

ITERATED SPINNING AND HOMOLOGY SPHERES

ALEXANDER I. SUCIU

ABSTRACT. Given a closed n -manifold M^n and a tuple of positive integers P , let $\sigma_P M$ be the P -spin of M . If $M^n \not\cong S^n$ and $P \neq Q$ (as unordered tuples), it is shown that $\sigma_P M \not\cong \sigma_Q M$ if either (1) $H_*(M^n) \not\cong H_*(S^n)$, (2) $\pi_1 M$ finite, (3) M aspherical, or (4) $n = 3$. Applications to the homotopy classification of homology spheres and knot exteriors are given.

In this paper we study manifolds obtained by iterated p -spinning. The process of spinning was introduced by Artin [2] and generalized to p -spinning by Epstein [5] and Cappell [3]. As noticed by the latter, $p+q$ -spinning is not in general the same as p -spinning followed by q -spinning. We expand on this remark of Cappell and show how to distinguish among the various iterated spins of a given manifold by means of their homology or the homology of their universal covers. As an application, we prove that successively spinning 3-manifolds in different ways produces manifolds with distinct homotopy types. Other applications concern homology spheres. For every $n \geq 5$, we construct homology n -spheres distinguished by their r th homotopy groups ($2 \leq r \leq \frac{n-1}{2}$), or by their first k -invariants. Finally, we discuss iterated spinning of knots.

1. SPUN MANIFOLDS

Let M^n , $n \geq 1$, be a closed, smooth n -manifold with basepoint $*$, and p a positive integer. Let D^n be a fixed embedded disk in M^n and let $M_0^n = M^n \setminus \text{int } D^n$ be a punctured copy of M . The p -spin of M is the closed, smooth $n+p$ -manifold

$$\sigma_p M = \partial(M_0 \times D^{p+1}) = M_0 \times S^p \cup_{S^{n-1} \times S^p} S^{n-1} \times D^{p+1}.$$

In other words, $\sigma_p M$ is obtained from $M \times S^p$ by performing surgery on $* \times S^p$ with trivial framing. If $n \geq 3$, then $\sigma_p M$ has the same fundamental group as M by Van Kampen's theorem. Another nice feature of p -spinning is that it respects connected sums.

Received by the editors April 5, 1988.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 57R19; Secondary 55Q52, 57Q45.

Key words and phrases. p -spinning, homology sphere, homotopy type.

Partially supported by a grant from Northeastern University's Research and Scholarship Development Fund.

Lemma 1.1 (Gordon [7]). $\sigma_p(M_1 \# M_2) \cong \sigma_p M_1 \# \sigma_p M_2$.

The process of p -spinning can of course be iterated. Given an r -tuple $P = (p_1, \dots, p_r)$, $p_i > 0$, we define the P -spin of M to be

$$\sigma_P M = \sigma_{p_1} \cdots \sigma_{p_r} M.$$

It is a closed, smooth manifold of dimension $n + |P|$, where $|P| = p_1 + \cdots + p_r$. We also have $\pi_1(\sigma_p M) \cong \pi_1 M$ (if $n \geq 3$) and $\sigma_p(M_1 \# M_2) \cong \sigma_p M_1 \# \sigma_p M_2$. The next lemma shows that $\sigma_p M$ depends only on the *unordered* tuple P .

Lemma 1.2. $\sigma_p \sigma_q M \cong \sigma_q \sigma_p M$.

Proof. Recall that $\sigma_p M$ is the result of surgery on $* \times S^p$ in $M \times S^p$. Therefore there is a cobordism $W_p(M) = M \times S^p \times I \cup h^{p+1}$ from $M \times S^p$ to $\sigma_p M$. Then $W_p(M) \times S^q \cong M \times S^p \times S^q \times I \cup h^{p+1} \cup h^{p+q+1}$ is a cobordism from $M \times S^p \times S^q$ to $\sigma_p M \times S^q$. Also $W_q(\sigma_p M) = \sigma_p M \times S^q \times I \cup h^{q+1}$ is a cobordism from $\sigma_p M \times S^q$ to $\sigma_q \sigma_p M$. Finally,

$$\begin{aligned} W_{q,p}(M) &= W_p(M) \times S^q \cup_{\sigma_p M \times S^q} W_q(\sigma_p M) \\ &= M \times S^p \times S^q \times I \cup h^{p+1} \cup h^{p+q+1} \cup h^{q+1} \end{aligned}$$

is a cobordism from $M \times S^p \times S^q$ to $\sigma_q \sigma_p M$. Now rearrange the handles to get

$$W_{q,p}(M) \cong M \times S^q \times S^p \times I \cup h^{q+1} \cup h^{q+p+1} \cup h^{p+1} = W_{p,q}(M),$$

a cobordism from $M \times S^q \times S^p$ to $\sigma_p \sigma_q M$. This proves the lemma. \square

If M is a manifold admitting a simple geometric description it is often not too difficult to identify $\sigma_p M$. For example, $\sigma_p S^n$ is S^{n+p} . We leave as an exercise for the reader to show $\sigma_p(\mathbb{RP}^n) = S^{n-1} \times_{\mathbb{Z}_2} S^{p+1}$ (where the generator of \mathbb{Z}_2 acts by (antipodal, reflection)), $\sigma_p(\mathbb{CP}^n) = \mathbb{CP}^{n-1} \times S^{p+2}$, and $\sigma_p(\mathbb{HP}^n) = \mathbb{HP}^{n-1} \times S^{p+4}$. What we really need is to identify the P -spins of product of spheres.

Lemma 1.3. (i) $\sigma_p(S^m \times S^n) \cong S^{m+p} \times S^n \# S^m \times S^{n+p}$.

(ii) $\sigma_P(S^m \times S^n) \cong \#_{R \subseteq P} S^{m+|R|} \times S^{n+|P|-|R|}$.

Proof. (i) By definition, $\sigma_p(S^m \times S^n) = \partial W$, where $W = (S^m \times S^n)_0 \times D^{p+1}$. As $(S^m \times S^n)_0 \cong D^{m+n} \cup h^m \cup h^n$, we have $W \cong D^{m+n+p+1} \cup h^m \cup h^n$. The attaching spheres for the handles of W form a link $S^{m-1} \cup S^{n-1} \subset S^{m+n+p}$ of codimension $p+2 > 2$. By general position this is the unlink. Thus

$$\begin{aligned} W &\cong (D^{m+n+p+1} \cup h^m) \natural (D^{m+n+p+1} \cup h^n) \\ &\cong S^m \times D^{n+p+1} \natural S^n \times D^{m+p+1}. \end{aligned}$$

This implies $\partial W \cong S^m \times S^{n+p} \# S^n \times S^{m+p}$.

(ii) Follows from (i) and Lemma 1.1. The connected sum is taken over all subsets R of P , including $R = P$ and $R = \emptyset$ (where $|\emptyset| = 0$). \square

In particular, we have a complete description of spins of 2-manifolds:

$$\sigma_p(S^2) = S^{p+2}, \quad \sigma_p(\#_1^g S^1 \times S^1) = \#_1^{2g} S^{p+1} \times S^1, \quad \sigma_p(\#_1^h \mathbb{RP}^2) = \#_1^h S^{p+1} \tilde{\times} S^1.$$

Spun 3-manifolds were studied in [13], where it is shown that if M^3 is oriented and geometric, then $\pi_1 M$ determines the oriented homotopy type of $\sigma_1 M$.

We are now in position to describe the universal cover of the P -spin of M^n , $n \geq 3$, in terms of the P -spin of the universal cover of M^n . This will generalize the description in Plotnick [10], done there for $n = 3$ and $p = 1$. First consider $\widetilde{\sigma_p M}$. Let $\pi = \pi_1 M \cong \pi_1(\sigma_p M)$. Then $\widetilde{\sigma_p M}$ is obtained by taking the cover $\widetilde{M} \times S^p$ of $M \times S^p$ and performing (equivariant) surgery on the $|\pi|$ lifts of $* \times S^p$. That is,

$$\widetilde{\sigma_p M} = \widetilde{M}_0 \times S^p \cup \coprod_{\pi} S^{n-1} \times S^p \coprod_{\pi} S^{n-1} \times D^{p+1}.$$

The first surgery yields $\sigma_p \widetilde{M}$. The other $|\pi| - 1$ surgeries are done on nullhomotopic S^p 's, so they add copies of $S^{n-1} \times S^{p+1}$. Therefore

$$\sigma_p \widetilde{M} \cong \sigma_p \widetilde{M} \# \#_1^{|\pi|-1} S^{n-1} \times S^{p+1}.$$

Two particular cases are worth mentioning:

$$\text{If } M = S^n / \pi, \text{ then } \sigma_p \widetilde{M} \cong \#_1^{|\pi|-1} S^{n-1} \times S^{p+1};$$

$$\text{If } M = \mathbb{R}^n / \pi, \text{ then } \sigma_p \widetilde{M} \cong S^{n-1} \times \mathbb{R}^{p+1} \# \#_1^{\infty} S^{n-1} \times S^{p+1}.$$

We now generalize to iterated spins.

Theorem 1.4. $\sigma_p \widetilde{M} \cong \sigma_p \widetilde{M} \# \#_1^{|\pi|-1} \#_{\emptyset \neq R \subseteq P} S^{n-1+|P|-|R|} \times S^{|R|+1}.$

Proof. We use the above formula, Lemmas 1.1 and 1.3, and induction on length (P) . If $P = (p, P')$ then

$$\begin{aligned} \sigma_p \widetilde{M} &= \sigma_p(\widetilde{\sigma_{P'} M}) \cong \sigma_p(\sigma_{P'} \widetilde{M}) \# \#_1^{|\pi|-1} S^{n-1+|P'|} \times S^{p+1} \\ &\cong \sigma_p(\sigma_{P'} \widetilde{M} \# \#_1^{|\pi|-1} \#_{\emptyset \neq R' \subseteq P'} S^{n-1+|P'|-|R'|} \times S^{|R'|+1}) \# \#_1^{|\pi|-1} S^{n-1+|P'|} \times S^{p+1} \\ &\cong \sigma_p(\sigma_{P'} \widetilde{M}) \# \#_1^{|\pi|-1} \#_{\emptyset \neq R' \subseteq P'} (S^{n-1+|P'|-|R'|} \times S^{|R'|+1} \\ &\quad \# S^{n-1+|P'|-|R'|-p} \times S^{|R'|+p+1}) \# \#_1^{|\pi|-1} S^{n-1+|P'|} \times S^{p+1} \\ &= \sigma_p \widetilde{M} \# \#_1^{|\pi|-1} \#_{\emptyset \neq R \subseteq P} S^{n-1+|P|-|R|} \times S^{|R|+1}. \quad \square \end{aligned}$$

2. HOMOLOGY OF $\sigma_p M$

In this section we compute the homology groups of $\sigma_p M$ and show how quite often they can be used to distinguish between $\sigma_p M$ and $\sigma_Q M$, if $P \neq Q$. Of course, if $|P| \neq |Q|$, then $\dim \sigma_p M \neq \dim \sigma_Q M$ and there is nothing to prove.

So from now on we will implicitly assume $|P| = |Q|$ whenever we compare $\sigma_P M$ to $\sigma_Q M$.

Lemma 2.1. $H_i(\sigma_P M) \cong H_i(M_0) \oplus \tilde{H}_{i-p}(M)$.

Proof. The Mayer-Vietoris sequence for the decomposition $\sigma_P M = M_0 \times S^p \cup_{S^{n-1} \times S^p} S^{n-1} \times D^{p+1}$ reduces to the exact sequences

$$\begin{aligned} 0 \rightarrow H_i(M_0 \times S^p) \rightarrow H_i(\sigma_P M) \rightarrow 0, \quad i \neq p, p+n-1, p+n; \\ 0 \rightarrow H_p(S^p) \rightarrow H_p(M_0 \times S^p) \rightarrow H_p(\sigma_P M) \rightarrow 0; \\ 0 \rightarrow H_{p+n}(\sigma_P M) \rightarrow H_{p+n-1}(S^{n-1} \times S^p) \rightarrow H_{p+n-1}(M_0 \times S^p) \\ \rightarrow H_{p+n-1}(\sigma_P M) \rightarrow 0. \end{aligned}$$

The result follows from the Künneth formula. \square

Theorem 2.2. $H_i(\sigma_P M) \cong H_i(M_0) \oplus \bigoplus_{\emptyset \neq R \subseteq P} \tilde{H}_{i-|R|}(M_0) \oplus \tilde{H}_{i-|P|}(M)$.

Proof. It suffices to prove that $\tilde{H}_i(\sigma_P M_0) \cong \bigoplus_{R \subseteq P} \tilde{H}_{i-|R|}(M_0)$. This is done by induction on length (P) . For $P = (p)$, this is immediate from Lemma 2.1. For $P = (p, P')$ we have

$$\begin{aligned} \tilde{H}_i(\sigma_P M_0) &= \tilde{H}_i(\sigma_P(\sigma_{P'} M_0)) \\ &\cong \tilde{H}_i(\sigma_{P'} M_0) \oplus \tilde{H}_{i-p}(\sigma_{P'} M_0) \\ &\cong \bigoplus_{R' \subseteq P'} \tilde{H}_{i-|R'|}(M_0) \oplus \bigoplus_{R' \subseteq P'} \tilde{H}_{i-p-|R'|}(M_0) \\ &= \bigoplus_{R \subseteq P} \tilde{H}_{i-|R|}(M_0). \quad \square \end{aligned}$$

In particular, if M^n is a *homology n -sphere*, that is if $H_*(M^n) \cong H_*(S^n)$, then $\sigma_P M$ is a homology $n+|P|$ -sphere. This means that we cannot distinguish homologically among the various iterated spins of a given homology sphere (unless they have different dimensions). For other manifolds though, we have the following:

Theorem 2.3. If $H_*(M^n) \not\cong H_*(S^n)$ and $P \neq Q$ (as unordered tuples), then $H_*(\sigma_P M) \not\cong H_*(\sigma_Q M)$.

Proof. By assumption we have $\tilde{H}_*(M_0^n) \not\cong \tilde{H}_*(D^n)$, so let i_0 be the smallest integer such that $\tilde{H}_{i_0}(M_0^n) \neq 0$. Arrange P and Q in ascending order and assume $P < Q$. Let j be the smallest integer such that $p_j < q_j$. Then write $P = (P', P'')$, $Q = (P', Q'')$, where $P' = (p_1 \leq \dots \leq p_{j-1})$ and $P'' = (p_j = \dots = p_{j+k-1} < p_{j+k} \leq \dots \leq p_r)$, for some $k \geq 1$. (If $j = 1$, $P = P''$, $Q = Q''$.) For $i < p_j + i_0$, one can check that $H_i(\sigma_P M) \cong H_i(\sigma_Q M)$. For $i = p_j + i_0$

Thus

$$H_i(\widetilde{\sigma_p M}) \cong \begin{cases} H_i(\widetilde{M_0}), & i < p+1, \\ H_i(\widetilde{M_0}) \oplus I\pi, & i = p+1, \\ H_i(\widetilde{M_0}) \oplus H_{i-p}(\widetilde{M_0}), & p+1 < i < p+n-1, \\ H_i(\widetilde{M}), & i \geq p+n-1. \quad \square \end{cases}$$

We are now ready to describe the homology of $\widetilde{\sigma_P M}$. For P a tuple we will use a generalized Kronecker delta given by $\Delta(i, P) = \sum_{\emptyset \neq R \subseteq P} \delta(i, |R|)$.

Theorem 3.2.

$$H_i(\widetilde{\sigma_P M}) \cong H_i(\widetilde{M_0}) \oplus \bigoplus_{\emptyset \neq R \subseteq P} \widetilde{H}_{i-|R|}(\widetilde{M_0}) \oplus \widetilde{H}_{i-|P|}(\widetilde{M}) \oplus (I\pi)^{\Delta(i-1, P)}.$$

Proof. It suffices to prove that $\widetilde{H}_i(\sigma_P M_0) \cong \bigoplus_{R \subseteq P} \widetilde{H}_{i-|R|}(\widetilde{M_0}) \oplus (I\pi)^{\Delta(i-1, P)}$. Again this is done by induction on length (P) . If $P = (p, P')$, then

$$\begin{aligned} \widetilde{H}_i(\widetilde{\sigma_P M_0}) &= \widetilde{H}_i(\sigma_p(\widetilde{\sigma_{P'} M_0})) \\ &\cong \widetilde{H}_i(\widetilde{\sigma_{P'} M_0}) \oplus \widetilde{H}_{i-p}(\widetilde{\sigma_{P'} M_0}) \oplus (I\pi)^{\delta(i-1, p)} \\ &\cong \left(\bigoplus_{R' \subseteq P'} \widetilde{H}_{i-|R'|}(\widetilde{M_0}) \oplus (I\pi)^{\Delta(i-1, P')} \right) \\ &\quad \oplus \left(\bigoplus_{R' \subseteq P'} \widetilde{H}_{i-p-|R'|}(\widetilde{M_0}) \oplus (I\pi)^{\Delta(i-p-1, P')} \right) \oplus (I\pi)^{\delta(i-1, p)} \\ &= \bigoplus_{R \subseteq P} \widetilde{H}_{i-|R|}(\widetilde{M_0}) \oplus (I\pi)^{\Delta(i-1, P)}. \quad \square \end{aligned}$$

We now would like to distinguish between $H_*(\widetilde{\sigma_P M})$ and $H_*(\widetilde{\sigma_Q M})$, provided $P \neq Q$. For that we adopt the notation from the proof of Theorem 2.3. $P = (P', P'')$, with $P' = (p_1 \leq \dots \leq p_{j-1})$, $P'' = (p_j = \dots = p_{j+k-1} < p_{j+k} \leq \dots \leq p_r)$ and $Q = (P', Q'')$, with $Q'' = (q_j \leq \dots \leq q_s)$, where $p_j < q_j$ and $k \geq 1$. For $i \leq p_j$, we have $H_i(\widetilde{\sigma_P M}) \cong H_i(\widetilde{\sigma_Q M})$, as can be seen by direct computation or by using Lemma 5.1 below. For $i = p_j + 1$ however, we have

Lemma 3.3. $H_{p_j+1}(\widetilde{\sigma_P M}) \cong H_{p_j+1}(\widetilde{\sigma_Q M}) \oplus (I\pi)^k$.

Proof. We compute

$$\begin{aligned}
H_{p_j+1}(\sigma_P \widetilde{M}) &= H_{p_j+1}(\sigma_{P''}(\sigma_{P'} M)) \\
&\cong H_{p_j+1}(\sigma_{P'} M_0) \oplus \bigoplus_{\emptyset \neq R'' \subset P''} \widetilde{H}_{p_j+1-|R''|}(\sigma_{P'} M_0) \\
&\quad \oplus \widetilde{H}_{p_j+1-|P''|}(\sigma_{P'} \widetilde{M}) \oplus (I\pi)^{\Delta(p_j, P'')} \\
&\cong H_{p_j+1}(\sigma_{P'} \widetilde{M}_0) \oplus \bigoplus_{\emptyset \neq R'' \subset P''} \bigoplus_{R' \subseteq P'} \widetilde{H}_{p_j+1-|R''|-|R'|}(\widetilde{M}_0) \\
&\quad \oplus \bigoplus_{R' \subset P'} \widetilde{H}_{p_j+1-|P''|-|R'|}(\widetilde{M}_0) \\
&\quad \oplus \widetilde{H}_{p_j+1-|P''|-|P'|}(\widetilde{M}) \oplus \bigoplus_{\emptyset \neq R'' \subseteq P''} (I\pi)^{\Delta(p_j-|R''|, P')} \oplus (I\pi)^{\Delta(p_j, P'')}.
\end{aligned}$$

For $\emptyset \neq R'' \subseteq P''$ and $R' \subseteq P'$ we have $p_j - |R''| \leq 0$, so $p_j + 1 - |R''| - |R'| \leq 1$. This implies $\widetilde{H}_{p_j+1-|R''|-|R'|}(\widetilde{M}_0) = \widetilde{H}_{p_j+1-|P''|-|P'|}(\widetilde{M}) = 0$. Also

$$\Delta(p_j, P'') = \sum_{\emptyset \neq R'' \subseteq P''} \delta(p_j, |R''|) = k$$

and

$$\Delta(p_j - |R''|, P') = \sum_{\emptyset \neq R' \subseteq P'} \delta(p_j - |R''|, |R'|) = 0,$$

as $p_j - |R''| \leq 0$ and $|R'| > 0$. Therefore

$$H_{p_j+1}(\sigma_P \widetilde{M}) \cong H_{p_j+1}(\sigma_{P'} \widetilde{M}_0) \oplus (I\pi)^k.$$

Similarly,

$$H_{p_j+1}(\sigma_Q \widetilde{M}) \cong H_{p_j+1}(\sigma_{P'} \widetilde{M}_0),$$

finishing the proof. \square

In order to be able to distinguish among the $\mathbb{Z}\pi$ -modules in Lemma 3.3 we need a cancellation theorem. Namely, given π a finitely generated group, and A, B finitely-generated $\mathbb{Z}\pi$ -modules, we want to conclude that $A \oplus B \cong A \Rightarrow B = 0$. Such a general result holds if A is projective. For then we have $(\mathbb{Z}\pi)^l \oplus B \cong (\mathbb{Z}\pi)^l$, which implies $B = 0$ by a theorem of Kaplansky [9]. But in our situation $A = H_{p_j+1}(\sigma_P \widetilde{M})$ need not be projective, so we will settle for two particular cases.

Corollary 3.4. *If $\pi_1 M$ is a finite, nontrivial group and $P \neq Q$, then $H_*(\sigma_P \widetilde{M}) \not\cong H_*(\sigma_Q \widetilde{M})$ as abelian groups.*

Proof. We have

$$\begin{aligned}
\text{rank}(H_{p_j+1}(\sigma_P \widetilde{M})) &= \text{rank}(H_{p_j+1}(\sigma_Q \widetilde{M})) + k(|\pi| - 1) \\
&> \text{rank}(H_{p_j+1}(\sigma_Q \widetilde{M})). \quad \square
\end{aligned}$$

Corollary 3.5. *If M is aspherical and $P \neq Q$, then $H_*(\sigma_P \widetilde{M}) \not\cong H_*(\sigma_Q \widetilde{M})$ as $\mathbb{Z}\pi$ -modules.*

Proof. Since \widetilde{M} is contractible, $\widetilde{H}_i(\widetilde{M}_0) = 0$ for $i \neq n-1$ and $\widetilde{H}_{n-1}(\widetilde{M}_0) \cong \mathbb{Z}\pi$. Thus:

$$\begin{aligned} H_{p_j+1}(\sigma_{P'} M_0) &\cong \bigoplus_{R' \subseteq P'} \widetilde{H}_{p_j+1-|R'|}(\widetilde{M}_0) \oplus (I\pi)^{\Delta(p_j, P')} \\ &\cong (\mathbb{Z}\pi)^{\delta(p_j-n+2, 0)+\Delta(p_j-n+2, P')} \oplus (I\pi)^{\Delta(p_j, P')}. \end{aligned}$$

As π is a Poincaré-duality group of dimension $n > 1$, $H^1(\pi, \mathbb{Z}\pi) = 0$. The cohomology exact sequence associated to the coefficient sequence $0 \rightarrow I\pi \rightarrow \mathbb{Z}\pi \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0$ yields $H^1(\pi, I\pi) \cong H^0(\pi, \mathbb{Z}) \cong \mathbb{Z}$. Therefore

$$\begin{aligned} \text{rank } H^1(\pi, H_{p_j+1}(\sigma_P \widetilde{M})) &= \text{rank } H^1(\pi, H_{p_j+1}(\sigma_Q \widetilde{M})) + k \\ &> \text{rank } H^1(\pi, H_{p_j+1}(\sigma_Q \widetilde{M})). \quad \square \end{aligned}$$

4. HOMOTOPY TYPE OF ITERATED SPINS

In view of the above results it seems reasonable to make the following:

Conjecture. If $M^n \not\cong S^n$ and $P \neq Q$, then $\sigma_P M \not\cong \sigma_Q M$.

It follows from 2.3, 3.4 and 3.5 that the conjecture is true if either

- (1) M^n is not a homology sphere, or
- (2) M^n is a homology sphere with $\pi_1 M$ finite, or
- (3) M^n is an aspherical homology sphere.

We showed in §3 how to reduce this problem to a cancellation theorem for $\mathbb{Z}\pi$ -modules. If $n = 3$ we can bypass this seemingly difficult algebraic question by using some geometric arguments.

Theorem 4.1. *If $M^3 \not\cong S^3$ and $P \neq Q$, then $\sigma_P M \not\cong \sigma_Q M$.*

Proof. Suppose $\sigma_P M \simeq \sigma_Q M$. By the above, we only have to consider the case when M is a homology 3-sphere. In particular, M is orientable. Then M admits a prime factorization $M = M_1 \# \cdots \# M_l$ (see [8]). The summands M_i are homology 3-spheres, with $\widetilde{M}_i \simeq S^3$ or $\widetilde{M}_i \simeq \mathbb{R}^3$. We may assume $\pi_1 M_i \neq 1$, as homotopy sphere summands do not affect the homotopy type of M or that of its spins. We may also assume $l \geq 2$, for otherwise $M = M_1$ is of type (2) or (3) and $\sigma_P M \not\cong \sigma_Q M$.

Now let p be a nonnegative integer so that $|P| + p = |Q| + p = 4m$, for some $m > 0$, and set $\overline{P} = (P, p)$, $\overline{Q} = (Q, p)$. Clearly $\overline{P} \neq \overline{Q}$ and $\sigma_{\overline{P}} M \simeq \sigma_{\overline{Q}} M$. By Lemma 1.1 there exists a homotopy equivalence

$$\sigma_{\overline{P}}(M_1) \# \sigma_{\overline{P}}(M_2 \# \cdots \# M_l) \simeq \sigma_{\overline{Q}}(M_1) \# \sigma_{\overline{Q}}(M_2 \# \cdots \# M_l)$$

between closed, orientable manifolds of dimension $4m + 3 \geq 7$. By Cappell's Splitting Theorem [4, Theorem 3] the homotopy equivalence is splittable, and

we get either

- (i) $\sigma_{\overline{P}}(M_1) \simeq \sigma_{\overline{Q}}(M_1), \quad \text{or}$
- (ii) $\sigma_{\overline{P}}(M_1) \simeq \sigma_{\overline{Q}}(M_2 \# \cdots \# M_l).$

In the first case we derive a contradiction as before. In the second case, we must have $l = 2$, for otherwise $\pi_1 M_1$ is a nontrivial free product. Hence $\sigma_{\overline{P}} M_1 \simeq \sigma_{\overline{Q}} M_2$. This implies $\pi_1 M_1 \cong \pi_1 M_2$ and thus $\widetilde{M}_1 \simeq \widetilde{M}_2$. It follows from the proof of Theorem 3.5 that $H_*(\sigma_{\overline{P}} \widetilde{M}_1) \not\cong H_*(\sigma_{\overline{Q}} \widetilde{M}_2)$ —a contradiction. \square

The above proof may be adapted to show that the Conjecture holds for connected sums of manifolds of type (1), (2) or (3). But of course not all manifolds of dimension > 3 can be so decomposed. For example, let Σ^3 be the Poincaré sphere and form $S^1 \times \Sigma^3 \# S^1 \times \Sigma^3$, with fundamental group

$$(t, a, b | a^3 = b^5 = (ab)^2, [t, a] = [t, b] = 1) \\ * (u, c, d | c^3 = d^5 = (cd)^2, [u, c] = [u, d] = 1).$$

Then perform surgery on the curves $t(cd)^2$ and $u(ab)^2$. The resulting manifold is a homology 4-sphere M^4 with

$$\pi_1 M^4 = \mathbb{Z}_2 \times \pi_1 \Sigma *_{\mathbb{Z}_2 \times \mathbb{Z}_2} \mathbb{Z}_2 \times \pi_1 \Sigma.$$

This group is not a free product, so $M \not\cong M_1 \# M_2$ (unless M_1 or M_2 is $\simeq S^4$). Moreover, the center $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not square-root closed in $\pi_1 M$, so none of Cappell's splitting theorems apply.

5. HOMOLOGY SPHERES

We now use the techniques of the previous sections to produce examples of homology spheres distinguished by certain homotopy type invariants. First recall a definition from [15]. Two connected CW-complexes K_1 and K_2 have the same p -type¹ if there is a cellular map $f: K_1^{(p+1)} \rightarrow K_2^{(p+1)}$ with $f|_{K_1^{(p)}}$ a homotopy equivalence. If K_1 and K_2 are of the same p -type, they have isomorphic homotopy groups π_i ($i = 1, \dots, p$) and k -invariants k_i ($i = 1, \dots, p-1$). Now let $P = (p_1, \dots, p_r)$, $Q = (q_1, \dots, q_s)$.

Lemma 5.1. $\sigma_P M$ and $\sigma_Q M$ have the same p -type, for

$$p = \min(p_1, \dots, p_r, q_1, \dots, q_s).$$

Proof. Recall $\sigma_P M = M_0 \times S^p \cup S^{n-1} \times D^{p+1}$. It follows that $\sigma_P M$ admits a CW-decomposition with $(\sigma_P M)^{(p)} \simeq (M_0)^{(p)}$. If $p < q$, we also have $(\sigma_Q M)^{(p+1)} \simeq (M_0)^{(p)}$. Thus $\sigma_Q M$ and $\sigma_P M$ have the same p -type. The lemma follows readily. \square

¹ This is called the $(p+1)$ -type in [15].

Theorem 5.2. *Given $p \geq 1$ and $n \geq 2p + 3$, there are homology n -spheres Σ_1^n and Σ_2^n of the same p -type, but with $\pi_{p+1}(\Sigma_1) \not\cong \pi_{p+1}(\Sigma_2)$ as abelian groups.*

Proof. Let $\Sigma^3 = \Sigma(2, 3, 5) = S^3/\pi$ be the Poincaré homology sphere, with $\pi = \pi_1 \Sigma$ the binary icosahedral group of order 120. Define

$$\Sigma_1^n = \sigma_{n-3} \Sigma^3, \quad \Sigma_2^n = \sigma_{n-p-3} \sigma_p \Sigma^3.$$

By the above lemma, Σ_1 and Σ_2 have the same p -type. By Theorem 1.4, their universal covers are

$$\begin{aligned} \tilde{\Sigma}_1 &= \#_1^{119} S^{n-2} \times S^2, \\ \tilde{\Sigma}_2 &= \#_1^{119} (S^{n-2} \times S^2 \# S^{n-p-1} \times S^{p+1} \# S^{n-p-2} \times S^{p+2}). \end{aligned}$$

As $p + 1 < n - p - 1 \leq n - 2$, we have

$$\begin{aligned} \pi_{p+1}(\Sigma_1) &\cong \pi_{p+1} \left(\bigvee_1^{119} (S^{n-2} \vee S^2) \right) \cong \bigoplus_1^{119} \pi_{p+1}(S^2), \\ \pi_{p+1}(\Sigma_2) &\cong \bigoplus_1^{119} (\pi_{p+1}(S^2) \oplus \pi_{p+1}(S^{p+1}) \oplus \pi_{p+1}(S^{n-p-2})). \end{aligned}$$

Therefore

$$\pi_{p+1}(\Sigma_2) \cong \begin{cases} \pi_{p+1}(\Sigma_1) \oplus \mathbb{Z}^{238}, & \text{if } n = 2p + 3, \\ \pi_{p+1}(\Sigma_1) \oplus \mathbb{Z}^{119}, & \text{if } n > 2p + 3. \quad \square \end{cases}$$

In particular, for $n \geq 5$ we get two homology n -spheres with isomorphic π_1 but distinct π_2 . In fact, we may get $n - 3$ such manifolds by setting $\Sigma_i^n = \sigma_{n-i-2} \sigma_1^{i-1} \Sigma^3$, $i = 1, \dots, n - 3$. Then

$$\pi_2(\Sigma_i^n) \cong \begin{cases} \mathbb{Z}^{119i}, & \text{if } i < n - 3, \\ \mathbb{Z}^{119(i+1)}, & \text{if } i = n - 3. \end{cases}$$

The next theorem provides examples with different $\mathbb{Z}\pi$ -module structure on π_2 . Whether there are homology 4-spheres with distinct π_2 (as groups or as $\mathbb{Z}\pi$ -modules) is still an open question (see [12]).

Theorem 5.3. *Given $n \geq 5$, there are homology n -spheres Σ_1^n and Σ_2^n with $\pi_1 \Sigma_1 \cong \pi_1 \Sigma_2$, $\pi_2 \Sigma_1 \cong \pi_2 \Sigma_2$ as abelian groups, but $\pi_2 \Sigma_1 \not\cong \pi_2 \Sigma_2$ as $\mathbb{Z}\pi_1$ -modules.*

Proof. Start with $\Sigma^3 = \Sigma(2, 3, 7) = \mathbb{R}^3/\pi$ —an aspherical Brieskorn homology sphere — and define Σ_1^n, Σ_2^n as in Theorem 5.2. By Lemma 3.3 and the Hurewicz isomorphism we have

$$\pi_2 \Sigma_1 \cong \mathbb{Z}\pi, \quad \pi_2 \Sigma_2 \cong \begin{cases} \mathbb{Z}\pi \oplus (I\pi)^2, & \text{if } n = 5, \\ \mathbb{Z}\pi \oplus I\pi, & \text{if } n > 5. \end{cases}$$

These are both free abelian groups of countable rank, but, by the proof of Corollary 3.5, are distinct as $\mathbb{Z}\pi$ -modules. \square

We conclude this section with examples of homology n -spheres distinguished by their k -invariants, generalizing those of [11] and [12]. Recall that the k -invariant of a CW-complex K is a cohomology class $k(K) \in H^3(\pi_1 K, \pi_2 K)$. It behaves nicely under p -spinning. Indeed, for $n \geq 3$ the inclusion map $M_0^n \rightarrow \sigma_p M^n$ induces $\pi_1 M_0 = \pi_1 \sigma_p M$ and $\pi_2 M_0 \subset \pi_2 \sigma_p M = \pi_2 M_0 \oplus I\pi$, if $p = 1$ and $\pi_2 M_0 = \pi_2 \sigma_p M$, if $p > 1$. Since continuous maps preserve k -invariants, it follows that

$$k(\sigma_p M) = \begin{cases} (k(M_0), 0), & \text{if } p = 1, \\ k(M_0), & \text{if } p > 1. \end{cases}$$

Notice also that for $n \geq 4$, $\pi_2 M_0 = \pi_2 M$ and so $k(M_0) = k(M)$.

Theorem 5.4. *For $n \geq 3$ and $N \geq 2$, there exist N homology n -spheres with isomorphic π_1 and π_2 (as $\mathbb{Z}\pi_1$ -modules), but with distinct k -invariants.*

Proof. We will construct homology n -spheres $\Sigma_1^n, \dots, \Sigma_N^n$ satisfying a stronger condition. Namely, for $i \neq j$, given any isomorphism $\alpha: \pi_1 \Sigma_i^n \rightarrow \pi_1 \Sigma_j^n$ and any α -homomorphism $\beta: \pi_2 \Sigma_i^n \rightarrow \pi_2 \Sigma_j^n$, we have $\alpha^*(k(\Sigma_j^n)) \neq \beta_*(k(\Sigma_i^n))$.

The existence of homology n -spheres Σ_i^n verifying the above condition is established for $n = 3$ in [11] and for $n = 4$ in [12]. For $n \geq 5$, set

$$\Sigma_i^n = \sigma_{n-4} \Sigma_i^4.$$

We claim Σ_i^n satisfy the above condition.

If $n = 5$, suppose there exist, for $i \neq j$, an isomorphism $\alpha: \pi_1 \Sigma_i^5 \cong \pi_1 \Sigma_j^5 \rightarrow \pi_1 \Sigma_j^5 \cong \pi_1 \Sigma_j^4$ and an α -homomorphism $\beta: \pi_2 \Sigma_i^5 \cong \pi_2 \Sigma_i^4 \oplus I\pi \rightarrow \pi_2 \Sigma_j^5 \cong \pi_2 \Sigma_j^4 \oplus I\pi$ such that $\alpha^*(k(\Sigma_j^5)) = \beta_*(k(\Sigma_i^5))$. Let β' be the restriction of β to $\pi_2(\Sigma_i^4)$. As $k(\Sigma_i^5) = (k(\Sigma_i^4), 0)$, we see that β' is an α -homomorphism (not necessarily an isomorphism, even if β is) satisfying $\alpha^*(k(\Sigma_j^4)) = \beta'_*(k(\Sigma_i^4))$. This is a contradiction.

If $n > 5$, then $\pi_2 \Sigma_i^n = \pi_2 \Sigma_i^4$ and $k(\Sigma_i^n) = k(\Sigma_i^4)$, so the proof is immediate. \square

Remark. One cannot get these high-dimensional homology spheres by directly spinning the three-dimensional ones. Indeed, for every i and j , $\sigma_p(\Sigma_i^3)$ is diffeomorphic to $\sigma_p(\Sigma_j^3)$. This is why we first had to construct Σ_i^4 before being able to continue spinning.

6. ITERATED SPUN KNOTS

Let $K = (S^n, S^{n-2})$, $n \geq 3$, be a smooth n -knot. The p -spin of K is the $n + p$ -knot $\sigma_p K = (\sigma_p S^n, \sigma_p S^{n-2})$. Cappell [3] shows that if $X(K) = S^n \setminus S^{n-2} \times D^2$ is the exterior of K , then $X(\sigma_p K) = S^p \times X(K) \cup (D^{p+1} \times S^{n-2} \setminus D^{p-n-1}) \times S^1$, identified along $S^p \times S^{n-2} \times S^1$. Moreover, if K is a standardly fibered knot, with fiber F_0^{n-1} , then $\sigma_p K$ is also standardly fibered, with fiber $(\sigma_p F)_0$. He then notices that in general $\sigma_{p+q} K \not\cong \sigma_q \sigma_p K$. For P a tuple, denote by $\sigma_P K$ the iterated P -spin of K . We then venture the following conjecture.

Conjecture. If K is a nontrivial knot and $P \neq Q$, then $\sigma_P K \not\cong \sigma_Q K$.

One approach to this problem is to study the $\mathbb{Z}\pi_1 X$ -module structure of the homology groups of $X(\widetilde{\sigma_P K})$ (of course $H_*(X(\sigma_P K)) \cong H_*(S^1)$ no longer provides any information). These groups can be computed using the description of $H_*(X(\sigma_P K))$ in [6] (see also [1, 5, 14]), which is similar to that of $H_*(\sigma_P \widetilde{M})$ in Lemma 3.1. The ensuing difficulties though are much the same as for closed manifolds. We thus will content ourselves with providing some evidence for the validity of the conjecture in case the knot is fibered.

First note that if K and K' are standardly fibered knots, with fibers F_0 and F'_0 , then $K \cong K'$ implies $F \simeq F'$. For if $Y(K) = X(K) \cup_{S^{n-2} \times S^1} D^{n-1} \times S^1$, then the universal abelian cover of Y is homotopy equivalent to F . It now follows from Theorem 2.3 that:

If K is a standardly fibered knot, with closed fiber not a homology sphere, then the conjecture is true.

In particular, if $K = (S^3, S^1)$ is fibered and nontrivial, then $P \neq Q$ implies $\sigma_P K \not\cong \sigma_Q K$. If K is a topologically nontrivial knot in S^4 , its closed fiber F^3 cannot be a homotopy S^3 by the work of Freedman. It follows from Theorem 4.1 that:

If $K = (S^4, S^2)$ is a standardly fibered knot not homeomorphic to the trivial knot, then the conjecture is true.

Acknowledgment. I wish to thank R. Daverman for raising some questions which led to the results in §5.

REFERENCES

1. J. J. Andrews and D. W. Sumners, *On higher-dimensional fibered knots*, Trans. Amer. Math. Soc. **153** (1971), 415–426.
2. E. Artin, *Zur Isotopie zweidimensionaler Flächen in R_4* , Abh. Math. Sem. Univ. Hamburg, **4** (1926), 174–177.
3. S. E. Cappell, *Superspinning and knot complements*, Topology of Manifolds (Proc. Inst. Univ. of Georgia, Athens, 1969), Markham, Chicago, Il., 1970, pp. 358–383.
4. —, *A splitting theorem for manifolds*, Invent. Math. **33** (1976), 69–170.
5. D. B. A. Epstein, *Linking spheres*, Proc. Cambridge Philos. Soc. **56** (1960), 215–219.
6. C. Mc A. Gordon, *Some higher-dimensional knots with the same homotopy groups*, Quart. J. Math. Oxford **24** (1973), 411–422.
7. —, *A note on spun knots*, Proc. Amer. Math. Soc. **58** (1976), 361–362.
8. J. Hempel, *3-manifolds*, Princeton Univ. Press, Princeton, N.J., 1976.
9. I. Kaplansky, *Fields and rings*, Chicago Lectures in Mathematics, Chicago Univ. Press, Chicago, Il., 1969.
10. S. P. Plotnick, *Equivariant intersection forms, knots in S^4 , and rotations in 2-spheres*, Trans. Amer. Math. Soc. **296** (1986), 543–575.
11. S. P. Plotnick and A. I. Suciu, *k-invariants of knotted 2-spheres*, Comment. Math. Helv. **60** (1985), 54–84.

12. A. I. Suciu, *Homology 4-spheres with distinct k -invariants*, Topology Appl. **25** (1987), 103–110.
13. ———, *The oriented homotopy type of spun 3-manifolds*, Pacific J. Math. **131** (1988), 393–399.
14. D. W. Sumners, *On an unlinking theorem*, Proc. Cambridge Philos. Soc. **71** (1972), 1–4.
15. J. H. C. Whitehead, *Combinatorial homotopy. I*, Bull. Amer. Math. Soc. **55** (1949), 213–245.

DEPARTMENT OF MATHEMATICS, NORTHEASTERN UNIVERSITY, BOSTON, MASSACHUSETTS 02115