TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 321, Number 1, September 1990

WEAKLY ALMOST PERIODIC FUNCTIONS AND THIN SETS IN DISCRETE GROUPS

CHING CHOU

ABSTRACT. A subset E of an infinite discrete group G is called (i) an R_W-set if any bounded function on G supported by E is weakly almost periodic, (ii) a weak p-Sidon set $(1 \leq p < 2)$ if on $l^1(E)$ the l^p-norm is bounded by a constant times the maximal C^*-norm of $l^1(G)$, (iii) a T-set if $xE \cap E$ and $Ex \cap E$ are finite whenever $x \neq e$, and (iv) an FT-set if it is a finite union of T-sets. In this paper, we study relationships among these four classes of thin sets. We show, among other results, that (a) every infinite group G contains an R_W-set which is not an FT-set; (b) countable weak p-Sidon sets, $1 \leq p < 4/3$ are FT-sets.

1. INTRODUCTION

Let G be an infinite discrete group, WAP(G) the algebra of weakly almost periodic (w.a.p.) functions on G. A subset E of G is called an R_W-set if every function in $l^\infty(G)$ which vanishes off E is w.a.p.; E is called a T-set if $E \cap xE$ and $E \cap Ex$ are finite whenever $x \in G$, $x \neq e$, the identity of G. It was first proved by W. Rudin [17] that T-sets and hence finite unions of T-sets are R_W-sets. However, they seem to constitute the only known R_W-sets in the literature. In §3 we show that every infinite group contains an R_W-set which is not a finite union of T-sets. R_W-sets have already been studied by W. Ruppert [18]. We need the following characterization of R_W-sets which is similar to a result of his: a subset E of G is an R_W-set if and only if it does not contain a set of the form $\{x_i y_j : i = 1, 2, \ldots, \ 1 \leq j \leq i\}$ or $\{x_i y_j : j = 1, 2, \ldots, \ 1 \leq i \leq j\}$ where $\{x_i\}$ and $\{y_i\}$ are two sequences of distinct elements in G.

For $1 \leq p < 2$, a subset E of G is called a weak p-Sidon set if there is a finite constant τ such that $\|f\|_p \leq \tau \|f\|_*$ whenever $f \in l^1(E)$ where $\|\cdot\|_*$ denotes the maximal C^*-algebra norm on $l^1(G)$. Weak 1-Sidon sets are called weak Sidon sets in Picardello [15] and, for abelian G, weak p-Sidon sets are just p-Sidon sets as defined in Edwards and Ross [7]. We show that if $1 \leq p < 4/3$ and E is a weak p-Sidon set then E contains no large squares. This generalizes

Received by the editors November 4, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 43A46, 43A60; Secondary 43A30, 43A07.
Key words and phrases. Discrete groups, weakly almost periodic functions, infinite triangles, large squares, large k-cubes, wide strips, T-sets, weak p-Sidon sets, R_W-sets.

0002-9947/90 \$1.00 + \$.25 per page

a result in [7] for abelian groups. Déchamps-Gondim [5] proved that countable Sidon sets in abelian groups are finite unions of T-sets. We are able to adopt her proof to show in §4 that if $1 \leq p < 4/3$ then countable weak p-Sidon sets are finite unions of T-sets. J. Bourgain [2] showed that Sidon sets, countable or not, in abelian groups are always finite unions of T-sets. It does not seem to be known whether his result holds for p-Sidon sets if $1 < p < 4/3$.

On the other hand, T-sets can be quite large. Indeed, §4 also contains the following result which improves a result of ours in [4]: every infinite G contains a T-set E such that, for each positive integer k, E has a subset A of the form $A = A_1 \cdots A_k = \{x_1 \cdots x_k : x_i \in A_i, i = 1, \ldots, k\}$ where $|A_i| = k$ and $|A| = k^k$. By a result of Johnson and Woodward [12], we then conclude that every infinite abelian group contains a T-set which is not a p-Sidon set for any $1 \leq p < 2$.

Definitions and general results on R_W-sets and weak p-Sidon sets are contained in §2.

2. Preliminaries and general results

Throughout this paper, G denotes an infinite discrete group, N the set of positive integers, and for a set A, $|A|$ the cardinality of A.

Definition 2.1. (a) If $\{x_i : i \in N\}$ and $\{y_j : j \in N\}$ are two sequences in G such that $(i, j) \to x_i y_j$ is a one-one mapping from N^2 into G then $S = \{x_i y_i : i, j \in N\}$ is called an infinite square in G and the sets $\{x_i y_j : i \in N, 1 \leq j \leq i\}$ and $\{x_i y_j : j \in N, 1 \leq i \leq j\}$ are called infinite triangles.

(b) If $A, A_i, i = 1, \ldots, k$, are subsets of G, $A = A_1 \cdots A_k$, $|A_i| = n$ and $|A| = n^k$ then A is called a k-cube of length n.

(c) If $C = AB$ or BA where A is infinite, $|B| = n$, and $(a, b) \to ab$ or ba is a one-one mapping from $A \times B$ to AB or BA, then C is called a strip of width n.

A subset E in G is said to contain large k-cubes if, for any given $n \in N$, E contains a k-cube of length n. E is said to contain wide strips if, for any given $n \in N$, E contains a strip of width n. A 2-cube is called a square in [4] and a 1-cube of length n is just a set with n elements. A k-cube of length 2 is also called a parallelepiped of dimension k; see Hare [11].

Lemma 2.2. (a) *If $E = AB$ where A and B are infinite subsets of G then E contains an infinite square.*

(b) *Suppose that $E \subset G$. If, for each $n \in N$, there exist subsets $A_1, \ldots, A_k$ of G such that $|A_i| = n$, $i = 1, \ldots, k$, and $A_1 \cdots A_k \subset E$ then E contains large k-cubes.*

(c) *If $E = AB$ where $A, B \subset G$, A is infinite and $|B| = n$, then there exists an infinite set $A_1 \subset A$ such that $A_1 B$ is a strip (of width n).*

Proof. (a) Suppose that we have chosen $A_n = \{a_1, \ldots, a_n\} \subset A$, $B_n = \{b_1, \ldots, b_n\} \subset B$ such that $|A_n B_n| = n^2$. Choose $a_{n+1} \in A \backslash A_n B_n B_n^{-1}$ and then choose $b_{n+1} \in B\backslash A_{n+1}^{-1}A_{n+1}B_n$ where $A_{n+1} = \{a_1, \dots, a_{n+1}\}$. Let $B_{n+1} = \{b_1, \dots, b_{n+1}\}$. Then $|A_{n+1}B_{n+1}| = (n+1)^2$. Thus, by induction, E contains an infinite square $\{a_ib_j : i, j \in N\}$.

(b) For $k = 2$, this result was proved in [13, p. 8]. In general, using the method of [13], it is not hard to show, by induction on k, that if $B_i \subset G$, $|B_i| = n^{2i-1} + 1$, $i = 1, \dots, k$, then there exist $A_i \subset B_i$ such that $|A_i| = n$ and $|A_1 \cdots A_k| = n^k$.

We omit the simple proof of (c).

Note that the proof of (a) also shows that if $\{a_i\}$ and $\{b_j\}$ are two sequences of distinct elements in G then $E = \{a_ib_j : i \in N, 1 \le j \le n\}$ contains an infinite triangle.

As usual, $l^\infty(G)$ denotes the space of bounded complex-valued functions on G with sup norm. For $f \in l^\infty(G)$ and $x \in G$, ${}_xf \in l^\infty(G)$ is defined by ${}_xf(y) = f(xy)$, $y \in G$. $f \in l^\infty(G)$ is said to be weakly almost periodic (w.a.p.) if the left orbit $O_L(f) = \{{}_xf : x \in G\}$ of f is relatively weakly compact in $l^\infty(G)$. WAP(G), the space of w.a.p. functions on G, is a translation invariant C^*-subalgebra of $l^\infty(G)$ and, by Ryll-Nardzewski's fixed point theorem [19], it has a unique two-sided invariant mean m_G. The following result of Grothendieck [10] is the basic tool in our study of w.a.p. functions.

Lemma 2.3 (Grothendieck's criterion). *$f \in l^\infty(G)$ is w.a.p. if and only if whenever $\{x_i\}$ and $\{y_j\}$ are two sequences in G and $\lim_i \lim_j f(x_iy_j)$ and $\lim_j \lim_i f(x_iy_j)$ exist, then they are equal.*

If E is a subset of G and A a subalgebra of $l^\infty(G)$ then $l^\infty(E)$ is said to reside in A, or, in short, E is an R_A-set, if whenever $f \in l^\infty(G)$ and f vanishes off E then $f \in A$. Clearly, the union of two R_A-sets is an R_A-set. For convenience, $R_{\mathrm{WAP}(G)}$-sets will be called R_W-sets. Since T-sets are R_W-sets (see [4, Lemma 3.2]), finite unions of T-sets are R_W-sets.

Proposition 2.4. *Let E be a subset of G. Then the following two conditions are equivalent:*

(1) *E is an R_W-set;*

(2) *if $\{a_i : i \in N\}$ is a sequence of distinct elements in G then both the sets*

$$A = \{x \in G : xa_i \text{ is eventually in } E\},$$

$$B = \{x \in G : a_ix \text{ is eventually in } E\}$$

are finite.

Proof. (1) $\Rightarrow$ (2). Suppose that B is infinite. Then there exists a sequence of distinct elements $\{b_j : j \in N\}$ in G such that for each j, $\{a_ib_j : i \in N\}$ is eventually in E. By replacing $\{a_i\}$ and $\{b_j\}$ by subsequences, we may assume that $\{a_ib_j : i, j \in N\}$ is an infinite square; see Lemma 2.2(a). Define $f \in l^\infty(G)$ by setting $f(a_ib_j) = 1$ if $a_ib_j \in E$ and $i \ge j$ and $f(x) = 0$ for all other $x \in G$. Then

$$\lim_i \lim_j f(a_i b_j) = 0, \quad \lim_j \lim_i f(a_i b_j) = 1,$$

and hence, by Lemma 2.3, $f \notin \mathrm{WAP}(G)$. Since f vanishes off E, by definition, E is not an R_W-set. Similarly, if A is infinite then E is not an R_W-set.

(2) $\Rightarrow$ (1). Suppose that (2) holds. Since $\mathrm{WAP}(G)$ is a norm closed linear space, to show that E is an R_W-set, it suffices to show that $\chi_A \in \mathrm{WAP}(G)$ for each $A \subset E$. By Lemma 2.3, it suffices to show that whenever $\{a_i\}$ and $\{b_j\}$ are two sequences in G such that

$$L_1 = \lim_j \lim_i \chi_A(a_i b_j), \quad L_2 = \lim_i \lim_j \chi_A(a_i b_j)$$

exist then $L_1 = L_2$. It is easy to see that if either $\{a_i\}$ or $\{b_j\}$ is eventually a constant then $L_1 = L_2$. Therefore, we only have to consider the case that $\{a_i\}$ and $\{b_j\}$ are sequences of distinct elements. We claim that in this case $L_1 = L_2 = 0$. Indeed, if, say, $L_1 = 1$ then there exists j_0 such that if $j \geq j_0$ then $\lim_i \chi_A(a_i b_j) = 1$, i.e., for $j \geq j_0$, $\{a_i b_j : i \in N\}$ is eventually in E. Therefore B is infinite, a contradiction.

Remarks. (1) In order to show that weak Sidon sets ae R_W-sets, we presented the above proposition at the 1982 Summer Meeting of the American Mathematical Society in Toronto; see Abstracts Amer. Math. Soc. **3** (1982), p. 353. Meanwhile, Ruppert has obtained, independently, several characterizations of R_W-sets in [18]. His condition (ii) in Theorem 7 of [18] is equivalent to our condition (2) above. For the sake of completeness, we include a proof of our proposition here.

(2) As usual, if $\{A_i\}$ is a sequence of sets then $\liminf A_i = \bigcup_{n=1}^{\infty}(\bigcap_{i=n}^{\infty} A_i)$. The above proposition states that E is an R_W-set if and only if $\liminf a_i E$ and $\liminf E a_i$ are finite for any sequence $\{a_i\}$ of distinct elements in G.

(3) It is not hard to see that the above proposition can be also stated as follows: a subset E of G is an R_W-set if and only if it does not contain infinite triangles.

Lemma 2.5. *If E is an R_W-set in an infinite group G, then $m_G(\chi_E) = 0$.*

Proof. Since $\chi_E \in \mathrm{WAP}(G)$, $m_G(\chi_E)$ is well defined. By Ryll-Nardzewski's fixed point theorem [19], $m_G(\chi_E) = c$ is the unique constant in the closed convex hull of $O_L(\chi_E)$. If c > 0, then there exists $\sum_{i=1}^{n} \lambda_i \chi_{x_i E} \in \mathrm{co}\, O_L(\chi_E)$ (the convex hull of $O_L(\chi_E)$) such that

$$\left\| \sum_{i=1}^{n} \lambda_i \chi_{x_i E} - c \right\|_{\infty} < \frac{c}{2}.$$

This implies that $\bigcup_{i=1}^{n} x_i E = G$. Since E is an R_W-set, so are $x_i E$, $i = 1, \ldots, n$. Therefore, $G = \bigcup_{i=1}^{n} x_i E$ is also an R_W-set and hence $\mathrm{WAP}(G) = l^{\infty}(G)$. This contradicts the well-known fact that $\mathrm{WAP}(G) \subsetneq l^{\infty}(G)$; see [3, p.

68]. (We can also argue as follows: since G clearly contains infinite triangles, by Proposition 2.4, it is not an R_W-set.)

A group G is said to be amenable if $l^\infty(G)$ has a left invariant mean μ: $\mu \in l^\infty(G)^*$, $\|\mu\| = 1$, $\mu \geq 0$ and $\mu(l_x f) = \mu(f)$ for all $f \in l^\infty(G)$ and $x \in G$. For example, solvable groups are amenable but nonabelian free groups are not amenable; see Pier [14]. If G is amenable, let $\mathrm{LIM}(G)$ be the set of all left invariant means on G and, for $E \subset G$, let $\overline{d}_l(E) = \sup\{\mu(\chi_E): \mu \in \mathrm{LIM}(G)\}$, the left upper density of E. For amenable G, if $\chi_E \in \mathrm{WAP}(G)$ then $m_G(\chi_E) = \overline{d}_l(E)$. Therefore, by the above lemma, if $\overline{d}_l(E) > 0$ then E is not an R_W-set. By Proposition 2.4, we obtain the following.

Corollary 2.6. *If G is an infinite amenable group, $E \subset G$ and $\overline{d}_l(E) > 0$ then E contains infinite triangles.*

Remark. Let $\mathbf{Z}$ be the additive group of integers. It is easy to construct a subset E of $\mathbf{Z}$ such that (i) E contains infinite triangles, (ii) $\overline{d}_l(E) = 0$ and (iii) E does not contain arithmetic progressions of length 3. Note that a celebrated result of E. Szemirédi [20] states that if $E \subset \mathbf{Z}$ and $\overline{d}_l(E) > 0$ then E contains arbitrarily long arithmetic progressions; see Furstenberg [9] for an ergodic theoretical proof of this result.

Let $C^*(G)$ be the completion of $l^1(G)$ with respect to the maximal C^*-norm $\|\cdot\|_*$: for $f \in l^1(G)$,

$$\|f\|_* = \sup\{\|\pi(f)\|: \pi \text{ a unitary representation of } G\},$$

where $\pi(f) = \sum\{f(x)\pi(x): x \in G\}$. Then the dual Banach space of $C^*(G)$ can be identified with $B(G)$ (the Fourier-Stieltjes algebra of G) which consists of coefficient functions of unitary representations of G. Let $B_\lambda(G)$ be the algebra of coefficient functions of unitary representations of G which are weakly contained in the left regular representation λ. Then $B_\lambda(G)$ can be identified with the dual Banach space of $C_\lambda^*(G)$, the C^*-algebra generated by $\{\lambda(f): f \in l^1(G)\}$. See Eymard [8], for definitions and results mentioned in this paragraph.

If E is a subset of G and $f \in l^1(E)$ then f will be identified with the function on G which equals f on E and is identically zero off E. For $1 \leq p < 2$, a subset E of G is called a weak p-Sidon (p-Sidon) set if there is a finite constant τ such that $\|f\|_p \leq \tau\|f\|_*$ ($\|f\|_p \leq \tau\|\lambda(f)\|$) for each $f \in l^1(E)$. Note that p-Sidon sets are always weak p-Sidon and if G is amenable then weak p-Sidon sets are p-Sidon, since, in this case $\|f\|_* = \|\lambda(f)\|$, $f \in l^1(G)$; see [8]. For abelian G, a weak p-Sidon set is just a p-Sidon set as defined by Edwards and Ross [7]. Note also that (weak) 1-Sidon sets are just (weak) Sidon sets as defined by Picardello [15]. Furthermore, a subset E of G is a weak Sidon set (Sidon set) if and only if $B(G)|E = l^\infty(E)$ ($B_\lambda(G)|E = l^\infty(E)$). We showed in [4] that weak Sidon sets do not contain large squares. This result can be strengthened somewhat with a minor change of the proof.

Proposition 2.7. *If* $1 \le p < 4/3$ *and* E *is a weak* p*-Sidon set in* G *then* E *does not contain large squares.*

Proof. The proof is similar to that of Proposition 3.4 of [4]. We will give only an outline here. Suppose that E contains large squares. Then for each n, choose a square $S = AB$ in E of length n, where $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$. Let (u_{ij}) be an $n \times n$ unitary matrix with complex entries and with $|u_{ij}| = 1/\sqrt{n}$. Let $g = \sum_{i,j=1}^{n} u_{ij}\delta_{a_i b_j}$ where for $t \in G$, δ_t denotes the function on G which equals 1 at t and zero elsewhere. Then

$$\|g\|_p = \left(\sum_{i,j=1}^{n} |u_{ij}|^p\right)^{1/p} = n^{2/p-1/2}.$$

On the other hand, as proved in [4], $\|g\|_* \le n$. Therefore, if E is a weak p-Sidon set, then $2/p - 1/2 \le 1$ or $p \ge 4/3$, a contradiction.

When G is abelian, the above result is due to Edwards and Ross [7, Corollary 2.7].

Corollary 2.8. *For* $1 \le p < 4/3$, *if* E *is a weak* p*-Sidon set in* G *then* E *is an* R_W*-set; in particular,* $\chi_E \in \mathrm{WAP}(G)$.

Proof. By Proposition 2.7, E does not contain large squares and hence it does not contain infinite triangles. By Proposition 2.4, E is an R_W-set.

Remarks. (1) If G is an infinite abelian group, then it contains an infinite square S such that S is a 4/3-Sidon set; see [7, Corollary 5.5]. By Proposition 2.4, S is not an R_W-set. Therefore the above result does not hold if $p \ge 4/3$.

(2) If the set E in the above corollary is countable, one can actually conclude that E is a finite union of T-sets; see §4.

(3) If G is abelian, a well-known result of Drury [6] states that if E is a Sidon set then $\chi_E \in B(G)^-$, the uniform closure of $B(G)$. Note that, for every infinite group G, $B(G)^-$ is properly contained in $\mathrm{WAP}(G)$; see [4].

Lemma 2.9. *For* $E \subset G$ *and* $n \in N$, *consider the following conditions:*

(i) E *is a union of* n T*-sets;*
(ii) E *contains no strips of width* $n + 1$;
(iii) *for any finite set* F *in* G, $\{x \in G\colon |xF \cap E| > n\}$ *and* $\{x \in G\colon |Fx \cap E| > n\}$ *are finite.*

Then (i) $\Rightarrow$ (ii) *and* (ii) $\Rightarrow$ (iii).

Proof. (i) $\Rightarrow$ (ii). Let $E = E_1 \cup \cdots \cup E_n$ where $E_1, \dots, E_n$ are T-sets. Suppose that E contains a strip C of width $n+1$, say $C = BA$ where $B = \{b_1, \dots, b_{n+1}\}$ and $A = \{a_1, a_2, \dots\}$. By replacing $A = \{a_j\}$ by a subsequence, if necessary, we may assume that, for each $1 \le i \le n+1$, $b_i A$ is contained in some E_k, $1 \le k \le n$. So there exist $i_1, i_2 \in \{1, 2, \dots, n+1\}$, $i_1 \ne i_2$, such that $b_{i_1}A \cup b_{i_2}A \subset E_{k_0}$ for some $1 \le k_0 \le n$. This contradicts the fact that E_{k_0} is a T-set.

(ii) $\Rightarrow$ (iii). Suppose that there exist a finite set F and a sequence of distinct elements $\{x_k\}$ in G such that $|x_k F \cap E| > n$ for all k. Then for each k there is a set $F_k \subset F$ such that $|F_k| = n+1$ and $x_k F_k \subset E$. Since F contains only finitely many subsets with cardinality $n+1$, there exists an infinite subset I of N and $F' \subset F$ such that, if $k \in I$, then $F_k = F'$. Then $\{x_k : k \in I\}F'$ is a strip of width $n+1$ contained in E.

For convenience, we call a set E in G an FT-set if it is a finite union of T-sets. It is not hard to see that a subset E of G is a T-set if and only if given any finite subset Δ of G there exists a finite subset F of E such that $x, y \in E \backslash F$ and $x \neq y$ imply $xy^{-1}, x^{-1}y \notin \Delta$. Therefore, in the terminology of [13, p. 112], a set E is a T-set if and only if it tends to infinity.

3. Existence of R_W-sets which are not FT-sets

As in [4], a subset E of G is said to be relatively dense if there exist finite sets X and Y such that $G = XEY$. We need the following result of ours in [4].

Lemma 3.1. *Let S be a relatively dense subset of G and F a finite subset of G, $e \notin F$. Then there exists a relatively dense subset E of S such that*

$$(xE \cap E) \cup (Ex \cap E) = \varnothing$$

for $x \in F$.

Lemma 3.2. *Let P be a relatively dense subset of G, say $G = XPY$ where X and Y are finite. Then for each positive integer n there exists a finite set E such that the set*

$$Q = \{x \in G : |xEb^{-1} \cap P| \geq n \text{ for some } b \in Y\}$$

is relatively dense in G. In particular, Q is infinite.

Proof. Choose any finite subset $E = \{z_1, \ldots, z_k\}$ of G such that $k = |E| = |X||Y|n$. Fix $x \in G$. Then, for each $1 \leq i \leq k$, xz_i can be written as $xz_i = a_i p_i b_i$ where $a_i \in X$, $b_i \in Y$, and $p_i \in P$. For $(a, b) \in X \times Y$, let

$$I(a, b) = \{i : 1 \leq i \leq k,\ a_i = a,\ b_i = b\}.$$

Then $\bigcup\{I_{(a,b)} : (a, b) \in X \times Y\} = \{1, 2, \ldots, k\}$, and hence

$$k = \sum\{|I_{(a,b)}| : (a, b) \in X \times Y\}.$$

Since $k = |X||Y|n$, there exists $(c, d) \in X \times Y$ such that $|I_{(c,d)}| \geq n$. If $i \in I_{(c,d)}$ then

$$c^{-1}xz_i d^{-1} = p_i \in (c^{-1}xEd^{-1}) \cap P$$

and the p_i's, $i \in I_{(c,d)}$, are distinct. Therefore, $c^{-1}x \in Q$. Thus $G = XQ$ and hence Q is relatively dense.

We are now ready to give the main result of this section.

Theorem 3.3. *Let G be an infinite group. Then there exists a subset D of G such that*

(a) *D is not an FT-set;*

(b) *D is an R_W-set.*

Proof. Without loss of generality, we may assume that G is countably infinite. Then there exists a sequence of finite symmetric subsets $\{F_n\}$ of G such that

$$e \in F_1 \subset F_2 \subset \dots, \quad \text{and} \quad G = \bigcup F_n .$$

(A set $B \subset G$ is symmetric if $B = B^{-1}$.) By Lemma 3.1, we can find a sequence of relatively dense subsets S_n of G such that $S_1 \supset S_2 \supset \cdots$, and

$$(xS_n \cap S_n) \cup (S_n x \cap S_n) = \varnothing, \quad \text{if } x \in F_n \backslash \{e\}. \tag{3.1}$$

For each n, choose finite sets X_n and Y_n such that $X_n S_n Y_n = G$. By Lemma 3.2, for each $n \in N$, there exists a finite set E_n such that

$$Q_n = \{x \in G : |xE_n b^{-1} \cap S_n| \geq n \text{ for some } b \in Y_n\}$$

is infinite.

Fix infinite subsets $N_1, N_2, \dots$ of N such that $N_i \cap N_j = \varnothing$, if $i \neq j$, and $N_1 \cup N_2 \cup \cdots = N$. Then for each $n \in N$ there is a unique positive integer $\sigma(n)$ such that $n \in N_{\sigma(n)}$.

Choose $t_1 \in Q_{\sigma(1)}$, $b_1 \in Y_{\sigma(1)}$ such that $|t_1 E_{\sigma(1)} b_1^{-1} \cap S_{\sigma(1)}| \geq \sigma(1)$. Suppose that we have chosen t_j, b_j in G, $j = 1, \dots, n$, such that the t_j's are distinct, $b_j \in Y_{\sigma(j)}$ and if $D_j = t_j E_{\sigma(j)} b_j^{-1} \cap S_{\sigma(j)}$, then $|D_j| \geq \sigma(j)$ and $F_j D_j F_j \cap (D_1 \cup \cdots \cup D_{j-1}) = \varnothing$, for $2 \leq j \leq n$. Now since $Q_{\sigma(n+1)}$ is infinite there exists $t_{n+1} \in Q_{\sigma(n+1)}$ such that

$$t_{n+1} \notin F_{n+1}(D_1 \cup \cdots \cup D_n) F_{n+1} Y_{\sigma(n+1)} E_{\sigma(n+1)}^{-1} \cup \{t_1, \dots, t_n\}. \tag{3.2}$$

Since $t_{n+1} \in Q_{\sigma(n+1)}$, there exists $b_{n+1} \in Y_{\sigma(n+1)}$ such that if $D_{n+1} = t_{n+1} \cdot E_{\sigma(n+1)} b_{n+1}^{-1} \cap S_{\sigma(n+1)}$ then $|D_{n+1}| \geq \sigma(n+1)$. By (3.2),

$$F_{n+1} D_{n+1} F_{n+1} \cap (D_1 \cup \cdots \cup D_n) = \varnothing.$$

Therefore, by induction, we can construct two sequences $\{t_n\}$ and $\{b_n\}$ in G such that the t_n's are distinct, $b_n \in Y_{\sigma(n)}$ and if $D_n = t_n E_{\sigma(n)} b_n^{-1} \cap S_{\sigma(n)}$ then

$$|D_n| = |D_n \cap S_{\sigma(n)}| \geq \sigma(n); \tag{3.3}$$

$$F_n D_n F_n \cap (D_1 \cup \cdots \cup D_{n-1}) = \varnothing, \qquad n \geq 2. \tag{3.4}$$

Since $D_n \subset S_{\sigma(n)}$, (3.1) implies that

$$\text{if } x \in F_{\sigma(n)} \backslash \{e\}, \text{ then } (xD_n \cap D_n) \cup (D_n x \cap D_n) = \varnothing. \tag{3.5}$$

Also note that, as a consequence of (3.4), we have

$$\text{if } x \in F_n, \ m \geq n \text{ and } l \neq m, \text{ then } (xD_m \cap D_l) \cup (D_m x \cap D_l) = \varnothing. \tag{3.6}$$

We claim that $D = \bigcup_{n=1}^{\infty} D_n$ satisfies conditions (a) and (b) in the statement of the theorem.

We will first prove that D satisfies (a). Fix $k \in N$. Note that if $n \in N_k$ then $b_n \in Y_k$. Since N_k is infinite and Y_k is finite, there exist an infinite subset I_k of N_k and an element $b \in Y_k$ such that if $n \in I_k$ then $b_n = b$. Let $F_k = E_k b^{-1}$. Then, for $n \in I_k$,

$$D \cap t_n F_k = D \cap t_n E_k b_n^{-1} \supset D_n ,$$

and hence $|D \cap t_n F_k| \geq |D_n| \geq k$. Since I_k is infinite and $\{t_n : n \in N\}$ is a sequence of distinct elements in G, by (i) $\Rightarrow$ (iii) of Lemma 2.9, we conclude that D is not a union of $k-1$ T-sets. Since $k \in N$ is arbitrary, D is not an FT-set, as claimed.

It remains to show that D satisfies (b). To this end, first let $x \in F_k \backslash \{e\}$, $k \geq 2$, be fixed. Then, by (3.6),

$$xD \cap D \subset \{(xD_1 \cup \cdots \cup xD_{k-1}) \cap D\} \cup \left\{ \bigcup_{m \geq k} (xD_m \cap D_m) \right\} . \tag{3.7}$$

Note that if $m \notin N_1 \cup \cdots \cup N_{k-1}$, i.e., $\sigma(m) \geq k$, then $x \in F_k \subset F_{\sigma(m)}$, and hence, by (3.5), $xD_m \cap D_m = \varnothing$. Therefore, (3.7) implies that

$$xD \cap D \subset F_x \cup \left(\bigcup \{xD_m \cap D_m : m \in N_1 \cup \cdots \cup N_{k-1}\} \right) , \tag{3.8}$$

where $F_x = (xD_1 \cup \cdots \cup xD_{k-1}) \cap D$ is a finite set. Now assume that $\{a_i : i \in N\}$ is a sequence of distinct elements in D, $x \in F_k \backslash \{e\}$ and $\{xa_1, xa_2, \dots\}$ is eventually contained in D; in other words $\{a_i\}$ is eventually contained in $x^{-1}D \cap D$. Since $x^{-1} \in F_k \backslash \{e\}$, by (3.8), $\{a_i\}$ is eventually contained in $\bigcup \{D_m : m \in N_1 \cup \cdots \cup N_{k-1}\}$. Similarly, we can prove that if $x \in F_k \backslash \{e\}$ and $\{a_i\}$ and $\{a_i x\}$ are both eventually contained in D then $\{a_i\}$ is eventually contained in $\{D_m : m \in N_1 \cup \cdots \cup N_{k-1}\}$.

Suppose that D is not an R_W-set. Then, by Proposition 2.4, there exist two sequences $\{a_i\}$, $\{y_j\}$ of distinct elements in G such that either

(I) $\{y_j a_i : i \in N\}$ is eventually contained in D for each j, or

(II) $\{a_i y_j : i \in N\}$ is eventually contained in D for each j.

By symmetry, we only have to consider case (I). By renaming the two given sequences, we may also assume that $y_1 = e$. Then, as demonstrated in the above paragraph, $\{a_i\}$ is eventually contained in $\bigcup \{D_m : m \in N_1 \cup \cdots \cup N_{k-1}\}$ for some fixed $k \geq 2$. By taking a subsequence, if needed, we may assume that $\{a_i\}$ is contained in $\bigcup \{D_m : m \in N_{k_0}\}$ for some fixed k_0. Assume that $a_i \in D_{m_i}$ where $m_i \in N_{k_0}$. We may further assume that the m_i's are distinct. For a fixed j, $j \neq 1$, since $\{a_i\}$ is eventually contained in $D \cap y_j^{-1} D$, by (3.7), $a_i \in y_j^{-1} D_{m_i} \cap D_{m_i}$ when i is sufficiently large. Thus, for each $l \in N$ there exists an i such that

$$\{a_i = y_1 a_i , y_2 a_i , \dots , y_l a_i\} \subset D_{m_i} .$$

This is impossible, since $|D_{m_i}| \le |E_{k_0}|$ for each i. Therefore, D is an R_W-set as claimed.

Remarks. (1) If G is an abelian group then the above proof can be simplified somewhat. Our result seems to be new even for $G = \mathbf{Z}$, the additive group of integers. However, for $\mathbf{Z}$ the set D can be constructed more explicitly as follows.

Write N as a disjoint union of infinite sets N_k, $k = 1, 2, \ldots$, and define $\sigma(n)$ as before. Define blocks of consecutive positive integers C_n, $n = 1, 2, \ldots$, inductively so that

$$\min C_{n+1} > \max C_n + n, \tag{3.9}$$

$$|C_n| = (\sigma(n) + 1)^2. \tag{3.10}$$

Assume that $C_n = \{t_n, t_n + 1, \ldots, t_n + (\sigma(n) + 1)^2 - 1\}$. Let

$$J_n = \{t_n, t_n + (\sigma(n) + 1), t_n + 2(\sigma(n) + 1), \ldots, t_n + \sigma(n)(\sigma(n) + 1)\}$$

and $D = \bigcup_{n=1}^{\infty} J_n$. Then D is an R_W-set but is not an FT-set.

(2) Let G^w be the weakly almost periodic compactification of the discrete group G. We can consider G as a subset of G^w. Then the multiplication on G can be extended to G^w which makes G^w a semigroup with separately continuous multiplication; cf. [3]. In particular, G acts on the compact space G^w by left multiplication. From the definition of R_W-sets, it is easy to see that a subset E of G is an R_W-set if and only if χ_E is w.a.p. and E^- (the closure of E in G^w) is the Stone-Čech compactification of E; see Ruppert [18]. $\omega \in G^w \backslash G$ is said to be strongly G-discrete if there is a neighborhood U of ω in $G^w \backslash G$ such that $xU \cap yU = \varnothing$ if $x, y \in G$, $x \neq y$. Note that if E is an FT-set and $\omega \in E^- \backslash G$ then ω is strongly G-discrete. On the other hand, if D is the R_W-set constructed in Theorem 3.3, then D^- is homeomorphic to βD and there exists $\omega \in D^- \backslash G$ such that ω is not strongly G-discrete.

(3) The set D constructed in Theorem 3.3 contains large squares, since it contains wide strips. Therefore the R_W-set D is not a weak Sidon set. Hence it implies the known result that $B(G)^-$ is properly contained in WAP(G) for every infinite group G; see [4].

4. Further results on FT-sets

Déchamps-Gondim proved in [5] that countable Sidon sets in abelian groups are FT-sets. She has actually obtained the following result in her proof: if E is a countable subset of an abelian group G and if E does not contain wide strips then E is an FT-set. Her proof, with some minor modifications, also works for nonabelian groups.

Theorem 4.1. *A countable subset E of a group G is an FT-set if and only if it does not contain wide strips.*

Proof. The "only if" part of the theorem is true no matter whether E is countable or not; see (i) $\Rightarrow$ (ii) in Lemma 2.9. We will now outline the proof of the "if" part in four steps. Assume that E is a countable subset of G which does not contain strips of width $n+1$.

(I) Given any finite set Δ in G, there exists a finite subset F of E such that

$$|x\Delta \cap (E\backslash F)| \leq n, \qquad x \in G. \tag{4.1}$$

Indeed, by (ii) $\Rightarrow$ (iii) in Lemma 2.9, the set $F' = \{x \in G: |x\Delta \cap E| > n\}$ is finite. Let $F = F'\Delta$. Then (4.1) holds. See also Lemme 6.1 of [5] and Lemma 8.8 of [13].

(II) Given any finite set $\Delta \subset G$, E can be written as a disjoint union $E = F \cup (\bigcup_{i\in I} F_i)$ where F is finite, for $i \in I$, $|F_i| \leq n$, and $F_i^{-1}F_j \cap \Delta = \varnothing$ if $i \neq j$.

This is Lemme 6.2 of [5]; see also Corollary 8.10 of [13]. We include an outline of its proof here. We may assume that Δ is symmetric and $e \in \Delta$. By (I) there exists a finite set F such that

$$|x\Delta^n \cap (E\backslash F)| \leq n \tag{4.2}$$

for all $x \in G$. A finite set of the form

$$\{x, xt_1, xt_1t_2, \dots, xt_1\cdots t_{k-1} = y\}$$

is called a Δ-chain in $E\backslash F$ if it is a subset of $E\backslash F$ and $t_1, \dots, t_{k-1} \in \Delta$. For $x, y \in E\backslash F$, we define $x \sim y$ if there is a Δ-chain from x to y. Then $\sim$ is an equivalence relation. Note that if $x \sim y$, $x \neq y$, then x and y can be linked by a Δ-chain $x_1 = x$, $x_2 = xt_1, \dots, x_k = xt_1\cdots t_{k-1} = y$ such that $x_1, \dots, x_k$ are distinct. By (4.2), $k \leq n$. Therefore if X is an equivalence class and $x_0 \in X$ then any element of X is of the form x_0t for some $t \in \Delta^n$. By (4.2) again, $|X| \leq n$. Clearly, if x and y are in different equivalence classes then $x^{-1}y \notin \Delta$. Thus the F_i's in (II) can be taken to be the equivalence classes of $\sim$.

(III) There exist E_i, $i = 1, \dots, n$, such that $E = E_1 \cup \cdots \cup E_n$ and each E_i satisfies the condition that $E_i \cap xE_i$ is finite if $x \in G$, $x \neq e$.

Follow the proof of Theorem 9.1 of [13]. But, unlike the proof there, the set E is not assumed to be symmetric and we apply (II) instead of Lemma 8.9 of [13]. Note that the countability of E is needed in the proof of (III).

(IV) E is an FT-set.

Write $E = \bigcup_{i=1}^n E_i$ as in (III). By symmetry, each E_i can be written as $E_i = \bigcup_{j=1}^n E_{ij}$ where, for each (i, j), $E_{ij}x \cap E_{ij}$ is finite whenever $x \neq e$. Therefore, $E = \bigcup_{i,j=1}^n E_{ij}$ and each E_{ij} is a T-set.

Corollary 4.2. *Assume that E is a countable weak p-Sidon set in a group G where $1 \leq p < 4/3$. Then E is an FT-set.*

Proof. By Proposition 2.7, E does not contain large squares and hence does not contain wide strips. By the above theorem, E is an FT-set.

Remarks. (1) As mentioned in §2, if $p \geq 4/3$ then the above corollary is not true.

(2) We do not know whether Theorem 4.1 or Corollary 4.2 holds for uncountable sets. Using completely different arguments, Bourgain [2, Corollaire 3.5] proved that Sidon sets in abelian groups are always FT-sets. However, his proof does not carry over to the case of p-Sidon sets if $p > 1$. A subset E of an abelian group G is said to be exactly p-Sidon if E is p-Sidon but is not q-Sidon for any $q < p$. Blei [1] proved that for any p, $1 < p < 2$, and for any infinite abelian group G, there exists a countable subset E of G such that E is exactly p-Sidon. If $p < 4/3$, by Corollary 4.2, his countable p-Sidon set is an FT-set.

We prove in [4] that, given any infinite group G, there exists a T-set E in G which contains large squares. This is the key step to show that $B(G)^- \nsubseteq \mathrm{WAP}(G)$ for any infinite group G; see [4]. (For abelian G this result is due to Rudin [17] and Ramirez [16].) It turns out that a T-set can even contain large k-cubes for any given k. To prove this we need the following refinement of Lemma 3.6 of [4].

Lemma 4.3. *There is a function $\alpha: N \times N \to N$ such that whenever $A = A_1 \cdots A_k$ is a k-cube of length $\alpha(k, n)$ and is contained in the union of two subsets E_1 and E_2 of a group G then there exist subsets B_i of A_i, such that $|B_i| = n$, $i = 1, \ldots, k$, and $B = B_1 \cdots B_k$ is contained in either E_1 or E_2.*

Proof. Let $\alpha(1, n) = 2n$. For $k \geq 2$, define $\alpha(k, n)$ inductively, by setting

$$\alpha(k, n) = 2n \binom{\alpha(k-1, n)}{n}^{k-1}$$

(For integers m, n, $0 \leq m \leq n$, $\binom{n}{m} = \frac{n!}{m!(n-m)!}$.) Then α is the function we want. We will prove this by induction on k.

A 1-cube of length n is just a finite set with n elements. If $A \subset E_1 \cup E_2$, $|A| = 2n = \alpha(1, n)$ then clearly there is a subset B of A such that $|B| = n$ and B is contained in either E_1 or E_2. Suppose that our result holds for $k-1$. Let $A = A_1 \cdots A_k$ be a k-cube of length $\alpha(k, n)$ and $A \subset E_1 \cup E_2$. Choose subsets A'_i of A_i, $i = 1, \ldots, k-1$, such that $|A'_i| = \alpha(k-1, n)$. For each $y \in A_k$,

$$A' = A'_1 \cdots A'_{k-1} \subset E_1 y^{-1} \cup E_2 y^{-1}.$$

Therefore, by inductive assumption, for each $y \in A_k$, there exists a $(k-1)$-cube $K(y) = A_1(y) \cdots A_{k-1}(y)$ of length n where $A_i(y) \subset A'_i$ and $K(y)$ is contained in either $E_1 y^{-1}$ or $E_2 y^{-1}$, or equivalently, $K(y)y$ is contained in either E_1 or E_2. Clearly, there exists a set $A'_k \subset A_k$ such that $|A'_k| = (1/2)\alpha(k, n)$ and either (i) $K(y)y \subset E_1$ for all $y \in A'_k$ or (ii) $K(y)y \subset E_2$ for all $y \in A'_k$. Suppose that (i) holds. Let $\{C_1, \ldots, C_l\}$ be the collection of subsets of A' of the form $A''_1 \cdots A''_{k-1}$ where $A''_i \subset A'_i$ and $|A''_i| = n$, $i = 1, \ldots, k-1$. Note that $l = \binom{\alpha(k-1,n)}{n}^{k-1}$. Let

$$D_i = \{y \in A'_k : K(y) = C_i\}, \qquad i = 1, \ldots, l.$$

Then $\bigcup_{i=1}^{l} D_i = A'_k$. Therefore, there exists some i_0 such that $|D_{i_0}| \geq n$; otherwise,

$$nl = \tfrac{1}{2}\alpha(k, n) = |A'_k| < nl, \quad \text{a contradiction}.$$

Choose $A''_k = \{y_1, \ldots, y_n\} \subset D_{i_0}$ and write $C_{i_0} = A''_1 \cdots A''_{k-1}$. Then $A''_1 \cdots A''_k \subset E_1$. This completes the proof of the lemma.

The above lemma implies that every relatively dense subset of an infinite group contains large k-cubes for each $k \in N$; see [4, p. 146]. As a consequence, we can follow the proof of Proposition 3.10 of [4] to obtain the following.

Theorem 4.4. *Let G be an infinite group. Then for each $n \in N$ there exists an n-cube K_n of length n in G such that $E = \bigcup_{n=1}^{\infty} K_n$ is a T-set.*

Johnson and Woodward [12] proved that if a subset E of an abelian group contains large k-cubes then it is not a p-Sidon set for any $p < 2k/(k+1)$. Therefore Theorem 4.4 has the following consequence.

Corollary 4.5. *Let G be an infinite abelian group. Then there exists a T-set E in G such that E is not a p-Sidon set for any $1 \leq p < 2$.*

We do not know whether the above corollary holds for general infinite groups.

References

1. R. C. Blei, *Combinatorial dimension and certain norms in harmonic analysis*, Amer. J. Math **106** (1984), 847–887.
2. J. Bourgain, *Propriétés de décomposition pour les ensembles de Sidon*, Bull. Soc. Math. France **111** (1983), 421–428.
3. R. B. Burckel, *Weakly almost periodic functions on semigroups*, Gordon and Breach, New York, 1971.
4. C. Chou, *Weakly almost periodic functions and Fourier-Stieltjes algebras of locally compact groups*, Trans. Amer. Math. Soc. **274** (1982), 141–157.
5. M. Déchamps-Gondim, *Ensembles de Sidon topologiques*, Ann. Inst. Fourier (Grenoble) **22** (1972), 51–79.
6. S. W. Drury, *Sur les ensembles de Sidon*, C. R. Acad. Sci. Paris Sér. I Math. **271** (1970), 162–163.
7. R. E. Edwards and K. A. Ross, *p-Sidon sets*, J. Funct. Anal. **15** (1974), 404–427.
8. P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France **92** (1964), 181–236.
9. H. Furstenberg, *Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions*, J. Analyse Math. **31** (1977), 204–256.
10. A. Grothendieck, *Critères de compacité dans les espaces fonctionnels généraux*, Amer. J. Math. **74** (1952), 168–186.
11. K. E. Hare, *Arithmetic properties of thin sets*, Pacific J. Math . **131** (1988), 143–155.
12. G. W. Johnson and G. S. Woodward, *On p-Sidon sets*, Indiana Univ. Math. J. **24** (1974–1975), 161–167.

13. J. M. López and K. A. Ross, Lecture Notes in Pure and Appl. Math.,, Dekker, New York, 1975.

14. J. Pier, *Amenable locally compact groups*, Wiley-Interscience, New York, 1984.

15. M. A. Picardello, *Lacunary sets in discrete noncommutative groups*, Bull. Un. Mat. Ital. **8** (1973), 494–508.

16. D. Ramirez, *Weakly almost periodic functions and Fourier-Stieltjes transforms*, Proc. Amer. Math. Soc. **19** (1968), 1087–1088.

17. W. Rudin, *Weak almost periodic functions and Fourier-Stieltjes transforms*, Duke Math. J. **26** (1959), 215–220.

18. W. A. F. Ruppert, *On weakly almost periodic sets*, Semigroup Forum **32** (1985), 267–281.

19. C. Ryll-Nardzewski, *On fixed points of semigroups of endomorphisms of linear spaces*, Proc. Fifth Berkeley Sympos. Math. Statist. and Prob., Vol. II, Part I, Theory of Probability, Univ. of California Press, Berkeley, 1966, pp. 55–61.

20. E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression*, Acta Arith. **27** (1975), 199–245.

Department of Mathematics, State University of New York at Buffalo, Buffalo, New York 14214