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THE 27-DIMENSIONAL MODULE FOR E_6, III

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ABSTRACT. This is the third in a series of five papers investigating the subgroup structure of the universal Chevalley group $G = E_6(F)$ of type E_6 over a field F and the geometry induced on the 27-dimensional FG-module V by the symmetric trilinear form f preserved by G. The series uses the geometry on V to describe and enumerate (up to a small list of ambiguities) all closed maximal subgroups of G when F is finite or algebraically closed.

The main result of this third paper is Theorem 3 in §1, which shows that the normalizer of any solvable subgroup of G not contained in $Z(G)$ stabilizes one of several structures on V. The action of G on these structures and the stabilizer in G of each structure are also described. In addition various secondary results are established. For example G-classes of elementary abelian p-subgroups are nearly enumerated when $p = 2$ or 3 and p is distinct from the characteristic of G.

Cohen, Liebeck, Saxl, and Seitz have announced a classification of the local maximal subgroups of finite exceptional groups of Lie type. Also the Lie theory gives a lot of information about local subgroups when F is algebraically closed or finite. The approach here is different and in general more naive and elementary. The theory of algebraic groups is not required and the emphasis is on describing subgroups of G concretely in terms of the representation of G on V and the geometry of V rather than in abstract group theoretic or Lie theoretic terms.

§1 contains a discussion of notation and terminology and a precise statement of the main theorem. We will refer extensively to results from parts I and II of this series [1]. Lemma x.y.z of part I will be referred to by the label I.x.y.z. There is a summary of some of the most important facts from parts I and II in §1.

1. THE E_6 SETUP

In the remainder of this paper we continue the hypotheses and notation of §§3 and 4 of part I. In particular, V is a 27-dimensional vector space over a

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field F with basis

$$X = (x_i, x_i', x_{ij} : 1 \le i, j \le 6, i < j),$$

and for $i < j$, $x_{ji} = -x_{ij}$. Further, f is the symmetric trilinear form on V whose monomials in X are

$$x_i x_j' x_{ij}, \quad 1 \le i, j \le 6, \ i \ne j,$$
$$x_{1d, 2d} x_{3d, 4d} x_{5d, 6d}, \qquad d \in \text{Coset},$$

where Coset is some set of coset representative for Alt_P in Alt. Alt is the alternating group on $\{1, \dots, 6\}$, and Alt_P is the stabilizer in Alt of the partition

$$P = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$$

of $\{1, \dots, 6\}$. Further, let $\mathscr{F} = (T, Q, f)$ be the 3-form defined by f, which makes X singular and brilliant (cf. the discussion in §2 of part I, particularly the definition of the notion of a 3-form and I.2.14).

Let $G = O(V, \mathscr{F})$ and $\Delta = \Delta(V, \mathscr{F})$ be the group of isometries and similarities of (V, f), respectively. (Similarities are maps g of V preserving f up to a scalar multiple $\lambda(g)$.) By I.5.4, G is the universal Chevalley group $E_6(F)$. Denote by $\Gamma = \Gamma(V, \mathscr{F})$ the group of semilinear maps g on V such that $f(xg, yg, zg) = \lambda(g) f(x, y, z)^{\sigma(g)}$ for all $x, y, z \in V$ and some $\lambda(g) \in F^\sharp$ and $\sigma(g) \in \text{Aut}(F)$. Write ξ for the transpose inverse map on $GL(V)$ defined by X and let ϕ be the polarity of V induced by X. Then ξ induces an automorphism of G nontrivial on the Dynkin diagram of G, and if F is finite or algebraically closed, then $\text{Aut}(G) = \Gamma\langle\xi\rangle / Z(\Delta)$.

In §2 of part I, we find that for $x \in V$, Q induces a quadratic form Q_x on V which has the f-induced form f_x as its associated bilinear form. Further, $x\Delta$ is defined to be the radical of f_x and for $U \le V$, $U\Delta = \bigcap_{x \in U} x\Delta$ and $U\theta$ consists of those v in V such that U is totally singular with respect to Q_v. Both $U\Delta$ and $U\theta$ are subspaces of V. We say U is *singular* if U is contained in $U\Delta$ and we say U is *brilliant* if U is contained in $U\theta$ and the cubic form T of $\mathscr{F}$ is trivial on U. A point $\langle v \rangle$ is *dark* if $T(v) \ne 0$.

For $1 \le k \le 6$ but $k \ne 5$, define

$$V_k = \langle x_i : 1 \le i \le 6 \rangle, \qquad V_k' = \langle x_i' : 1 \le i \le k \rangle.$$

Further, define

$$V_{15} = \langle x_{ij}; i, j \rangle, \qquad U_5 = \langle V_4, x_5 \rangle, \qquad V_5 = \langle V_4, x_{56} \rangle,$$
$$V_{10} = \langle U_5, x_{6i}; 1 \le i \le 5 \rangle, \qquad V_{12} = V_6 + V_6',$$

and

$$V_9 = \langle x_1, x_{16}, x_6' : x_2', x_6, x_{26} : x_{12} x_1', x_2 \rangle.$$

Denote by $\mathscr{V}_i$ the set of conjugates of V_i under G for $1 \le i \le 6$, and $i = 9, 10, 12$. Define $G_i = N_G(V_i)$ and $\Delta_i = N_\Delta(V_i)$ for $i \in \{1, 2, 3, 5, 10, 6\}$.

Let $L = SL(V_6)$ and represent L on V_6' so that the map $x_i \mapsto x_i'$ is an FL-isomorphism. Represent L on V_{15} so that the map $x_{ij} \mapsto x_i^* \wedge x_j^*$ is an FL-isomorphism of V_{15} with the exterior square of the dual of V_6, where $(x_i^* : i)$ is the dual basis to $(x_i : i)$. Thus we may regard L as a subgroup of $SL(V)$. By I.3.2, L is a subgroup of G.

Denote by $\widehat{X}$ the set of points generated by elements of X and let H_0 and $H(\Delta)$ denote the subgroups of G and Δ fixing each point in $\widehat{X}$, respectively. There is a description of the elements in H_0 and $H(\Delta)$ in §3 of part I.

For $M \leq G$, write $R(M)$ for the unipotent radical of M. If q is a prime power and p a prime with $(q, p) = 1$, then $d_q(p)$ denotes the least positive integer d with $q^d \equiv 1 \bmod p$.

Given a positive integer i, write $\zeta_i(U)$ for the subspace of V generated by the subspaces $W\Delta$, as W varies over the i-dimensional subspaces of U. Write $\zeta(U)$ for $\zeta_1(U)$ and $V(U)$ for $\zeta_2(U)$.

A line l is *hyperbolic* if l is generated by singular points A and B with $A \nleq B\Delta$. In that event A and B are the unique singular points in l. By I.6.4, each line of V generated by singular points is singular or hyperbolic, and G is transitive on singular lines and on hyperbolic lines.

By I.6.6 and I.6.7, if U is a nonsingular subspace contained in some member of $\mathscr{V}_{10}$, then U is contained in a unique member of $\mathscr{V}_{10}$, which we denote by $\Phi(U)$. Thus each hyperbolic line $\langle x, y\rangle$ is in a unique member $\Phi(x, y)$ of $\mathscr{V}_{10}$.

An n-tuple $(A_i : i)$ of singular points is *special* if each of the lines $A_i + A_j$ is hyperbolic and $A_i \nleq (A_j + A_k)\theta$ for all distinct i, j, k. A plane is *special* if it is generated by a special triple. By I.6.9, G is transitive on special planes. Write $\mathscr{U}_3$ for the set of special planes.

Define a subgroup M of Γ to be *brilliant* if M stabilizes some member of $\mathscr{V}_i$, $i = 1, 2, 3, 5, 6, 10, 9$, or 12. By Theorem 1 in part II, if U is a nontrivial brilliant subspace of V generated by singular points then $N_\Gamma(U)$ is a brilliant subgroup of G.

For $U \leq V$, write $\mathscr{V}_i(U)$ for the set of members of $\mathscr{V}_i$ contained in U. For $U \in \mathscr{V}_6$ let $\mathrm{Op}(U)$ consist of those $Z \in \mathscr{V}_6$ such that $Z \cap V(U) = 0$.

Given an extension K of F, write U^K for $K \otimes_F U$ when U is a subspace of V. Denote by $\mathscr{F}^K$ the form induced on V^K by $\mathscr{F}$ and let $G^K = O(V^K, \mathscr{F}^K)$.

A 9-*decomposition* of V is a set S of nine special planes such that for each A, B in S, A is incident to B and $\Sigma(A, B)$ is in S, in the notation of part II, §3. A 3-*decomposition* of V is a decomposition $V = A_1 \oplus A_2 \oplus A_3$ such that for some 9-decomposition S of V, $A_i = A + B + \Sigma(A, B)$ for some $A, B \in S$. By II.3.5, G is transitive on 9-decompositions and 3-decompositions of V. Also each member of a 3-decomposition is in $\mathscr{V}_9$ and conversely by II.3.6, each member of $\mathscr{V}_9$ is in a unique 3-decomposition of V.

A subspace U of V is *totally dark* if each point of U is dark. Suppose F is finite. A *twisted special plane* is a totally dark plane U such that for some cubic extension K of F, U^K is a special plane. By 7.3, G is transitive on twisted special planes. A *twisted 9-decomposition* of V is a set S of nine twisted special planes such that for some cubic extension K of F, S^K is a 9-decomposition of V. By 7.4, G is transitive on twisted 9-decompositions of V. A *twisted 9-subspace* is a 9-dimensional subspace U of V such that $U \notin \mathscr{V}_9$ but $U^K \in \mathscr{V}_9$ for some quadratic extension K of V. Write $\mathscr{U}_9$ for the set of twisted 9-subspaces of V. By 3.3, G is transitive on $\mathscr{U}_9$.

Let $\mathscr{U}_6$ consist of the 6-dimensional subspaces $U = \langle v_1, v_2, v_3, w_1, w_2, w_3\rangle$ of V such that (v_1, v_2, v_3, v_4) is a special 4-tuple and for $s \in S_3$, w_{1s} is the projection of v_4 on $\langle v_{2s}, v_{3s}\rangle\Delta$ with respect to the decomposition

$$V = \langle v_1, v_2, v_3\rangle \oplus \langle v_1, v_2\rangle\Delta \oplus \langle v_1, v_3\rangle\Delta \oplus \langle v_2, v_3\rangle\Delta.$$

By 4.5.1, G is transitive on $\mathscr{U}_6$.

If k is a cubic extension of F, a *twisted 3-decomposition* of V is a k-space structure on V whose stabilizer in G is irreducible on V and is the stabilizer in G of a 3-decomposition of V^k. If F is finite, then by 7.5, Δ is transitive on twisted 3-decompositions of V.

If F contains an element of order 3 then an *exotic E_{81}-subgroup* of G is a subgroup $E \cong E_{81}$ of G such that $Z(G) \leq E$, but E is contained in no Cartan subgroup of G. We find in 8.3 that, if F is finite or algebraically closed, then Δ is transitive on its exotic E_{81}-subgroups.

We now define our set $\mathscr{C}$ of natural structures on V. Define $\mathscr{C}_{\mathrm{ALG}}$ to consist of members of $\mathscr{V}_i$, $i = 1, 2, 3, 5, 6, 10, 12, \mathscr{U}_3, \mathscr{U}_6$, dark points, 3-decompositions, conjugates of $\widehat{X}$ when $|F| > 2$, and exotic E_{81}-subgroups of G when F contains an element of order 3. If F is algebraically closed, let $\mathscr{C} = \mathscr{C}_{\mathrm{ALG}}$. If F is finite, let ${}^2\mathscr{C}_{\mathrm{ALG}}$ consist of members of $\mathscr{U}_9$, twisted special planes, twisted 9-decompositions, and twisted 3-decompositions. Further, set $\mathscr{C} = \mathscr{C}_{\mathrm{ALG}} \cup {}^2\mathscr{C}_{\mathrm{ALG}}$.

The following theorem is perhaps the main result of this paper:

Theorem 3. *Let F be finite or algebraically closed and K a closed solvable subgroup of G such that $K \nleq Z(G)$. Then:*

(1) *$N_\Gamma(K)$ stabilizes some member of $\mathscr{C}$.*

(2) *Δ is transitive on each type of structure in $\mathscr{C}$.*

(3) *Either $B = N_{\mathrm{Aut}(G)}(K)$ acts on $N_G(S)$ for some $S \in \mathscr{C}$ or $N_\Gamma(K)$ acts on some maximal parabolic P of G. For $b \in B - \Gamma$, $P \cap P^b$ is a B-invariant Levi factor of P and P, P^b is conjugate to G_1, $N_G(\Phi(x_6', x_{16}))$.*

2. Root groups

In this section we continue the hypotheses and notation of §1. Denote by $\mathscr{X}$ the set of all conjugates of the root group $\{X(t) : t \in F\}$ (cf. part I, §3). Recall $\mathscr{V}_6$ denotes the set of all conjugates of V_6. Then $\mathscr{V}_6$ is the set of 6-dimensional singular subspaces of V and for $Y \in \mathscr{X}$, $[V, Y] \in \mathscr{V}_6$. Conversely, if $U \in \mathscr{V}_6$, there is a unique member Y of $\mathscr{X}$ with $U = [V, Y]$. Denote this subgroup by $\mathscr{X}(U)$. Further, for $M \leq G$, write $\mathscr{X}(M)$ for $\mathscr{X} \cap M$. Similarly for $U \leq V$, write $\mathscr{V}_6(U)$ for $U \cap \mathscr{V}_6$.

(2.1) *G_6 has five orbits $\mathscr{V}_6^i$, $1 \leq i \leq 5$, on $\mathscr{V}_6$. These may be described as follows:*

(1) $\mathscr{V}_6^1 = \{V_6\}$.

(2) *$\mathscr{V}_6^2$ consists of those $U \in \mathscr{V}_6$ with* $\dim(U \cap V_6) = 3$, $U = \langle V_3, x_{45}, x_{46}, x_{56}\rangle \in \mathscr{V}_6^2$ *and* $U \leq V(V_6)$.

(3) *$\mathscr{V}_6^3$ consists of those $U \in \mathscr{V}_6$ with $U \cap V_6$ a point,* $U = \langle x_1, x_1', x_{26}, x_{36}, x_{46}, x_{56}\rangle \in \mathscr{V}_6^3$ *and $U \cap V(V_6)$ is a hyperplane of U.*

(4) *$\mathscr{V}_6^4$ consists of those $U \in \mathscr{V}$ with* $\dim(U \cap V(V_6)) = 3$, $U = \langle x_{45}, x_{46}, x_{56}, V_3'\rangle \in \mathscr{V}_6^4$ *and* $U \cap V_6 = 0$.

(5) *$\mathscr{V}_6^5$ consists of those $U \in \mathscr{V}_6$ with $U \cap V(V_6) = 0$. $V_6' \in \mathscr{V}_6^5$.*

Proof. By I.1.2, it suffices to show $\mathrm{Wyl}(X)_6$ has five orbits on the set $\mathscr{U}$ of U in $\mathscr{V}_6$ with $U = \langle U \cap X\rangle$. Let $V_6 \neq U \in \mathscr{U}$. By I.3.6.6, $d = \dim(U \cap V_6) \leq 3 \geq \dim(U \cap V_6') = e$. As $\mathrm{Wyl}(X)_6$ induces S_6 on $\{\langle x_i\rangle : 1 \leq i \leq 6\}$, we may take $U \cap V_6 = V_d$ if $d \neq 0$. Let $A = \langle x_{45}, x_{46}, x_{56}\rangle$ and $B = \langle x_{45}, x_{46}, x_{43}\rangle$. If $d = 3$, then by I.3.6.5, V_6 and $V_3 + A$ are the unique members of $\mathscr{U}$ containing V_3. Thus by transitivity of G on $\mathscr{V}_3$ and I.1.2, each member of $\mathscr{V}_3$ generated by members of X is in just two members of $\mathscr{U}$. Further, we may take $d \leq 2$, and then by symmetry take $e \leq 2$. If $d = 2$, then by I.3.6.4, $e = 0$. Thus $\dim(U \cap V(V_6)) \geq 4$, so conjugating in $\mathrm{Wyl}(X)_6$, we may take A or B to be contained in U. As V_6 and $V_3 + A$ are the members of $\mathscr{U}$ containing A, we are done in the former case, and as the two members of $\mathscr{U}$ containing B are $\mathrm{Wyl}(X)_6$-conjugate to the space in (3), we are done in the latter.

For $U = (V_6)^g \in \mathscr{V}_6$, write $\mathscr{V}_6^i(U)$ for $(\mathscr{V}_6^i)^g$. Further, if $Y = \mathscr{X}(U)$, write $\mathscr{X}_i(Y)$ for the set of groups $\mathscr{X}(W)$ as W varies over $\mathscr{V}_6^i(U)$. Write $\mathrm{Op}(U)$ for $\mathscr{V}_6^5(U)$ and $\mathrm{Op}(Y)$ for $\mathscr{X}_5(Y)$.

(2.2) *Let $Z \in \mathscr{X}$ and $U = [V, Z]$. Then:*

(1) *$\mathscr{X}_2(Z) = \mathscr{X}(R(N_G(Z))) - \{Z\}$. Further if $Y \in \mathscr{X}_2(Z)$, then $\langle Z, Y\rangle = Z \times Y$ is partitioned by $\mathscr{X}(ZY)$ and $N_G(ZY)$ acts transitively on $\mathscr{X}(ZY)$.*

(2) *$\mathscr{X}_3(Z)$ consists of those $Y \in \mathscr{X}$ commuting with Z but not contained in $R(N_G(X))$. If $Y \in \mathscr{X}_3(Z)$, then $\langle Z, Y\rangle = Z \times Y$ and $\mathscr{X}(ZY) = \{Z, Y\}$.*

(3) *$\mathscr{X}_4(Z)$ consists of those $Y \in \mathscr{X}$ such that $\langle Z, Y\rangle$ is unipotent but non-abelian. For $Y \in \mathscr{X}_4(Z)$, $[Z, Y] \in \mathscr{X}_2(Z) \cap \mathscr{X}_2(Y)$ and $\langle Z, Y\rangle$ is special with $\langle Z, Y\rangle/[Z, Y] \cong F^2$.*

(4) $\mathrm{Op}(Z)$ *consists of those $Y \in \mathscr{X}$ such that $\langle Z, Y\rangle$ is not unipotent. For $Y \in \mathrm{Op}(Z)$, $\langle Z, Y\rangle$ is a conjugate of $SL_2(\langle x_1, x_1'\rangle)$.*

Proof. Conjugating in G, we may assume $Z \leq L$. Now $\mathscr{X}_i(Z) \cap L$ is nonempty for each i and the structure of $\langle Z, Y_i\rangle$ is visible in L.

(2.3) *G_6 has three orbits on singular points with the following representatives:*

(1) $V_1 \leq V_6$.

(2) $\langle x_{12}\rangle \in V(V_6) - V_6$.

(3) $\langle x_6'\rangle \in V - V(V_6)$, $(x_6')\Delta \cap V_6 = \langle x_6\rangle$.

Proof. Use I.1.2 and argue as in the proof of 2.1.

(2.4) *Let $A = V_6 + V_6'$. For $v = \sum a_i x_i \in V_6$ define $v' = \sum a_i x_i'$. Then:*

(1) *For $a \in F$, define $U(a) = \{x + ax' : x \in V_6\}$. Then $S = \{U(a), V_6' : a \in F\}$ is a partition of the singular vectors of A.*

(2) *If W is a singular subspace of A of dimension at least 2, then either W is contained in a unique member of S or $W = \langle x, x'\rangle$ for some $0 \neq x \in V_6$.*

(3) *$SL_2(\langle x_1, x_1'\rangle)$ is transitive on $\mathscr{V}_6(A)$, and $\mathscr{V}_6(A) = S$.*

Proof. Let $y = r + s'$ be a singular vector in A, where $0 \neq r, s \in V_6$. Notice $r\Delta \cap A = \langle V_6, r'\rangle$. Then r, s', and y are singular, so $s' \in V_6' \cap r\Delta = \langle r'\rangle$. Hence $s = ar$ for some $a \in F^\sharp$ and (1) is established. Further this shows $\langle r, r'\rangle$ and V_6 are the maximal singular subspaces of $r\Delta \cap A$. Therefore, if W is a singular subspace of A through r of dimension at least 2, then $W \leq V_6$ or $W = \langle r, r'\rangle$. Next it is easy to check $U(a)$ is a conjugate of V_6 under K. Hence (1) and the observation of the previous sentence establish (2), and (3) is also evident.

(2.5) *Let T be the set of lines in the dual space V_6^* of V_6, S the set of singular points of V_{15}, and define $\Xi : T \to S$ by*

$$U\Xi = \langle u \wedge v : U = \langle u, v\rangle\rangle,$$

subject to the identification of V_{15} with $V_6^* \wedge V_6^*$ as in § 1. Then:

(1) *Ξ is a bijection.*

(2) *L is transitive on S and T, and Ξ commutes with the actions of $LH(\Delta)$ on S and T.*

(3) *$U\Xi \leq (W\Xi)\Delta$ if and only if $U \cap W \neq 0$.*

(4) *For $x \in V_6$ and $U \in T$, $U\Xi \in x\Delta$ if and only if $x \in \ker(U)$.*

Proof. Recall from I.3.7 that $G_6 = RLH_0$, where $R = R(G_6)$. Now G_6 is transitive on singular points in $V(V_6) - V_6$ by 2.3. So as $V(V_6) = V_6 + V_{15}$ with $[R, V(V_6)] = V_6$ and V_{15} is $LH(\Delta)$-invariant, it follows that LH_0 is transitive on S. As H_0 fixes a singular point in V_{15}, also L is transitive on S.

Thus, as V_{15} is FL-isomorphic to $V_6^* \wedge V_6^*$ via $x_{ij} \mapsto x_i^* \wedge x_j^*$, Ξ is a permutation equivalence of the actions of L on S and T. Further, $H(\Delta)$ acts on $\langle x_1^*, x_2^*\rangle$ and $\langle x_{12}\rangle$, so Ξ is even $LH(\Delta)$-equivariant. Thus (1) and (2) are established. Next from (1) and (2), L is rank 3 on S, so as $\langle x_{12}, x_{13}\rangle$ is singular but $\langle x_{12}, x_{34}\rangle$ is not, (3) holds. Similarly, $x_3 \in x_{12}\Delta$ but $x_1 \notin x_{12}\Delta$, so (4) holds.

(2.6) *Let $A = V_6 + V_6'$ and adopt the notation of 2.4. Let y be a singular vector such that $y\Delta \cap V_6 = \langle x\rangle$ and $y\Delta \cap V_6' = \langle z'\rangle$, for some $z \in V_6$. Then:*

(1) *$y = az + bx' + u$ for some $a, b \in F^\sharp$, and $u \in V_{15} \cap \langle x, z, x', z'\rangle\Delta$.*

(2) *If $\langle z\rangle = \langle x\rangle$, then $az + bx'$ and u are singular. Further, if U is the member of $\mathscr{V}_6(A)$ containing $az+bx'$, we have $y \in V(U)$ but $y \notin V(W)$ for $W \in \mathscr{V}(A) - \{W\}$. Indeed y is conjugate under LH_0 to $x_1 + x_1' + x_{34}$.*

(3) *If $\langle z\rangle \neq \langle x\rangle$, then $az + bx'$ and u are nonsingular, $y \notin V(W)$ for any $W \in \mathscr{V}(A)$, and y is conjugate to $x_1 + x_2' + x_{34} - x_{56}$ under L.*

Proof. Write $y = r+s'+u$, $r, s \in V_6$, $u \in V_{15}$. Then $y \in x\Delta \leq \langle V(V_6), x'\rangle$, so $\langle s\rangle = \langle x\rangle$. Similarly, $\langle z\rangle = \langle r\rangle$. As $r \in V_6 \leq x\Delta$ and $y, s' \in x\Delta$, also $u \in x\Delta$. Thus $u \in x\Delta \cap V_{15} \leq x'\Delta$. Therefore (1) holds.

Suppose $\langle z\rangle = \langle x\rangle$. Then $r + s'$ and y are singular, and $r+s' \in u\Delta$, so u is singular. By 2.5, L is transitive on pairs α, β where α and β are singular points in V_6 and V_{15}, respectively, and $\alpha \in \beta\Delta$. Thus we may take $r = ax_1$, $s = bx_1$, and $u = cx_{34}$. Conjugating in H_0 we may take a, b, c to be all 1. By 2.4, $r+s'$ is in a unique member U of $\mathscr{V}_6(A)$; indeed $U = \{t+t' : t \in V_6\}$. Then visibly $y\Delta \cap U = x_{34}\Delta \cap U = \langle x_i + x_i' : i \neq 3, 4\rangle$, so $y \in V(U)$. On the other hand for $U \neq W \in \mathscr{V}_6(A)$, $y\Delta \cap W = W \cap \langle x_1, x_1'\rangle$ is a point, so $y \notin V(W)$. Hence (2) is established.

So take $\langle z\rangle \neq \langle x\rangle$. Then $r + s'$ is nonsingular while y is singular and $u \in (r+s')\Delta$. Hence u is nonsingular. For $W \in \mathscr{V}_6(A)$, $V(W) = W + V_{15}$, so as the projection of y on A is nonsingular, $y \notin V(W)$. Conjugating in L, we may take $r = x_1$ and $s = x_2$. Let $w = x_{12}$. Then

$$0 = Q_w(y) = Q_w(x_1 + x_2) + Q_w(u) = 1 + Q_w(u),$$

so $Q_w(u) = -1$. Further, $B = \langle x_1, x_2, x_1', x_2'\rangle$ is hyperbolic with $B\Delta$ a nondegenerate 6-dimensional subspace of $\Phi(B)$ with respect to Q_w, and $M = C_L(w) \cap N_L(B)$ induces a subgroup of $O(B\Delta, Q_w)$ containing $\Omega(B\Delta, Q_w)$. So M is transitive on vectors $u \in B\Delta$ with $Q_w(u) = -1$. Thus conjugating in M, we may take $u = x_{34} - x_{56}$, completing the proof.

(2.7) *Let $A = V_6 + V_6'$. Then $N_G(A) = LKH_0$ has four orbits on singular points of V with the following representatives:*

(1) *$V_1 \leq A$, $V_1\Delta \cap A = \langle V_6, x_1'\rangle$.*

(2) *$\langle x_{12}\rangle \leq V_{15}$, $x_{12}\Delta \cap A = \langle x_i, x_i' : i > 2\rangle$.*

(3) *$\langle x\rangle$, where $x = x_1 + x_{34}$ has nontrivial singular projections on A and V_{15}, $x \in V(W)$ for a unique $W \in \mathscr{V}_6(A)$; namely $W = V_6$ is the member of $\mathscr{V}_6(A)$ containing the projection x_1 of x on A, $x\Delta \cap A = \langle x_1, x_2, x_5, x_6, x_1'\rangle$.*

(4) *$\langle z\rangle$, where $z = x_1 + x_2' + x_{34} - x_{56}$ has nonsingular projections on A and V_{15}, $z \notin V(W)$ for any $W \in \mathscr{V}_6(A)$, $z\Delta \cap A = \langle x_1', x_2, x_1 - x_2'\rangle$.*

Proof. From 2.4, KL is transitive on the singular points of A, and L is transitive on the singular points of V_{15}. So we may assume $y = w + u$ is singular with $w \in A^\sharp$ and $u \in (V_{15})^\sharp$. If $y \in V(W_i)$ for distinct $W_i \in \mathscr{V}(A)$, $i = 1, 2$, then $y \in V(W_1) \cap V(W_2) = V_{15}$. Thus without loss, y is not in

$V(V_6)$ or $V(V_6')$. Thus we are in the setup of 2.6, and that lemma completes the proof, together with some straightforward calculations.

(2.8) *Let S be a set of singular points of V, $U = \langle S \rangle$, and $Y \in \mathscr{X}$. Then Y acts on U if and only if for each $s \in S - V([V, Y])$, $s\Delta \cap [V, Y] \leq U$.*

Proof. Notice $[s, Y] = s\Delta \cap [V, Y]$ for each singular s in $V - C_V(Y)$.

3. TWISTED 9-SUBSPACES

In this section we continue the hypotheses and notation of §1.

(3.1) *Adopt the notation of* 2.4. *Let $\bar{l} = \langle y_1, y_2 \rangle$ be a singular line, such that $y\Delta \cap W$ is a point for all $y \in \bar{l}^{\sharp}$ and all $W \in \mathscr{V}_6(A)$. Let π and π' be the projections on V_6 and V_6', respectively, with respect to the decomposition $V = V_6 \oplus V_6' \oplus V_{15}$. Assume $\langle \pi(y_2)' \rangle = \langle \pi'(y_1) \rangle$. Let $l = \langle \pi(y_1), \pi(y_2) \rangle$. Then:*

(1) $l' = \langle \pi'(y_1), \pi'(y_2) \rangle$.

(2) *$B = l + l' + \bar{l}$ is a 6-dimensional subspace of some conjugate of V_{10}, with (B, Q_z) nondegenerate of Witt index 2 for each $z \in V - B\theta$.*

Proof. Conjugating in G and appealing to 2.7, we may take $y_1 = x_2 + x_1' + u_1$ with $x_1, x_2 \in u_1\Delta$ and $u_1 \in V_{15}$. Similarly we may take $y_2 = x_1 + x' + u_2$ for some $x \in V_6$, since $\langle \pi(y_2)' \rangle = \langle \pi'(y_1) \rangle$. Suppose $x \notin \langle x_1, x_2 \rangle$. Then conjugating in L, we may assume $x = x_3$. Now as $\bar{l}$ is singular, $0 = f(x_{23}, y_1, y_2) = 1 + f(x_{23}, u_1, u_2)$, so $f(x_{23}, u_1, u_2) \neq 0$. But $u_1, u_2 \in V_{15} \cap x_1\Delta$ and hence are orthogonal with respect to $f_{x_{23}}$, a contradiction. Therefore (1) is established.

Let $D = l + l'$. By (1), $D = \langle x_1, x_2, x_1', x_2' \rangle$ and $u_1, u_2 \in D\Delta$, so $B \leq \Phi(D)$, a conjugate of V_{10}. Let $z \in V - B\theta$. By hypothesis (and 2.7) for $y \in \bar{l}^{\sharp}$, $\pi(y) + \pi'(y)$ is nonsingular. Thus the projection A of $\bar{l}$ on D is a definite 2-subspace of D with respect to Q_z. Thus the projection of $\bar{l}$ on $D\Delta \cap B = \langle u_1, u_2 \rangle$ is onto, and $\langle u_1, u_2 \rangle$ is also definite. Hence (B, Q_z) is of Witt index 2. So (2) is established.

(3.2) *Let $U \in \mathrm{Op}(W)$ for all $W \in \mathscr{V}_6(A)$. Let $M = \langle \mathscr{X}(V_6), \mathscr{X}(V_6'), \mathscr{X}(U) \rangle$. Then there exists an M-invariant special triple (s_1, s_2, s_3) from V_{15} such that $A + U = B_{12} + B_{31} + B_{32}$, where:*

(1) *$B_{12} = \langle s_1, s_2 \rangle\Delta \cap (A + U) = l_1 + l_2 + l_3$, and $l_1 = V_6 \cap \langle s_1, s_2 \rangle\Delta$, $l_2 = V_6' \cap \langle s_1, s_2 \rangle\Delta$, and $l_3 = U \cap \langle s_1, s_2 \rangle\Delta$ are all lines.*

(2) *For $i = 1, 2$, $B_{3i} = \langle s_3, s_i \rangle\Delta \cap (A + U) = \bigoplus_j (l_j\Delta \cap \langle s_3, s_i \rangle\Delta)$.*

(3) *B_{12} is a nondegenerate 6-dimensional subspace of Witt index 2 with respect to Q_{s_3}.*

(4) *$A + U$ is determined up to conjugation in $N_G(L)$ by the similarity type of (B_{12}, Q_{s_3}).*

Proof. In proving 3.2, we continue the notation of 3.1. Notice that as $U \in \mathrm{Op}(W)$ for all $W \in \mathscr{V}_6(A)$, $\pi : U \to V_6$ is an isomorphism by 2.6. Thus for each $u \in U^\sharp$, the hypotheses of 3.1 are satisfied for the line $\bar{l}(u)$ of U such that $\bar{l}(u) = \langle u, \pi^{-1}(t)\rangle$ and $t' = \pi'(u)$. Hence the relation $u \sim w$ if $\bar{l}(u) = \bar{l}(w)$ is an equivalence relation on $U^\sharp$ by 3.1.1.

For $v \in V_6$, let $l(v) = \bar{l}(\pi^{-1}(v))\pi$. As $\pi : U \to V_6$ is an isomorphism and $\sim$ is an equivalence relation on $U^\sharp$, the relation $v \sim w$ if $l(v) = l(w)$ is an equivalence relation on $V_6^\sharp$.

Let $\bar{l} = \bar{l}(u) \leq U$. Without loss, $l = \pi(\bar{l}) = V_2$. As $\sim$ is an equivalence relation on $V_6^\sharp$, $l(x_3) \cap l = 0$. Hence conjugating in $N_L(l) \cap N_L(\langle x_3\rangle)$, we may take $l(x_3) = \langle x_3, x_4\rangle$. Similarly we may take $l(x_5) = \langle x_5, x_6\rangle$.

Let $C_i = \Phi(l(x_i))$. Then $B \leq C_1$ by 3.1. So for $u \in l'$ or $\bar{l}$, $u\Delta \cap l \neq 0$, so $u\Delta \cap V_6 \leq l$. Thus $[\mathscr{X}(V_6), u] = u\Delta \cap V_6 \leq B$, so $\mathscr{X}(V_6)$ acts on B. Similarly, M acts on B. So M acts on $\Phi(B) = C_1$. Similarly M acts on C_i, for $i = 1, 3, 5$. Then M acts on $C_1 \cap C_3 = \langle x_{56}\rangle$, and then as M is generated by unipotent elements, M fixes x_{56}.

Let $s_1 = x_{12}$, $s_2 = x_{34}$, and $s_3 = x_{56}$. Define B_{ij} and l_i as in the statement of 3.2. By the previous paragraph (s_1, s_2, s_3) is an M-invariant special triple. As $l \leq \langle s_2, s_3\rangle\Delta$ and l' and $\bar{l}$ are M-conjugate to l, $B \leq B_{12}$. By symmetry $\dim(B_{3i}) \geq 6$, for all $i = 1, 2$. So as $B_{12} \cap B_{3i} = 0$ and $\dim(A + U) = 18$, $B = B_{12}$ and $A + U$ is the direct sum of the B_{ij}. In particular (1) and (3) are established. Further, $l\Delta \cap \langle s_3, s_i\rangle\Delta$ is a line for $i = 1, 2$, so as $B_{3i} \cap U \leq l\Delta \cap \langle s_3, s_i\rangle\Delta$ and is of dimension at least 2, we conclude that (2) holds.

Let $\tilde{U} \in \mathscr{V}_6$, such that $\tilde{U} \in \mathrm{Op}(W)$ for all $W \in \mathscr{V}_6(A)$. We have shown there is $g \in N_G(A)$ with $(A+\tilde{U})g = D_{12}+D_{31}+D_{32}$, where $D_{ij} = \langle s_i, s_j\rangle\Delta \cap (A+\tilde{U})$ and $A \cap D_{12} = \langle l, l'\rangle$; without loss $g = 1$. Let N be the stabilizer in $N_G(A)$ of (s_1, s_2, s_3). If (B, Q_{s_1}) is similar to (D_{23}, Q_{s_1}), then D_{23} is conjugate to B in N. Then by (2), $A + U$ is conjugate to $A + \tilde{U}$. Conversely, if $(A + U)y = A + \tilde{U}$ for some $y \in N_G(A)$, then replacing y by yx for suitable $x \in L$, we may assume y fixes each $\langle s_i\rangle$. Hence $By = D_{23}$, and y preserves Q_{s_1} up to a scalar, so (B, Q_{s_1}) is similar to (D_{23}, Q_{s_1}). So (4) is established.

(3.3) *Let* $U \in \mathrm{Op}(W)$ *for all* $W \in \mathscr{V}_6(A)$ *and let* $M = \langle\mathscr{X}(V_6), \mathscr{X}(V_6'), \mathscr{V}_6(U)\rangle$. *Then*:

(1) *There exists a quadratic Galois extension* K *of* F *such that* $C_G(M) \cong SL_3(K)$ *and* $M \cong SU_3(K/F)$.

(2) $V = A \oplus U \oplus C_V(M)$ *with* $C_V(M) = \bigcap_{W \in \mathrm{Op}(A)} V(W) \in \mathscr{U}_9$ *and* $A + U = C_V(M)\theta$.

(3) $A + U$ *is the* K*-tensor product of the natural modules for* M *and* $C_G(M)$ *as a* $KMC_G(M)$*-module.*

(4) $K \otimes_F C_V(M)$ *is isomorphic as a* $C_G(M)K$*-module to* $N \otimes N^\sigma$, *where* N *is the natural* $KC_G(M)$*-module and* $\langle\sigma\rangle = \mathrm{Gal}(K/F)$.

(5) *$N_G(A+U)$ is $MC_G(M)$ extended by a graph automorphism.*

(6) *If F is finite, then each member of $\mathscr{U}_9$ is of the form $C_V(\overline{M})$ for some $\overline{M} = \langle B_1, B_2, B_3\rangle$ with $B_i \in \mathscr{X}$, $B_2 \in \mathrm{Op}(B_1)$, and $B_3 \in \mathrm{Op}(B)$ for all $B \in \mathscr{X}(\langle B_1, B_2\rangle)$.*

Proof. Adopt the notation of the proofs of 3.1 and 3.2. Let C be the subspace of B orthogonal to $l+l'$. Then there exists a quadratic Galois extension K of F such that (C, Q_{s_1}) is similar to (K, N_F^K). Moreover (C, Q_{s_1}) determines the similarity type of (B, Q_{s_1}). Let $\langle\sigma\rangle = \mathrm{Gal}(K/F)$. Consider the K-form $\mathscr{F}^K$ on $V^K = K\otimes_F V$ and regard G as the fixed points of the field automorphism σ on the isometry group G^K of $(V^K, \mathscr{F}^K)$ and V as the fixed points of σ on V^K. There is a 3-decomposition $V^K = A_1\oplus A_2\oplus A_3$ such that σ induces a graph-field automorphism on $M_1 = C_{G^K}(A_1)$ and interchanges $M_2 = C_{G^K}(A_2)$ and $M_2 = C_{G^K}(A_3)$. Thus the stabilizer in G of this 3-decomposition is J_1J_2, where $J_1 = C_{M_1}(\sigma) \cong SU_3(K/F)$ and $J_2 = C_{M_2M_3}(\sigma) \cong SL_3(K)$. Similarly $V = I_1 \oplus I_2$, where $I_1 = C_{A_1}(\sigma) = C_V(J_1)$ is isomorphic to $N\otimes N^\sigma$ as a KJ_2-module and $I_2 = C_{A_2+A_3}(\sigma)$ is isomorphic to the tensor product of the natural modules for J_1 and J_2 as a KJ_1J_2-module. Finally $N_G(J_1J_2)$ is J_1J_2 extended by a graph-field automorphism.

Next J_1 is generated by root groups Y_i, $1\le i\le 3$, of G such that $Y_2 \in \mathrm{Op}(Y_1)$ and $Y_3\in \mathrm{Op}(Y)$ for all root groups Y in $\langle Y_1, Y_2\rangle$. So without loss $[Y_1, V] = V_6$ and $[Y_2, V] = V_6'$. Let $\widetilde{U} = [Y_3, V]$. Then $I_2 = A+\widetilde{U}$. We may assume the special triple (s_1, s_2, s_3) is contained in I_1, so that $\widetilde{B} = I_2\cap\langle s_1, s_2\rangle\Delta$ is of dimension 6 and J_1-invariant. As all irreducible submodules of I_2 are natural modules for J_1, $\widetilde{B}$ is a natural KJ_1-module.

Then J_1 preserves a hermitian symmetric sesquilinar K-form h on $\widetilde{B}$. Further each J_1-invariant quadratic F-form on $\widetilde{B}$ is of the form h_a for some $a\in K^\sharp$, where $h_a(v) = \mathrm{Tr}_F^K(h(av, v))$. Thus, each such form is either of Witt index 3 or of Witt index 2 with the orthogonal complement to a hyperbolic 4-subspace similar to (K, N_F^K). In particular by 3.2, $(\widetilde{B}, Q_{s_1})$ must be of the latter sort, and hence I_1 is $N_G(A)$-conjugate to $A+U$ by 3.2.4.

Finally, if F is finite and $E\in\mathscr{U}_9$, then there is a 3-decomposition $V^K = E_1+E_2+E_3$ with $E^K = E_1$. Now σ preserves the unique 3-decomposition of V^K containing E^K, so either $E_2\sigma = E_3$ or $E_i\sigma = E_i$ for each i. In the former case, (6) holds and we are done by remarks above. In the latter, σ induces a field automorphism on $C_{G^K}(E_i)$ for each i, and then $C_{E_i}(\sigma)\in\mathscr{V}_9$, contrary to the hypothesis.

(3.4) *If F is finite, then G is transitive on $\mathscr{U}_9$.*

Proof. This follows from 3.2.4 and 3.3, since up to isometry there is a unique 6-dimensional orthogonal space over F with Witt index 2.

4. Special planes

In this section $\pi = \langle x_1, x_{16}, x_6'\rangle$ is a special plane. Let $v_1 = x_1$, $v_2 = x_{16}$, and $v_3 = x_6'$. Let $W_i = \langle v_j, v_k\rangle\Delta$ for $\{i, j, k\} = \{1, 2, 3\}$. Then $W = W_1 + W_2 + W_3 = \pi\theta$.

(4.1) *Let $w = \Sigma w_i \in W$ with $w_i \in W_i$. Then w is singular if and only if w_i is singular and $w_i \in w_j\Delta$ for all i, j.*

Proof. This is II.2.2.

(4.2) *Let $w = w_1 + w_2 + w_3$ with $w_i \in W_i$. Then:*

(1) *$y = x_1 + w$ is singular if and only if w_i is singular for all i and one of the following holds:*

(a) *$w_1 = 0$ and $w_2 \in w_3\Delta$. In this case $\langle y, x_1\rangle$ is singular.*

(b) *$l = \langle w_2, w_3\rangle$ is hyperbolic, $0 \neq \langle w_1\rangle = l\Delta \cap W_1$, and $Q_v(x_1 + w_1) = -Q_v(w_2 + w_3)$ for $v \in V - l\theta$.*

(2) *$C = C_G(\pi)$ has three orbits on points of type* (1a) *with representatives $\langle x_6\rangle$, $\langle x_2\rangle$, and $\langle x_6 + x_2\rangle$.*

(3) *C is transitive on points of type* (1b) *with representative satisfying $w_1 = x_2'$, $w_2 = x_1'$, $w_3 = x_2$.*

(4) *In any event* $\mathrm{Rad}(\langle y, \pi\rangle) = \langle w\rangle$, *so $\langle y, v_1, v_2\rangle$ and $\langle y, v_1, v_3\rangle$ are not special.*

Proof. Assume y is singular. $Q_{x_1}(y) = Q_{x_1}(w_1)$, so w_1 is singular. $Q_{v_i}(y) = Q_{v_i}(x_1 + w_i)$ for $i \neq 1$, and $w_i \in x_1\Delta$, so w_i is singular. If $w_1 = 0$, then as $w_2 + w_3 \in x_1\Delta$, $w_2 + w_3$ is singular, so $w_2 \in w_3\Delta$. So assume $w_1 \neq 0$.

If $w_2 \in w_3\Delta$, then $w_2 + w_3$ is singular in $y + \Phi(k)$, where $k = \langle x_1, w_1\rangle$. Thus by I.7.4, $y \in \Phi(k)$. But $\Phi(k) \cap (W_2 + W_3) \leq w_1\Delta$, impossible as $w_1 + x_1$ is nonsingular and hence not in $w_1\Delta$.

As $w_2 + w_3 \in x_1\Delta$, $0 = Q_u(y) = Q_u(w_2 + w_3) = f(u, w_2, w_3)$ for each $u \in W_1 \cap w_1\Delta$. But as $l = \langle w_2, w_3\rangle$ is hyperbolic, $W_1 \cap l\theta$ is a hyperplane of W_1. So $W_1 \cap l\theta = W_1 \cap w_1\Delta$, and hence (4) holds.

The proofs of (2) and (3) are easy. Finally in either case $0 = f(y, y, v_i) = f(x_1 + w, x_1 + w, v_i) = f(w, w, v_i)$, and $0 = Q_{v_i}(y) = Q_{v_i}(w)$, so $\langle w\rangle = \mathrm{Rad}(\langle y, \pi\rangle)$.

(4.3) *Let $w = w_1 + w_2 + w_3$ with $w_i \in W_i$. Then:*

(1) *If $y = v_1 + v_2 + w$ is singular, then $\langle w_1, w_2\rangle$ and $\langle w_1 + w_2, v_1 + v_2 + w_3\rangle$ are singular with $Q_{v_3}(w_3) = -1$.*

(2) *$C_G(\pi)$ has two orbits on such singular points with representatives satisfying $w_1 = w_2 = 0$, $w_3 = x_2 - x_{26}$ and $w_1 = x_2'$, $w_2 = x_1'$, $w_3 = x_2 - x_{26}$, respectively.*

(3) $\mathrm{Rad}(\langle y, \pi\rangle) = 0$.

(4) *$\langle v_1, v_3, y\rangle$ and $\langle v_2, v_3, y\rangle$ are special but $\langle v_1, v_2, y\rangle$ is not.*

Proof. Assume y is singular. Then $0 = Q_{v_3}(y) = Q_{v_3}(v_1 + v_2 + w_3) = 1 + Q_{v_3}(w_3)$. Hence $z = v_1 + v_2 + w_3$ is singular.

Similarly for $i = 1, 2$, $Q_{v_i}(y) = Q_{v_i}(w_i)$, so w_i is singular. Also for $u \in W_3 \cap w_3\Delta$, $0 = Q_u(y) = Q_u(w_1 + w_2) = f(u, w_1, w_2)$. Thus if $\langle w_1, w_2 \rangle$ is hyperbolic, then as in the proof of 13.2, w_3 is singular, a contradiction. So $w_1 + w_2 = y - z$, y and z are singular. Hence $\langle y, z \rangle$ is singular. So (1) is established.

As $C = C_G(\pi)$ is transitive on vectors w_3 in W_3 with $Q_{v_3}(w_3) = -1$, C is transitive on the set of singular points of type $\langle y \rangle$ with $w_1 = w_2 = 0$. So assume $\langle y \rangle$ is in the set S of such points with $w_1 \neq 0$. As

$$w_1\Delta = \langle v_2, v_3 \rangle \oplus \bigoplus_{i=1}^{3} (W_i \cap w_1\Delta),$$

but $z \in (w_1 + w_2)\Delta$, we have $w_2 \neq 0$. Conjugating in C, we may take $w_1 = x_2'$ and $w_2 = x_1'$. Let $e = w_1 + w_2$. Then

$$e\Delta = \langle V_6', x_{ij}, x_{1i} - x_{2i}, x_1 + x_2 : i, j \neq 1, 2 \rangle .$$

Then $z = (v_2 - x_{26}) + (v_1 + x_2) + (x_{26} - x_2) + w_3$ with the first two terms in $e\Delta$. Thus $(x_{26} - x_2) + w_3 \in W_3 \cap e\Delta = A = \langle x_{36}, x_{46}, x_{56} \rangle$, so $w_3 \in (x_2 - x_{26}) + A$. As C_e is transitive on this coset, (2) is established.

Next $Q_{v_3}(w) = Q_{v_3}(w_3) = -1$, so $\mathrm{Rad}(\langle y, \pi \rangle) = 0$. Part (4) is easily checked.

(4.4) *Let* $w = w_1 + w_2 + w_3$ *with* $w_i \in W_i$ *and* $a, b \neq 0$. *Then*:

(1) $y = v_1 + av_2 + bv_3 + w$ *is singular if and only if* $Q_{v_1}(w_1) = -ab$, $Q_{v_2}(w_2) = -b$, $Q_{v_3}(w_3) = -a$, $f(w_1, w_2, w_3) = 2ab$, *and* $\langle w_3 \rangle = A\Delta \cap W_3$, *where*

$$A = \{u \in W_3 : f(u, w_1, w_2) = 0\} .$$

(2) $C_G(\pi)$ *is transitive on the set of singular points* $\langle y \rangle$ *of* V *of this form. Further one such point has* $w_1 = x_2' - abx_{12}$, $w_2 = x_1' + bx_6$, *and* $w_3 = x_2 - ax_{26}$.

(3) $\langle y, v_i, v_j \rangle$ *is special for each* $i \neq j$.

(4) $\langle v_i \rangle$, $1 \leq i \leq 3$, *and* $\langle y \rangle$ *are the singular points in* $\langle y, \pi \rangle$.

Proof. Assume y is singular. Let a_i denote the coefficient of v_i in y. Then $0 = Q_{v_i}(y) = a_j a_k + Q_{v_i}(w_i)$. In particular w_i is nonsingular. If $u \in D = W_3 \cap w_3\Delta$, then $0 = Q_u(y) = f(u, w_1, w_2)$. So $D \leq A$. But as w_i is nonsingular, conjugating in $C = C_G(\pi)$, we may take $w_1 = x_2' - abx_{12}$ and $w_2 = x_1' + bx_6$. Then we find $A = W_3 \cap d\Delta$ is a hyperplane of W_3, where $d = x_2 - ax_{26}$. It follows that $w_3 = \varepsilon d$, $\varepsilon = +1$ or -1.

Next for each i, $0 = Q_{w_i}(y) = f(a_i v_i, w_i, w_i) + f(w_1, w_2, w_3) = -2ab + f(w_1, w_2, w_3)$, while $f(w_1, w_2, \varepsilon d) = 2\varepsilon ab$. Thus $\varepsilon = +1$, and (1) and (2) hold. Also (3) is easy to check.

Notice that as each w_i is nonsingular, the first three lemmas in this section show that if $\langle s \rangle$ is a singular point in $\langle y, \pi \rangle$ not in $\langle \pi \rangle$, then the projection of s on $\langle v_i \rangle$ is nontrivial for each i. Then (2) and an easy calculation imply (4).

(4.5) *Let* $y = v_1 + v_2 + v_3 + w$, *where* $w = w_1 + w_2 + w_3$ *and* $w_1 = x_2' - x_{12}$, $w_2 = x_1' + x_6$, *and* $w_3 = x_2 - x_{26}$. *Let* $U = \langle \pi, y \rangle$. *Then*:

(1) Δ *is transitive on ordered special* 4-*tuples of points.*

(2) $(\langle v_1 \rangle, \langle v_2 \rangle, \langle v_3 \rangle, \langle y \rangle)$ *is a special* 4-*tuple of points.*

(3) *The singular points in* U *are* $\langle v_i \rangle$ *and* $\langle y \rangle$.

(4) $U\theta = \langle w_i \Delta \cap (\pi\theta) : 1 \leq i \leq 3 \rangle$ *is of dimension* 21.

(5) *Let* $\pi(w) = \langle w_1, w_2, w_3, \pi \rangle$. *Then* $\pi(w)\theta = U\theta$, *and if* $\mathrm{char}(F) \neq 2$ *then* $V = \pi(w) \oplus U\theta$.

(6) $C_G(U) = C_G(\pi(w))$.

(7) *The singular points in* $\pi(w)$ *are* $\langle v_{1\sigma} \rangle$, $\langle v_{1\sigma} + a^2 v_{2\sigma} + a w_{3\sigma} \rangle$, *and* $\langle v_1 + a^2 v_2 + b^2 v_3 + ab w_1 + b w_2 + a w_3 \rangle$, $a, b \in F^\sharp$, $\sigma \in \mathrm{Sym}(\{1, 2, 3\})$.

Proof. First (2) follows from 4.4.3. Next, as G is transitive on special 3-tuples, each special 4-tuple of points is conjugate to $(\langle v_1 \rangle, \langle v_2 \rangle, \langle v_3 \rangle, \langle v_4 \rangle)$ for some singular v_4. By earlier results in this section, $v_4 = v_1 + a v_2 + b v_3 + w'$ for some $w' \in W$. As $H(\Delta)$ induces a full diagonal group on π, conjugating in $H(\Delta)$, we may take $a = b = 1$. Then by 4.4, $\langle v_4 \rangle$ is conjugate to $\langle y \rangle$ under $C_G(\pi)$. So (1) is established. Notice (3) is just a restatement of 4.4.4.

Next $U\theta = \bigcap_{i=1}^3 A_i$ where $A_i = \{w \in W : f(w, y, v_i) = 0\}$. Further $A_1 = \{w \in W : f(w, w_1, v_1) = 0\} = (w_1 \Delta \cap W_1) + W_2 + W_3$, so (4) holds.

Of course $\pi(w)\theta \leq U\theta$, as $U \leq \pi(w)$. Further, $U\theta \leq w_i\theta$ for each i and $w_i \in v_j \Delta$ for $i \neq j$, so $\langle w_i, v_j \rangle\theta = w_i\theta$. We saw in the previous paragraph that $U\theta \leq A_i \leq (w_i, v_i)\theta \cap w_i\theta = \langle w_i, v_i \rangle\theta$. Next by 4.4.1, $(w_3 \Delta \cap W) \leq \langle w_1, w_2 \rangle\theta$, while of course $(w_i)\Delta \cap W \leq \langle w_i, w_j \rangle\theta$. Thus $U\theta \leq \langle w_1, w_2 \rangle\theta$ by (4). Finally by symmetry, $U\theta \leq \langle w_i, w_j \rangle\theta$ for all i, j. Hence $U\theta \leq \pi(w)\theta$, so (5) holds.

$C_G(U)$ acts on $W_i = \langle v_j, v_k \rangle\Delta$, and hence fixes the projection w_i of y on W_i. Thus (6) holds. Finally (7) follows from previous lemmas in this section.

5. CERTAIN 6-DIMENSIONAL SUBSPACES OF V

In this section the hypotheses and notation of §4 are continued. In addition let

$$w_1 = x_2' - x_{12}, \quad w_2 = x_1' + x_6, \quad w_3 = x_2 - x_{26},$$

and $\pi(w) = \langle \pi, w_1, w_2, w_3 \rangle$. Let α be the element of $GL(V)$ such that

$$\alpha = (x_1, x_6', x_{16})(x_1', x_{21}, x_2)(x_2', x_{62}, x_6)(x_3', -x_3, x_{45})(x_{13}, x_{63}, x_{23}) \cdot (x_4', -x_4, x_{53})(x_{14}, x_{64}, x_{24})(x_5', -x_5, x_{34})(x_{15}, x_{65}, x_{25}).$$

In §6 of [3] it is shown that $\alpha \in G$. Evidently α acts on π and α has cycle (w_1, w_3, w_2). So α also acts on $\pi(w)$.

Indeed let $\Omega = N_{O(V,f)}(\pi)^\infty$. Then $\Omega \cong \mathrm{Spin}_8^+(F)$ acts naturally on W_1 and α induces a triality automorphism on Ω, so W_2 and W_3 are the two conjugates of the natural module under triality. It turns out $C_\Omega(\alpha) \cong G_2(F)$, a fact we will prove in a moment.

The next lemma follows from 6.2, 6.3, and 6.4 in [3].

(5.1) (1) *Let* $\varepsilon \in GL(V)$ *centralize* v_i *and have cycles*

$$(x_2', x_{21})(x_{62}, x_2)(x_6, x_1')(x_3', x_{13})(-x_3, x_{63})(x_{45}, x_{23}) \cdot (x_4', x_{14})(-x_4, x_{64})(x_{53}, x_{24})(x_5', x_{15})(-x_5, x_{65})(x_{34}, x_{25}).$$

Then $\varepsilon \in C_G(\alpha)$ *and* ε *fixes each* w_i.

(2) *Let* $\gamma, \delta \in GL(V)$ *fix* $v_i, x_1', x_{21}, x_2, x_2', x_{62}$, *and* x_6, *let* γ *have cycles*

$$(x_3', x_4', x_5')(x_3, x_4, x_5)(x_{45}, x_{53}, x_{34}) \cdot (x_{13}, x_{14}, x_{15})(x_{63}, x_{64}, x_{65})(x_{23}, x_{24}, x_{25}),$$

let δ *invert* $x_3, x_3', x_{13}, x_{23}, x_{36}$, *and* x_{45}, *and let* δ *have cycles*

$$(x_4', -x_5')(x_4, -x_5)(x_{14}, -x_{15})(x_{24}, -x_{25})(x_{46}, -x_{56})(x_{34}, x_{35}).$$

Then $\langle\gamma, \delta\rangle \leq C_G(\langle\alpha, \varepsilon\rangle)$ *and* δ *inverts* γ.

(3) *The group* $\langle\varepsilon, \gamma, \delta\rangle$ *is isomorphic to* $Z_2 \times S_3$, *is transitive on*

$$\{\langle x_i'\rangle, \langle x_{1i}\rangle : 3 \leq i \leq 5\},$$

and fixes w_1 *and* $\langle x_2' + x_{12}\rangle$.

Now let $X_1 = \{x_2' + x_{12}, x_i', x_{1i} : 3 \leq i \leq 5\}$, $U_1 = \langle X_1\rangle$, *and define a 3-linear form* h *on* U_1 *by* $h(x, y, z) = f(x, y\alpha, z\alpha^2)$.

(5.2) *The form* h *is an alternating trilinear form on* U_1.

Proof. As f is symmetric and α preserves f, h is symmetric with respect to each odd permutation of its three indices. So it suffices to show $h(x, y, y) = 0$ for all $x \in X_1$ and $y \in U_1$. Hence by 5.1.3, it suffices to check this equation for $x = x_2' + x_{12}$ and $x = x_3'$. But an easy calculation shows $h'_{x_2+x_{12}} = h(x_2' + x_{12}, *, *)$ and h_{x_3} are alternating and indeed:

(5.3) (1) $h'_{x_2+x_{12}} = \sum_{i=3}^{5} x_i' x_{1i}$.

(2) $h'_{x_3} = x_{13}(x_2' + x_{12}) + x_5' x_4'$.

Now let β be the element denoted by θ in §6 of [3]. Then a calculation shows β fixes $x_1', x_{2\sigma}$, and $x_{1,2\sigma}$, and β has cycles

$$(x_{2\sigma,3}, x_{2\sigma,4}, -x_{2\sigma,3}, -x_{2\sigma,4})(x_{2\sigma}', x_{5\sigma,6\sigma}, -x_{2\sigma}', -x_{5\sigma,6\sigma})(x_3, x_{13}) \cdot (x_4, x_{14})(x_1, x_{34}, -x_1, -x_{34})$$

for $\sigma \in \langle(2, 5, 6)\rangle$, the subgroup of odd permutations in the symmetric group on $\{2, 5, 6\}$.

Let $Z = V_6\beta$ and $Z' = V_6'\beta$. Then we have root groups $\mathscr{X}(Z) = \{X_Z(t) : t \in F\}$ and $\mathscr{X}(Z') = \{X_{Z'}(t) : t \in F\}$ with $K_Z = \langle\mathscr{X}(Z), \mathscr{X}(Z')\rangle \cong SL_2(F)$. Recall $X_{Z'}(t)$ fixes each element of Z' and $V_{15}\beta$, while $X_{Z'}(t) : x_i\beta \mapsto x_i\beta + tx_i'\beta$.

We calculate that $\dim(Z\alpha^r \cap Z\alpha^s) = 1$ for $\alpha^r \neq \alpha^s$, so by 2.1 and 2.2, $[K_{Z\alpha^r}, K_{Z\alpha^s}] = 1$. Thus if we define

$$g(t) = X_Z(t)X_{Z\alpha}(t)X_{Z\alpha}2(t), \qquad g'(t) = X_{Z'}(t)X_{Z'\alpha}(t)X_{Z'\alpha^2}(t),$$

then $Y = \{g(t) : t \in F\} \cong \mathscr{X}(Z) \cong F$ and $K' = \langle Y, Y' \rangle \cong (S)L_2(F)$ with $K' \leq C_G(\alpha)$. It is easy to check that $g(t)$ and $g'(t)$ fix x_1 and w_1, so as K' centralizes α, K' centralizes $\pi(w)$.

Next by construction, $C = C_G(\langle \alpha, \pi(w) \rangle)$ acts on U_1 and preserves h. Also C preserves the quadratic form $Q_1 = Q_{x_1}$ on U_1. By 5.3, h is the $G_2(F)$ alternating trilinear form on U_1 and Q_1 is its associated quadratic form, described in §2 of [2]. By 2.11 in [2], $O(U_1, h, Q_1) = \langle K', \gamma, \delta, \varepsilon \rangle$, and then by 3.4 in [2], $O(U_1, h, Q_1) \cong G_2(F)$. We summarize these results in:

(5.4) $C_G(\langle \alpha, \pi(w) \rangle) = \langle K', \gamma, \delta, \varepsilon \rangle = O(U_1, h, Q_1) \cong G_2(F)$.

(5.5) *Let* $M = C_G(\langle \alpha, \pi(w) \rangle)$, $y = v_1 + v_2 + v_3 + w$. *Then*:

(1) $C_V(C_G(\langle \pi, y \rangle)) = \pi(w)$.

(2) *If* $\mathrm{char}(F) \neq 2$, *then* $V = \pi(w) \oplus \pi(w)\theta$, $\pi(w) = C_V(M)$, *and* $\pi(w)\theta$ *is the direct sum of the three natural isomorphic* FM*-modules* U_1, $U_1\alpha$, *and* $U_1\alpha^2$.

(3) *If* $\mathrm{char}(F) = 2$, *then* $V = \pi(w) + W$ *with* $\pi(w) = C_V(M)$, $\pi(w) \cap W = \langle w_1, w_2, w_3 \rangle$, *and* W *is the direct sum of the three isomorphic indecomposable* FM*-modules* W_1, W_2, *and* W_3.

Proof. By 4.5.6, $C_G(\langle \pi, y \rangle) = C_G(\pi(w))$, so as $M \leq C_G(\pi(w))$, (1) holds as $C_V(M) = \pi(w)$. If $\mathrm{char}(F) \neq 2$, then by 4.5.5, $V = \pi(w) \oplus \pi(w)\theta$, and by 4.5.4, $\pi(w)\theta = U_1 \oplus U_1\alpha \oplus U_1\alpha^2$. Thus (2) holds. Part (3) is similar.

(5.6) *For* $a, b \in F^\sharp$, *let* $k(a, b) = l(a_1, \ldots, a_6)h(t)$, *where* $a = a_i$, $2 \leq i \leq 5$, $b = a_1$, $t = ba^2$, *and* $a_6 = (ba^4)^{-1}$. *Then*:

(1) $k(a, b) = aI$ *on* W_1, bI *on* W_2, *and* $(ab)^{-1}I$ *on* W_3.

(2) $v_1k(a, b) = a^{-2}v_1$, $v_2k(a, b) = b^{-2}v_2$, *and* $v_3k(a, b) = (ab)^2v_3$.

(3) $C_{H_0}(\pi(w)) = \{l(a_1, \ldots, a_6)h(t) : a_1 = a_2 = a_6 = t = 1\}$.

(4) *Let* $H(\pi(w)) = \{k(a, b) : a, b \in F^\sharp\}$. *Then*

$$N_{H_0}(\pi(w)) \cap \bigcap_{i=1}^{3} N_G(\langle v_i \rangle) = C_{H_0}(\pi(w)) \times H(\pi(w))$$
$$= \{l(a_1, \ldots, a_6)h(t) : a_1a_2^2 = t, a_1 = t^2a_6\}.$$

(5) $H(\pi(w))$ *is the subgroup of* G *inducing scalar action on* $\langle v_i \rangle$ *and* $U_1\alpha^i$, $1 \leq i \leq 3$.

Proof. Parts (1) and (2) are straightforward calculations; see §3 of part I for the definitions of $l(a)$ and $h(t)$. Further the subgroup of G fixes each $\langle v_i \rangle$ and $\pi(w)$ fixes each $\langle w_i \rangle$. Then (3) is easy and (3) and I.3.12 imply (4).

(5.7) (1) *$N_\Delta(\pi(w))$ is transitive on ordered special 4-tuples of points contained in $\pi(w)$.*

(2) *If U and Ug are special 4-subspaces of $\pi(w)$ for some $g \in \Delta$, then $g \in N_\Delta(\pi(w))$.*

(3) *Let $\mathscr{P}$ be the set of singular points in $\pi(w)$ and for distinct $C, D \in \mathscr{P}$ let $\mathscr{L}(C, D) = \mathscr{P} \cap \Phi(\langle C, D\rangle)$. Let $\mathscr{L}$ be the set of all $\mathscr{L}(C, D)$, C, D distinct in $\mathscr{P}$. Then $(\mathscr{P}, \mathscr{L})$ is a projective plane over F.*

(4) *Let $M = C_G(\langle \alpha, \pi(w)\rangle)$. Then $M = C_G(\pi(w))$ and $C_G(M) \cong SL_3(F)$ acts as $PSL_3(F)$ on $(\mathscr{P}, \mathscr{L})$. $H(\pi(w))$ is the Cartan subgroup of $C_G(M)$ fixing $\langle v_i\rangle$, $1 \le i \le 3$.*

(5) *A triple (A, C, D) from $\mathscr{P}$ is special if and only if $\{A, C, D\}$ is linearly independent in $(\mathscr{P}, \mathscr{L})$.*

(6) *$N_G(\pi(w)) = C_G(M) \times M \cong SL_3(F) \times G_2(F)$.*

(7) *If $\mathrm{char}(F) \neq 2$, then $C_G(M)$ is irreducible on $\pi(w)$ and $N_G(\pi(w))$ is irreducible on $\pi(w)\theta$.*

(8) *If $\mathrm{char}(F) = 2$, then $C_G(M)$ is irreducible on $\langle w_1, w_2, w_3\rangle = \pi(w)_0$ and on $\pi(w)/\pi(w)_0$. Further $N_G(\pi(w))$ is indecomposable on $\pi(w)\theta$.*

Proof. First $\pi(w)\theta \le U\theta$ for each special 4-subspace U of $\pi(w)$, so by 4.5, (1) and (5), $U\theta = \pi(w)\theta$. By 4.5.1, $U = \langle \pi, y\rangle d$ for some $d \in \Delta$. By 5.5, $\pi(w)d = C_V(C_G(U)) \le C_V(M) = \pi(w)$, so $d \in N(\pi(w))$. Thus (1) and (2) hold.

From 4.5.7, each pair of points in $\mathscr{P}$ is contained in some special 4-tuple of $\pi(w)$, so $Y = N_\Delta(\pi(w))$ is 2-transitive on $\mathscr{P}$ by (1). Similarly, we find each triple of points not contained in a member of $\mathscr{L}$ is in some special 4-tuple of $\pi(w)$, so Y is transitive on such triples and (5) holds. Finally with 4.5.7, $H(\pi(w))$ is regular on the set T of points of $\mathscr{P}$ not in $\mathscr{L}(v_i, v_j)$ for any $i \neq j$, so Y is transitive on pairs (C, D), (R, S) with $R, S \notin \mathscr{L}(C, D)$ and $S \notin \mathscr{L}(C, R), \mathscr{L}(D, R)$. So as $\mathscr{L}(v_1, v_2) \cap \mathscr{L}(v_3, y)$ contains a point, it follows that distinct members of $\mathscr{L}$ intersect in a point. Now (3) is easy. Moreover we see $PSL_3(F)$ is induced on $(\mathscr{P}, \mathscr{L})$ by $I = \langle H(\pi(w))^Y\rangle$.

Let $X = C_G(\pi(w))$. Then X acts on W_i, so by 5.6.1, $[X, H(\pi(w))] \le C_X(W) = C_G(V) = 1$. Thus, $I \le C_G(X) \le C_G(M)$. But from the description in 5.5 of the action of M on V, $C_{GL(\pi(w)\theta)}(M) \cong GL_3(F)$, so as I induces $PSL_3(F)$ on $(\mathscr{P}, \mathscr{L})$, $SL_3(F) \le I \le C_G(X) \le C_G(M) \le GL_3(F)$. Further by 5.6.5, $H(\pi(w))$ is a Cartan subgroup of $C_G(M)$, so $SL_3(F) \cong I = C_G(X) = C_G(M)$. Then $\alpha \in C_G(M) = I$, so $X \le C_G(\alpha) = M$. Finally by 5.6.4, $Y = XM$, so (4) and (6) are established. Parts (7) and (8) are now easy, given 5.5.

(5.8) (1) *Each member of $\mathscr{U}_6$ is G-conjugate to $\pi(w)$.*

(2) *If (v_1, v_2, v_3) is a special triple, u_i is a nonsingular vector in $\langle v_1, v_i\rangle\Delta$ for $i = 2, 3$, and $\langle u_1\rangle = A\Delta \cap \langle v_2, v_3\rangle\Delta$, where*

$$A = \{u \in \langle v_2, v_3\rangle\Delta : f(u, u_2, u_3) = 0\},$$

then $\langle v_1, v_2, v_3, u_1, u_2, u_3\rangle \in \mathscr{U}_6$.

Proof. Recall the definition of $\mathscr{U}_6$ in §1. Observe 4.5.1 implies (1), while 4.4.1 and 4.4.2 imply (2).

6. SEMISIMPLE SUBGROUPS

In this section K is an abelian subgroup of G such that V is a semisimple FK-module. Denote by $\mathscr{P}$ the set of pairs (P, U) such that $U \in \mathscr{V}_{10}$, P is a singular point, and $P \nleq U\theta$. Define F to be *n-good* if there exist no definite n-dimensional orthogonal spaces over F.

(6.1) (1) *If K fixes $U \in \mathscr{V}_{10}$, then K fixes a member of $\mathscr{P}$ containing U.*
(2) *If K fixes a singular point P, then K fixes a member of $\mathscr{P}$ containing P.*

Proof. Assume K fixes $U \in \mathscr{V}_{10}$. As K is semisimple, K fixes a point P with $V = U\theta + P$. As there exists a unique singular point in $(U + P) - U$, we may take P singular. Hence (1) holds.

Part (2) is the dual of (1). That is $K\xi$ fixes the unique member $P\phi\Phi$ of $\mathscr{V}_{10}$ in $P\phi$, and hence by (1) a singular point Q with $(Q, P\phi\Phi) \in \mathscr{P}$. Then K fixes $Q\phi\Phi$ and $(P, Q\phi\Phi) \in \mathscr{P}$.

(6.2) *If $(P, U) \in \mathscr{P}$ and S is a hyperbolic basis for (U, Q_U), then the pointwise stabilizer in G of $\{P, \langle s\rangle : s \in S\}$ is a Cartan subgroup of G.*

Proof. This follows from I.3.12.

(6.3) *Assume all eigenvalues of elements of K are in F. Then*:
(1) *If $K^3 \neq 1$, then K fixes an element of $\mathscr{P}$.*
(2) *If K fixes a member of $\mathscr{P}$, then K^2 is contained in a Cartan subgroup of G.*
(3) *If* char$(F) \neq 2$ *and F is* 4*-good, then each involution in G acts nontrivially on some singular point of V.*
(4) *Assume* char$(F) \neq 2$ *and $K^2 = 1$. Then either K is contained in a Cartan subgroup of G or K centralizes some special plane π and each singular point of V fixed by K, and K fixes no singular point in $\pi\theta$.*

Proof. As K is abelian semisimple with all eigenvalues in F, $V = \bigoplus_{\lambda\in\Lambda} V_\lambda$ for some $\Lambda \subseteq \mathrm{Hom}(K, F)$. Observe that if $\lambda \in \Lambda$ with $\lambda^3 \neq 1$, then V_λ is brilliant. Thus K fixes a brilliant point $B \leq V_\lambda$. If B is singular, then K fixes some member of $\mathscr{P}$ by 6.1. If B is nonsingular, then K fixes $\Phi(B)$ and then again by 6.1, K fixes a member of $\mathscr{P}$. So in any event (1) is established.

Assume K fixes $(P, U) \in \mathscr{P}$. Then $P \leq V_\beta$, for some $\beta \in \Lambda$ and $U = \bigoplus_{\alpha\in A} U_\alpha$ for some $A \subseteq \Lambda$. Further, either U_α is singular or $\beta = \alpha^{-2}$. Indeed if $\beta \neq \alpha^{-2}$, then $U_\alpha + U_{(\alpha\beta)^{-1}}$ is hyperbolic. Hence by 6.2, either K is contained in a Cartan subgroup or there exists $\alpha \in A$ with $\beta = \alpha^{-2}$. We may assume the latter.

Let A_0 be the set of weights α with $\beta = \alpha^{-2}$. If $A_0 = \{\alpha\}$, then as U and $U_\gamma + U_{(\gamma\beta)^{-1}}$ are hyperbolic for $\gamma \notin A_0$, we conclude U_α is hyperbolic. But then K is contained in a Cartan subgroup by 6.2. In general, if $\alpha, \lambda \in A_0$, then for $k \in K$, $\alpha(k^2) = \alpha(k)^2 = \beta^{-1}(k) = \lambda(k)^2 = \lambda(k^2)$. Thus, $A_0(K^2)$ is of order one, so (2) is established.

To prove (3) and (4) we may assume $\mathrm{char}(F) \neq 2$ and $K^2 = 1$. By (1), K fixes $(P, U) \in \mathscr{P}$. Suppose some $t \in K$ inverts P. Then $[U, t]$ and $C_U(t)$ are singular by a remark above, and hence each is of rank $\dim(U)/2 = 5$. So U_α is singular for all $\alpha \in A$, and hence K is contained in a Cartan subgroup. Hence we may assume K fixes each singular point it fixes.

Claim K fixes a singular point of U. For K acts on $P\Delta$, so either $[P\Delta, K] = 0$ or K is nontrivial on some point Q in $P\Delta$. In the latter case Q is nonsingular by assumption. In the former K fixes some nonsingular point Q of $P\Delta$. So in any event K fixes some nonsingular point Q in $P\Delta$. Now K acts on $U \cap \Phi(Q)$, which is nontrivial and singular by I.7.3, establishing the claim.

As K fixes a singular point Q of U, it also fixes a singular point R in $U - Q\Delta$. Then $P + Q + R$ is a special plane centralized by K, which we may take to be the plane π of §4. Define W_i, $1 \leq i \leq 3$, as in §4.

If K takes some singular point S_1 in W_1, then K acts on $W_i \cap S_1\Delta$ for $i = 2, 3$. Thus K centralizes $W_i \cap S_1\Delta$, as that space is singular by II.2.1. Similarly for $S \in \mathscr{V}_1(W_2 \cap S_1\Delta)$, K centralizes $W_1 \cap S\Delta$. But from II.2.1, $S_1\Delta \cap W_1 = \langle S\Delta \cap W_1 : S \in \mathscr{V}_1(W_2 \cap S_1\Delta)\rangle$, so K centralizes W_1. Thus K centralizes W_i for each i, so $K = 1$.

Thus we may assume K fixes no singular points of W_i. Then by 4.1, K fixes no singular point in $W = \pi\theta$. So (4) is established and it remains to prove (3). Thus we may take $K = \langle t\rangle$ of order 2 and assume F is 4-good. Now $W_1 = C_{W_1}(t) \oplus [W_1, t]$. As t fixes no singular point of W, $C_{W_1}(t)$ and $[W_1, t]$ are definite with respect to Q_{v_1}. This is impossible as F is 4-good and $\dim(W_1) = 8$. So (3) is established.

If $\mathrm{char}(F) \neq 2$ we distinguish two classes I_{12} and I_{16} of involutions of G. Namely, I_{12} consists of those involutions t of G such that $[V, t] = A + B$ and $C_V(t) = V(A) \cap V(B)$ for some $A \in \mathscr{V}_6$ and $B \in \mathrm{Op}(A)$. Thus $\dim([V, t]) = 12$, for $t \in I_{12}$. Next I_{16} consists of those involutions t of G such that for some $(P, U) \in \mathscr{P}$, $C_V(t) = P + U$ and $[V, t] = P\Delta \cap U\theta$. Define $c(t) = P$ and $\Phi(t) = U$ for such an involution t. Notice $\dim([V, t]) = 16$ for $t \in I_{16}$.

(6.4) *Let* $\mathrm{char}(F) \neq 2$ *and* Y *be an elementary abelian* 2*-subgroup of* H_0. *Then* $Y \leq E = O_2(\mathrm{Wyl}(X))$ *and identifying* E *with the* 6*-dimensional orthogonal space over* $GF(2)$ *of Witt index* 2, *we have*:

(1) *If* $|Y| = 2$, *then either* $Y \in I_{16}$ *and* Y *is singular in* E, *or* $Y \in I_{12}$ *and* Y *is nonsingular in* E.

(2) *If* $Y = \{1, t_1, t_2, t_3\}$ *is of order* 4, *then up to conjugation in* $\mathrm{Wyl}(X)$ *one of the following holds*:

(a) *Y is singular, $(c(t_1), c(t_2), c(t_3))$ is a special triple, and* $\Phi(t_i) = \Phi(t_j, t_k)$;

(b) *Y is hyperbolic,* $c(t_1) = \langle x_6' \rangle$, $\Phi(t_1) = V_{10}$, $c(t_2) = \langle x_1' \rangle$, $\Phi(t_2) = \Phi(x_2, x_{12})$, *and* $[V, t_1t_2] = A + B$, *where* $A = \langle x_1, x_1', x_{i6} : 2 \le i \le 5 \rangle$ *and* $B = \langle x_6, x_6', x_{1i} : 2 \le i \le 6 \rangle$;

(c) *Y is totally isotropic but nonsingular,* $c(t_1) = \langle x_{16} \rangle$, $\Phi(t_1) = \Phi(x_1, x_6')$, $[V, t_2] = V_6 + V_6'$, *and* $[V, t_3] = A + B$, *with A, B as in* (b); *or*

(d) *Y is definite,* $[V, t_1] = V_6 + V_6'$, $[V, t_2] = \langle x_i, x_r', x_{ij}, x_{rs} : 1 \le i, j \le 3, 4 \le r, s \le 6 \rangle$, *and* $[V, t_3] = \langle x_i', x_r, x_{ij}, x_{rs} : 1 \le i, j \le 3, 4 \le r, s \le 6 \rangle$.

(3) *If $1 \ne Y$, then $N_G(Y)$ is brilliant or stabilizes a special plane or normalizes $\widehat{X}$.*

Proof. As $E \cong E_{64}$ contains all involutions in H_0, $Y \le E$. Recall

$$\mathrm{Wyl}(X)/E \cong O_6^-(2)$$

induces the full orthogonal group on the orthogonal space E. In particular, $\mathrm{Wyl}(X)$ has two orbits on $E^\sharp$ consisting of singular and nonsingular points of E. Further, from the Schult presentation for $\mathrm{Wyl}(X)$ in §3 of part I, we have a bijection $s \mapsto \hat{x}(s)$ of the singular points s of E with $\widehat{X}$ such that for $e \in E$, e centralizes $\hat{x}(s)$, if and only if s and e are orthogonal in E. Thus, $Y \le E$ centralizes $n(Y^\perp)$ members of $\widehat{X}$, where $n(Y^\perp)$ is the number of singular points of E in the subspace $Y^\perp$ of E orthogonal to Y. In particular, as $n(e^\perp) = 11, 15$ for e singular, nonsingular, respectively, e is in I_{16}, I_{12} for e singular, nonsingular, respectively. Hence (1) is established. Further, if e is singular, then $c(e) = \hat{x}(e)$ and $\Phi(e) = \langle \hat{x}(a) : e \ne a \in e^\perp \rangle$.

Next $\mathrm{Wyl}(X)$ has four orbits on lines of E: totally singular, totally isotropic but not totally singular, hyperbolic, and definite. From this remark and the discussion in the previous paragraph, one can check that (2) holds.

It remains to prove (3). If Y is a singular point, then $M = N_G(Y)$ stabilizes $c(Y)$, so M is brilliant. If Y is a nonsingular point, then M stabilizes $[V, Y] \in \mathscr{V}_{12}$, so again M is brilliant. If Y is a line and not definite, then M stabilizes $\langle c(y) : y \in I_{16}(Y) \rangle$, which is a special plane, a hyperbolic line, or a singular point of V. If Y is a definite line, then $C_V(Y) \in \mathscr{V}_9$. Similar arguments handle the remaining cases.

(6.5) *If $\mathrm{char}(F) \ne 2$ and F is 4-good, then I_{12} and I_{16} are the two classes of involutions of G.*

Proof. This follows from 6.3 and 6.4.1.

(6.6) *Define π and W_i as in* § 4 *and let $t \in I_{12} \cap C_G(\pi)$. Then $[W_i, t]$ is a 4-dimensional hyperbolic subspace of W_i for each i.*

Proof. As $t \in I_{12}$, $[V, t] = A + B$, $A \in \mathscr{V}_6$, $B \in \mathrm{Op}(A)$. As $[t, \pi] = 0$, $A + B \le W = \pi\theta$. For $a \in A$, $a = a_1 + a_2 + a_3$, $a_i \in W_i$. By 4.1, a_i is singular and $a_i \in a_j\Delta$. Also for $c \in A$, $a+c$ is singular, so $a_i + c_i = (a+c)_i$ is singular and hence $c_i \in a_i\Delta$. Thus the projection A_i of A on W_i is singular. Claim $\dim(A_i) = 2$ and $A = A_1 + A_2 + A_3$. If not, then $\dim(A_i) > 2$ for some i, say $i = 1$. Now $A_1 \leq [V, t]$, so $A \leq D \in \mathscr{V}_6(A+B)$ by 2.4. By II.2.1, $A_1\Delta = \langle v_2, v_3, U_1, w_2, w_3 \rangle$ where $w_i \in W_i$ and $U_1 = A_1\Delta \cap W_1$ is of rank 5. Then D is a hyperplane of $\langle U_1, w_2, w_3 \rangle$. So $D_1 = D \cap U_1$ is a hyperplane of U_1. Then by II.2.1, $D_1\Delta \cap W$ is of rank 5, a contradiction.

(6.7) *Assume* $\mathrm{char}(F) \neq 2$, *F is 4-good, and $K \leq G$ with $K^2 = 1$ but K is contained in no Cartan subgroup of G. Let $A = \mathrm{Aut}(G)$. Then*:

(1) $N_A(K) \leq N_A(N_G(U))$ *for some* $U \in \mathscr{P}, \mathscr{U}_9, \mathscr{U}_6$, *or* $\mathscr{U}_3$.

(2) *There is a special plane $\pi = \langle v_1, v_2, v_3 \rangle$ of V centralized by K such that $K = K_0 \times J$ where $K_0^\sharp = I_{16} \cap K$, $|K_0| \leq 4$, $|J| \leq 8$, and J centralizes a member of $\mathscr{U}_6$ containing π.*

(3) *If $K \cong E_4$, then $K^\sharp \subseteq I_{12}$ and $C_V(K) \in \mathscr{U}_9$.*

Proof. Adopt the notation of §4. By 6.3.4, we may assume $[K, \pi] = 0$, K centralizes each singular point it fixes, and K fixes no singular point in $W = \pi\theta$. Let t_i be the element of I_{16} with $c(t_i) = \langle v_i \rangle$ and $\Phi(t_i) = \Phi(v_j, v_k)$, for $i \not\equiv j, k$.

Suppose $t \in I_{16} \cap K$. Then $v_i \in C_V(t) = c(t) + \Phi(t)$, so $c(t) = \langle v_i \rangle$ or $v_i \in \Phi(t)$. As $\Phi(t)$ is brilliant, $\pi \nleq \Phi(t)$, so $c(t) = \langle v_i \rangle$ for some i and $v_j, v_k \in \Phi(t)$, for $i \neq j, k$. Hence $\Phi(t) = \Phi(v_j, v_k)$. That is $t = t_i$. Hence

(a) $K = K_0 \times J$, where $K_0 \leq \langle t_1, t_2 \rangle$ and $K_0^\sharp = I_{16} \cap K$.

(b) J fixes no singular point of W.

For if J fixes singular $\langle w \rangle \leq W$, then the projection w_i of w on W_i is singular and $\langle w_i \rangle$ is fixed by K, since K_0 fixes every point of W_i. Notice (b) and the hypothesis that F is 4-good imply $|J| \geq 4$. Thus:

(c) If $K \cong E_4$, then $K = J$.

(d) If there exist distinct i, j with $C_{W_i}(J) \neq 0 \neq C_{W_j}(J)$, then J centralizes some member of $\mathscr{U}_6$.

Let $w_k \in C_{W_k}(J)$, $k = 2, 3$. Define a map α of the points of W_3 to the points of W_1 as in II.2.3 with respect to $z = w_2$, and let $\langle w_1 \rangle = \langle w_3 \rangle\alpha$. By II.2.3, α is $C_G(\langle \pi, w_2 \rangle)$-equivariant, so J acts on $\langle w_1 \rangle$. Then as J centralizes w_2 and w_3 and $f(w_1, w_3, w_3) \neq 0$, J centralizes w_1. But by 5.8, $\langle \pi, w_1, w_2, w_3 \rangle \in \mathscr{U}_6$.

(e) If $E_4 \cong E \leq J$, then E has four rank 2 eigenspaces on W_i.

Indeed let $E = \langle a, b \rangle$. Then by 6.6, $[W_i, a]$ and $C_{W_i}(a)$ are hyperbolic 4-spaces. As the same holds for b and ab, remark (e) holds.

We can now complete the proof of 6.7. To prove (2) it suffices to take $K_0 = 1$. Also if $K_0 \neq 1$, then by (c), $|K| > 4$ and we need only establish (1) in this case. If $K_0 = \langle t_1, t_2 \rangle$, then $\pi = C_V(K_0)$, while if $K_0 = \langle t_1 \rangle$, then $C_V(K_0) = c(t_1) + \Phi(t_1)$, so $N_A(K)$ acts on $N_G(\pi)$ or $N_G(c(t_1), \Phi(t_1))$ by 6.8 below.

So assume $K = J$. By 6.5, K contains a 4-group E. Suppose $k \in K - E$ and let $K_2 = \langle E, k \rangle$. For $a \in E^\sharp$, $\langle a, k \rangle$ has four rank 2 eigenspaces on W_i by (e), so $C_S(k)$ is a point for each eigenspace S of E on W_i. Thus $C_{W_i}(K_2) \neq 0$, so by (d), $U = C_W(K_2) \in \mathscr{U}_6$. Then by 5.7.6, $N_G(U) = G_1 \times G_2$ with $K_2 \leq G_2 \cong G_2(F)$. As $m_2(G_2(F)) = 3$, $K_2 = C_K(U)$ and $K = K_1 \times K_2$ with $K_1 = K \cap G_1$. But from 5.6, each element of $K_1^\sharp$ is in I_{16}, so $K = K_2$ and $N_A(K)$ acts on $N_G(U)$ by 6.8.

Thus we may take $K = E$ and it remains to prove (3). By (e), each eigenspace of K on W_i is of rank 2. So as K centralizes each singular point, it fixes and centralizes no singular point of W, and each of these eigenspaces is definite. As $K = J$, $K^\sharp \subseteq I_{12}$. In particular, $\dim([V, K]) = 18$, $\dim(C_V(K)) = 9$, and if $K = \langle t, r \rangle$, then $[V, K] = [V, t] \oplus A$, where $A \in \mathscr{V}_6([V, r])$. Further $[W_i, K]$ is 6-dimensional of Witt index 2, so $A \in \mathrm{Op}(B)$ for all $B \in \mathscr{V}_6([V, t])$ and hence $C_V(K) \in \mathscr{U}_9$ by 3.3.2. So the proof of 6.7 is at last complete.

(6.8) *Let $I \leq Y \leq G$ and $A = \mathrm{Aut}(G)$. Then*:

(1) *If Y is the unique member of Y^G containing I and $Y^G = Y^A$, then $N_A(I) \leq N_A(Y)$.*

(2) *If $U = C_V(I)$, $Y = C_G(U)$, and $Y^G = Y^A$, then $N_A(I) \leq N_A(Y)$.*

Proof. Part (1) is easy and implies (2).

7. Cubic extensions

In this section we continue the notation and terminology of §1.

(7.1) *Let U be a totally dark subspace of V. Then*:

(1) *If F is algebraically closed, then* $\dim(U) \leq 1$.

(2) *If F is finite, then* $\dim(U) \leq 3$.

Proof. If U is totally dark, then the cubic polynomial T has no zeros on $U^\sharp$. For example, if x, y are linearly independent in U, then for all $t \in F$, $0 \neq T(tx+y) = t^3T(x) + t^2Q(y, x) + tQ(x, y) + T(y)$, so F is not algebraically closed. Hence (1) holds. If F is finite, we appeal to Theorem 3 of [5, p. 5], which says that as T has no zeros on $U^\sharp$, $3 = \deg(T) \geq \dim(U)$.

(7.2) *Let K be a cyclic cubic Galois extension of F and $\langle \sigma \rangle = \mathrm{Gal}(K/F)$. Then*:

(1) *$\mathscr{E} = (N, P, h)$ is a 3-form on K over F, where $N = N_F^K$, $P(a, b) = \mathrm{Tr}_F^K(a(b\sigma)(b\sigma^2))$, and $h(a, b, c) = \mathrm{Tr}_F^K(a(b\sigma)(c\sigma^2) + a(c\sigma)(b\sigma^2))$.*

(2) *K is totally dark with respect to $\mathscr{E}$.*

(3) *$\Delta(K, \mathscr{E})$ is $K^\sharp$ acting by multiplication extended by σ. Further $O(K, \mathscr{E}) = \{k \in K : N(k) = 1\}$ extended by σ.*

(4) *$(K^K, \mathscr{E}^K)$ is a special plane.*

(5) *If $\overline{\mathscr{E}}$ is a 3-form on a plane U over K generated by a special triple (s_1, s_2, s_3), $\alpha \in O(U, \overline{\mathscr{E}})$ with $s_i\alpha = s_{i+1}$, and $\tau = \sigma\alpha$, then $(C_U(\tau), \overline{\mathscr{E}}\,\mathrm{Tr}_F^K)$ is isometric to $(K, \mathscr{E})$.*

Proof. Assume the hypotheses of (5). Then $\overline{\mathscr{E}}\,\mathrm{Tr}_F^K$ is a 3-form on $C_U(\tau)$. For $a \in K$, let $s(a) = as_1 + (a\sigma)s_2 + (a\sigma^2)s_3$. Then $s\mathscr{E}\,\mathrm{Tr}_F^K = \mathscr{E}$, so (1) and (5) hold.

As $N(a) \neq 0$ for $a \in K^\sharp$, (2) holds. Next the proof of (4). As $\mathscr{E}^K$ is nontrivial, it suffices to show K^K is generated by singular vectors. Let $r \in K$ with $r^3 \notin F$ and set $k = r/r\sigma$. Thus $N(k) = 1$. Let $\alpha : K \to K$ be translation by k; we will see below that $\alpha \in O(K, \mathscr{E})$. Thus α induces $\alpha^K \in O(K^K, \mathscr{E}^K)$ and as k is an eigenvalue of α, α^K has eigenvalues $k, k\sigma, k\sigma^2$. Let u_1, u_2, u_3 be the corresponding eigenvectors. Then $P^K(u_i, u_j) = P^K(u_i\alpha^K, u_j\alpha^K) = k\sigma^i(k\sigma^j)^2P^K(u_i, u_j)$. Thus it remains only to observe $k\sigma^i(k\sigma^j)^2 \neq 1$, so that u_j is singular. Hence (4) is established.

For $k \in K$, let α_k be translation by k. Check that $\alpha_k \in \Delta(K, \mathscr{E})$ with $\lambda(\alpha_k) = N(k)$. Thus $H = \{\alpha_k : N(k) = 1\} \leq O(K, \mathscr{E})$. Also $\sigma \in O(K, \mathscr{E})$. Conversely, let $M = \Delta(K, \mathscr{E})$. Then $M \cong M^K \leq \Delta(K^K, \mathscr{E}^K) = S_3\mathrm{wr}K^\sharp$ by (4). Indeed M^K is the fixed points of τ on $\Delta(K^K, \mathscr{E}^K)$, and hence is $\{\alpha_k : k \in K^\sharp\}$ extended by σ. So (3) holds.

(7.3) *Let F be finite of order q, K a cubic extension of F, and U a totally dark subspace of V. Then*:

(1) *If U is a line, then U^K has exactly three brilliant points, but no singular points.*

(2) *If U is a plane and U^K is special, then U is determined up to conjugacy in Δ, $C_G(U) \cong {}^3D_4(F)$, and $N_G(U)/C_G(U)$ is a subgroup of the cyclic group of order $q^2 + q + 1$ extended by an automorphism of order 3.*

Proof. Assume $U = \langle x, y\rangle$ is a line. Then $T(ax + y) = a^3T(x) + a^2Q(y, x) + aQ(x, y) + T(y) = r(a)$, $r \in F[t]$. So $r(t)$ is irreducible and K is its splitting field. Hence r has three roots in K, so U^K has three brilliant points. Let $\langle u\rangle, \langle v\rangle$ be two such points. As U^K is not brilliant, to prove (1) we may take v singular, and $Q(v, u) \neq 0$. Then $T(bu + v) = b^2Q(v, u) \neq 0$ for $b \in F^\sharp$, contradicting three brilliant points in U^K. Hence (1) holds.

Assume the hypotheses of (2), and let $U^K = \pi$ and $\langle\sigma\rangle = \mathrm{Gal}(K/F)$. Adopt the notation of §4 in discussing π. Note GV is the fixed points of σ on G^KV^K. As G^K is transitive on special planes, the number of orbits of G on planes Z of V with Z^K special is the number of orbits of $N_{G^K}(\pi)$ on the set of G^K conjugates of σ acting on π.

If σ fixes $\langle v_i\rangle$, then σ fixes some $u_i \in \langle v_i\rangle^\sharp$, so U is special. Hence σ permutes $(\langle v_i\rangle : 1 \leq i \leq 3)$ transitively, so $\sigma = g\tau$, where τ induces a field automorphism on $\Omega = C_{G^K}(\pi)^\infty \cong \mathrm{Spin}_8^+(K)$, and g induces an automorphism in the coset of triality. As Ω is transitive on such elements, we have transitivity of Δ on the set D of totally dark planes Z with Z^K special, once we show C is transitive on elements of order 3 in σC, where C is the subgroup of $H(\Delta)$

inducing scalar multiplication on W_3. For this, it suffices to show $\langle \sigma, C\rangle$ has $Z_{3^m} \operatorname{wr} Z_3$ for a Sylow 3-group.

Now $g = h(t)l(a_1, \dots, a_6)\delta(s) \in C$ induces bI on W_3, if and only if $a_i = a$ is independent of i for $1 < i < 6$, $t = a^{-2}(a_6)^{-1}$, $s = bta^{-1}$, and $a_1 a_6 a^4 = 1$. Thus $g = g(a, a_1, s)$ is determined by a, a_1, s, and $C \cong (K^\sharp)^3 = K^\sharp \times K^\sharp \times K^\sharp$. Define $\alpha : (K^\sharp)^3 \to C$ by $(u, v, w)\alpha = g((uv)^{-1}, v, u^1 w)$. Then α is an isomorphism with inverse $g(a, a_1, s) \mapsto ((aa_1)^{-1}, a_1, s(aa_1)^{-1})$. Further $(b_1, b_2, b_3)\alpha$ has eigenvalues b_i on W_i. Then $C = C_1 \times C_2 \times C_3$, where $C_i = \{g_i(k) : k \in K^\sharp\}$ and $g_1(k) = (k, 1, 1)\alpha$, $g_2(k) = (1, k, 1)\alpha$, and $g_3(k) = (1, 1, k)\alpha$. As σ has cycle (W_1, W_2, W_3), $g_i(k)^\sigma = g_{i+1}(k)$. Hence $\langle C, \sigma\rangle$ has Sylow 3-group $Z_{3^m} \operatorname{wr} Z_3$, as desired.

Finally as σ induces a graph-field automorphism on Ω, ${}^3D_4(q) \cong C_\Omega(\sigma) = C_G(U)$ and by 7.2, $N_G(U)/C_G(U)$ is a subgroup of $O(U, \mathscr{F})$, which is a cyclic group of order $q^2 + q + 1$ extended by an automorphism of order 3.

(7.4) *Let F be finite of order q, p a prime with $d_q(p) = 3$, and J an elementary abelian p-subgroup of G. Let $V = A_1 \oplus A_2 \oplus A_3$ be a 3-decomposition of V. Then*:

(1) *$J \le P \in \mathrm{Syl}_p(G)$, and up to conjugation in G, $P = P_1 \times P_2 \times P_3$ with $P_i \in \mathrm{Syl}_p(M_i)$, $M_i = C_G(A_i) \cong SL_3(F)$, and P_i is cyclic.*

(2) *Each member of the set S of FP-irreducibles on V is a twisted special plane and distinct members are nonequivalent FP-modules, so $V = \bigoplus_{A\in S} A$. Indeed S is a twisted 9-decomposition of V.*

(3) *$C_G(P) = \langle t\rangle T_1T_2T_3Z(M_1M_2M_3)$ with $T_i = C_{M_i}(P)$ a maximal torus of M_i and $\langle t\rangle$ inducing a diagonal automorphism on each M_i.*

(4) *$N_G(P)/C_G(P) \cong GL_2(3)/3^{1+2}$ acts 2-transitively on S as $GL_2(3)/E_9$ and $N_\Gamma(P) = N_\Gamma(S)$.*

(5) *For each $A, B \in S$ there is $\Sigma(A, B) \in S$ with $A + B + \Sigma(A, B) \in \mathscr{V}_9$.*

(6) *$N_G(J)$ preserves a 3-decomposition α of G or $N_G(J) \le N_G(S)$ or $N_G(J) \le N_G(A)$ for some $A \in S$. Further $N_{\mathrm{Aut}(G)}(J)$ acts on $N_G(B)$ for $B = \alpha, S, A$, in the respective case.*

Proof. Let $M_0 = M_1M_2M_3$. As $|M_0| = |G|_p$, (1) holds. Further $A_i = C_V(P_i)$, so $C_G(P) \le M_0\langle t\rangle$ where $\langle t\rangle$ induces a diagonal automorphism on M_i centralizing $T_i = C_{M_i}(P_i)$, a maximal torus of M_i. Thus (3) holds.

Let K be a cubic extension of F and $\langle\sigma\rangle = \mathrm{Gal}(K/F)$. Then $q^3 \equiv 1 \bmod p$, so $P \le H_0^K$ by 6.3. Indeed $G = C_{(G^K)}(\tau)$, where τ induces a field automorphism on M_i^K and $\tau = \sigma\rho$, where σ induces a field automorphism on G^K centralizing $\mathrm{Wyl}(X)$ and $\rho = \alpha\gamma$ is of order 3 in $\mathrm{Wyl}(X)$ with α, γ as in §6 of [3].

Further, each orbit of ρ on $\widehat{X}$ generates a special plane π_i, and $C_V K(\tau)$ is the direct sum of the planes $U_i = C_{\pi_i}(\tau)$, with U_i a twisted special plane and P irreducible on U_i. Indeed $\{\pi_i : i\}$ is a 9-decomposition of V^K, so $S = \{U_i : i\}$ is a twisted 9-decomposition of V. This establishes (2).

Next $C_{\mathrm{Wyl}(X)}(\tau) = C \cong GL_2(3)/3^{1+2}$ is transitive on the orbits of ρ on $\widehat{X}^K$ with kernel of order 3 and hence $C \leq N_G(P)$ with C acting 2-transitively on S as $GL_2(3)/E_9$. $N_\Gamma(S) \leq N_{(\Gamma^K)}(S^K)$ and as $U_i^K = \pi_i$ is a special plane, $N_{(\Gamma^K)}(S^K)$ acts on $\widehat{X}^K$. Indeed $N_G(S^K) = CC_{H_0}K(\tau)$, establishing (4).

For $A, B \in S$, A^K is a special plane and B^K is incident to A^K in the sense of part II, §3. Let $\Sigma(A, B)$ be the member of S with $\Sigma(A, B)^K = \Sigma(A^K, B^K)$ in the notation of part II, §3. Then (5) holds by part II, §3 and the definition of $\mathscr{V}_9$.

Claim $N_G(J)$ is irreducible on V or fixes some 3-decomposition or some member of S. For if $N_G(J)$ is not irreducible on V, then $N_G(J)$ acts on some proper nontrivial subspace U of V. So $U = B_1 \oplus \cdots \oplus B_m$ for some $B_i \in S$. If $m = 1$, then $U \in S$ and the claim holds. If $m = 2$, then $U = A \oplus B$ for some $A, B \in S$ and we let $U(A, B) = A + B + \Sigma(A, B)$. Then $N_G(J)$ acts on $U\theta = \langle D \in S : D \nleq U(A, B)\rangle$ and hence preserves the 3-decomposition through $U(A, B)$. So again the claim holds.

So assume $m \geq 3$. Replacing K by $K\xi$ if necessary, we may assume $m \leq 4$. So $m = 3$ or 4. Now C has two orbits on triples from S with representatives $(A, B, \Sigma(A, B))$ and (A, B, D), $D \nleq U(A, B)$. If $U = U(A, B)$, then certainly the claim holds. If $m = 3$ and $U \neq U(A, B)$, then $U = A \oplus \theta(B \oplus \theta D)$ and as each member of S is totally dark,

$$A \cup B \cup D = \{u \in U : \operatorname{codim}(u\Delta^U) \leq 3\}.$$

Thus $N_G(J)$ acts on $\{A, B, D\}$ and hence also on S (cf. II.3.5.3).

So $m = 4$. If $U = U(A, B) + D$, then $U = U(A, B) \oplus \theta D$ and

$$D = \{u \in U : \operatorname{codim}(u\Delta^U) \leq 3\}$$

is $N_G(J)$-invariant. Thus no triple among our four is of type $(A, B, \Sigma(A, B))$. But then $U = ((A \oplus \theta B) \oplus \theta D) \oplus \theta E$, so that $N_G(U)$ permutes $\{A, B, D, E\}$ and hence also S. Thus the claim is established.

It remains to prove (6). Suppose $N_G(J) \leq N_G(\alpha)$, for some 3-decomposition $\alpha = \{A_1, A_2, A_3\}$. Then $J_1J_2J_3 \operatorname{char} N_G(J)$, where J_i is the projection of J on M_i, so without loss $J_i \leq J$. If $J = J_1$ or J_1J_2, then $E(C_G(J)) = M_2M_3$ or M_3, so $N_A(J) \leq N_A(N_G(\alpha))$. So $J = P$ and then $N_A(J) \leq N_A(N_G(S))$ as S is the unique 9-decomposition fixed by P. Moreover, we see P is not characteristic in $N_G(J)$.

Assume next that $N_G(J) \leq N_G(U)$ for some $U \in S$. Again we may take $J_1J_2 \leq J$, where J_2 is the projection of J on $D = E(C_G(U))$ and $J_1 = C_P(D)$. If $J = J_1$, then $N_A(J) \leq N_A(D) = N_A(N_G(U))$, so we may assume $J_2 \neq 1$. Similarly we may assume $J_2 = P \cap O_p(N_D(J_2))$. By the previous paragraph, $|J_2| = p$, so $E(N_D(J)) = D_1 \cong SL_3(q)$. But then $D_1 = C_G(A)$ for some $A \in \mathscr{V}_9$, and 6.8 completes the proof.

So $N_G(J)$ is irreducible on V. Then $C_V(J) = 0$. Assume J is noncyclic. Then $V = \bigoplus_I C_V(I)$ as I varies over the set Ξ of hyperplanes I of J with $C_V(I) \neq 0$. Further, for each $I \in \Xi$, either $C_V(I) \in S$ or $|I| = p$ and $C_V(I) \in \mathscr{V}_9$. For example $C_J(A_1) = C_J(\pi + \pi')$, where π and π' are incident special planes with $A_1 = \pi + \pi' + \Sigma(\pi, \pi')$. So by 2-transitivity of $N_G(P)$ on S, $C_V(I) = A + B + \Sigma(A, B) \in \mathscr{V}_9$, if I is of order p and centralizes $A, B \in S$.

Now if $C_V(I) \in S$ for all $I \in \Xi$, then $\Xi = S$ is $N_G(J)$-invariant, contradicting P not characteristic in $N_G(J)$. So as $N_G(J)$ is irreducible on V, $\Xi = \{A_1, B, C\}$, $A_1, B, C \in \mathscr{V}_9$. In particular, $C_G(J) \leq N_G(A_1) \leq N_G(\alpha)$ and as $O_p(C_G(J)) \leq J$, $J = J_1 J_2 J_3$, where J_i is the projection of J on M_i. Then as $J \neq P$, we have a contradiction as above.

So J is of order p. As $N_G(J)$ is irreducible on V, $V^K = U \oplus U^\sigma \oplus U^{\sigma^2}$ for some KJ-module U admitting $N_G(J)$ irreducibly. Thus U is an eigenspace for J with eigenvalue λ, say. Then λ^q and λ^{q^2} are the eigenvalues of J on U^σ and U^{σ^2}, respectively. As $p \neq 3$, these eigenspaces are brilliant. Claim $U^\sigma + U^{\sigma^2} \leq U\theta$; then U is singular, contradicting $\dim(U) = 9$. For if $x \in U$, $y \in U^{\sigma^i}$, $i = 1$ or 2, with $0 \neq Q(y, x)$, then $1 = \lambda^{q^i+2}$. Hence $q^i + 2 \equiv 0 \bmod p$. So q or $q^2 \equiv -2 \bmod p$. Then $0 \equiv q^2 + q + 1 \equiv 3 \bmod p$, contradicting $p > 3$.

(7.5) *Let F be finite of order q and p a prime with $d_q(p) = 9$. Then:*

(1) *A Sylow p-subgroup P of G is cyclic.*

(2) *$N_G(\Omega_1(P))$ is the split extension of a cyclic group of order $q^6 + q^3 + 1$ by an automorphism of order 9.*

(3) *$N_G(\Omega_1(P))$ is contained in the stabilizer Y of a unique twisted 3-decomposition of V.*

(4) *Let $k = GF(q^3)$. Then Y is the extension of $L_3(k)$ by a field automorphism of order 3 and V isomorphic as a $GF(k)E(Y)$-module to $N \otimes N^{*\delta}$, where N is the natural $kE(Y)$-module and $\langle\delta\rangle = \mathrm{Gal}(k/F)$.*

(5) *Δ is transitive on twisted 3-decompositions of V.*

Proof. Let K be an extension of F with $|K : F| = 9$. Let $\langle\sigma\rangle = \mathrm{Gal}(K/F)$. Observe $\mathrm{Wyl}(X)$ has a unique class of elements of order 9 with representative ρ, where ρ has the following cycles on X:

$$(x_1, x_{56}, x_3', x_6', x_{32}, -x_4, x_{16}, x_{41}, x_{34}),$$
$$(x_2, x_{46}, x_{45}, x_1', x_{52}, x_4', x_{21}, x_{31}, -x_5),$$
$$(-x_3, x_2', x_{42}, x_{53}, x_{62}, x_{51}, x_5', x_6, x_{36}).$$

Indeed in the notation of §6 of [3], $\rho^3 = \alpha\gamma$, where α is introduced at the beginning of §6 of [3] and γ is defined in 6.3 of [3]. In the notation of [3], $E = Q_8^3 = E_1 * E_2 * E_3$ is the central product of three $\alpha\gamma$-invariant quaternion groups $E_1 = \langle d, e_1 d_2 d_6\rangle$, $E_2 = \langle d_6 d_2, d d_2\rangle$, and $E_3 = \langle d_3 d_4, d_3 d_5\rangle$ with ρ

permuting the groups E_i transitively and

$$\rho = (d, d_6d_2, d_3d_4, e_1d_2d_6, dd_2, d_4d_5, d_2d_6e_1d, d_6dz, d_5d_3).$$

Let $\tau = \sigma\rho$ and $g \in C_{H_0^K}(\tau)$. Then g has eigenvalue $\lambda(x) = \lambda(x, g)$ on $\langle x\rangle \in \widehat{X}$ and

$$(*) \qquad \lambda(x\tau^k) = \lambda(x)\sigma^k = \lambda(x)^{q^k}.$$

In particular, if $\lambda = \lambda(x_1)$, then $\lambda(x_6') = \lambda^{q^3}$, $\lambda(x_{16}) = \lambda^{q^6}$, and

$$1 = f(x_1, x_6', x_{16}) = f(x_1g, x_6'g, x_{16}g) = \lambda^{(q^6+q^3+1)}.$$

Therefore

$$(**) \qquad \lambda^{(q^9-1)/(q^3-1)} = 1.$$

Similarly $1 = f(x_1, x_4', x_{14}) = \lambda^{(q^7+1)}\lambda(x_4')$, so $\lambda(x_4') = \lambda^{-(q^7+1)}$. Then $\lambda(x_{12}) = \lambda(x_4')^q = \lambda^{-(q(q^7+1))}$ and $1 = f(x_1, x_2', x_{12}) = \lambda(x_2')\lambda^{(1-q(q^7+1))}$, so $\lambda(x_2') = \lambda^{(qq(q^7+1)-1}$. As $\langle x_1\rangle$, $\langle x_4'\rangle$, and $\langle x_2'\rangle$ are representatives for the three orbits of ρ on $\widehat{X}$, we see that $(*)$ and $\lambda = \lambda(x_1)$ determine g uniquely. Then by $(**)$, $C_{(H_0^K)}(\tau) = \langle h_0\rangle$, where h_0 and $\lambda_0 = \lambda(h_0, x_1)$ are of order $q^6 + q^3 + 1$. Further ρ induces an automorphism of order 9 on h_0 and $\langle h_0, \rho\rangle \le C_GK(\tau)$.

Next, τ induces a field automorphism on Δ^K, so by Lang's Theorem, τ is conjugate to σ under Δ^K. So there is a conjugate $\langle h, r\rangle$ of $\langle h_0, \rho\rangle$ in $G = C_GK(\sigma)$. Now $|G|_p$ divides $q^6 + q^3 + 1$ as $d_q(p) = 9$, so P is cyclic and we may take $P \le \langle h\rangle$.

Thus (1) holds and to prove (2) we must show $\langle h, r\rangle = N_G(\Omega_1(P))$. Let g be of order p in h_0; we must show $\langle h_0, \rho\rangle = N_GK(\langle g\rangle) \cap C(\tau)$. We have shown $\langle h_0\rangle = C_{H_0^K}(\tau)$. Further ρ is self-centralizing in $\mathrm{Wyl}(X)$, so $\langle h_0, \rho\rangle = C_{\mathrm{Wyl}(X)H_0^K}(\tau)$. Thus to prove (2) it suffices to show $N_GK(\langle g\rangle)$ acts on $\widehat{X}$. This will be the case if all eigenvalues of g are distinct, so assume not.

$N_{\mathrm{Wyl}(X)}(\langle\rho\rangle)$ is transitive on $\widehat{X}$, so without loss $\lambda = \lambda(x)$ for some $\langle x\rangle \in \widehat{X} - \{V_1\}$. Then $\langle x\rangle$ is not in $V_1\langle\rho\rangle$, as the eigenvalues in each $\langle\rho\rangle$-orbit are distinct. Indeed we may assume $x \in x_1\Delta$: For if U is the λ-eigenspace of g, then U contains at most one point from each $\langle\rho\rangle$-orbit on $\widehat{X}$, so $U = \langle x_1, x\rangle$ or $\langle x_1, x, y\rangle$. Further if $\langle x_1, x\rangle$, $\langle x_1, y\rangle$, and $\langle x, y\rangle$ are hyperbolic, then U is a hyperbolic line or special plane, so $\widehat{X}$ is the set of singular $\langle g\rangle$-invariant points of V^K and $N_GK(\langle g\rangle)$ acts on $\widehat{X}$ as required.

We now check that x cannot be a point of $x_1\Delta$ not in $V_1\langle\rho\rangle$. For example, if $x = x_5$ then $\lambda = \lambda(x_5) = \lambda^{-q^3(q^7+1)}$, so $0 \equiv q^3(q^7 + 1) + 1 \bmod p$. But $q^3(q^7 + 1) + 1 \equiv q + q^3 + 1 \equiv q - (q^6 + 1) + 1 = q(1 - q^5) \bmod p$. Thus $q^5 \equiv 1 \bmod p$, contradicting $d_q(p) = 9$. Similar arguments eliminate the other possibilities.

Thus (2) is at last established and it remains to prove (3) through (5). Let k be a cubic extension of F in K and $\langle\delta\rangle = \mathrm{Gal}(k/F)$. Let t be of order 3 in G transitive on a 3-decomposition $V^k = A_1 + A_2 + A_3$ of V^k. Let $\beta = \delta t$, $M_i = C_G k(A_i)$, and $M = M_1M_2M_3$. Then $M_i \cong SL_2(k)$ and M is described in II.3.6. Notice β permutes the subgroups M_i transitively and $C_M(\beta) = Y_0 \cong L_3(k)$ with t inducing a field automorphism of order 3 on Y_0. Let $Y = Y_0\langle t\rangle$. Then Y_0 contains a subgroup I of order $q^6 + q^3 + 1$ and $N_Y(I)$ is the extension of I by an automorphism of order 9. So by (2), Y contains the normalizer in $C_{(G^k)}(\beta) \cong G$ of an element of $C_{(G^k)}(\beta)$ of order p as claimed.

It remains to show the representation of Y on V is as claimed. But A_3 is isomorphic to $N \otimes N^*$ as an M_1M_2-module with respect to the isomorphism $t: M_1 \to M_2$, so A_3 is isomorphic to $N \otimes N^{*\delta}$ as a Y_0-module, with A_1 and A_2 the Galois conjugates of A_3 under δ. Hence (4) holds.

As $\Omega_1(P)$ is weakly closed in $N_G(\Omega_1(P))$ and $N_G(P) \leq N_G(Y)$, Y is the unique member of Y^G containing $\Omega_1(P)$. So (5) and 6.8 imply (3). Part (3) holds as $\mathrm{Hom}_{E(Y)}(V, S^2(V^*)) \cong F$.

8. Semisimple subgroups over algebraically closed and finite fields

In this section we continue the notation and terminology of §6. In particular, K is a semisimple abelian subgroup of G. In addition assume F is either algebraically closed or finite.

(8.1) *If K is of order 3 and F has an element of order 3, then K is contained in a Cartan subgroup of G.*

Proof. Some eigenspace of K is of dimension at least 4, so by 7.1, g fixes a brilliant point. Then apply 6.1 and 6.3.2.

(8.2) *Assume F contains an element ω of order 3. Then there are three conjugacy classes of 9-subgroups $\langle g, Z(G)\rangle$ of G containing $Z(G)$:*

(1) *Type 1 in which g has two 6-dimensional eigenspaces U_1 and U_2 and one 15-dimensional eigenspace U_3. Indeed $U_1 \in \mathscr{V}_6$, $U_2 \in \mathrm{Op}(U_1)$, and $U_3 = V(U_1) \cap V(U_2)$. For example, take $g = l(\omega, \omega^{-1}, 1^4)$.*

(2) *Type 2 in which g has three 9-dimensional eigenspaces U_i. There exists a special triple (v_1, v_2, v_3) with $U_i = \langle v_i, v_j\Delta \cap v_k\Delta\rangle$. For example, take $g = l((\omega)^2, (\omega^{-1})^2, 1^2)$.*

(3) *Type 3 in which g has three 9-dimensional eigenspaces U_i and $V = U_1 \oplus U_2 \oplus U_3$ is a 3-decomposition of V. For example take $g = l((\omega)^3, (\omega^{-1})^3)$.*

Proof. By 8.1 we may take $g \in H_0$. Let E be the subgroup of elements of order 3 in H_0. Then $E \cong E_{3^6}$ and $\mathrm{Wyl}(X)/O_2(\mathrm{Wyl}(X))$ acts faithfully on E, centralizes $Z(G)$, and acts as $O_5(3)$ on $E/Z(G)$. Thus $\mathrm{Wyl}(X)$ has three orbits on $(E/Z(G))^\sharp$, with representatives $g_iZ(G)$, $1 \le i \le 3$, such that $q(g_iZ(G)) = 0, 1, -1$, where q is the quadratic form on $E/Z(G)$ preserved by $\mathrm{Wyl}(X)$. Hence, the first assertion of the lemma is established. Moreover, the three 9-groups listed in (1) through (3) are visibly not conjugate in G, and hence are representatives for the G-classes.

(8.3) *Assume F contains an element of order 3 and let $Z(G) \le A \le G$ with $A \cong E_{3^n}$, but A is not contained in a Cartan subgroup of G. Then each member of $A - Z(G)$ is of Type 3, and either*

(1) *F is finite of order $q \equiv 4, 7 \bmod 9$, $A \cong E_{27}$ or E_{81}, and $N_{\mathrm{Aut}(G)}(A)$ acts on the stabilizer of a twisted 9-decomposition or 3-decomposition of V, respectively; or*

(2) *A is an exotic E_{81}-subgroup of G, all eigenspaces of A are dark points, $N_G(A)$ is transitive on the eigenspaces of A, $A = C_G(A)$, and $N_G(A)/A$ is the centralizer in $SL(A)$ of $Z(G)$. Moreover Δ is transitive on exotic E_{81}-subgroups of G.*

Proof. Assume A is not contained in a Cartan subgroup of G. By 8.2, $|A| \ge 27$. By 6.1 and 6.3.2, A fixes no brilliant point. However, if $g \in A$ is of Type 1, then g has a singular eigenspace, and hence so does A. Similarly, if $g \in A$ is of Type 2, then each eigenspace of g has a distinguished singular point fixed by $C_G(g)$, so again A fixes a singular point. Thus each member of $A - Z(G)$ is of Type 3.

Let $g_0 \in A - Z(G)$. Define $V = A_1 \oplus A_2 \oplus A_3$ to be the 3-decomposition of V of part II, §4, and adopt notation as in that section. We may take A_i, $1 \le i \le 3$, to be the eigenspaces of g_0. Let $g \in A - \langle Z(G), g_0\rangle$ and set $K = \langle Z(G), g_0, g\rangle$. Suppose for each i, g fixes no brilliant point of A_i. By part II, §4, we may regard A_3 as the tensor product $B_1 \otimes B_2$ of natural modules B_i for $M_i \cong SL_3(F)$, with the restriction of f to A_3 the tensor product of the alternating trilinear forms on B_1 and B_2 preserved by M_1 and M_2. Further $C_G(g_0)$ induces $M_1M_2H_1$ on A_3; here $h \in H_1$ acts as $(a, 1, 1)$ on B_1 and $(a^{-1}, 1, 1)$ on B_2 for some $a \in F^\sharp$.

Now if g fixes points P_1 and P_2 in B_1 and B_2, then $P_1 \otimes P_2$ is a singular point of A_3 fixed by g, contrary to assumption. This forces F to be finite of order $q \equiv 4, 7 \bmod 9$ and $g \in M_0$ centralizes a torus T_i of M_i of order q^2+q+1. Then $C_G(K) = T\langle n\rangle K$, where $T = T_1T_2T_3$ and $n = n_1n_2n_3$, with n_i of order 3 in M_i acting on T_i. In particular, if $K = A$, then $N_G(A) \le N_G(P)$, where $P \in \mathrm{Syl}_p(T)$ for any prime p divisor $p \ne 3$ of $q^2 + q + 1$. Hence $N_{\mathrm{Aut}(G)}(A)$ acts on the stabilizer of a twisted 9-decomposition of V by 7.4.

So assume $K \ne A$; then we may take $A = \langle K, n\rangle$. Next replacing n by a suitable member of $n\langle g_0, Z(G)\rangle$ if necessary, we may take $n \in M_0^y$, where $g = g_0^y$. Further, n is centralized by a Sylow 3-group R of $N_{M_0}(K_0)$ and by a Sylow 3-group R' of $N_{M_0^y}(K_0)$, and $\langle R, R'\rangle$ is transitive on the subgroups of order 3 in K distinct from $Z(G)$. So $\langle Z(G), n\rangle = K_1 = \bigcap_x (M_0^x \cap A)$, as x

varies over $N_G(A)$. Thus $N_{\mathrm{Aut}(G)}(A) \leq N_{\mathrm{Aut}(G)}(K_1)$ and hence $N_{\mathrm{Aut}(G)}(A)$ acts on the stabilizer of the 3-decomposition defined by K_1.

Thus (1) holds, unless $K = \langle g, g_0, Z(G)\rangle$ is in a Cartan subgroup of G for each $g \in A - \langle g_0, Z(G)\rangle$; so we assume such a containment for all such g. In particular, $|A| \geq 81$. Also $g_0 \in E \leq H_0$ with $E \cong E_{3^6}$ and $\widetilde{E} = E/Z(G)$ admits a quadratic form preserved by Wyl(X). Indeed the subspace $\widetilde{E}_0$ of $\widetilde{E}$ orthogonal to $\tilde{g}_0$ is $E_0 = E_1E_2E_3$, where $E_i = M_i \cap E$. That is $E_0 = E \cap M_0$.

Now K is in an M_0-conjugate of E which we may take to be E. If $g \notin E_0$, then K contains an element not of Type 3, as $\langle \tilde{g}, \tilde{g}_0\rangle$ is not totally singular. Hence $K \leq E_0$. Thus $g = g_1g_2g_3$, g_i of order 1 or 3 in E_i. As M_i is transitive on elements of order 3 in $M_i - Z(M_i)$, it follows that $N_G(M_0)$ has three orbits $\mathscr{O}_i$ on 27-subgroups $\langle Z(M_0), e\rangle$ with e in some M_0-conjugate of E_0; namely $\langle Z(M_0), e_i\rangle \in \mathscr{O}_i$, if e_i projects nontrivially on i members of $M_j/Z(M_j)$, $j = 1, 2, 3$. An easy calculation shows e_i is of Type i. Thus, as g is of Type 3, $K \in \mathscr{O}_3$ and $g_i \in M_i - Z(M_i)$ for each i. Therefore, $A \cap M_1M_2 = Z(M_0)$, so the projection $A/Z(M_0) \mapsto M_3/Z(M_3)$ is an injection. Then as $m_3(M_3/Z(M_3)) = 2$, $|A| = 81$.

Next, Aut(M_0) is transitive on the set J of subgroups A of M_0 over $Z(M_0)$ isomorphic to E_{81} such that each element in $A - Z(M_0)$ projects on an element of M_i of order 3; further, for $A \in J$, $N_{M_0}(A)/A \cong SL_2(3)/E_{81}$ or Q_8/E_{81} for F containing or not containing an element of order 9, respectively. It follows that with $g = g_0^x$, we have $\langle N_{M_0}(A), N_{M_0^x}(A)\rangle/A$ is the centralizer in $SL(A)$ of $Z(G)$. Thus Aut$_G(A)$ is this stabilizer. Also $V = \bigoplus_{B\in S} C_V(B)$, where S is the set of hyperplanes of A missing $Z(G)$. As $|S| = 27$ and $N_G(A)$ is transitive on S, each space $C_V(B)$, $B \in S$, is a point, and these 27 points are the eigenspaces of A. Each is dark, as A is not in a Cartan subgroup.

Next, if F is algebraically closed, then M_0 is transitive on J, so A is determined up to conjugacy in G. So assume F is finite and let $\overline{F}$ be the algebraic closure of F, $\overline{G} = E_6(\overline{F})$, and regard G as $C_{\overline{G}}(\sigma)$, σ a Frobenius map. Then $N_{\overline{G}\langle\sigma\rangle}(A) = \langle\sigma\rangle \times N_G(A)$, so $\langle\sigma, Z(G)\rangle$ is the unique $\overline{G}$-conjugate of $\langle\sigma, Z(G)\rangle$ in $C_{\overline{G}\langle\sigma\rangle}(N_G(A))$. Hence as $\overline{G}$ is transitive on exotic E_{81}-subgroups of $\overline{G}$, it follows that $\Delta = N_{\overline{G}}(\langle\sigma, Z(G)\rangle)$ is transitive on exotic E_{81}-subgroups of G.

(8.4) *Let F be finite of order q and p a prime with $d_q(p) > 1$. Assume K is an elementary abelian p-group with $C_V(K)$ nontrivial and totally dark. Then either*

(1) *$C_V(K)$ is a point, or*

(2) *$C_V(K)$ is a twisted special plane.*

Proof. Let $U = C_V(K)$, k a cubic extension of F, and $\langle\sigma\rangle = \mathrm{Gal}(k/F)$. By 7.1, $\dim(U) \leq 3$. If U is a line, then by 7.3, U^k has three brilliant points P_i, none of which is singular. Now by 6.1, K fixes a singular point S of

V^k. As $S \not\leq U^k = C_Vk(K)$, $d_q(p) = 3$. Then by 6.3, K is contained in a Cartan subgroup of G^k. But then $C_{V^k}(K)$ is generated by singular points, a contradiction.

So take U to be a plane. Again by 7.3 and 6.1, K fixes a member (P, W) of $\mathscr{P}$. Then by 6.3, if $d_q(p) = 3$, then U^k is generated by singular points. So U^k is a special plane and hence U is a twisted special plane. So take $d_q(p) \neq 3$. Then $[K, P] = 0$. As U is totally dark, σ moves P, so $W = \langle P\langle\sigma\rangle\rangle$ is U^k or a line. In the first case, U^k is a special plane and we are done. In the second as W is σ-invariant, $W = Z^k$ for some line Z in U. But also W contains the three singular images of P under $\langle\sigma\rangle$, so W is singular. Thus Z is brilliant, a contradiction.

(8.5) *Let F be finite of order q and $p \neq$ char(F) a prime. Assume K is an elementary abelian p-group. Then either*

(1) $C_V(K) \neq 0$, *or*

(2) $d_q(p) = 1, 3$, *or* 9.

Proof. First V is the direct sum of irreducible FK-submodules W_i, $1 \leq i \leq r$, with $W_i \leq C_V(K)$ or $\dim(W_i) = d_q(p)$. Hence, if $C_V(K) = 0$, then $27 = \dim(V) = rd_q(p)$, so $d_q(p) = 3^i$, $0 \leq i \leq 3$. As $((q^{27} - 1)/(q^9 - 1), |G|) = 3$, $d_q(p) \neq 27$.

(8.6) *Let $U \leq I = \langle x_6', V_{10}\rangle$. Then one of the following holds*:

(1) $N_\Gamma(U) \leq \Gamma_{10}$.

(2) $U \cap V_{10}$ *and is singular and* $N_\Gamma(U)$*-invariant.*

(3) U *is a special plane and* $U = \langle x_6', s, t\rangle$ *for some singular* $s, t \in V_{10}$.

(4) $U = \langle x_6', s\rangle$ *is a hyperbolic line for some singular* $s \in V_{10}$.

(5) U *is a line and contains exactly three brilliant points, none of which is singular.*

Proof. Let $W = U \cap V_{10}$ and $g \in N_\Gamma(U) - G_{10}$. Then $Z = W \cap Wg$ is singular by I.7.2. As $\dim(U/W) \leq 1$, either $Z = W$ or $U = W + Wg$, and we may assume the latter. As $S = \langle x_6'\rangle$ is the unique singular point in $I - V_{10}$, either $\langle\mathscr{V}_1(U)\rangle = Z$ or $S \leq Wg$ and $U = W \oplus S$.

Assume $\langle\mathscr{V}_1(U)\rangle = Z$. If $Z \neq 0$, then $W \leq Z\Delta \geq Wg$, so $U \leq Z\Delta \leq V_{10}\theta$. But then $U \leq V_{10}\theta \cap I = V_{10}$, a contradiction. If $Z = 0$, then $\dim(U) = 2$ and $U = W \oplus Wg$. Here one can check (5) holds.

So take $U = W \oplus S$ with $S \leq Wg$. As S is the unique singular point in $Wg - Z$, either $Z = 0$ and $Wg = S$, or Z is a point and Wg a hyperbolic line. Then (4) or (3) holds in the respective case.

(8.7) *Let F be finite of order q and p a prime with $d_q(p) > 1$. Assume K is an elementary abelian p-group. Then one of the following holds*:

(1) $C_V(K)$ *is totally dark.*

(2) *$N_\Gamma(K)$ acts on a maximal parabolic.*
(3) *$C_V(K)$ is a special plane.*
(4) *$C_V(K) \in \mathscr{V}_9$ or $\mathscr{U}_9$.*

Proof. Assume otherwise and let $U = C_V(K)$. Then by 6.1, K fixes $(P, \Phi) \in \mathscr{P}$. As $d_q(p) > 1$, $P \leq U$. If $U_0 = \langle PN_\Gamma(K) \rangle \leq P\Delta$, then U_0 is singular, so (2) holds. Thus we may assume $Pg \not\leq P\Delta$ for some $g \in N_\Gamma(U)$. Then the projection S of Pg on Φ is singular and K centralizes a hyperbolic line in Φ through S, so we can take the special plane π of §4 to be contained in U; adopt the notation of that section. Next $\pi \neq U$ by hypothesis, so we may take $0 \neq W_2 \cap U$. If $C_V(K) \leq \pi + W_2$, then (2) holds by 8.6. So we may also take $0 \neq W_3 \cap U$. If $W_2 \cap U$ contains a singular point S, then U contains a hyperbolic line in W_2 through S. So in any event, there is a nonsingular $w_i \in U \cap W_i$, $i = 2, 3$. Define $\alpha(w_3)$ as in II.2.3 with respect of $z = w_2$. Then $w_1 = \alpha(w_3) \in U$ and $\langle \pi, w_1, w_2, w_3 \rangle = U_6 \in \mathscr{U}_6$.

So $K \leq C_G(U_6) = M$. By 5.4, $M \cong G_2(F)$. Hence by 15.1 in [2], one of the following holds:

(a) $d_q(p) = 2$ and $K \leq T_1^+$.
(b) $d_q(p) = 6$ and $K \leq T_2^+$.
(c) $d_q(p) = 3$ and $K \leq T_2^-$.

Here T_i^ε is a maximal torus of M of order $(q+1)^2$, $(q^3+1)/(q+1)$, $(q^3-1)/(q-1)$, in (a), (b), (c), respectively, and these tori are described in §15 of [2]. In particular, T_i^ε is contained in a subgroup M_ε of M isomorphic to $SL_3^{-\varepsilon}(F)$ and generated by root groups with $C_V(K) = C_V(M_\varepsilon)$ of rank 9. That is $C_V(K) \in \mathscr{U}_9$ or $\mathscr{V}_9$.

9. Overgroups of Cartan subgroups

In this section we continue the hypotheses and notation of §1. In addition, assume F is finite or algebraically closed and $|F| > 2$.

(9.1) (1) *$\hat{X}$ is the set of H_0-invariant points of V.*
(2) *Distinct members of $\hat{X}$ are not FH_0-isomorphic.*
(3) *$U \leq V$ is H_0-invariant, if and only if $U = \langle U \cap X \rangle$.*

Proof. Let $H_1 = C_{H_0}(V_1)$. Claim $V_1 = C_V(H_1)$. For $l(a)h(t) : x_1 \mapsto t^{-1}a_1x_1$, so $l(a)h(t) \in H_1$ if and only if $a_1 = t$. Hence if $a_1 = 1$, then $l(a) \in H_1$. Now for each $i > 1$, $l(a)$ does not centralize x_i, x_i', or x_{ij} for suitable a with $a_1 = 1$. Further if $a_1 \neq 1$, then $l(a)h(a_1)$ moves x_1'. So the claim is established.

Next by the claim and transitivity of $\mathrm{Wyl}(X)$ on $\hat{X}$, $P = C_V(C_{H_0}(P))$ for each $P \in \hat{X}$. Hence (2) holds. Then (2) implies (1) and (3).

(9.2) *Suppose* $K \leq H_0$ *with* $K \nleq Z(G)$ *and* $N_\Gamma(K)$ *is irreducible on* V. *Then either*

(1) $N_\Gamma(K) \leq N_\Gamma(\widehat{X})$, *or*

(2) $K^9 = 1$, $\Omega_1(K) \cong E_9$, *and* $N_\Gamma(\Omega_1(K))$ *is the stabilizer of a 3-decomposition of* V.

Proof. Let $V = \bigoplus_{\lambda\in\Lambda} V_\lambda$ be the eigenspace decomposition of V under K and $M = N_\Gamma(K)$. As $K \nleq Z(G)$, $|\Lambda| > 1$. As M is irreducible on V we conclude M is transitive on $\{V_\lambda : \lambda \in \Lambda\}$, $N_M(V_\lambda)$ is irreducible on V_λ for each $\lambda \in \Lambda$, and $\dim(V_\lambda) = 1, 3$, or 9. Pick $\alpha \in \Lambda$ and let $U = V_\alpha$. By 9.1, $U = \langle U \cap X\rangle$.

If $\dim(U) = 1$, then by 9.1, $\{V_\lambda : \lambda \in \Lambda\} = \widehat{X}$, so (1) holds.

Suppose $\dim(U) = 3$. As $N_M(U)$ is irreducible on U and $U = \langle U \cap X\rangle$, U is singular or a special plane. In the latter case $N_M(U)$ permutes the three singular points $\widehat{X} \cap U$ in U, so again $M \leq N_G(\widehat{X})$.

So take U singular. Let $D = \bigcap_{\lambda\in\Lambda} N_G(V_\lambda)$. Then $H_0 \leq D \trianglelefteq M$. If D is not irreducible on U, then U is the sum of three singular points permuted transitively by $N_M(U)$. By 9.1 these points are $\widehat{X} \cap U$, so again M acts on $\widehat{X}$. Thus we may assume D is irreducible on U. Thus D is irreducible on V_λ for all $\lambda \in \Lambda$.

Let $\alpha' \in \Lambda$ with $V_{\alpha'} \nleq \zeta(U)$. Then as D is irreducible on $V_{\alpha'}$, $V = V_{\alpha'} \oplus \zeta(U)$ and then $\alpha = (\alpha')'$. Now M permutes the pairs $\{V_\lambda, V_{\lambda'}\}$, $\lambda \in \Lambda$, contradicting $|\lambda| = 9$ odd.

So $\dim(U) = 9$. Suppose $K^3 \neq 1$. Then U is brilliant. As U is of dimension 9, U is not singular, so there exists $y \in U$, $\beta \in \Lambda$, $x \in X \cap V_\beta$ with $Q(x, y) \neq 0$. Then $0 \neq Q(x, y) = Q(xk, yk) = \beta(k)\alpha(k)^2 Q(x, y)$ for $k \in K$. So $\beta = \alpha^{-2}$. Similarly $\alpha^4 = (\alpha^{-2})^{-2} \in \Lambda$, and as $K^3 \neq 1$, $\alpha^4 \neq \alpha$. So $\Lambda = \{\alpha, \alpha^{-2}, \alpha^4\}$. Further $\alpha^{-8} = (\alpha^4)^{-2} \in \Lambda$, so $\alpha^{-8} = \alpha$ and hence $K^9 = 1$.

So replacing K by K^3, we may assume that either $K^3 = 1$ or K is cyclic and $K^3 = Z(G)$. Assume $K^3 = 1$ and let $k \in K - Z(G)$. Then U is contained in an eigenspace for k so all eigenspaces of k have a dimension divisible by 9. Hence by 8.2, U is an eigenspace for k and indeed as $N_M(U)$ is irreducible on U, 8.2 says $\Xi = \{V_\lambda : \lambda \in \Lambda\}$ is a 3-decomposition of V, $K \cong E_9$, and $N_G(K)$ is the stabilizer of Ξ.

Finally, assume $K = \langle k\rangle \cong Z_9$ and $Z(G) \leq K$. Let B be the subgroup of H_0 consisting of elements b with $b^3 \in Z(G)$. Then $D = E \times \langle b_0\rangle$, with $E \cong E_{3^6}$ the subgroup of elements of order 3 in H_0 and Eb_0 the set of elements cubing to some fixed generator of $Z(G)$. Now $B/Z(G)$ is dual to E as a Wyl(X)-module, so in particular as $E/Z(G)$ is a 5-dimensional orthogonal space for Wyl(X), Wyl(X) has two orbits on points of $B - E$. So there are two classes of cyclic subgroups $\langle j\rangle$ of order 9 with $\langle j^3\rangle = Z(G)$. One can check that representatives for these classes are $l(a)$, $l(b)$, where $a_i = \alpha$, $1 \leq i \leq 5$,

$a_6 = \alpha^4$, and $b_i = \alpha$, $1 \leq i \leq 4$, $b_5 = b_6 = \alpha^{-2}$. Further for $l(a)$:

$$\begin{aligned} V_\alpha &= \langle x_i, x_i' : 1 \leq i \leq 5 \rangle && \text{is of dimension } 10, \\ V_{\alpha^4} &= \langle x_6, x_6', x_{i6} : 1 \leq i \leq 5 \rangle && \text{is of dimension } 7, \\ V_{\alpha^{-2}} &= \langle x_{ij} : 1 \leq i \leq j \leq 5 \rangle && \text{is of dimension } 10, \end{aligned}$$

while for $l(b)$:

$$\begin{aligned} V_\alpha &= \langle x_i, x_i', x_{i5}, x_{i6} : 1 \leq i \leq 4 \rangle && \text{is of dimension } 16, \\ V_{\alpha^{-2}} &= \langle x_5, x_6, x_5', x_6', x_{ij} 1 \leq i \leq 4 \rangle && \text{is of dimension } 10, \\ V_{\alpha^4} &= \langle x_{56} \rangle . \end{aligned}$$

So $N_\Gamma(K)$ is not irreducible, but if $K = \langle l(a) \rangle$, then $C_G(K)$ is the Levi factor $N_G(\langle x_6, x_6' \rangle) \cap N_G(\langle x_{16}, \ldots, x_{56} \rangle)$, while if $K = \langle l(b) \rangle$, then $C_G(K)$ is the Levi factor $N_G(\langle x_{56} \rangle) \cap N_G(\Phi(x_5, x_6'))$.

(9.3) *If F is finite or algebraically closed and k is of order 9 in G with $Z(G) = \langle k^3 \rangle$, then F contains an element of order 9 and $C_G(k)$ is conjugate to one of the Levi factors $G_1 \cap N_G(\Phi(x_6', x_{16}))$ or $G_2 \cap N_G(\langle x_{12}, x_3', \ldots, x_6' \rangle)$.*

Proof. If F contains an element of order 9, we showed this near the end of the proof of the preceding lemma. So assume F is finite of order q and q is not congruent to $1 \bmod 9$. Let K be a cubic extension of F; then K is of order $q^3 \equiv 1 \bmod 9$, so we can take $k \in C_{H_0^K}(\tau)$ for some field automorphism τ of G^K. Then $\tau = \sigma g$ for some $g \in \mathrm{Wyl}(X)$, where σ is the field automorphism determined by X. Hence $k^\sigma = k^q$, so $k = k^\tau = k^{\sigma g} = k^{qg}$, and hence $k^{g^{-1}} = k^q$. But k has eigenvalues $\alpha, \alpha^4, \alpha^{-2}$, with α of order 9, while $q \equiv 4$ or $-2 \bmod 9$. So g permutes the three eigenvalues of g transitively, whereas we saw at the end of the proof of the previous lemma that the eigenspaces of k are not of the same dimension.

Theorem 9.4. *Let K be a subgroup of H_0 not contained in $Z(G)$. Then one of the following holds*:

(1) *$N_\Gamma(K)$ is brilliant.*
(2) *$N_\Gamma(K)$ acts on some special plane.*
(3) *$N_\Gamma(U)$ acts on $\widehat{X}$.*

In the remainder of this section, let $M = N_\Gamma(K)$ and assume M does not satisfy the conclusion of Theorem 9.4. Then by 9.2, M is not irreducible on V, so we may choose $0 \neq U$ to be a proper irreducible M-subspace of V. Indeed replacing K by $K\xi$ if necessary, we may assume $\dim(U) \leq 13$. Notice that if K is finite, then we may take K to be an elementary abelian p-group by 9.3. We establish Theorem 9.4 in a sequence of lemmas.

(9.5) *Each member of* $X \cap U$ *is in a special triple in* $X \cap U$.

Proof. Let $x \in X \cap U$. As M is not brilliant, neither is U. As M is irreducible on U, $U = \langle xN_G(U) \rangle$. Thus $U \not\leq x\Delta$, so by 9.1, there is $y \in X \cap U - x\Delta$. If $U \leq \langle x, y \rangle \theta$ for all such y, then $x \in U\theta$. Then as $U = \langle xN_G(U) \rangle$, U is brilliant, contradicting M not brilliant. So U contains the unique member of $\widehat{X} - \langle x, y \rangle \theta$ by 9.1, completing the proof.

By 9.5, U contains a special plane. Usually we will take the special plane π of §4 to be contained in U and adopt the notation of §4. In particular, v_i, W, W_i are as in §4. As 9.4.2 does not hold, $U \neq \pi$.

(9.6) $C_V(K) = 0$.

Proof. If not we may take $U \leq C_V(K)$. As $U \neq \pi$, $X \cap W$ is nonempty. Now if $C_V(K) \leq \pi + W_i$ for some i, then M is brilliant by 8.6, a contradiction. So $X \cap W_i$ is nonempty for at least two $i \in \{1, 2, 3\}$. Hence if for each distinct i, j and each $x \in W_i \cap C(K)$, $x\Delta \cap X \cap W_j$ is empty, 9.5 implies $U = \pi + \pi'$ for some special plane π' incident with π. But then M fixes the unique member of $\mathscr{V}_9$ containing $\pi + \pi'$, contradicting M not brilliant. Therefore we may take $x_2, x_2' \in C_V(K)$. Thus $K \leq K_2$, where

$$K_2 = C_{H_0}(\langle \pi, x_2, x_2' \rangle) = \langle l(1, 1, a_3, a_4, a_5, a_3^{-1}a_4^{-1}a_5^{-1}) : a_i \in F^\sharp \rangle .$$

Now $C_V(K_2) \in \mathscr{V}_9$, so as $N(K_2) \cap N(H_0)$ is transitive on $X - C_V(K_2)$, we may take $x_3 \in C_V(K)$. Thus $K \leq K_3 = \{l(1, 1, 1, a, a^{-1}, 1) : a \in F^\sharp\} \cong F^\sharp$ and $C_V(K_3) \in \mathscr{V}_{15}$, contradicting M not brilliant.

(9.7) K *is not a* 3-*group*.

Proof. If so by an earlier remark we may take K elementary abelian. Also we may take $Z(G) \leq K$. Now K is contained in the group $E \cong E_{3^6}$ of H_0 generated by elements of order 3. Let $\widetilde{E} = E/Z(G)$ and $Y = \mathrm{Wyl}(X)/O_2(\mathrm{Wyl}(X))$. As we observed in the proof of 8.2, we may regard $\widetilde{E}$ as a 5-dimensional orthogonal space over $GF(3)$ and Y as $O_5(3)$. By a Frattini argument, $M = C(K)N_M(H_0)$, so we may choose $\widetilde{K}$ to be minimal subject to being nontrivial and $N_Y(\widetilde{K})$-invariant. Hence $\widetilde{K}$ is nondegenerate or totally singular in the orthogonal space $\widetilde{E}$.

Let $E_0 = L \cap E$. The quadratic form q on $E = 0$ is $q(l(\omega^{e_1}, \dots, \omega^{e_6})) = \sum_i e_i^2$, where $\omega \in F$ is of order 3. Using this observation, we calculate that if $\widetilde{I}$ is a singular line in $\widetilde{E}$, then the eigenspaces for I form a 9-decomposition of V and hence if $I \leq K$, then $M \leq N(H_0)$, a contradiction. So the Witt index of $\widetilde{K}$ is at most 1. Now by 8.2, $\widetilde{K}$ is not a point, so $\widetilde{K}$ is a nondegenerate line or plane or a nondegenerate 4-space of Witt index 1. In each case we find that for some m, the eigenspaces of K of dimension m generate a subspace of V which is brilliant or a special plane.

Next $U = \bigoplus_{\lambda \in \Lambda} U_\lambda$ is the direct sum of eigenspaces U_λ for some set $\Lambda \leq \mathrm{Hom}(K, F^\sharp)$ of weights. As M is irreducible on U, M is transitive on Λ. Let $m = |\Lambda|$ and $a = \dim(U_\lambda)$, $\lambda \in \Lambda$. By 9.6 and transitivity of M on Λ, $1 \notin \Lambda$. Similarly, if $\lambda \in \Lambda$ with $\lambda^3 = 1$, then $K/C_K(U)$ is an elementary abelian 3-group, so as $M \leq N(C_K(U))$, replacing K by $C_K(U)$ if necessary, we can assume either $K^3 = 1$ or $C_V(K) \neq 0$, contradicting 9.6 or 9.7. Thus:

(9.8) *For* $\lambda \in \Lambda$, $\lambda^3 \neq 1$, *and hence* U_λ *is brilliant.*

(9.9) $m \geq 3$.

Proof. By 9.8, $m \geq 1$, so assume $m = 2$; say $U = U_\alpha \oplus U_\beta$. Let (x, y, z) be a special triple in $X \cap U$. Without loss $x, y \in U_\alpha$. Then by 9.8, $z \in U_\beta$, so $\alpha^2\beta = 1$. By symmetry $\alpha\beta^2 = 1$, so $\alpha^3 = 1$, contrary to 9.8.

(9.10) $N_{\mathrm{Wyl}(X)}(U)$ *is transitive on* $\widehat{X} \cap U$.

Proof. As $N_\Gamma(H_0)$ is the product of $\mathrm{Wyl}(X)$ with the pointwise stabilizer of $\widehat{X}$, it suffices to show $N_M(H_0)$ is irreducible on U. As H_0 fixes each U_λ, $N_M(H_0)$ is transitive on Λ, so it suffices to show $N_I(H_0)$ is irreducible on $Z = U_\lambda$, where $I = N_M(Z)$. As $C_M(Z)H_0$ is transitive on $H_0^G \cap C_M(Z)H_0$, by a Frattini argument it suffices to show $I^Z \cap N(H_0^Z)$ is irreducible on Z.

Now $\dim(Z) = a$ with $ma \leq 13$ and $m \geq 3$, so $a \leq 4$. If $a = 1$, then the result is trivial. As I is irreducible on Z, Z is singular, a hyperbolic line or 4-space, a special plane, or a conjugate of $\langle x_1, x_1', x_2, x_2' \rangle$. If Z is a hyperbolic line or special plane, then M acts on $\widehat{X} \cap U$. If Z is singular, then H_0^Z is the full diagonal group on Z and hence I^Z is an irreducible subgroup of $GL(Z)$ containing the diagonal group, so $N_I Z(H_0^Z)$ is irreducible on Z. A similar argument works in the remaining case where $I^Z = O_4^+(F)$ and H_0^Z is a Cartan group of I^Z.

(9.11) *If* $\dim(U/x\Delta^U) = 2$ *for some* $x \in X \cap U$, *then* U *is the direct sum of special planes transitively permuted by* M *and* $\dim(U) = 9$ *or* 12.

Proof. By 9.10, $\dim(U/x\Delta^U) = 2$ for all $x \in X \cap U$. Thus $U = \bigoplus_{A \in R} \theta A$ for some set R of H_0-invariant special planes transitively permuted by M (cf. 1.7 in [2]; the same proof works when f is symmetric since members of X are singular). Thus $\dim(U) = 3k$ and as $3 < \dim(U) \leq 13$, $k = 2, 3, 4$. Without loss π and $\pi' = \langle x_2', x_{26}, x_6 \rangle$ are in R. If $R = \{\pi, \pi'\}$, then M acts on $U + \sum(\pi, \pi') \in \mathscr{V}_9$, contradicting M not brilliant. So $\dim(U) = 9$ or 12.

(9.12) $\dim(U) \geq 9$.

Proof. We may take $\pi \leq U$ and adopt the notation of §4. By 9.11, we may assume $\dim(U/x\Delta^U) > 2$ for each $x \in X \cap U$. Thus $|W_i \cap \widehat{X} \cap U| \geq 2$ for each $i = 1, 2, 3$. Hence $\dim(U) \geq 9$.

Let $N = N_{\mathrm{Wyl}(X)}(U)$ and $N^* = N^{(\widehat{X}\cap U)}$. Then N^* is a section of $W(E_6) \cong O_6^-(2)$, so $|N^*| = 2^i 3^j 5^k$. Hence as N is transitive on $\widehat{X}$, $\dim(U) \leq 13$, and $\dim(U) \geq 9$, we have

(9.13) $\dim(U) = 9, 10$, *or* 12.

(9.14) N^* *is a* $\{2, 3\}$*-group and* $\dim(U) = 9$ *or* 12.

Proof. If not we may take $n^* \in N^*$, where n is the element of $L \cap \mathrm{Wyl}(X)$ induced by $(1, 2, 3, 4, 5)$. Hence n^* has orbits

$$\{x_6\}, \{x_6'\}, \{x_1, \ldots, x_5\}, \{x_1', \ldots, x_5'\}, \qquad Y_1 = \{x_{16}, \ldots, x_{56}\}$$
$$Y_2 = \{x_{12}, x_{23}, x_{34}, x_{45}, x_{51}\}, \qquad Y_3 = \{x_{13}, x_{24}, x_{35}, x_{14}, x_{25}\}$$

on $\widehat{X}$. By 9.12, $\dim(U) = 9, 10$, or 12, so n has orbit structure $5^2, 1^1, 5^2$ on $\widehat{X} \cap U$. We conclude $\widehat{X} \cap U = Y_1 \cup Y_2$ or $Y_1 \cup Y_3$. But now for $x \in Y_1$, $\dim(U/x\Delta^U) = 2$, contradicting 9.11.

Next $W(E_6)$ has three classes of elements of order 3 with representatives μ_i, $1 \leq i \leq 3$, where μ_3 has nine orbits of length 3 on $\widehat{X}$ forming a 9-decomposition of V, and μ_1 and μ_2 are induced by the elements $(1, 2, 3)$ and $(1, 2, 3)(4, 5, 6)$ in $L \cap \mathrm{Wyl}(X)$, respectively.

(9.15) $\mu_3 \notin N^*$.

Proof. Assume $\mu_3 \in N^*$. We may assume the orbits of μ_3 make of the 9-decomposition S of II.3.5.3. $N_{W(E_6)}(\langle\mu_3\rangle)$ induces $GL_2(3)/E_9$ acting 2-transitively on the nine orbits of μ_3 on $\widehat{X}$. Now argue as in the proof of 7.4.

(9.16) *Either* $\langle\mu_i\rangle$ *or* $\langle\mu_1, \mu_2\rangle$ *is a Sylow* 3*-group of* N^*, *and if* $\mu_1 \in N^*$ *fixes a point of* $\widehat{X} \cap U$, *then* $\langle\mu_1, \mu_2\rangle$ *is Sylow and* $\dim(U) = 12$.

Proof. A 3-subgroup of $W(E_6)$ containing no conjugate of μ_3 is conjugate to $\langle\mu_1\rangle$, $\langle\mu_2\rangle$, or $\langle\mu_1, \mu_2\rangle$. The remaining remark follows from 9.14 and the transitivity of N^* on $\widehat{X} \cap U$.

(9.17) $\mu_1 \notin N^*$.

Proof. Assume $\mu_1 \in N^*$. Notice $\langle x_i, x_i', x_{ij} : i, j \in \{1, 2, 3\}\rangle = A \in \mathscr{V}_9$.

As $N_{W(E_6)}(\langle\mu_1\rangle)$ is transitive on the orbits of μ_1 of length 3, we may take $V_3 \leq U$. By 9.5, either $V_3' + \langle x_{12}, x_{13}, x_{23}\rangle \leq U$ or we may take x_4' and $\sigma = \langle x_{14}, x_{24}, x_{34}\rangle$ in U. In the first case $A \leq U$ and we find A is M-invariant, contradicting M not brilliant.

So x_4', σ are in U. By 9.16, $\dim(U) = 12$ and $\mu_2 \in N^*$. But now the image $V_1 + \langle x_4', x_5', x_6'\rangle + A$ of V_2, x_4', σ under μ_2 is contained in U, contradicting $\dim(U) = 12$.

By 9.16 and 9.17, $\langle \mu_2 \rangle$ is Sylow in N^*. So by transitivity of N^* on $\widehat{X} \cap U$ and 9.14, $\dim(U) = 12$. Let $A_1 = \langle x_i, x_i', x_{ij} : i, j \in \{1, 2, 3\} \rangle$, $A_2 = \langle x_{ir} : i \in \{1, 2, 3\}, r \in \{4, 5, 6\} \rangle$, and $A_3 = \langle x_r, x_r', x_{rs} : r, s \in \{4, 5, 6\} \rangle$. Then $\alpha = \{A_1, A_2, A_3\}$ is a 3-decomposition of V and μ_2 acts on A_i. Indeed the orbits of μ_2 on $\widehat{X} \cap A_2$ and $\widehat{X} \cap A_3$ are singular planes, while the orbits on $\widehat{X} \cap A_2$ are special planes. As the normalizer of μ_2 is transitive on the singular orbits, we may take $V_3 \leq U$. Then by 9.5, either $V_3' + \langle x_{12}, x_{13}, x_{23} \rangle$ or $B = \langle x_4', x_5', x_6', x_{14}, x_{25}, x_{26} \rangle$ is contained in U. In the first case M acts on A, contradicting M not brilliant. So the latter holds and by 9.5 we may take $U = V_3 + B + \langle x_{16}, x_{24}, x_{35} \rangle$. Now we calculate that $U\theta = \langle \widehat{X} - (U + P) \rangle$, $P = \langle x_{15}, x_{26}, x_{34} \rangle$. Then M acts on $C = U + U\theta$, so $M\xi$ acts on $C\phi = P$, for our final contradiction. Thus Theorem 9.4 is at last established.

(9.18) *If $I \leq \mathrm{Aut}(G)$ and $I \cap \Gamma$ acts on some maximal parabolic of G, then one of the following holds:*

(1) *I acts on some proper parabolic of G.*

(2) *I acts on some conjugate of L.*

(3) *I acts on some conjugate of the Levi factor $G_1 \cap N_G(\Phi(x_{16}, x_6'))$.*

Proof. Assume $J = I \cap \Gamma$ acts on some maximal parabolic P, but none of the conclusions of the lemma hold. Let $b \in I - J$. Then I acts on $Y = P \cap P^b$. If $R(Y) \neq 1$, then (1) holds by the Borel-Tits theorem, so take $R(Y) = 1$. Then by §2.8 in [4], Y is a Levi factor of P and P^b. It follows that either (3) holds or up to conjugation $P = G_2$ and $P^b = N_G(U)$, $U = \langle x_{12}, x_3', \ldots, x_6' \rangle$. But then $N_B(Y)$ acts on some conjugate of L.

(9.19) *Let K be a subgroup of H_0 not contained in $Z(G)$ and let $B = \mathrm{Aut}(G)$. Then one of the following holds:*

(1) *$N_\Gamma(K)$ acts on some maximal parabolic of G, or*

(2) *$N_B(K) \leq N_B(N_\Gamma(\alpha))$ for some 3-decomposition α of V, some $\alpha \in \widehat{X}^G$, or some $\alpha \in \mathscr{V}_{12}$ or $\mathscr{U}_3$.*

Proof. Assume otherwise and let $M = N_\Gamma(K)$. Suppose first M is brilliant. As M acts on no maximal parabolic of G, M acts on a member of $\mathscr{V}_{12}$ or $\mathscr{V}_9$. Assume M acts on L, and let $SL_2(F) \cong J \leq C_G(L)$. We have a representation of ML on a 6-dimensional F-space U. As M acts on no maximal parabolic of L, M is irreducible on U. Then as $H_0 \leq M$, one of the following holds: (a) $L \leq M$; (b) $L \cap M \leq N(H_0)$; (c) $L_1 L_2 \trianglelefteq L \cap M$ with $L_i \cong SL_3(F)$; (d) $L_1 L_2 L_3 \trianglelefteq L \cap M$ with $L_i \cong SL_2(F)$. In (a), $L \trianglelefteq N_B(K)$, so (2) holds. In (c), $L_1 L_2$ preserves a unique 3-decomposition, so (2) holds. In (d), $JL_1L_2L_3 = J_0 \trianglelefteq N_B(K)$ and $C_V(J_0)$ is a special plane, so (2) holds.

Thus we may assume (a) holds. Similarly, either $J \leq M$ or $J \cap M \leq N(H_0)$. In the latter case, $H_0 \trianglelefteq N_B(K)$, so (2) holds. In the former $J \trianglelefteq N_B(K)$ and again (2) holds.

So assume M acts on $A_1 \in \mathscr{V}_9$ and let $\alpha = \{A_1, A_2, A_3\}$ be the 3-decomposition containing A_1 and $M_i = C_G(A_i)$. As above, $M_i \not\leq M$ and $M_1 \cap M$ and $M_2M_3 \cap M$ do not both act on H_0. We conclude some element of M induces a graph automorphism on M_1 and $SL_2(F) \cong L_1 \trianglelefteq M_1 \cap M$. But $N_\Gamma(L_1) = N_\Gamma(U)$ for some $U \in \mathscr{V}_{12}$.

Thus M is not brilliant, so by 9.4 we may assume M acts on the special plane π of §4. Let $J = C_G(\pi)$, so that $J \cong \mathrm{Spin}_8^+(F)$. Again $J \not\leq M$ nor does $M \cap J$ act on H_0. As M acts on no maximal parabolic, some $m \in M$ is transitive on the three singular points in π and hence induces triality on J. Then as M acts on no proper parabolic of J, we conclude there is $Y \trianglelefteq N_B(K)$ with Y the central product of one, three, or four copies of $SL_2(F)$, leading us to the usual contradiction.

10. The proof of Theorem 3

In this section we assume the hypotheses and notation of §1. In addition, assume F is algebraically closed or finite. Using results in previous sections, we establish Theorem 3. First a lemma:

(10.1) *Let K be an extraspecial 3-subgroup of G of exponent 3 with $Z(K) = Z(G)$. Then*:

(1) $|K| = 27$.

(2) *If $N_G(K)$ is irreducible on $K/Z(K)$, then all elements of $K - Z(G)$ are of Type 2 and $N_{\mathrm{Aut}(G)}(K)$ acts on $N_G(U)$ for some U in $\mathscr{U}_6$, $\mathscr{V}_9$, or $\mathscr{U}_9$.*

Proof. Let $g \in K - Z(G)$ and $Z(G) = \langle z \rangle$. Then there is $h \in K$ with $\langle g, h \rangle = K_0$ extraspecial of order 27, and we may take $[g, h] = z$. As $Z(G)$ is of order 3, F contains an element ω of order 3 and $z = \omega I$. Let $V_i(g)$ be the ω^i-eigenspace of g, with the index i read mod 3. Then $V_{i+1}(g)h = V_{i+1}(g^h) = V_{i+1}(gz) = V_i(g)$, so $\langle h \rangle$ is transitive on the eigenspaces of g. In particular, $\dim(V_i(g)) = 9$ for each i, so by 8.1, g is of Type 2 or 3.

We will show $O^{3'}(C_G(K_0))K_0$ has a subgroup H of index at most 3 such that $H = K_0 \times H_1$ with $H_1 \trianglelefteq C_G(K_0)$. It will follow from this claim that K_0 is a maximal extraspecial 3-subgroup of G, and hence $K = K_0$; in particular, (1) will be established. For, if $K \neq K_0$, then $K = K_0K_1$ with $K_1 = C_K(K_0)$ extraspecial. Now there is $1 \neq x \in H_1 \cap K_1$, and as K_1 is extraspecial there is $y \in K_1$ with $[y, x] = z$. Thus $z \in [y, H_1] \leq H_1$, contradicting $K_0 \cap H_1 = 1$.

Suppose first that g is of Type 3. Adopt the notation of part II, §§3 and 4. We may assume that $A_i = V_i(g)$. Thus h permutes the subgroups M_i transitively, so $C_{M_0}(h) = Z(G) \times H_1$ with $H_1 \cong L_3(F)$. Further $O^{3'}(C_G(g)) = M_0H_0$, so K_0H_1 is of index at most 3 in $O^{3'}(C_G(K_0))K_0$, establishing our claim in this case.

In addition we claim in this case that $N_G(K_0)$ is not irreducible on $K_0/Z(G)$. For otherwise some $x \in N_G(K_0)$ induces an automorphism of order 4 on K_0.

Then x^2 inverts $K_0/Z(G)$, so without loss, x^2 inverts h and g. Hence by II.3.6, x^2 interchanges M_2 and M_3 and induces a graph automorphism on M_1. So x^2 induces a graph automorphism on H_1. This is impossible, as $H_1 = E(C_G(K_0)) \trianglelefteq N_G(K_0)$, whereas a graph automorphism of $L_3(F)$ is not a square in $\mathrm{Aut}(L_3(F))$.

So we may assume all elements of $K_0 - Z(G)$ are of Type 2. In particular, we may adopt the notation of §4 and assume $V_i(g) = \langle v_i, W_i \rangle$. Let $\Omega = C_G(\pi)^\infty$, so that $\Omega \cong \mathrm{Spin}_8^+(F)$. Then h is transitive on the singular points of π and induces a triality automorphism on Ω. Define α as in §5 and let $K_2 = \langle g, \alpha \rangle$. Then $K_2 \cong 3^{1+2}$ and $O^{3'}(C_G(K_2)) = G_2 \cong G_2(F)$ by 5.4. Then from the main theorem of [2], there is an element β of order 3 in G_2 such that $O^{3'}(C_{G_2}(\beta)) = G_3 \cong SL_3^\varepsilon(F)$ and G_3 is generated by three root groups of G. Moreover up to conjugation in $C_G(g)$, $h = \alpha$ or $\alpha\beta$; let $H_1 = G_2$ or G_3 in the respective case. Then the claim is established, so $K = K_0$.

Finally $H_1 = E(C_G(K)) \trianglelefteq N_{\mathrm{Aut}(G)}(K)$. Let $U = C_V(H_1)$. Then $U \in \mathscr{U}_6$, if $H_1 = G_2$ by 5.4 and 5.5, while if $H_1 = G_3$, then as $G_3 \cong SL_3^\varepsilon(F)$, H_1 is generated by three root groups of G, so $U \in \mathscr{V}_9$ or $\mathscr{U}_9$ for $\varepsilon = +$ or $-$, respectively. Now 6.8 completes the proof.

Now to the proof of Theorem 3. Part (2) has already been established in various lemmas. For most types T of structures in $\mathscr{C}$, §1 indicates the lemma in which the transitivity of Δ on T is established. To prove parts (1) and (3), we begin a short series of reductions. Let K be a closed solvable subgroup of G with $K \not\leq Z(G)$, and assume K does not satisfy the conclusion of Theorem 3. Subject to this constraint, pick K so that the length of the derived series for K is minimal.

(10.2) *K is abelian.*

Proof. Assume K is not abelian. Then $[K, K] \neq 1$ and by minimality of the derived length $d(K)$ of K, $[K, K] = Z(G)$. So K is nilpotent of class 2. By minimality of $d(K)$, $Z(K) = Z(G)$. But as $Z(G)$ is of order 3 and K is of class 2, $K^3 \leq Z(K)$. Thus $Z(G) = \Phi(K)$ and K is a finite extraspecial 3-group. Proceeding by induction on the order of K, K is of exponent 3 and $N_G(K)$ is irreducible on $K/Z(G)$. Now 10.1 supplies a contradiction and completes the proof.

(10.3) $R(K) = 1$.

Proof. If not by the Borel-Tits theorem, $N_{\mathrm{Aut}(G)}(K)$ acts on a proper parabolic.

(10.4) *K is semisimple on V.*

Proof. This follows, as K is abelian with $R(K) = 1$ and F is perfect.

(10.5) *K is not contained in a Cartan subgroup of G.*

Proof. This follows from 9.18 and 9.19.

(10.6) (1) *F is finite of order q.*

(2) *K is an elementary abelian p-group for some prime p with $d_q(p) > 1$.*

Proof. If F is finite, then by induction on the order of K we may take K to be a p-group for some prime p and either K is elementary abelian or $p = 3$ and $K \cong Z_9$ with $Z(G) = K^3$. The last case is out by 9.3. So if F is finite of order q, then K is an elementary abelian p-group. Hence if $d_q(p) = 1$, then by 6.3 and 10.5, $p = 2$ or 3. Similarly if F is algebraically closed, then by 6.3 and 10.5, $K^6 = 1$. Then K is finite and by induction on the order of K, K is an elementary abelian p-group for $p = 2$ or 3.

So in any event, we may assume K is a finite elementary abelian p-group for $p = 2$ or 3, and if $p = 3$, then F contains an element of order 3. Now 6.7 supplies a contradiction when $p = 2$, while 8.3 supplies a contradiction when $p = 3$.

(10.7) *$C_V(K) = 0$ and $d_q(p) = 3$ or 9.*

Proof. By 8.5 we may assume $U = C_V(K) \neq 0$. Then by 8.7, 6.8, and 9.18, U is totally dark. Finally 8.4 and 6.8 complete the proof.

If $d_q(p) = 3$, then 7.4 supplies a contradiction. If $d_q(p) = 9$, then 7.5 supplies a contradiction. Thus the proof of Theorem 3 is complete.

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