" and "2." Note that $i, j \in \{1, 2\}$ and $i \neq j$ in these identities.

| Conway Polynomial $\nabla(z_1, z_2)$ | Twisted Alexander Polynomial $P(v, z)$ |
|---|---|
| 1. $\nabla({}_i\times_i) - \nabla({}_i\times_i) = z_i\nabla({}_i\asymp_i)$ | $v^{-1}P({}_i\times_i) - vP({}_i\times_i) = zP({}_i\asymp_i)$ |
| 2. $\nabla({}_i\times_j) + \nabla({}_i\times_j)$ $= \left\{\left[z_1z_2 + \sqrt{(z_1^2+4)(z_2^2+4)}\right]\Big/2\right\}\nabla({}_i\asymp_j)$ | $v^{-2}P({}_i\times_j) + v^2P({}_i\times_j)$ $= (2+z^2)P({}_i\asymp_j)$ |
| 3. $\nabla({}_i\times_j) + \nabla({}_i\times_j)$ $= \left\{\left[-z_1z_2 + \sqrt{(z_1^2+4)(z_2^2+4)}\right]\Big/2\right\}\nabla({}_i\asymp_j)$ | $v^{-1}P({}_i\times_j) + vP({}_i\times_j)$ $= (v^{-1}+v)P({}_i\asymp_j)$ |
| 4. $\nabla_{J\#_iK} = z_i\nabla_J\nabla_K$ | $P_{J\#_iK} = P_JP_K$ |
| 5. $\nabla({}_1\circ\circ{}_2) = 0$ | $P({}_1\circ\circ{}_2) = (v^{-1}-v)z^{-1}$ |
| 6. $\nabla(\circ_i) = z_i^{-1}$ | $P(\circ_i) = 1$ |
| 7. $\nabla({}_1\text{Hopf}_2) = 1$ | $P({}_1\text{Hopf}_2) = (v-v^3)z^{-1} + vz$ |
| 8. $\nabla({}_1\text{Hopf}_2) = -1$ | $P({}_1\text{Hopf}_2) = (v^{-3}-v^{-1})z^{-1} - v^{-1}z$ |

Each crossing in a diagram has a *sign* of $\pm 1$ depending on whether it is right or left handed, respectively. (The crossing of $D_+$ shown in Figure 1.1 is right handed.) The *chromatic* linking number, $l(L)$, of a dichromatic link $L$ is the sum of the signs of the 1-over-2 crossings in any diagram of $L$. This is exactly $\mathrm{lk}(L_1, L_2)$, the sum of the individual linking numbers between the components of $L_1$ and the components of $L_2$.

An important class of 1-trivial dichromatic links is the set of 2-bridge links. We shall adopt the following notation. If $w = a^{n_1}b^{m_1}\cdots a^{n_k}$ is a word in the letters $a$ and $b$, let $H(w)$ be the 2-bridge link shown in Figure 1.3. The twists are chosen so that the chromatic linking number is $\sum n_i$ and the writhe of the 2-component, as pictured in Figure 1.3, is $2\sum m_i$. If $K$ and $L$ are 2-bridge links corresponding to the words $g$ and $h$, then we may also denote their 1-connected sum as $K \#_1 L = H(g \#_1 h)$. We shall usually write $w$ instead of $H(w)$. Except when we are explicitly discussing words rather than links, this convenience should not give rise to any confusion.

Suppose that $L$ is an $i$-trivial dichromatic link. Then we may move $L$ by an ambient isotopy in $\mathbf{R}^3 \cup \{\infty\}$ until the $i$-component is the $z$-axis union the point at infinity, oriented downward. If we now project the link into the $x$–$y$ plane we are left with a diagram of only the $j$-sublink, $j \neq i$, in the punctured plane $\mathbf{R}^2 - \{0\}$. We call such a diagram an *$i$-punctured* diagram. It is shown in [HP] that two such diagrams represent the same $i$-trivial link if and only if they are related by a finite sequence of Reidemeister moves in $\mathbf{R}^2 - \{0\}$. Of course, such moves could never reverse the orientation of the $i$-component, so it is important that we have adopted the convention that it be oriented downward.

We shall call a diagram ($i$-punctured or ordinary) *descending* if it is possible to traverse the components (in the direction of their orientation) in some order and starting from some point on each component so that each crossing is reached for the first time on the overcrossing strand. Clearly any diagram may be made descending by changing crossings of the diagram. If an $i$-punctured diagram is descending, it is not hard to see that it represents an $i$-connected sum of 2-bridge links. If, additionally, each component descends from a basepoint located as far away as possible from the puncture (that is, the basepoint lies on the boundary of the unbounded region in the complement of that component), then the diagram represents an $i$-connected sum of $(2, 2k)$-torus links with parallel orientations.

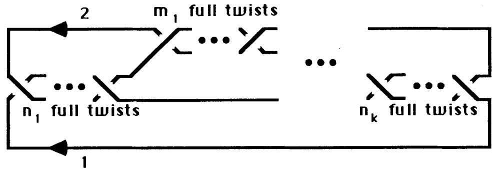



FIGURE 1.3

The process of *resolving* a link into simpler links by repeatedly changing and smoothing crossings is by now well documented and should be familiar to the reader. We shall call an ordered triple of diagrams $(D_+, D_-, D_0)_i$ a *chromatic skein triple of diagrams*, or more simply a *diagram triad*, if $D_-$ is obtained from $D_+$ by changing a single right-handed pure $i$-colored crossing and $D_0$ is obtained by smoothing that crossing. (The diagrams may be ordinary or punctured diagrams.) We shall call $(L_+, L_-, L_0)_i$ a *chromatic skein triple of links*, or more simply a *link triad*, if there exists a diagram triad $(D_+, D_-, D_0)_i$ representing the link triad in the obvious way. Link triads may be joined to form trees of links. Such a tree is a *resolving tree* of a link $L$ if one of its outermost links is $L$ and all other outermost links belong to some previously chosen set of *elementary links* whose values of a given invariant are preassigned.

In the well-known case of computing the twisted Alexander polynomial of a monochromatic link, the unknot can serve as the sole elementary link, its value being arbitrarily assigned to be 1. The resolution of a dichromatic link requires a far more complicated set of elementary links. In particular, since the changing and smoothing of pure colored crossings preserves the chromatic linking number $l$, there must be at least one elementary link in each linking number class. A reasonable set of representatives of the linking number classes is the set of $(2, 2l)$-torus links with parallel orientation. Clearly any dichromatic link can be resolved into 1-trivial links by changing and smoothing only pure 1-colored crossings. By passing to 1-punctured diagrams and then making these diagrams descending, we can further resolve 1-trivial links into 1-connected sums of torus links. Finally, any unlink summand can be eliminated in the presence of other summands. To see this last point, imagine a triad obtained by first introducing a 1-gon by means of a Type I Reidemeister move somewhere along a 2-colored component and then changing and smoothing this crossing. Obviously, the previous argument can be made with the color 1 replaced with the color 2. Thus we have proved that each of the sets

$$\mathscr{E}_i = \{H(a^{n_1} \#_i \cdots \#_i a^{n_k}) \mid n_r \neq 0\} \cup \{H(1)\}$$

for $i = 1$ or $i = 2$ is elementary.

A third elementary set of interest is the following set of 2-bridge links:

$$\mathscr{B} = \{H(a^{k_1} b a^{k_2} b \cdots b a^{k_n}) \mid k_r \neq 0\} \cup \{H(1)\}.$$

To see that $\mathscr{B}$ is elementary it suffices to show that every link in $\mathscr{E}_1$, for example, can be resolved into elements of $\mathscr{B}$. But the triad

$$(a^{k_1} b a^{k_2} \#_1 \cdots \#_1 a^{k_n}, a^{k_1+k_2} \#_1 \cdots \#_1 a^{k_n}, a^{k_1} \#_1 a^{k_2} \#_1 \cdots \#_1 a^{k_n})_2$$

shows how one can successively eliminate 1-connected summands in favor of elements of $\mathscr{B}$.

Summarizing, we have

**Lemma 1.1.** *The sets of dichromatic links $\mathscr{E}_1$, $\mathscr{E}_2$, and $\mathscr{B}$ are each elementary.*

Of course, the choice of an elementary set of links is by no means unique. For example, if $\mathscr{E}$ is elementary and $\mathscr{E} \subset \mathscr{F}$, then $\mathscr{F}$ is also elementary. An extreme case is the one where $\mathscr{E}$ is the set of all links. However, it is of more interest to consider elementary sets which are minimal. In §3 we prove that both $\mathscr{E}_1$ and $\mathscr{E}_2$ are minimal.

Using Lemma 1.1, we see that to compute $\nabla(z_1, z_2)$ relative to $\mathscr{E}_1$, for example, one must eventually compute the Conway polynomial for 1-connected sums of $(2, 2k)$-torus links. Since the Conway polynomial obeys a connected sum rule (Property 4 in Table 1.2), it remains only to compute the values of the torus links themselves. But these values are uniquely determined by the "clasp rule" (Property 2 in Table 1.2) together with the values 0, 1, and $-1$ of the unlink and right-handed and left-handed Hopf links, respectively.

The twisted Alexander polynomial is actually an invariant of monochromatic links, but we shall consider it as an invariant of dichromatic links by simply ignoring the coloring of a link. As a dichromatic invariant it can be computed in a fashion similar to that just described for the Conway polynomial. Again, the values of $P(v, z)$ for the elementary links are uniquely determined by a connected sum rule and a clasp rule.

We may now state the basic goal of this paper, which is to investigate the following question.

*Question* 1.2. Let $\mathscr{E}$ be some elementary set of dichromatic links, for example, $\mathscr{E}_1$, $\mathscr{E}_2$, or $\mathscr{B}$, and for each $E \in \mathscr{E}$ let $[E]$ be an indeterminate. What are necessary and sufficient relations among $\{v_1, v_2, z_1, z_2, \{[E]\}_{E\in\mathscr{E}}\}$ so that there exists a well-defined invariant $\Omega$ of ambient isotopy classes of oriented dichromatic links uniquely determined by the following identities?

$$(1.1) \qquad v_1^{-1}\Omega({}_1\times{}_1) - v_1\Omega({}_1\times{}_1) = z_1\Omega({}_1\asymp{}_1),$$

$$(1.2) \qquad v_2^{-1}\Omega({}_2\times{}_2) - v_2\Omega({}_2\times{}_2) = z_2\Omega({}_2\asymp{}_2),$$

$$(1.3) \qquad \Omega(E) = [E] \quad \text{for all } E \in \mathscr{E}.$$

Table 1.2 shows that two definite possibilities are the Conway polynomial $\nabla(z_1, z_2)$ and the twisted Alexander polynomial $P(v, z)$. Thus, part of our interest in Question 1.2 is motivated by our desire to answer the following, more specific, question.

*Question* 1.3. In particular, is there an invariant $\Omega$ as described in Question 1.2 which is a mutual extension of both the twisted Alexander polynomial and the Conway polynomial?

We shall prove in §4 that the answer to Question 1.3 is no. In §3 we provide a complete answer to Question 1.2 in the case of $i$-trivial dichromatic links. (Of course, in this case, we must not allow the use of rule (1.1) if $i = 1$ or (1.2) if $i = 2$ in order to avoid leaving the class of $i$-trivial links.) For the case of all dichromatic links we provide, in §4, only a partial answer to Question 1.2. In particular, we find some necessary, but insufficient, relations.

Some remarks on our formulation of equations (1.1) and (1.2) are in order. We call these identities *skein relations* or *crossing rules*. Apparently more general are the relations

$$A_i\Omega({}_i\times_i) - B_i\Omega({}_i\times_i) = C_i\Omega({}_i\asymp_i), \qquad i = 1, 2.$$

However, if $A_iB_i \neq 0$ this relation may be symmetrized by dividing by $(A_iB_i)^{1/2}$. The substitutions $v_i = (B_i/A_i)^{1/2}$ and $z_i = C_i/(A_iB_i)^{1/2}$ then return us to (1.1) and (1.2). The two-term skein relations in which $A_i = 0$ or $B_i = 0$ appear to be uninteresting. We shall also make the blanket assumption that $z_i \neq 0$.

We shall say that link invariant $\alpha$ *extends* link invariant $\beta$ if $\alpha$ takes different values on two links whenever $\beta$ takes different values on the same two links. If $\alpha$ can distinguish a pair of links which $\beta$ cannot, then we say that $\alpha$ is a *proper* extension. For example, the twisted Alexander polynomial is a proper extension of both the Jones polynomial and the one-variable Alexander polynomial. Two link invariants are called *independent* if neither extends the other and *equivalent* if each extends the other.

## 2. The uncoupled invariant

As we have already said, there exist dichromatic link invariants which satisfy equations (1.1) and (1.2) provided one introduces relations among the variables $v_i$ and $z_i$ and among the elementary values $\{[E]\}$. These are the Conway and twisted Alexander polynomials. But if no relations are introduced among $\{v_1, v_2, z_1, z_2\}$ there still exists an invariant of dichromatic links satisfying (1.1) and (1.2). This invariant is

$$\mathscr{U}(L) = P_{L_1}(v_1, z_1)P_{L_2}(v_2, z_2)[a^{l(L)}]$$

where $l(L)$ is the chromatic linking number of $L$ and $[a^l]$ is an indeterminate. If $E$ and $F$ are two elementary links in $\mathscr{B}$ with chromatic linking number $l$, then we have set $[E] = [F] = [a^l]$. So while no relations have been introduced among $\{v_1, v_2, z_1, z_2\}$, many relations have been introduced among the elementary values $\{[E]\}_{E\in\mathscr{B}}$. (If we worked over the elementary set $\mathscr{E}_1$ instead of $\mathscr{B}$, for example, the relations among the elementary values would be different.) Since the components of the $(2, 2l)$-torus link $a^l$ are themselves unknotted, we see that the indeterminates $\{[a^l]\}$ are simply the values taken by $\mathscr{U}$ on the torus links (as well as all other links of two trivial components with linking number $l$). It is not hard to verify that $\mathscr{U}$ does indeed satisfy equations (1.1) and (1.2). We call $\mathscr{U}$ the *uncoupled* invariant. Note that $\mathscr{U}$ cannot distinguish any pair of 2-bridge links having the same chromatic linking number $l$. Since this is not the case for either the Conway polynomial or the twisted Alexander polynomial, we see that $\mathscr{U}$ extends neither. Similarly, $\nabla(z_1, z_2)$ cannot distinguish any pair of split links (since it is zero for such links) while $\mathscr{U}$ can. Finally, it is easy to produce a pair of links which $P(v, z)$ cannot distinguish while $\mathscr{U}$ can. For example, consider a link of three components colored in two different ways and with different chromatic linking numbers in each case. Thus neither $\nabla(z_1, z_2)$ nor $P(v, z)$ extends $\mathscr{U}$. However, we show in §5 that for $i$-trivial links, $\Omega^i$ is a proper extension of each of these three invariants.

The remainder of this section is devoted to proving the following theorem.

**Theorem 2.1.** *Suppose $\Omega$ is an invariant of dichromatic links satisfying the skein relations* (1.1) *and* (1.2). *If $v_1^2 \neq v_2^2$ then*

$$\Omega(L) = P_{L_1}(v_1, z_1)P_{L_2}(v_2, z_2)\Omega(a^{l(L)}).$$

*In particular, if there are no relations among $\{v_1, v_2, z_1, z_2\}$, then $\Omega$ can be derived from $\mathscr{U}$ by substituting for each indeterminate $[a^l]$ the value $\Omega(a^l)$.*

*Proof.* In order to prove the theorem it suffices to do so in the case where $L$ is a 1-connected sum of $(2, 2l)$-torus links. This is because Lemma 1.1 guarantees that every dichromatic link can be resolved into such links. Furthermore, it is easy to show that if the theorem holds for any two links of a triad, it holds for the third link as well.

Therefore, assume that $L = a^{k_1} \#_1 a^{k_2} \#_1 \cdots \#_1 a^{k_n}$. If $n = 1$, the result is obvious since both $L_1$ and $L_2$ are unknotted. Proceeding inductively, we consider a tree of four link triads, the first of which is

$$(a^{k_1}ba^{k_2} \#_1 \cdots \#_1 a^{k_n}, a^{k_1+k_2} \#_1 \cdots \#_1 a^{k_n}, a^{k_1} \#_1 a^{k_2} \#_1 \cdots \#_1 a^{k_n})_2.$$

Since 2-bridge links are interchangeable, we may redraw the link $a^{k_1}ba^{k_2}$ with the colors reversed. If we carry the other components of $a^{k_1}ba^{k_2} \#_1 \cdots \#_1 a^{k_n}$ along, we can then generate the triad

$$(a^{k_1}ba^{k_2} \#_1 \cdots \#_1 a^{k_n}, a^{k_1+k_2} \#_1 \cdots \#_1 a^{k_n}, a^{k_1} \#_2 (a^{k_2} \#_1 \cdots \#_1 a^{k_n}))_1.$$

If we then "put back" the crossing we just changed and smoothed, but with opposite handedness, we obtain the triad

$$(a^{k_1+k_2} \#_1 \cdots \#_1 a^{k_n}, a^{k_1}b^{-1}a^{k_2} \#_1 \cdots \#_1 a^{k_n}, a^{k_1} \#_2 (a^{k_2} \#_1 \cdots \#_1 a^{k_n}))_1.$$

Finally, we can once again interchange the components of $a^{k_1}b^{-1}a^{k_2}$, carrying along the remaining components of $a^{k_1}b^{-1}a^{k_2} \#_1 \cdots \#_1 a^{k_n}$. We can then form the fourth triad

$$(a^{k_1+k_2} \#_1 \cdots \#_1 a^{k_n}, a^{k_1}b^{-1}a^{k_2} \#_1 \cdots \#_1 a^{k_n}, a^{k_1} \#_1 a^{k_2} \#_1 \cdots \#_1 a^{k_n})_2.$$

Imagine these four triads joined along

$$a^{k_1}ba^{k_2} \#_1 \cdots \#_1 a^{k_n}, \quad a^{k_1} \#_2 (a^{k_2} \#_1 \cdots \#_1 a^{k_n}), \quad \text{and} \quad a^{k_1}b^{-1}a^{k_2} \#_1 \cdots \#_1 a^{k_n},$$

respectively, so that the outermost vertices of the tree are two copies of $a^{k_1} \#_1 a^{k_2} \#_1 \cdots \#_1 a^{k_n}$ and four copies of $a^{k_1+k_2} \#_1 \cdots \#_1 a^{k_n}$. We can therefore attempt to solve for $\Omega(a^{k_1} \#_1 a^{k_2} \#_1 \cdots \#_1 a^{k_n})$ in terms of $\Omega(a^{k_1+k_2} \#_1 \cdots \#_1 a^{k_n})$. This will be possible if $v_1^{-1}v_2 - v_1v_2^{-1} \neq 0$, for it is possible to combine the four skein relations arising from the four triads to obtain

$$\begin{aligned}(v_1^{-1}v_2 - v_1v_2^{-1})z_2\Omega(a^{k_1}\#_1 a^{k_2}\#_1 \cdots \#_1 a^{k_n}) \\ = (v_1^{-1}v_2 - v_1v_2^{-1})(v_2^{-1} - v_2)\Omega(a^{k_1+k_2}\#_1 \cdots \#_1 a^{k_n}).\end{aligned}$$

Hence if $v_1^2 \neq v_2^2$ we have

$$\Omega(a^{k_1}\#_1 a^{k_2}\#_1 \cdots \#_1 a^{k_n}) = (v_2^{-1} - v_2)z_2^{-1}\Omega(a^{k_1+k_2}\#_1 \cdots \#_1 a^{k_n}).$$

Now applying our inductive hypothesis gives

$$\begin{aligned}\Omega(a^{k_1}\#_1 a^{k_2}\#_1 \cdots \#_1 a^{k_n}) &= (v_2^{-1} - v_2)z_2^{-1}[(v_2^{-1} - v_2)z_2^{-1}]^{n-2}\Omega(a^{k_1+\cdots+k_n}) \\ &= P_{L_1}(v_1, z_1)P_{L_2}(v_2, z_2)\Omega(a^{l(L)}). \quad \square\end{aligned}$$

## 3. Defining $\Omega^i$ for $i$-trivial links

As we saw in the last section, unless relations are introduced among the variables $\{v_1, v_2, z_1, z_2\}$, any invariant satisfying the skein relations must simply be a derivative of the uncoupled invariant $\mathscr{U}$. We shall prove in this section that quite the reverse is true if one restricts to the class of $i$-trivial links. Indeed, in this case, without introducing any relations among $\{v_1, v_2, z_1, z_2\}$, there exists an invariant $\Omega^i$ of $i$-trivial links which satisfies the skein relations. Of course, since we are restricting our attention to $i$-trivial links, we cannot consider both skein relations (1.1) and (1.2). For example, if $i = 1$ we must exclude relation (1.1) since smoothing a pure 1-colored crossing produces a link which is not 1-trivial. Thus $\Omega^i$ will employ only the variables $v_j$ and $z_j$, $j \neq i$, and the elementary values $\{[E]\}_{E\in\mathscr{E}}$. We shall work exclusively with $\mathscr{E} = \mathscr{E}_i$. We shall prove that both $\nabla(z_1, z_2)$ and $P(v, z)$ can be derived from $\Omega^i$.

We will define $\Omega^i$ so that for an arbitrary link $L$ the invariant $\Omega^i(L)$ will equal a linear combination of the values of the elementary links with coefficients in $\mathbf{Z}[v_j^{\pm 1}, z_j^{\pm 1}]$, $j \neq i$. The values of $\Omega^i$ for the elementary links $\mathscr{E}_i$ will remain as indeterminates denoted by $\Omega^i(E) = [E]$, where $E \in \mathscr{E}_i$. If $E$ is the specific elementary link $E = H(a^{n_1}\#_i \cdots \#_i a^{n_k})$, we sometimes write $[a^{n_1}\#_i \cdots \#_i a^{n_k}]$ in place of $[E]$. Note that the integers $n_r$ need not be distinct and that their order is irrelevant.

**Theorem 3.1.** *There exists a unique isotopy invariant*

$$\Omega^i(L) \in \mathbf{Z}[v_j^{\pm 1}, z_j^{\pm 1}, \{[E]\}_{E\in\mathscr{E}_i}], \qquad j \neq i,$$

*of i-trivial dichromatic links L satisfying the following properties*:

1. *Crossing rule*:

$$v_j^{-1}\Omega^i({}_j\times_j) - v_j\Omega^i({}_j\times_j) = z_j\Omega^i({}_j\asymp_j), \qquad j \neq i.$$

2. *Initial data*:

$$\Omega^i(E) = [E] \quad \textit{for all } E \in \mathscr{E}_i .$$

*Proof*. For notational convenience, we will prove Theorem 3.1 with $i = 1$, the case where $i = 2$ being completely symmetric. Moreover, since every occurrence of $v$ and $z$ is then subscripted with a "2," we *shall omit these subscripts* for the remainder of this section!

Since the proof of Theorem 3.1 is rather long, we begin with a brief outline of what will follow. Our overall plan is to define $\Omega^1$ first for 1-punctured diagrams and to show that it is preserved by Reidemeister moves in $\mathbf{R}^2$–$\{0\}$. Thus $\Omega^1$ is actually an invariant of 1-trivial links. It will then be a simple matter to pass to ordinary diagrams rather than just 1-punctured diagrams.

To define $\Omega^1$ for 1-punctured diagrams we proceed in several steps. In Part 1 we define $\Omega^1$ for descending 1-punctured diagrams having any number of crossings. Such a diagram $D$ represents a 1-connected sum of 2-bridge links and hence corresponds to a collection of words in $a$ and $b$. We first describe how to obtain these words from the diagram and then how to operate on these words to arrive at $\Omega^1(D)$. If $D$ is a 1-punctured diagram having no crossings, then it is descending with respect to every possible choice of basepoints and ordering of the components. Thus $\Omega^1(D)$ is defined. This begins an inductive argument based on the number of crossings in a 1-punctured diagram.

In Part 2 we assume, inductively, that $\Omega^1$ has been defined for all 1-punctured diagrams having fewer than $n$ crossings and that, moreover, for such diagrams $\Omega^1$ satisfies four properties. Using this assumption, together with the definition of $\Omega^1$ for descending diagrams obtained in Part 1, we define $\Omega^1$ for an arbitrary 1-punctured diagram having $n$ crossings. The definition involves making several choices, which we show, in Part 3, are irrelevant. To complete the inductive step we must still show that $\Omega^1$ continues to satisfy the four properties of the inductive hypothesis.

After completing Part 3 we will have defined $\Omega^1$ for all 1-punctured diagrams and know that it satisfies the four properties of the inductive hypothesis. From one of these properties it will follow immediately that $\Omega^1$ is preserved by Reidemeister moves in $\mathbf{R}^2$–$\{0\}$ and hence that it actually represents an invariant of 1-trivial links.

To conclude the proof, it will remain to show that Property 1 of Theorem 3.1 holds for ordinary diagrams as well as for 1-punctured diagrams. This is done in Part 4.

*Part* 1. Defining $\Omega^1$ for descending diagrams. Let $D$ be a pointed and ordered 1-punctured diagram. That is, the components have been ordered and each has been endowed with a chosen basepoint. Assume additionally that $D$ is descending with respect to this choice of pointing and ordering. Hence $D$ represents a 1-connected sum of 2-bridge links. Our first task is to determine which 2-bridge links comprise $D$. To do this we introduce the following construction.

Let $A$ and $B$ be two "branch cuts" for each component $K$ of $D$. That is, $A$ and $B$ are disjoint closed rays embedded in $\mathbf{R}^2$ which emanate from the origin and basepoint $b$ of $K$, respectively, and which meet $K$ transversely in a finite number of points. ("$A$" stands for "axis" and "$B$" stands for "basepoint.") Orient $A$ and $B$ away from their endpoints and assign a $\pm 1$ to each intersection of $A$ or $B$ with $K$ as follows: Replace each intersection of $B - \{b\}$ with $K$ by a crossing of $B$ over $K$ and assign $+1$ for a right-handed crossing, $-1$ for a left-handed crossing. Similarly, replace each intersection of $A$ with $K$ by a crossing of $A$ under $K$ and assign $+1$ for a right-handed crossing, $-1$ for a left-handed crossing. Now as we traverse $K$, starting at the basepoint $b$ and traveling in the direction of its orientation, we may write down a word $w$ in the letters $a$ and $b$ as follows: If $A$ is crossed we record $a^{\pm 1}$ according to whether the crossing is positive or negative, and if $B$ is crossed we similarly record $b^{\pm 1}$. If neither $A$ nor $B$ is crossed we write the empty, or trivial, word 1. Note that every exponent appearing in $w$ is $\pm 1$, unless $w = 1$. These will be the only kinds of words we consider. In particular, *it is not necessary* to consider the free group generated by $a$ and $b$ or the usual equivalence relation on such words.

Each component $K_i$ of $D$ determines a word $w_i$ in $a$ and $b$ as described above. Clearly $D$ represents the link $H(w_1 \#_1 \cdots \#_1 w_k)$. The following lemma describes how the choice of branch cuts influences the words $w_1, \ldots, w_k$.

**Lemma 3.2.** *Let $D$ be a 1-punctured diagram with exactly one component $K$ which is descending with respect to some basepoint $b$. Let $A$ and $B$ be branch cuts for $K$ and let $w$ be the associated word. If $A'$ and $B'$ are two other branch cuts for $K$ (with respect to the same basepoint of course) with associated word $w'$, then $w'$ can be obtained from $w$ by a finite sequence of the following operations*:

1. *Introduce or delete a leading or trailing $b$ or $b^{-1}$.*
2. *Introduce or delete $bb^{-1}$, $b^{-1}b$, $aa^{-1}$, or $a^{-1}a$ from somewhere within the word.*

*Proof*. There exists an isotopy taking $A$ and $B$ to $A'$ and $B'$ keeping $K$ fixed. This isotopy may be accomplished so that the branch cuts change relative to $K$ by a finite sequence of moves of the type shown in Figure 3.1. It is now a simple matter to check that these moves affect the word $w$ as claimed. □

We now return to the problem of defining $\Omega^1(D)$, where $D$ is a pointed and ordered diagram which is descending with respect to its pointing and ordering. We shall do this by assigning a value $f(s)$ to the collection $s = \{w_1, \ldots, w_k\}$ of words determined by the components of $D$ as described above. It will turn out that $f(s)$ is invariant under the operations described in Lemma 3.2, so the choice of branch cuts will not influence the outcome.

Let $\mathscr{S}$ be the set of all finite nonempty unordered collections of (not necessarily distinct) words in the letters $a$ and $b$, including the trivial word 1.

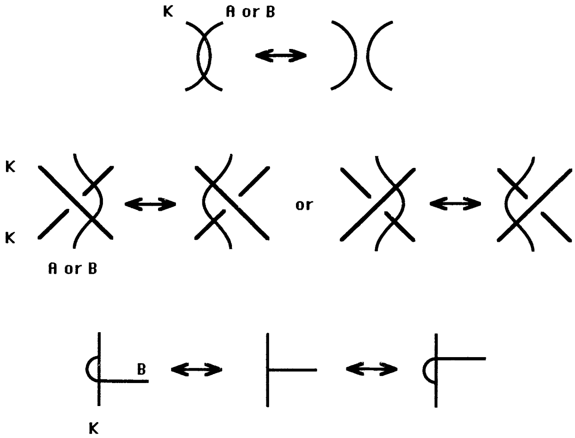



FIGURE 3.1

Moreover, assume that each word in $a$ and $b$ uses exponents of only $\pm 1$, so that $\{aaaa^{-1}, abba, 1\} \in \mathscr{S}$ but $\{a^2, ab^2a, a^2a^{-2}\} \notin \mathscr{S}$. Define an *elementary word* to be either the trivial word 1 or a word that does not involve $b$ and has all its exponents equal to 1 or all its exponents equal to $-1$. Finally, let the *elementary collections* $\overline{\mathscr{E}} \subset \mathscr{S}$ be $\{1\}$ together with all collections of nontrivial elementary words. These correspond, of course, to the elementary links.

We will define a function $f : \mathscr{S} \to \mathbf{Z}[v^{\pm 1}, z^{\pm 1}, \{[e]\}_{e \in \overline{\mathscr{E}}}]$ such that $f(e) = [e]$ if $e \in \overline{\mathscr{E}}$ and otherwise $f(s)$ is a linear combination of the elementary values with coefficients in $\mathbf{Z}[v^{\pm 1}, z^{\pm 1}]$. In other words, the elementary collections serve as indeterminates in terms of which all other collections are expressed.

**Theorem 3.3.** *There exists a unique function* $f : \mathscr{S} \to \mathbf{Z}[v^{\pm 1}, z^{\pm 1}, \{[e]\}_{e \in \overline{\mathscr{E}}}]$ *satisfying the following properties*:

1. *If* $e \in \overline{\mathscr{E}}$ *then* $f(e) = [e]$.
2. $f(\{1, w_1, \dots, w_k\}) = (v^{-1} - v)z^{-1} f(\{w_1, \dots, w_k\})$ *for all* $\{w_1, \dots, w_k\} \in \mathscr{S}$.
3. $f(\{gb^{\pm 1}h, w_1, \dots, w_k\})$
$$= v^{\pm 2} f(\{gh, w_1, \dots, w_k\}) \pm v^{\pm 1} z f(\{g, h, w_1, \dots, w_k\})$$
*for all* $\{g, h, w_1, \dots, w_k\} \in \mathscr{S}$.
4. *If* $s'$ *is obtained from* $s$ *by deleting* $aa^{-1}$, $a^{-1}a$, $bb^{-1}$ *or* $b^{-1}b$ *from somewhere within a word of* $s$, *then* $f(s) = f(s')$.
5. *If* $s'$ *is obtained from* $s$ *by deleting a leading or trailing* $b^{\pm 1}$ *from one of the words of* $s$, *then* $f(s) = f(s')$.

Remember that elements of $\mathscr{S}$ are *unordered* collections of words. Thus the operations indicated in properties 2 and 3 may be applied to any word in the collection, not just the "first" word.

*Proof*. We may write $\mathscr{S}$ as $\mathscr{S} = \bigcup \mathscr{S}_n$ where $\mathscr{S}_n$ are all collections of $\mathscr{S}$ having $n$ or fewer total appearances of $b$ or $b^{-1}$. If $s \in \mathscr{S}_0$ then define $f(s)$ as follows. First replace each nontrivial word of $s$ with $a^n$, where $n$ is the sum of all the exponents in that word. If $n = 0$ write 1 instead of $a^0$. Now return to an element of $\mathscr{S}$ by replacing $a^n$ with a string of $n$ $a$'s if $n > 0$ or a string of $n$ $a^{-1}$'s if $n < 0$. Next, eliminate excess 1's, if any, by using property 2. Now what remains is an elementary collection $e$, so assign it the value $[e]$. This defines $f$ uniquely for $\mathscr{S}_0$, and furthermore $f$ satisfies properties 1–5.

Now suppose inductively that $f$ has been defined for $\mathscr{S}_n$ and continues to satisfy properties 1–5. Let $s \in \mathscr{S}_{n+1}$. Choose an appearance of $b^{\pm 1}$, say $b^\beta$, in one of the words of $s$ and define

$$\begin{aligned} f(s) &= f(\{\dots, gb^\beta h, \dots\}) \\ &= v^{2\beta} f(\{\dots, gh, \dots\}) + \beta v^\beta z f(\{\dots, g, h, \dots\}). \end{aligned}$$

If $b^\beta$ is actually at the beginning of the word, then $g$ will not appear in $s$ and we interpret the right-hand side of this equation by replacing $gh$ with $h$ in the first term and $g$ with 1 in the second term. Similar considerations are made if $b^\beta$ appears at the end of the word or comprises the entire word. The collections on the right side of this equation are in $\mathscr{S}_n$, hence their values are defined and satisfy properties 1–5 above. To see that $f(s)$ is well defined, we must show that the choice of which appearance of $b^{\pm 1}$ to eliminate is immaterial. So suppose that

$$s = \{\dots, gb^\beta h, \dots, g'b^\gamma h', \dots\}.$$

If we first eliminate $b^\beta$ and then $b^\gamma$ we obtain

$$\begin{aligned} f(s) &= v^{2\beta} f(\{\dots, gh, \dots, g'b^\gamma h', \dots\}) \\ &\quad + \beta v^\beta z f(\{\dots, g, h, \dots, g'b^\gamma h', \dots\}). \\ &= v^{2\beta+2\gamma} f(\{\dots, gh, \dots, g'h', \dots\}) \\ &\quad + \gamma v^{2\beta+\gamma} z f(\{\dots, gh, \dots, g', h', \dots\}) \\ &\quad + \beta v^{\beta+2\gamma} z f(\{\dots, g, h, \dots, g', h', \dots\}) \\ &\quad + \beta\gamma v^{\beta+\gamma} z^2 f(\{\dots, g, h, \dots, g', h', \dots\}). \end{aligned}$$

But the reader may easily check that if we first eliminate $b^\gamma$ and then $b^\beta$ we obtain the same result. A similar computation holds if both $b^\beta$ and $b^\gamma$ are in the same word. Thus $f(s)$ is well defined.

It remains to complete the inductive step, that is, to show that $f$ continues to satisfy properties 1–5. If $s \in \mathscr{S}_{n+1}$ and $s'$ is obtained from $s$ by deleting $aa^{-1}$ or $a^{-1}a$ from within some word of $s$, then it is easy to see that $f(s) = f(s')$.

It is more interesting if $s'$ is obtained by deleting $bb^{-1}$ or $b^{-1}b$. For example,

$$\begin{aligned} f(s) &= f(\{\dots, gbb^{-1}h, \dots\}) \\ &= v^2 f(\{\dots, gb^{-1}h, \dots\}) + vzf(\{\dots, g, b^{-1}h, \dots\}) \\ &= f(\{\dots, gh, \dots\}) - vzf(\{\dots, g, h, \dots\}) \\ &\quad + v^{-1}zf(\{\dots, g, h, \dots\}) - z^2 f(\{\dots, g, 1, h, \dots\}) \\ &= f(s') - [vz - v^{-1}z + z^2(v^{-1} - v)z^{-1}]f(\{\dots, g, h, \dots\}) \\ &= f(s'). \end{aligned}$$

Thus the extension of $f$ to $\mathcal{S}_{n+1}$ continues to satisfy property 4 of the inductive hypothesis. We leave verification that $f$ satisfies the other properties to the reader. □

*Remark* 3.4. As already mentioned, the words associated to a descending diagram by means of the branch cuts may be taken to have only exponents of $\pm 1$. We have proved Theorem 3.3 with this assumption and we could continue to deal only with words of this kind. However, it is obviously more convenient to write $a^{10}$ for $aaaaaaaaaa$. Therefore, we shall henceforth make full use of the usual equivalence relation on words. This allows one, for example, to replace property 3 of Theorem 3.3 with

$$\begin{aligned} 3'.\ f(\{gb^n h, w_1, \dots, w_k\}) &= v^{2\,\mathrm{sgn}(n)} f(\{gb^{n-\mathrm{sgn}(n)}h, w_1, \dots, w_k\}) \\ &\quad + \mathrm{sgn}(n) v^{\mathrm{sgn}(n)} zf(\{g, h, w_1, \dots, w_k\}) \\ &\qquad \text{for all } n \in \mathbf{Z}, \text{ where } \mathrm{sgn}(n) = n/|n|. \end{aligned}$$

We leave it to the reader to check that no ambiguities can arise by operating in this greater generality.

Another useful notational convenience is to write $\{\dots, kw, \dots\}$ in place of the collection $\{\dots, w, w, \dots, w, \dots\}$ where the same word $w$ appears $k$ times. Thus, instead of writing $\{aaa^{-1}bbba, a, a, 1, 1\}$, we may write $\{ab^3a, 2a, 2\}$.

*Remark* 3.5. If $s = \{s_1, \dots, s_j\}$ and $t = \{t_1, \dots, t_k\}$ are elements of $\mathcal{S}$, we shall denote their *concatenation* by $s, t = \{s_1, \dots, s_j, t_1, \dots, t_k\}$. (Remember that the elements of $\mathcal{S}$ are collections of not necessarily distinct words.)

If $f(s) = \sum p_i[e_i]$ and $f(t) = \sum q_j[d_j]$ where $p_i, q_j \in \mathbf{Z}[v^{\pm 1}, z^{\pm 1}]$ and $e_i$, $d_j \in \overline{\mathcal{E}}$, it is not difficult to show that

$$f(s, t) = \sum p_i q_j f(e_i, d_j).$$

Even though $e_i$ and $d_j$ are both elementary, their concatenation $e_i, d_j$ need not be, since one or both of $e_i$ or $d_j$ might equal $\{1\}$.

Finally, we may use Theorem 3.3 to define $\Omega^1(D)$, where $D$ is a pointed and ordered diagram which is descending with respect to its pointing and ordering. Namely, first choose branch cuts for each component of $D$ and use these to determine an associated word in the letters $a$ and $b$. This determines a collection

$s \in \mathscr{S}$ for which we may compute $f(s)$ as in Theorem 3.3. Because of Lemma 3.2 and properties 4 and 5 of Theorem 3.3, $f(s)$ does not depend on the choice of branch cuts. Now derive $\Omega^1(D)$ from $f(s)$ by replacing each occurrence of the elementary value $[e]$ in $f(s)$ by the corresponding elementary value $[E]$. For example, $[a^{-2}, 2a^{-1}, a, a^2]$ is replaced by $[a^{-2} \#_1 2a^{-1} \#_1 a \#_1 a^2]$.

*Part* 2. Defining $\Omega^1$ for arbitrary 1-punctured diagrams. We shall now return to our overall program of defining $\Omega^1(D)$ for an arbitrary 1-punctured diagram $D$. As already mentioned, we shall do so by inducting on the number of crossings in the diagram. If $D$ has no crossings, then $D$ is descending with respect to every possible choice of pointing and ordering. Moreover, the collection $s$ of words associated to $D$ does not depend on the pointing and ordering. Thus the definition given in Part 1 for descending pointed and ordered diagrams may be applied to produce a unique value for $D$. In particular, if $D$ represents a 1-connected sum of $r$ right-handed Hopf links, $s$ left-handed Hopf links, and $t$ (two-component) unlinks, then

$$\Omega^1(D) = \begin{cases} ((v^{-1} - v)z^{-1}))^{t-1}[1] & \text{if } r + s = 0, \\ ((v^{-1} - v)z^{-1}))^t[sa^{-1} \#_1 ra] & \text{if } r + s > 0. \end{cases}$$

Now suppose, inductively, that $\Omega^1$ has been uniquely defined for all 1-punctured diagrams having less than $n$ crossings and, for such diagrams, satisfies the following properties:

1. *Crossing rule*: If $(D_+, D_-, D_0)$ is a triad of 1-punctured diagrams, then
$$v^{-1}\Omega^1(D_+) - v\Omega^1(D_-) = z\Omega^1(D_0).$$
2. *Descending diagram rule*: If $D$ is a 1-punctured diagram which is descending with respect to some choice of ordering and basepoints, and $s$ is the associated collection of words relative to some choice of branch cuts, then $\Omega^1(D) = f(s)$ as defined in Theorem 3.3.
3. *Invariance under Reidemeister moves*: If $D$ and $D'$ are 1-punctured diagrams and $D'$ is obtained from $D$ by a Reidemeister move in $\mathbf{R}^2-\{0\}$ which does not increase the number of crossings, then $\Omega^1(D') = \Omega^1(D)$.
4. *Invariance under moving separated subdiagrams*: Suppose $D$ is a 1-punctured diagram and $d_1$ and $d_2$ are two disjoint disks in $\mathbf{R}^2-\{0\}$ with $\partial d_1$ and $\partial d_2$ disjoint from the components of $D$. If $D'$ is obtained from $D$ by exchanging $d_1$ with $d_2$, then $\Omega^1(D) = \Omega^1(D')$.

Note that our definition of $\Omega^1(D)$ in the case where $D$ has no crossings does indeed satisfy these properties. (Properties 1 and 3 are vacuously satisfied.)

Now suppose that $D$ is a 1-punctured diagram with $n$ crossings. In order to define $\Omega^1(D)$ we proceed as follows. First choose some ordering of the components of $D$ together with a basepoint on each component. Next choose a sequence in which to change crossings in order to arrive at a diagram which is descending relative to this choice of ordering and basepoints. Accompanying each changing of a crossing is a smoothing of that crossing. Thus the process generates a binary tree $T$ which we call a *resolution* of $D$. The diagrams of $T$ that have $n$ crossings are all pointed and ordered, while those that result from the smoothings, and hence have $n-1$ crossings, are considered unpointed and unordered. Let $D_i$, $1 \leq i \leq m-1$, be the diagrams created as a result of the smoothings and $D_m$ be the descending diagram at the end of the resolution obtained by changing the crossings. For $i < m$ each $D_i$ has $n-1$ crossings and so $\Omega^1(D_i)$ exists by assumption. Furthermore, $D_m$ is a pointed and ordered diagram which is descending with respect to its pointing and ordering. Therefore define $\Omega^1(D_m)$ as in Part 1. We may now use the crossing rule together with the tree $T$ and all the values $\Omega^1(D_i)$ to assign a value of $\Omega^1$ to $D$. We must now show that $\Omega^1(D)$ is well defined.

*Part* 3. Proving that $\Omega^1$ is well defined. We must show that the definition of $\Omega^1$ given in Part 2 does not depend on the choice of ordering of the components, the basepoints chosen for each component, or the order in which crossings are changed to reach a descending diagram. Note that after a descending diagram $D_m$ is reached, more choices are still made in the computation of $\Omega^1(D_m)$, namely the choice of branch cuts used to determine the collection of words associated to $D_m$ and then the order in which $b$'s are eliminated from these words. However, we have already shown that these choices are irrelevant.

*Step* 1. The order in which the crossings of $D$ are changed to reach the descending diagram is immaterial.

The proof of this proceeds exactly as in [H] since only the crossing rule is employed.

*Step* 2. The choice of basepoints is immaterial.

The proof of this step is nearly the same as that given in [H], the difference being due to the fact that our elementary links are now 2-bridge links rather than unlinks. Proceeding as in [H], it suffices to consider the case where $D$ is descending and consists of a single component and $D'$ is obtained from $D$ by moving the basepoint forward past one crossing. It must appear as shown in Figure 3.2.

There are two cases: either a right- or left-handed crossing is involved. Consider a right-handed crossing and the tree shown in Figure 3.3.

Now $D_1$ is descending with respect to its basepoint and $D_2$ is a two-component link diagram which is descending with respect to the choice of pointing and ordering shown.

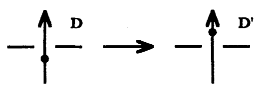



FIGURE 3.2

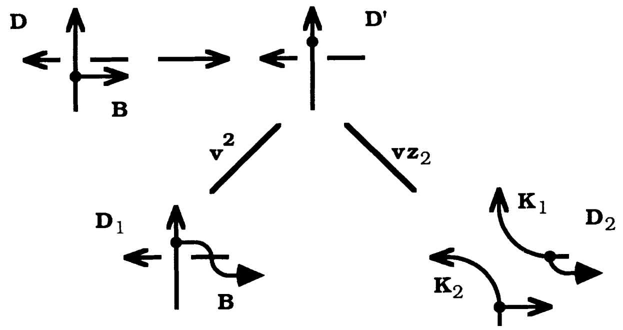



FIGURE 3.3

Let $A$ and $B$ be branch cuts for $D$ with the beginning of $B$ appearing as shown in the figure. Suppose $w = gh$ is the word associated with $D$, where $g$ is that part of $w$ determined by $D$ up to the crossing in question and $h$ is the latter part of $w$. So $\Omega^1(D) = f(\{gh\})$. But we may choose branch cuts for $D_1$ which coincide with $A$ and $B$ except for the very beginning of $B$ and conclude that $\Omega^1(D_1) = f(\{gb^{-1}h\})$. Similarly, $\Omega^1(D_2) = f(\{g, h\})$. Now $\Omega^1(D')$ is given by

$$\begin{aligned}\Omega^1(D') &= v^2\Omega^1(D_1) + vz\Omega^1(D_2)\\ &= v^2 f(\{gb^{-1}h\}) + vzf(\{g, h\})\\ &= v^2[v^{-2}f(\{gh\}) - v^{-1}zf(\{g, h\})] + vzf(\{g, h\})\\ &= f(\{gh\})\\ &= \Omega^1(D).\end{aligned}$$

*Step* 3. The choice of ordering of the components is immaterial.

Again proceeding as in [H], it suffices to consider the case where $D$ is a diagram with $n$ crossings which is descending with respect to some choice of pointing and ordering.

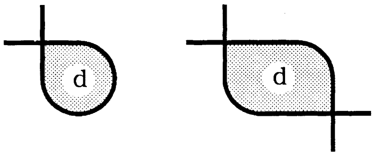
If there are no crossings between the components, then clearly the choice of ordering is immaterial since the diagram remains descending if the ordering is changed. Our strategy now is to reduce to this case by performing Reidemeister moves in $\mathbf{R}^2-\{0\}$, each of which does not increase the number of crossings. We must then prove that such moves preserve the value of $\Omega^1$. This is the same as the strategy employed in [H] but is complicated here by the fact that the Reidemeister moves must take place in $\mathbf{R}^2-\{0\}$ rather than $\mathbf{R}^2 \cup \{\infty\}$.

We begin by considering projections in $\mathbf{R}^2-\{0\}$ rather than diagrams. We say that a projection $P$ can be *strongly reduced* to the projection $P'$ if $P$ can be transformed to $P'$ by a finite sequence of Reidemeister moves in $\mathbf{R}^2-\{0\}$, none of which increase the number of double points, together with moving separated subprojections.

**Lemma 3.4** (1- and 2-gon clearing lemma). *Let $P$ be a link projection in $\mathbf{R}^2-\{0\}$ and suppose one of the following is true.*

(1) *A strand of $P$ forms a 1-gon which bounds a disk $d$ in $\mathbf{R}^2-\{0\}$ as shown below.*

(2) *Two strands of $P$ form a 2-gon with distinct vertices which bounds a disk $d$ in $\mathbf{R}^2-\{0\}$ as shown below.*




*Then $P$ can be strongly reduced to a projection where $d$ is* clear; *that is, no strands of $P$ meet the interior of $d$. Hence $d$ can be eliminated by a Type* I *or* II *Reidemeister move and $P$ can be strongly reduced to a projection having fewer double points.*

*Proof*. First move any separated subprojetions out of $d$ if necessary. We now begin with a special case of 2. Suppose that $d$ contains no 1-gons or 2-gons. Then the strands of $P$ that lie in $d$ form a braid from one side of $d$ to the other. If this braid is trivial, we may push the strands out of either end of $d$ using Type III moves. Since $d$ has two distinct vertices this really does clear $d$. If the braid is nontrivial, we may first use Type III moves to push the crossings of the braid out one side or the other of $d$.

Now suppose that $d$ contains no 1-gons. By successively clearing and eliminating innermost 2-gons, we may arrive at the previous case.

We shall now prove case 1. If $d$ contains no 1-gons, then any strand crossing $d$ forms a 2-gon with $\partial d$. This 2-gon contains no 1-gons and so may be cleared and eliminated. If $d$ contains 1-gons, then by successively clearing and eliminating innermost 1-gons we may arrive at the previous case.

Finally, we return to the general case of 2. By successively clearing and eliminating innermost 1-gons inside $d$ we may arrive at a previous case. □

**Lemma 3.5.** *Let $P$ be a projection in $\mathbf{R}^2-\{0\}$ having at least two components. Then $P$ can be strongly reduced in $\mathbf{R}^2-\{0\}$ to a disconnected projection.*

*Proof*. By using Lemma 3.4 we may assume that $P$ contains no 1- or 2-gons which bound disks in $\mathbf{R}^2-\{0\}$. Now let $A$ be a branch cut from $\{0\}$ to $\infty$ which meets $P$ transversely in as few points as possible. We claim that each component of $P$ must cross $A$ in only one direction (if they cross $A$ at all). For suppose not. Then there exists a path $\alpha$ in $P$ that leaves and returns to the same side of $A$ and otherwise does not meet $A$. Since $P$ contains no bounding 1-gons, this path is simple. Consider the disk $d$ bound by $\alpha$ and $\beta$, that part of $A$ between $\partial\alpha$. Since $P$ contains no bounding 2-gons, any strand of $P$ that enters $d$ via $\alpha$ must exist via $\beta$. Thus if we replace $\beta$ with a slight push-off of $\alpha$ we may produce a branch cut with at least two fewer points of intersection with $P$.

We may assume that every component of $P$ meets $A$. For if not, then such a component must be disjoint from all other components and bound a disk in $\mathbf{R}^2-A$. Thus we may safely ignore these components by picking them up and moving them far away from the origin. Hence $P$ is isotopic as a projection to a braid which is braided around the origin. We claim that Type III Reidemeister moves will now suffice to separate the components.

Think of $P$ as built from a union of concentric circles centered at the origin and with radii $1, 2, \dots, m$ by inserting crossings between adjacent circles, with one crossing in each sector $2\pi j/n \leq \theta \leq 2\pi(j+1)/n$, where $n$ is the number of crossings. Let $r$ be the maximum radius at which more than one component is present. If all the components are disjoint, no such radius exists and we let $r = 0$. If we consider all the components at radius $r$, at most one of them can attain a larger radius. If one does, then call that component $J$. If none do, then let $J$ be any of the components at radius $r$. Our goal is to push all components other than $J$ at radius $r$ down to lower radii. Let $K$ be such a component.

The circle of radius $r$ meets $P$ in a finite union of disjoint circular segments, some of which belong to $K$. Choose a segment belonging to $K$ and consider the braid in the sector $S$ containing this segment. It must appear as in Figure 3.4. Now $\beta$ may be empty since $r$ may equal $m$. Clearly, if $\alpha$ is trivial than $P$ contains a bounding 2-gon. So assume that $\alpha$ is nontrivial. Write $\alpha$ as $\alpha = \alpha_1\alpha_2$, where $\alpha_1$ is a maximal braid of the form $\alpha_1 = \sigma_{r-1}\sigma_{r-2}\cdots\sigma_{r-k}$.

Suppose first that $\alpha_2 = 1$. In this case a single Type III Reidemeister move can be used to move the segment of $K$ at radius $r$ down to radius $r-2$. This lowers the number of segments in $K$ which lie at radius $r$.

If $\alpha_2 \neq 1$, let $\sigma_j$ be the first crossing of $\alpha_2$. If $j < r-k-1$, we may reduce the length of $\alpha_2$ by isotoping $\sigma_j$ out the "top" of the sector $S$. If $j = r-k-1$, $\alpha_1$ is not maximal. If $j = r-k$, $P$ contains a bounding 2-gon. If $j > r-k$, then again a single Type III Reidemeister move can be used to reduce the length of $\alpha_2$.

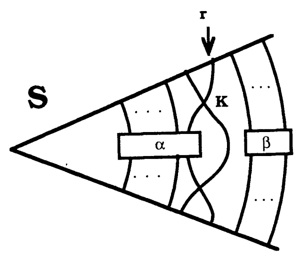



FIGURE 3.4

Thus the number of segments of $K$ which lie at radius $r$ may be reduced until none remain. Repeating the argument, we may eventually move all components other than $J$ down to smaller radii, thus reducing $r$. □

**Lemma 3.6.** *Let $D_1$ be a pointed and ordered 1-punctured diagram having $n$ crossings and which is descending with respect to its pointing and ordering. Suppose that*

(a) *$D_2$ is obtained from $D_1$ by a single Reidemeister move in $\mathbf{R}^2-\{0\}$ which does not increase the number of crossings and which furthermore takes place in a disk not containing the basepoint of $D_1$, or*
(b) *$D_2$ is obtained from $D_1$ by moving separated subdiagrams.*

*In either case, let $\tilde{D}_i$ be obtained from $D_i$ by reordering the components according to some nontrivial permutation $\sigma$. Then $\Omega^1(D_1) = \Omega^1(D_2)$ and $\Omega^1(\tilde{D}_1) = \Omega^1(\tilde{D}_2)$.*

*Proof*. The proof of case b is obvious, so consider case a. Since $D_1$ is descending and the Reidemeister move takes place away from the basepoints, it follows that $D_2$ is descending too. To compute $\Omega^1$ for $D_1$ and $D_2$ we use the descending diagram rule. However, when choosing branch cuts for the various components of $D_1$ or $D_2$ we may choose the cuts to lie away from the site of the Reidemeister move and hence to be the same for both $D_1$ and $D_2$. Thus $\Omega^1(D_1) = \Omega^1(D_2)$.

Now consider $\tilde{D}_1$ and a Type I move. Suppose the move will eliminate crossing $c$. Since $D_1$ is descending we do not need to change $c$ to make $\tilde{D}_1$ descending. Thus we may create a resolution $T_1$ of $\tilde{D}_1$ in which $c$ is never changed. Let $T_2$ be the resolution of $\tilde{D}_2$ obtained from $T_1$ by eliminating $c$ from each diagram of $T_1$. If $D_1^i, \dots, D_m^i$ are the outermost diagrams of $T_i$ then $\Omega^1(D_j^1) = \Omega^1(D_j^2)$ for $j < m$, since these diagrams have less than $n$ crossings, and for $j = m$ by the previous case, since these diagrams are descending. Hence $\Omega^1(D_1) = \Omega^1(D_2)$.

Now suppose we perform a Type II move, eliminating crossings $c$ and $d$. Either both crossings need to be changed to make $\tilde{D}_1$ descending or neither does. If neither does the proof proceeds as above. If both do, then begin to resolve $\tilde{D}_1$ by changing $c$ and $d$ first. Let $D_1^1$ result from smoothing $c$ and $D_2^1$ result from first changing $c$ and then smoothing $d$. Finally, let $D_3^1$ result from changing both $c$ and $d$. Now $\Omega^1(D_1^1) = \Omega^1(D_2^1)$ and since $c$ and $d$ have opposite signs we have $\Omega^1(\tilde{D}_1) = \Omega^1(D_3^1)$. But $\Omega^1(D_3^1) = \Omega^1(\tilde{D}_2)$ by the previous case.

Finally, consider a Type III move. If none of the crossings need to be changed to make $\tilde{D}_1$ descending, then proceed as before. Otherwise, the argument is similar to those given above. We shall prove one case and leave the others to the reader.

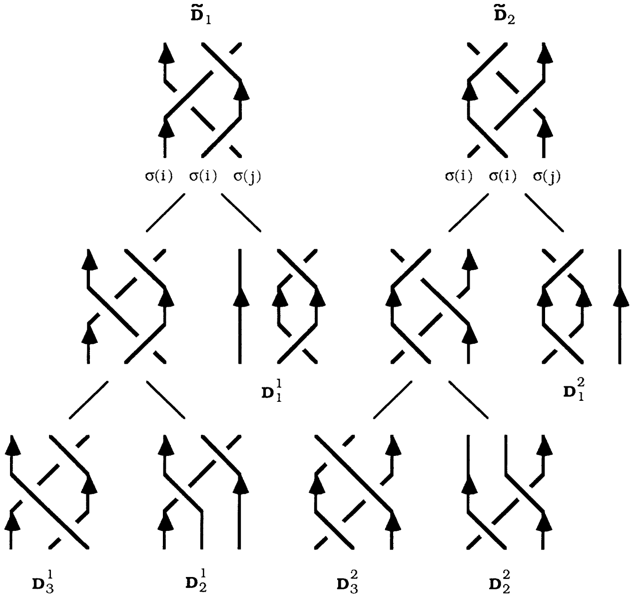



FIGURE 3.5. Here $i < j$ but $\sigma(i) > \sigma(j)$

Suppose that the three strands involved in the Reidemeister move belong to only two components and that two of the crossings of $\tilde{D}_1$ need to be changed. We begin to resolve $\tilde{D}_1$ and $\tilde{D}_2$ as shown in Figure 3.5. Clearly $\Omega^1(D_j^1) = \Omega^1(D_j^2)$ for $j = 1, 2$ and furthermore for $j = 3$ by the previous case. Hence $\Omega^1(\tilde{D}_1) = \Omega^1(\tilde{D}_2)$. □

We are now ready to prove Step 3. Let $D_1$ be a pointed and ordered 1-punctured diagram having $n$ crossings which is descending with respect to its pointing and ordering. let $\tilde{D}_1$ be obtained from $D_1$ by reordering the components according to some permutation $\sigma$. We want to prove that $\Omega^1(D_1) = \Omega^1(\tilde{D}_1)$. Let $P_1$ be the projection of $D_1$ or equivalently of $\tilde{D}_1$. By Lemma 3.5 there exists a sequence of projections $P_1, \ldots, P_m$ where each $P_i$ strongly reduces to $P_{i+1}$ either by a single Reidemeister move or by moving separated subprojections, and $P_m$ has no double points between components.

We would like to transform, by means of Reidemeister moves or by moving separated subdiagrams, both $D_1$ and $\tilde{D}_1$ through a sequence of diagrams having projections $\{P_i\}$. But this may not be possible since a Type II or Type III Reidemeister move performed in a projection $P$ may be "locked" in a diagram $D$ having $P$ as its projection.

Nevertheless, we shall create sequences $\{D_i\}$ and $\{\tilde{D}_i\}$ beginning with $D_1$ and $\tilde{D}_1$ of pointed and ordered diagrams such that

1. Each $D_i$ is descending with respect to its pointing and ordering.
2. Each $\tilde{D}_i$ is obtained from $D_i$ by reordering the components according to $\sigma$.

3. Both $D_i$ and $\widetilde{D}_i$ have projection $P_i$.
4. For each $i$, if $\Omega^1(D_{i+1}) = \Omega^1(\widetilde{D}_{i+1})$ then $\Omega^1(D_i) = \Omega^1(\widetilde{D}_i)$.

Thus $\Omega^1(D_m) = \Omega^1(\widetilde{D}_m)$ since $P_m$ is a disconnected projection and from this it follows that $\Omega^1(D_1) = \Omega^1(\widetilde{D}_1)$.

Hence it remains only to construct the sequences $\{D_i\}$ and $\{\widetilde{D}_i\}$. Suppose this has been done up to $i = j$. Now $P_{j+1}$ is obtained from $P_j$ by a single Reidemeister move which does not increase the number of crossings, or by moving a separated subprojection. Clearly the latter can be mimicked in both $D_j$ and $\widetilde{D}_j$, so suppose that $P_j$ is altered by a Reidemeister move. Suppose this move takes place inside the disk $d$. Suppose further that $D_j$ has no basepoints in $d$. Then the Reidemeister move can be carried out in both $D_j$ and $\widetilde{D}_j$ to give $D_{j+1}$ and $\widetilde{D}_{j+1}$. Now properties 1, 2, and 3 are clearly true and property 4 follows from Lemma 3.6.

If $D_j$ has basepoints in $d$, then let $D_j^1$ be obtained from $D_j$ by moving the basepoint out of $d$ and let $\widetilde{D}_j^1$ be obtained from this by reordering the components. Now by Step 2 we have $\Omega^1(D_j) = \Omega^1(D_j^1)$ and $\Omega^1(\widetilde{D}_j) = \Omega^1(\widetilde{D}_j^1)$. Let $D_j^2$ be obtained from $D_j^1$ by changing crossings so that $D_j^2$ is descending with respect to its pointing and ordering. Again let $\widetilde{D}_j^2$ be obtained from this by reordering of the components. Now since the diagrams produced by smoothing crossings in $D_j^1$ and $\widetilde{D}_j^1$ have one less crossing, it follows that if $\Omega^1(D_j^2) = \Omega^1(\widetilde{D}_j^2)$ then $\Omega^1(D_j^1) = \Omega^1(\widetilde{D}_j^1)$. The diagrams $D_j^2$ and $\widetilde{D}_j^2$ now satisfy the earlier case. □

This completes the proof that $\Omega^1$ is well defined for 1-punctured diagrams having $n$ crossings. It now remains to complete the inductive step, that is, to show that $\Omega^1$ continues to satisfy the four properties assumed in the inductive hypothesis. The only nontrivial property is the third one. However, the proof is similar to that given in Lemma 3.6 with the additional freedom of being able to choose basepoints and ordering as we wish. We leave this as an advanced exercise for the reader.

Having completed the inductive step, we may now conclude that $\Omega^1$ is defined for all 1-punctured diagrams and satisfies the four properties listed in the inductive hypothesis. The third property implies that $\Omega^1$ is preserved by Reidemeister moves in $\mathbf{R}^2-\{0\}$. Thus $\Omega^1$ is actually a 1-trivial dichromatic link invariant, since two 1-punctured diagrams represent the same 1-trivial dichromatic link if and only if they are related by Reidemeister moves in $\mathbf{R}^2-\{0\}$.

*Part* 4. Passing to ordinary diagrams. If $L$ is a 1-trivial dichromatic link, it is convenient to work with ordinary diagrams of $L$ in $\mathbf{R}^2 \cup \{\infty\}$ rather than 1-punctured diagrams. This improvement to ordinary link diagrams is all that remains to prove Theorem 3.1.

To verify property 1 of Theorem 3.1 let $(L_+, L_-, L_0)$ be a triad of dichromatic link diagrams in $\mathbf{R}^2$. There exists a triad $(D_+, D_-, D_0)$ of 1-punctured diagrams which respectively represent the links $(L_+, L_-, L_0)$. Now the crossing rule for 1-punctured diagrams implies the crossing rule for ordinary diagrams of 1-trivial dichromatic links. This completes the proof of Theorem 3.1. □

**Corollary 3.7.** *The elementary sets* $\mathscr{E}_1$ *and* $\mathscr{E}_2$ *are both minimal.*

*Proof*. Suppose $E \in \mathscr{E}_1$ but $\mathscr{E}_1 - \{E\}$ is minimal. Then there exists a resolution $T$ of $E$ into links of $\mathscr{E}_1 - \{E\}$. Now Theorem 3.1 guarantees that any resolution of $E$ will produce the elementary value $[E]$. But clearly the resolution $T$ will not. □

For any 3-manifold $M$ one can define the *skein module* $\mathscr{S}(M)$ as follows. Let $R = \mathbf{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ and denote by $\mathscr{L}$ the free $R$-module generated by the isotopy classes of all oriented links in $M$. Let $\mathscr{R}$ be the submodule generated by the skein relations

$$x[L_+] + y[L_-] + z[L_0],$$

where $(L_+, L_-, L_0)$ is a skein triple in $M$ defined in the obvious way and $[L]$ is the class of $L$ in $\mathscr{L}$. Then the skein module of $M$ is defined as $\mathscr{S}(M) = \mathscr{L}/\mathscr{R}$. The reader is referred to [P$_1$] for a general treatment of skein modules.

Since 1-trivial dichromatic links correspond to oriented monochromatic links in $S^1 \times D^2$, we have the following corollary to Theorem 3.1.

**Corollary 3.8.** *The skein module* $\mathscr{S}(S^1 \times D^2)$ *is free on an infinite set of generators. These generators may be taken as the links in* $S^1 \times D^2$ *corresponding to the elementary dichromatic links* $\mathscr{E}_i$.

## 4. General dichromatic skein invariants

In this section we refocus our attention on Question 1.2. Because of Theorem 2.1 we shall investigate possible invariants $\Omega$ which satisfy the skein relations (1.1) and (1.2) with $v_1^2 = v_2^2$. In particular, we shall assume that $v_1 = v_2 = v$. Note that choosing $-v_1 = v_2 = v$ cannot possibly yield anything different, as replacing $z_1$ with $-z_1$ returns us to the first case.

Unlike the situation of $i$-trivial links, we show that, in general, there must exist further relations among the indeterminates. This provides a partial answer to Question 1.2. As a corollary we obtain a negative answer to Question 1.3.

**Theorem 4.1.** *Suppose* $\Omega$ *is an invariant of dichromatic links satisfying the skein relations*

$$v^{-1}\Omega({}_i\times_i) - v\Omega({}_i\times_i) = z_i\Omega({}_i\asymp_i), \qquad i = 1, 2.$$

*Then either*

(i) $v^2 = 1$, *and* $\Omega$ *does not extend the twisted Alexander polynomial, or*

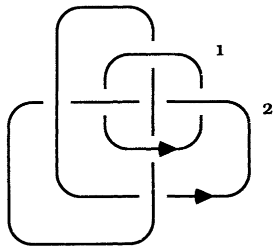



FIGURE 4.1. The link $7_6^2$

(ii) $z_1^2 = z_2^2$, *and* $\Omega$ *does not extend the Conway polynomial, or*

(iii) $\Omega(aba^{-1}) = \Omega(1)$, *and* $\Omega$ *extends neither.*

*Furthermore, none of these relations are sufficient. However, in each case, there does exist an invariant of dichromatic links satisfying these relations (among others), namely the Conway polynomial, the twisted Alexander polynomial, and the uncoupled invariant, respectively.*

**Corollary 4.1** (Answer to Question 1.3). *There does not exist an invariant* $\Omega$ *of dichromatic links which satisfies the skein relations* (1.1) *and* (1.2) *and which is a mutual extension of both* $\nabla(z_1, z_2)$ *and* $P(v, z)$.

*Proof of Theorem* 4.1. Let $L$ be the link $7_6^2$ in Rolfsen's tables [R], oriented and colored as shown in Figure 4.1.

Since $7_6^2$ is both 1- and 2-trivial, we may compute $\Omega$ relative to the elementary links $\mathscr{B}$ in two very different ways. If we change only 1-colored crossings and employ equation (1.1), we obtain

$$\Omega(7_6^2) = (1 + v^{-4} + v^{-4}z_1^2)\Omega(1) - (v^{-2} + v^{-4} + v^{-4}z_1^2)\Omega(aba^{-1}) + (v^{-2} + v^{-4})\Omega(a^2ba^{-2}) - v^{-4}\Omega(ababa^{-2}).$$

On the other hand, we may also compute $\Omega$ by changing only 2-colored crossings and employing equation (1.2). This yields

$$\Omega(7_6^2) = (1 - v^{-2} - v^{-2}z_2^2)\Omega(1) + (v^{-2} + v^{-2}z_2^2)\Omega(aba^{-1}).$$

If $\Omega$ is an invariant of dichromatic links, these two expressions must be equal. Equating them, we arrive at

$$\begin{aligned}(v^{-2} + v^{-4})\Omega(a^2ba^{-2}) - v^{-4}\Omega(ababa^{-2}) &= -(v^{-2} + v^{-2}z_2^2 + v^{-4} + v^{-4}z_1^2)\Omega(1) \\ &\quad + (2v^{-2} + v^{-2}z_2^2 + v^{-4} + v^{-4}z_1^2)\Omega(aba^{-1}).\end{aligned}$$

Finally, consider repeating the entire calculation above, starting with $7_6^2$ colored in the opposite way. Since 2-bridge links are interchangeable, reversing their coloring has no effect. Hence we obtain a formula identical to the one above except that $z_1$ and $z_2$ are interchanged. Equating the right-hand sides of these equations and factoring gives

$$0 = v^{-4}(v^2 - 1)(z_1^2 - z_2^2)(\Omega(aba^{-1}) - \Omega(1)).$$

This implies that either $v^2 = 1$, $z_1^2 = z_2^2$, or $\Omega(aba^{-1}) = \Omega(1)$.

However, if we set $v^2 = 1$ the initial two expressions will still be unequal. Nor will setting $z_1^2 = z_2^2$ or $\Omega(aba^{-1}) = \Omega(1)$ reconcile these calculations. Thus, while these relations are necessary, they are not sufficient.

If $v^2 = 1$, it is not hard to show that $\Omega$ is zero for any split link. Simply consider a triad $(K \#_i J, K \#_i J, K \cup J)$ obtained by changing and smoothing a crossing located at a half twist in the band connecting $K$ to $J$. Thus, in this case, $\Omega$ cannot extend $P(v, z)$.

If $z_1^2 = z_2^2$, then $\Omega$ is unaffected by color reversal. This is because the skein relations do not depend on the coloring and neither do the values of the elementary 2-bridge links, since they are interchangeable. Thus, in this case, $\Omega$ cannot extend the Conway polynomial.

Finally, if the Whitehead link $aba^{-1}$ and the unlink 1 cannot be distinguished by $\Omega$, then $\Omega$ extends neither the twisted Alexander polynomial or the Conway polynomial since both of these invariants can distinguish these links. □

We shall not address Question 1.2 any further in this paper. Yet it remains an interesting question if there exist invariants satisfying the relations given in Theorem 4.1 which are proper extensions of $\nabla$, $P$, and $\mathscr{U}$, respectively. Certainly no amount of computations employing the skein relations can ever lead to relations among elementary links having different chromatic linking numbers. Thus it is hard to imagine how assuming $v = 1$, for example, will force $\Omega$ to collapse all the way to the Conway polynomial, where the clasp rule relates links of different chromatic linking.

## 5. Invariants derived from $\Omega^i$ and some of their properties

The initial data for $\Omega^i$ consists of an infinite set of indeterminates, namely the values of the elementary links $\mathscr{E}_i$. An invariant $W^i$ having only a finite number of indeterminates may be derived from $\Omega^i$ by introducing relations among $\{[E]\}_{E\in\mathscr{E}_i}$. We shall do this in such a way that $W^i$ then satisfies a clasp rule. Such is the case for both the Conway polynomial and the twisted Alexander polynomial, and it is this fact that provides the motivation for collapsing $\Omega^i$ to $W^i$.

While $W^i$ appears to be a radical simplification of $\Omega^i$, it still contains the invariants $\mathscr{U}$, $\nabla$, and $P$ as special cases. Furthermore, there exists an invariant $\hat{d}^i$ which can be derived from $W^i$ which is completely analogous to the Jones polynomial. In fact, the existence of $\hat{d}^i$ can be quickly established using the analog of the Kauffman bracket. This is done in [HP]. Moreover, the Jones polynomial $V_L$ can be derived from $\hat{d}_L^i$. Figure 5.1 depicts the relationship between these $i$-trivial link invariants.

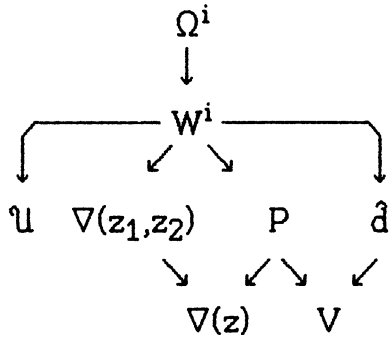



FIGURE 5.1. Invariants of $i$-trivial links with arrows indicating how one extends another

In this section we define $W^i$ and $\hat{d}^i$ and describe a few of their basic properties. We give examples illustrating their utility in studying various link symmetries.

The following theorem describes the invariant $W^i$.

**Theorem 5.1.** *There exists a unique invariant $W^i$ of i-trivial dichromatic links which may be derived from $\Omega^i$ and which satisfies the following properties:*
*Crossing rule*:

$$v^{-1}W^i({}_j\times_j) - vW^i({}_j\times_j) = z_2 W^i({}_j\asymp_j), \qquad j \neq i.$$

*Clasp rule*:

$$v^{-1/2}x^{-1}W^i({}_k \between {}_j) + v^{1/2}xW^i({}_k \between {}_j) = (v^{-1/2}h_+ + v^{1/2}h_-)\lambda^{-1}W^i({}_k \asymp {}_j), \quad j \neq k.$$

*Anticlasp rule*:

$$v^{1/2}x^{-1}W^i({}_k \between {}_j) + v^{-1/2}xW^i({}_k \between {}_j) = (v^{1/2}h_+ + v^{-1/2}h_-)\lambda^{-1}W^i({}_k \asymp {}_j), \quad j \neq k.$$

*Sum rule*:

$$W^i(K \#_i J) = (v^{-1} - v)z_j^{-1}\lambda^{-1}W^i(K)W^i(J), \qquad j \neq i.$$

*Initial data*:

$$W^i({}_1 \infty\, {}_2) = xh_+, \qquad W^i({}_1 \circ\circ\, {}_2)\lambda, \qquad W^i({}_1 \infty\, {}_2) = x^{-1}h_-.$$

*Proof*. Beginning with the invariant $\Omega^i$, we first introduce new variables $x$, $h_+$, $h_-$, and $\lambda$ and then make substitutions for the elementary values $[E]$ as follows. Let

$$[a] = x^{-1}h_+ \quad \text{and} \quad [a^{-1}] = xh_-.$$

Next, let $[a^n]$ for $n$ different from 1 or $-1$ be determined recursively by the formula

$$v^{-1/2}x^{-1}[a^{n+1}]+v^{1/2}x[a^{n-1}]=(v^{-1/2}h_+ + v^{1/2}h_-)\lambda^{-1}[a^n].$$

Note that this gives $[1]=\lambda$. Finally, let an arbitrary elementary value be given by

$$[a^{n_1}\#_i\cdots\#_i a^{n_k}]=((v^{-1}-v)/z_j\lambda)^{k-1}[a^{n_1}]\cdots[a^{n_k}], \qquad j\neq i.$$

This process determines a unique substitution for each elementary value. Let $W^i$ be the invariant derived from $\Omega^i$ by making these substitutions. It is now straightforward to check that $W^i$ satisfies the properties listed in the theorem. Alternatively, one could repeat the proof of Theorem 3.1 working with $W^i$ instead of $\Omega^i$. □

**Theorem 5.2.** *The analog of the Jones polynomial for i-trivial links,* $\hat{d}^i(A_j, h_j)$, *can be derived from* $W^i$ *by setting* $v_j=A_j^{-4}$, $z_j=A_j^{-2}-A_j^2$, $x=A_j^{-6}$, $h_+=h_-=h_j$, *and* $\lambda=1$.

The invariant $\hat{d}^i(A_j, h_j)$ can be quickly defined from a "bracket" function of $i$-punctured diagrams similar to Kauffman's bracket function for ordinary diagrams. Briefly, one begins with an invariant $\langle\rangle_i$ of an unoriented $i$-punctured diagram defined by the following properties.

1. $\langle\cdot\bigcirc\rangle_i=1$;
2. $\langle\odot\rangle_i=h_j$;
3. $\langle\times\rangle_i=A_j\langle\asymp\rangle_i+A_j^{-1}\langle)(\rangle_i$;
4. $\langle\cdot\bigcirc K\rangle_i=-(A_j^2+A_j^{-2})\langle\cdot K\rangle_i$, $K\neq\varnothing$;
5. $\langle\odot K\rangle_i=-(A_j^2+A_j^{-2})h_j\langle\cdot K\rangle_i$, $K\neq\varnothing$.

Then for an unoriented $i$-trivial link $L$ define

$$d^i(L)(A_j, h_j)=(-A_j^3)^{-\,\mathrm{sw}(D)}\langle D\rangle_i$$

where $D$ is any $i$-punctured diagram of $L$ and $\mathrm{sw}(D)$ is the self-writhe of $D$. Finally, if $L$ is an oriented $i$-trivial link, let $|L|$ denote $L$ stripped of its orientation and $\mathrm{lk}(L)$ equal the sum of the linking numbers between every pair of components of $L$. Then

$$\hat{d}^i(L)=(-A_j^3)^{-2\,\mathrm{lk}(L)}d^i(|L|).$$

Details can be found in [HP].

The invariant $W^i$ and those derived from it satisfy multiplicative rules for connected sums which we have listed in Table 1.2 and in Theorem 5.1. The invariant $\Omega^i$ nearly does and can be made to do so, without any loss of generality, by adopting some notational conventions. To see this we first state the following theorem, the proof of which we leave to the reader.

**Theorem 5.3.** *Let $L$ be an $i$-trivial dichromatic link and $M$ a $j$-colored monochromatic link, $j \neq i$.*

(i) *If $L_i$ and $L_j$ are split (separated by a 2-sphere), then $\Omega^i(L) = P_{L_j}(v_j, z_j)[1]$.*

(ii) *If $L \amalg M$ denotes the split union of $L$ and $M$, then $\Omega^i(L \amalg M) = (v_j^{-1} - v_j)z_j^{-1}P_M(v_j, z_j)\Omega^i(L)$.*

(iii) $\Omega^i(L \#_j M) = P_M(v_j, z_j)\Omega^i(L)$.

(iv) *If $K$ is an $i$-trivial link and $\Omega^i(L) = \sum \sigma_r[E_r]_i$ and $\Omega^i(K) = \sum \tau_s[F_s]_i$, where each $E_r$ and $F_s$ are elementary links in $\mathscr{E}_i$ and $\sigma_r$ and $\tau_r$ are polynomials in $v_j^{\pm 1}$ and $z_j^{\pm 1}$, then $\Omega^i(L \#_i K) = \sum \sigma_r \tau_s \Omega^i(E_r \#_i F_s)$.*

*Note that $E_r \#_i F_s$ may not be elementary since one or both summands might be trivial.*

**Corollary 5.4.** *If we agree to let $[E]_i[F]_i = \Omega^i(E \#_i F)$ for all $E, F \in \mathscr{E}_i$ and $[1] = (v_j^{-1} - v_j)z_j^{-1}$, then $\Omega^i(L \#_i K) = \Omega^i(L)\Omega^i(K)$. There is no loss of generality in doing this. If we further agree to expand the class of $i$-trivial links to include the $i$-colored unknot (which strictly speaking is not dichromatic since the $j$-sublink is empty) and define $\Omega^i(\circ_i) = 1$, then property* (i) *of Theorem* 5.3 *becomes a special case of property* (ii).

We now list some of the basic properties of $W^i$.

**Theorem 5.5.** *Let $L$ be an $i$-trivial dichromatic link with chromatic linking number $l = \mathrm{lk}(L_1, L_2)$. Then*

(i) *$W^i$ detects $l$. In particular we have*

$$W^i(L) = x^l(W^i(L)|_{x=1}).$$

(ii) *The uncoupled invariant may be derived from $W^i$ by setting $x = 1$ and $\lambda = h_+^{1/2}h_-^{1/2}$. In particular,*

$$W^i(L)|_{x=1, \lambda=h_+^{1/2}h_-^{1/2}} = h_+^{(1+l)/2}h_-^{(1-l)/2}P_{L_j}(v, z_j), \qquad j \neq i.$$

(iii) *The variable $x$ is redundant. That is, $W^i(L)$ and $W^i(L)|_{x=1}$ are equivalent invariants. In particular, $W^i(L)$ may be recovered from $W^i(L)|_{x=1}$.*

(iv) *The twisted Alexander polynomial of $L_j$, $j \neq i$, can be derived from $W^i$ by*

$$W^i(L)|_{x=1, \lambda=1, h_+=1, h_-=1} = P_{L_j}(v, z_j).$$

(v) *The sum of the exponents of $h_+$, $\lambda$, and $h_-$ in each term of $W^i$ is* 1.

(vi) *The twisted Alexander polynomial of $L$ can be derived from $W^i(L)$ by setting $z_j = z$, $\lambda = (v^{-1} - v)/z$, $x = v^{3/2}$, $h_+ = v^{-1/2}z + v^{-1/2}z^{-1} - v^{3/2}z^{-1}$, and $h_- = -v^{1/2}z - v^{1/2}z^{-1} + v^{-3/2}z^{-1}$.*

(vii) *The Conway polynomial of* $L$ *can be derived from* $W^i$ *by three successive substitutions. First set* $x = 1$, $h_+ = v^{1/2}$, *and* $h_- = v^{-1/2}\alpha\lambda - v^{-1/2}$, *where* $\alpha$ *is a new variable, and simplify the results. Next set* $\lambda = (v^{-1} - v)/z_1 z_2$ *and simplify the results. Finally, set* $v = 1$ *and* $\alpha = [z_1 z_2 + ((z_1^2+4)(z_2^2+4))^{1/2}]/2$.

*Proof*. Clearly (i) is true for the unlink and both the right- and left-handed Hopf links. Now if it is true for any two of three links related by changing or smoothing either a crossing or a clasp, then it is not hard to prove that it is true for the third as well. A similar proof may be given for property (ii).

Properties (iii) and (iv) follow immediately from the first two properties.

Note that Property (v) is true for the unlink and the right- and left-handed Hopf links. By inducting on $n$ and using the clasp rule, it is a simple matter to prove that it is true for $(2, 2l)$-torus links. The connected sum rule now implies that elementary links satisfy property (v). Finally, if two links in a triad satisfy property (v), then so does the third.

It is not hard to verify the last two properties. Simply make these substitutions and compare with Table 1.2. □

If $L$ is an $i$-trivial link, let $\overline{L}$ denote the mirror image of $L$ obtained by reflecting $L$ through a plane. Let $-L$ denote the link obtained by reversing all the orientations of $L$, and denote by $L_{-i}$ the link obtained by reversing only the orientations of the $i$-sublink. Note that $-L_{-i} = L_{-j}$, $j \neq i$. Finally, let $\widehat{L}$ denote the $j$-trivial link, $j \neq i$, obtained from $L$ by reversing the colors of $L$. The following theorem relates the values of these various links. We omit the proofs, which are all straightforward and similar in spirit.

**Theorem 5.6.** *Let* $L$ *be an* $i$*-trivial link. Then the following are true.*

(i) $\Omega^i(L) = \Omega^i(-L)$.

(ii) $\Omega^i(\overline{L})$ *is obtained from* $\Omega^i(L)$ *by replacing* $v$ *with* $v^{-1}$ *and, in each elementary value* $[E]$, *replacing* $a$ *with* $a^{-1}$.

(iii) $W^i(\overline{L})$ *is obtained from* $W^i(L)$ *by replacing* $v$ *with* $v^{-1}$, $z_j$ *with* $z_j^{-1}$, $x$ *with* $x^{-1}$, $h_+$ *with* $h_-$, *and* $h_-$ *with* $h_+$.

(iv) $W^i(L_{-j})$ *is obtained from* $W^i(L)$ *by replacing* $x$ *with* $x^{-1}$, $h_+$ *with* $h_-$, *and* $h_-$ *with* $h_+$.

**Theorem 5.7.** *Suppose* $L$ *is an* $i$*-trivial link and* $j \neq i$. *Then* $\Omega^j(\widehat{L})$ *is obtained from* $\Omega^i(L)$ *by replacing* $v_j$ *with* $v_i$, $z_j$ *with* $z_i$, *and each elementary value* $[a^{k_1} \#_i \cdots \#_i a^{k_n}]$ *with* $[a^{k_1} \#_j \cdots \#_j a^{k_n}]$. *Similarly,* $W^j(\widehat{L})$ *is obtained from* $W^i(L)$ *by replacing* $v_j$ *with* $v_i$ *and* $z_j$ *with* $z_i$.

**Corollary 5.8.** *If* $L$ *is both* 1- *and* 2*-trivial and interchangeable, that is,* $L = \widehat{L}$, *then* $\Omega^j(L)$ *is obtained from* $\Omega^i(L)$ *by replacing* $v_j$ *with* $v_i$, $z_j$ *with* $z_i$, *and each elementary value* $[a^{k_1} \#_i \cdots \#_i a^{k_n}]$ *with* $[a^{k_1} \#_j \cdots \#_j a^{k_n}]$. *Similarly,* $W^j(L)$ *is obtained from* $W^i(L)$ *by replacing* $v_j$ *with* $v_i$ *and* $z_j$ *with* $z_i$.

Thus, $\Omega^1(L)$ and $\Omega^2(L)$ can sometimes be used to show that a link is not interchangeable. For example, they may be used to show that the link $7^2_6$, as oriented in Figure 4.1, is not interchangeable. However, even the Conway polynomial can detect this. (So can $\hat{d}^1$ and $\hat{d}^2$, as shown in [HP].) Even as an unoriented link $7^2_6$ is not interchangeable. This can be seen by considering how each component *wraps* around the other. In general, if $L$ is an $i$-trivial link, let $\text{wrap}^i(L)$ denote the minimal geometric intersection number of the $j$-sublink of $L$ with any disk that spans the $i$-component of $L$.

**Theorem 5.9.** *Let* $L$ *be an i-trivial link with*

$$\Omega^i(L) = \sum \sigma_r(v_j, z_j)[a^{k_{r,1}} \#_i \cdots \#_i a^{k_{r,n_r}}].$$

*Then*

$$\max_r\{|k_{r,1}| + \cdots + |k_{r,n_r}|\} \le \text{wrap}^i(L).$$

*Moreover, these quantities are equal* mod 2.

*Proof.* Suppose $\text{wrap}^i(L) = m$. Then there exist an $i$-punctured diagram of $L$ and a branch cut from the puncture to infinity meeting $L$ in exactly $m$ points. If $D$ has no crossings, $L$ must be an $i$-connected sum of $m$ Hopf links since $m$ is minimal. In this case the result is obviously true. Assume now that $D$ has $k$ crossings but that the result is true for any link having an $i$-punctured diagram with fewer than $k$ crossings. By placing basepoints on each component of $D$ as far away as possible from the puncture and making $D$ descending with respect to this choice of basepoints, we arrive at a descending diagram where once again the theorem is clearly true. But the diagrams produced by smoothing crossings all have $k-1$ crossings and so by assumption satisfy the theorem. Finally, using the crossing rule to combine these polynomials will produce a value for $D$ which also satisfies the theorem. □

Computing $\Omega^1$ and $\Omega^2$ for the link $7^2_6$ yields

$$\Omega^1(7^2_6) = (2 - v_2^{-2} + z_2^2 + v_2^{-2} z_2^2)[1]_1 + (v_2^{-1} z_2 + v_2^{-1} z_2^3)[a \#_1 a^{-1}]_1$$

and

$$\begin{aligned}\Omega^2(7^2_6) = {} & (v_1^{-4} - v_1^{-2} z_1^2 + v_1^{-4} z_1^2)[1]_2 - (2v_1^{-1} z_1 + v_1^{-3} z_1 + v_1^{-3} z_1^3)[a \#_2 a^{-1}]_2 \\ & + v_1^{-3} z_1 [a^2 \#_2 a^{-2}]_2 - v_1^{-2} z_1^2 [a \#_2 a \#_2 a^{-2}]_2 .\end{aligned}$$

Here we have subscripted the elementary values with $i = 1, 2$ to especially emphasize that these indeterminates correspond to elementary links taken from different elementary sets. Thus the values of $\Omega^1$ and $\Omega^2$ of $7^2_6$ show that the components wrap at least 2 and 4 times, respectively, about each other. The

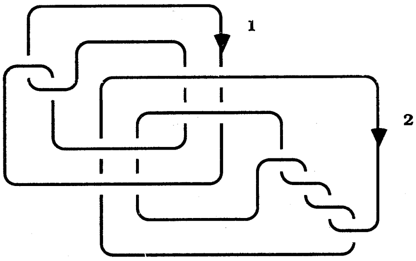



FIGURE 5.2

wrapping numbers are in fact equal to 2 and 4, as can be seen from specific diagrams of the link. The Conway polynomials also provides lower bounds on the wrapping numbers and these can be used to show that $7_6^2$ is not interchangeable.

The following example, however, exhibits a noninterchangeable boundary link. Since the Conway polynomial vanishes for boundary links, we cannot use it as a tool to study interchangeability in this case.

**Example 5.10.** Let $L$ be the dichromatic boundary link shown in Figure 5.2 Then one can compute

$$\begin{aligned}\Omega^1(L) = {} & [v_2^4(2-v_2^4) + v_2^2(1+v_2^2)(1-v_2^2)(1+2v_2^2)z_2^2 + v_2^4(1+v_2^2)(1-v_2^2)z_2^4][1] \\ & + [2v_2^3(1+v_2^2)(1-v_2^2)z_2 - 2v_2^3(1+v_2^2)^2 z_2^3 - v_2^5(1+v_2^2)z_2^5][a^{-1} \#_1 a] \\ & + v_2^5(1+v_2^2)z_2^3[a^{-2} \#_1 a^2] + v_2^4(1+v_2^2)z_2^4[a^{-1} \#_1 a^{-1} \# a^2] \\ & + v_2^3(1+v_2^2)z_2^3[a^{-1} \#_1 a^{-1} \#_1 a \#_1 a].\end{aligned}$$

Computing $\Omega^2(L)$ is much more laborious than computing $\Omega^1(L)$, but one can compute the terms of the form $cv_1^n[1]$ more easily. These are $(v_1^2+v_1^8-v_1^{10})[1]$. Hence, by Corollary 5.8, $L$ is not interchangeable. □

We close with the following example of two 2-bridge links which have the same value of $\Omega^1$ (or $\Omega^2$). This example was shown to us by J. H. Przytycki.

**Example 5.11.** The polynomial $\Omega^1$ (or $\Omega^2$) cannot distinguish the following two 2-bridge links:

$$L_1 = H(ab^{-1}aba^{-1}bab^{-1}a^{-1}ba^{-1}), \quad L_2 = H(a^{-1}bab^{-1}aba^{-1}ba^{-1}b^{-1}a).$$

To see that $\Omega^1(L_1) = \Omega^1(L_2)$ it is sufficient to consider skein triples involving the "middle" $b$. Moreover, it follows from the classification of 2-bridge links that $L_1$ and $L_2$ are different. □

## REFERENCES


[BLM] R. Brandt, W. B. R. Lickorish, and K. Millett, *A polynomial invariant for unoriented knots and links*, Invent. Math. **84** (1986), 563–573.

[C] J. H. Conway, *An enumeration of knots and links and some of their related properties*, Computational Problems in Abstract Algebra, Proc. Conf. Oxford (J. Leech,, eds.), Pergamon Press, New York, 1967, pp. 329–358.

[FYHLMO] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett, and A. Ocneanu, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc. **12** (1985), 239–249.

[Ha] R. Hartley, *The Conway potential function for links*, Comment. Math. Helv. **58** (1983), 365–378.

[Ho] C. F. Ho, *A new polynomial for knots and links—preliminary report*, Abstracts Amer. Math. Soc., **6** (1985), 300.

[H] J. Hoste, *A polynomial invariant of knots and links*, Pacific J. Math. **124** (1986), 295–320.

[HP] J. Hoste and J. H. Przytycki, *An invariant of dichromatic links*, Proc. Amer. Math. Soc. **105** (1989), 1003–1007.

[J] V. F. R. Jones, *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. (2) **126** (1987), 335–388.

[$K_1$] L. Kauffman, *State models and the Jones polynomial*, Topology **26** (1987), 395–407.

[$K_2$] ——, *New invariants in the theory of knots*, Amer. Math. Monthly **95** (1988), 195–242.

[Ki] M. E. Kidwell, *On the two-variable Conway potential function*, Proc Amer. Math. Soc. **98** (1986), 485–494.

[LM] W. B. R. Lickorish and K. Millett, *The new polynomial invariants of knots and links*, Math. Mag. **61** (1988), 3–23.

[M] H. Morton, *Seifert circles and knot polynomials*, Math. Proc. Cambridge Philos. Soc. **99** (1986), 107–109.

[N] Y. Nakanishi, *Fox's congruence classes and Conway's potential function of knots and links*, preprint.

[$P_1$] J. H. Przytycki, *Skein modules of 3-manifolds*, Bull. Polon. Acad. Sci. Math. (to appear).

[$P_2$] ——, *Survey on recent invariants in classical knot theory*, Warsaw University, 1986.

[PT] J. H. Przytycki and P. Traczyk, *Invariants of links of Conway type*, Kobe J. Math. **4** (1987), 115–139.

[R] D. Rolfsen, *Knots and links*, Publish or Perish, 1976.

[T] V. G. Turaev, *Reidemeister torsion in knot theory*, Russian Math. Surveys **41** (1986), 119–182.



DEPARTMENT OF MATHEMATICS, POMONA COLLEGE, CLAREMONT, CALIFORNIA 91711

DEPARTMENT OF MATHEMATICS, U.S. NAVAL ACADEMY, ANNAPOLIS, MARYLAND 21402 (Current address of M. E. Kidwell)

*Current address* (Jim Hoste): Department of Mathematics, Pitzer College, Claremont, California 91711