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ON THE BIHOMOGENEITY PROBLEM OF KNASTER

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ABSTRACT. The author constructs a locally connected, homogeneous, finite-dimensional, compact, metric space which is not bihomogeneous, thus providing a compact counterexample to a problem posed by B. Knaster around 1921.

0. INTRODUCTION

A topological space X is said to be *homogeneous* if for every two points p and q in X there exists a homeomorphism $h: X \to X$ such that $h(p) = q$. X is said to be bihomogeneous if for every two points p and q in X there exists a homeomorphism $h: X \to X$ such that $h(p) = q$ and $h(q) = p$. Around 1921, B. Knaster asked the question of whether every homogeneous space is bihomogeneous, and shortly after that C. Kuratowski (see [6]) described an example of a non-locally-compact, homogeneous subset of the plane, which is not bihomogeneous. In 1930, D. van Danzig asked whether homogeneity implies bihomogeneity for continua; see [10]. A locally compact, homogeneous, nonbihomogeneous, metric space was found by H. Cook in the early 1980s; see [3].

This paper contains an example of a seven-dimensional, homogeneous, nonbihomogeneous, locally connected, compact metric space. G. S. Ungar proved that certain homogeneity type properties imply local connectedness; see [9]. However, [5] and this paper show that a locally connected homogeneous continuum may lack some stronger but still very simple homogeneity properties.

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1. PRELIMINARIES

All spaces considered in this paper are metric and all maps are continuous. By S^n, E^n, B^n, and $\overline{B}^n$ we mean the n-dimensional sphere, the Euclidean n-space, the n-dimensional open ball, and the n-dimensional closed ball respectively.

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Let P and Q be two disjoint, closed subsets in a compact space X, and let $g: P \to Q$ be a homeomorphism. Let $\sim$ be an equivalence relation on X such that $p \sim q$ iff $p = q$, or if $p \in P$ and $q \in Q$ then $g(p) = q$, or if $q \in P$ and $p \in Q$ then $g(q) = p$. The space of equivalence classes with the quotient topology will be denoted by X/g.

In our applications of homology theory, we use either the singular or Čech homology groups with integral coefficients. For basic concepts of homotopy theory we refer the reader to [4].

Throughout this paper, M will denote the universal Menger curve as described in R. D. Anderson's paper [1, p. 321]. M is a subset of the cube $\{(x, y, z) \in E^3 : x, y, z \in [0, 1]\}$ such that the intersection of M with each of the faces of the cube is homeomorphic to Sierpiński's plane curve.

In [1], Anderson proved that M is homogeneous, and that every 1-dimensional continuum with no local cut points and no open subsets embeddable in the plane is homeomorphic to M. Furthermore, from results in [1 and 2], it follows that if U is an open connected subset of M, and $p, q \in U$, then there exists a homeomorphism $h: M \to M$ such that $h(p) = q$, and $h(v) = v$ for $v \in M - U$.

In [5], the authors employ the fact that continua which are Cartesian products with one or more factors homeomorphic to M admit few homeomorphisms. A similar idea is used here in the form of Lemmas 1 and 2 whose proofs are analogous to those of Theorems 2 and 1 in [5].

Lemma 1. *Let $X = X_1 \times X_2$, where X_i is homeomorphic to M for $i = 1, 2$. Let $U_i \subset X_i$ be a connected open set for $i = 1, 2$. If $\varphi: U_1 \times U_2 \to X$ is an open embedding, then $\varphi = \varphi_1 \times \varphi_2$, where either (1) $\varphi_1: U_1 \to X_1$ and $\varphi_2: U_2 \to X_2$, or (2) $\varphi_1: U_1 \to X_2$ and $\varphi_2: U_2 \to X_1$.*

Proof. Let $\pi_i: X \to X_i$ be the projection. Suppose that (u, v_1) and (u, v_2) are two distinct points in $U_1 \times U_2$. Let $\varphi((u, v_1)) = (x_1, y_1)$ and $\varphi((u, v_2)) = (x_2, y_2)$. Suppose that $x_1 \neq x_2$ and $y_1 \neq y_2$. Let $V_1 \subset X_1$ and $V_2 \subset X_2$ be such that $V_1 \times V_2$ is a neighborhood of (x_1, y_1) contained in $\varphi(U_1 \times U_2)$.

There exists a nonsingular loop $f: S^1 \to U_1$ such that $u \in f(S^1)$, and if $f_i: S^1 \to X$ is defined by $f_i(s) = (f(s), v_i)$ for $i = 1, 2$, then $\varphi \circ f_1(S^1) \in V_1 \times V_2$, and for $i = 1, 2$, we have $\pi_i \circ \varphi \circ f_1(S^1) \cap \pi_i \circ \varphi \circ f_2(S^1) = \varnothing$. Since $f_1(S^1)$ is a retract of $U_1 \times U_2$, then $\varphi \circ f_1(S^1)$ is a retract of $\varphi(U_1 \times U_2)$, and hence a retract of $V_1 \times V_2$. Therefore $\varphi \circ f_1$ is essential, which implies that at least one of the maps $\pi_i \circ \varphi \circ f_1$ is essential. Suppose that $\pi_1 \circ \varphi \circ f_1$ is essential. Since U_2 is arcwise connected, then f_1 and f_2 are homotopic, and hence $\pi_1 \circ \varphi \circ f_1$ and $\pi_1 \circ \varphi \circ f_2$ are homotopic. However, no two essential and disjoint loops in the Menger curve are homotopic. Therefore either $x_1 = x_2$ or $y_1 = y_2$.

If $x_1 = x_2$, then for any $v_3 \in U_2$ we have $\varphi((u, v_3)) = (x_1, y_3)$. Since a similar fact can be shown for points in $U_1 \times U_2$ with equal second coordinate, then $\varphi = \varphi_1 \times \varphi_2$, where $\varphi_1: U_1 \to X_1$ and $\varphi_2: U_2 \to X_2$.

If $y_1 = y_2$ then $\varphi = \varphi_1 \times \varphi_2$, where $\varphi_1: U_1 \to X_2$ and $\varphi_2: U_2 \to X_1$. □

Lemma 2. *Let $X = X_1 \times X_2$, where X_1 is homeomorphic to $M \times M$, and X_2 is a continuum whose every point has a closed neighborhood which is an absolute retract. For $i = 1, 2$, let $U_i \subset X_i$ be an open set and let U_2 be connected. If $\varphi: U_1 \times U_2 \to X$ is an open embedding, then for every $u \in U_1$ there exists an $x \in X_1$ such that $\varphi(\{u\} \times U_2) \subset \{x\} \times X_2$.*

Proof. Suppose that (u, v_1) and (u, v_2) are points in $U_1 \times U_2$. Let $\varphi((u, v_1)) = (x_1, y_1)$ and let $\varphi((u, v_2)) = (x_2, y_2)$. Suppose that $x_1 \neq x_2$. Let $V_1 \subset X_1$ and $V_2 \subset X_2$ be such that $V_1 \times V_2$ is a neighborhood of (x_1, y_1) contained in $\varphi(U_1 \times U_2)$, and V_2 is an absolute retract. Let $\pi_1: X \to X_1$ be the projection.

There exists an embedding $f: S^1 \times S^1 \to U_1$ such that $u \in f(S^1 \times S^1)$, $f(S^1 \times S^1)$ is a retract of X_1, and if $f_i: S^1 \times S^1 \to X$ is defined by $f_i(s) = (f(s), v_i)$ for $i = 1, 2$, then $\pi_1 \circ \varphi \circ f_1(S^1 \times S^1) \cap \pi_1 \circ \varphi \circ f_2(S^1 \times S^1) = \varnothing$, and $\varphi \circ f_1(S^1 \times S^1) \subset V_1 \times V_2$. Note that f_1 and f_2 are homotopic. Since $f_1(S^1 \times S^1)$ is a retract of $U_1 \times U_2$, $\varphi \circ f_1(S^1 \times S^1)$ is a retract of $\varphi(U_1 \times U_2)$, and hence $\varphi \circ f_1(S^1 \times S^1)$ is a retract of $V_1 \times V_2$. Let $g: X_1 \to X$ be an embedding defined by $g(p) = (p, y_1)$. Using the Čech homology and the induced homomorphism, we have $0 \neq (\varphi \circ f_1)_*(a) = (g \circ \pi_1 \circ \varphi \circ f_1)_*(a) = (g \circ \pi_1 \circ \varphi \circ f_2)_*(a)$, where a is a generator of $H_2(S^1 \times S^1)$.

Therefore, there are two 2-dimensional nontrivial homologous Čech cycles with disjoint carriers in X_1. By [7, p. 246], the dimension of X_1 is greater than 2, which is a contradiction. □

2. The twisted products

Denote by (r, θ, z) the cylindrical coordinates of a point in E^3. Let μ be an embedding of the Menger curve M in E^3 defined by $\mu(x, y, z) = (r, \theta, z)$, where $r = x + 1$, $\theta = \frac{2\pi}{9}y$, and $z = z$, for every $(x, y, z) \in M$. Let $f(r, \theta, z) = (r, \theta + \frac{2\pi}{9}, z)$ be the rotation about the z-axis through the angle of $\frac{2\pi}{9}$. Put $A_0 = \mu(M)$, $A_k = f^{(k)}(A_0)$, where $f^{(k)}$ is the kth iteration of f, and put $A = \bigcup_{i=0}^{8} A_i$.

Clearly, A is invariant under f, and $f_1 = f|_A$ is a periodic homeomorphism of A onto itself. By [1], A is homeomorphic to M. Cylindrical coordinates (r, θ, z) will be used to denote a point in A, and Cartesian coordinates $(\overline{x}, \overline{y}, \overline{z})$ will be used to denote a point in M. If $p = (a, m) \in A \times M$ is a point, where $a = (r, \theta, z)$ and $m = (\overline{x}, \overline{y}, \overline{z})$, then p may be denoted by $(r, \theta, z, \overline{x}, \overline{y}, \overline{z})$.

For every $\alpha \in [0, 1]$, put $M_\alpha = \{(\overline{x}, \overline{y}, \overline{z}) \in M: \overline{z} = \alpha\}$. Let $g_1: M_1 \to M_0$ be the homeomorphism taking $(\overline{x}, \overline{y}, 1)$ onto $(\overline{x}, \overline{y}, 0)$. Let $g_2: A \times M_1 \to A \times M_0$ be defined by $g_2(a, m) = (f_1(a), g_1(m))$. Define B as the quotient space $(A \times M)/g_2$.

Thus the continuum B, a twisted product of A and M, is obtained from the Cartesian product $A \times M$ by pasting the "top" $A \times M_1$ to the "bottom" $A \times M_0$. Points in B will be denoted in the same manner as the corresponding points in $A \times M$ for which $\overline{z} \neq 1$.

Define $f_2 \colon B \to B$ by $f_2(a, m) = (f_1(a), m)$ for $a \in A$ and $m \in M - M_1$. Clearly, f_2 is a periodic homeomorphism of period 9.

Lemma 3. *For every two points $p, q \in B$ there exists a homeomorphism $h \colon B \to B$ such that $h(p) = q$ and $h \circ f_2 = f_2 \circ h$.*

Proof. First, we shall show that for every two points $p = (r_p, \theta_p, x_p, \overline{x}_p, \overline{y}_p, \overline{z}_p)$ and $q = (r_q, \theta_q, x_q, \overline{x}_q, \overline{y}_q, \overline{z}_q)$ there exists a homeomorphism $h_1 \colon B \to B$ such that $h_1(p) = (r_p, \theta_p, z_p, \overline{x}_q, \overline{y}_q, \overline{z}_q)$ and $h_1 \circ f_2 = f_2 \circ h_1$.

Let M/g_1 be the space homeomorphic to M obtained from M by identifying, in a similar fashion as above, the point $(\overline{x}, \overline{y}, 1)$ with the point $(\overline{x}, \overline{y}, 0)$ for $\overline{x}, \overline{y} \in [0, 1]$. For $\alpha \in [0, 1)$, denote by $\widetilde{M}_\alpha$ the subset of M/g_1 corresponding to $M_\alpha \subset M$. The point in M/g_1 corresponding to the point $(\overline{x}, \overline{y}, \overline{z}) \in M$, where $\overline{z} \neq 1$, will be denoted by $(\overline{x}, \overline{y}, \overline{z})$.

For every $\alpha \in [0, 1)$, put $B_\alpha = \{(a, m) \in B \colon m \in M_\alpha\}$. The map $\Psi_\alpha \colon B - B_\alpha \to A \times (M/g_1 - \widetilde{M}_\alpha)$ defined by

$$\Psi_\alpha(r, \theta, z, \overline{x}, \overline{y}, \overline{z}) = \begin{cases} ((r, \theta, z), (\overline{x}, \overline{y}, \overline{z})) & \text{if } 0 \le \overline{z} < \alpha, \\ ((r, \theta + \frac{2\pi}{9}, z), (\overline{x}, \overline{y}, \overline{z})) & \text{if } \alpha < \overline{z} < 1, \end{cases}$$

is a homeomorphism.

There exists a number α_0 and there exists a connected open subset $U \subset M/g_1$ containing $(\overline{x}_p, \overline{y}_p, \overline{z}_p)$ and $(\overline{x}_q, \overline{y}_q, \overline{z}_q)$ such that $U \cap \widetilde{M}_{\alpha_0} = \varnothing$. There exists a homeomorphism $k_1 \colon M/g_1 \to M/g_1$ taking $(\overline{x}_p, \overline{y}_p, \overline{z}_p)$ onto $(\overline{x}_q, \overline{y}_q, \overline{z}_q)$, and not moving points outside U. Let $\overline{k}_1 \colon A \times (M/g_1 - \widetilde{M}_{\alpha_0}) \to A \times (M/g_1 - \widetilde{M}_{\alpha_0})$ be such that

$$\overline{k}_1((r, \theta, z), (\overline{x}, \overline{y}, \overline{z})) = ((r, \theta, z), k_1(\overline{x}, \overline{y}, \overline{z})),$$

and define $\overline{h}_1 \colon B \to B$ by setting

$$\overline{h}_1(v) = \begin{cases} \Psi_{\alpha_0}^{-1} \circ \overline{k}_1 \circ \Psi_{\alpha_0}(v) & \text{if } v \notin B_{\alpha_0}, \\ v & \text{if } v \in B_{\alpha_0}. \end{cases}$$

Hence,

$$\overline{h}_1(p) = \Psi_{\alpha_0}^{-1} \circ k_1 \circ \Psi_{\alpha_0}(p) = (r_p, \theta_p + \varepsilon \tfrac{2\pi}{9}, z_p, \overline{x}_q, \overline{y}_q, \overline{z}_q),$$

where $\varepsilon \in \{0, 1, -1\}$. Put

$$h_1(v) = \begin{cases} \overline{h}_1(v) & \text{if } \varepsilon = 0, \\ f_2^{-1} \circ \overline{h}_1(v) & \text{if } \varepsilon = 1, \\ f_2 \circ \overline{h}_1(v) & \text{if } \varepsilon = -1. \end{cases}$$

For any $v = (r, \theta, z, \overline{x}, \overline{y}, \overline{z}) \in B$, $h_1(v) = ((r, \theta+\delta\frac{2\pi}{9}, z), k_1(\overline{x}, \overline{y}, \overline{z}))$, where $\delta \in \{0, 1, -1\}$. Hence $h_1 \circ f_2(v) = ((r, \theta+(\delta+1)\frac{2\pi}{9}, z), k_1(\overline{x}, \overline{y}, \overline{z})) = f_2 \circ h_1(v)$.

Next, we shall show that there exists a homeomorphism $h_2: B \to B$ such that $h_2(r_p, \theta_p, z_p, \overline{x}_q, \overline{y}_q, \overline{z}_q) = q$, and $h_2 \circ f_2 = f_2 \circ h_2$.

If $f_2^{(i)}(r_p, \theta_p, z_p) = (r_q, \theta_q, x_q)$ for some i, then set $h_2 = f_2^{(i)}$. Otherwise, there is an open connected set $U \subset A$ containing (r_p, θ_p, z_p) and (r_q, θ_q, z_q), and such that the sets U, $f(U), \dots, f^{(8)}(U)$ are pairwise disjoint. There is a homeomorphism $k_2: A \to A$ which is the identity outside U taking (r_p, θ_p, z_p) onto (r_q, θ_q, z_q). Define h_2 by

$$h_2(v) = \begin{cases} (f^{(i)} \circ k_2 \circ (f^{(i)})^{-1}(a), m) & \text{if } a \in f^{(i)}(U), \\ v & \text{if } a \notin \bigcup_{j=1}^{9} f^{(j)}(U), \end{cases}$$

where $v = (a, m)$ and $i = 1, \dots, 9$.

Clearly, $h_2 \circ f_2 = f_2 \circ h_2$.

Finally, put $h = h_2 \circ h_1$. □

Let n be a positive integer, and let N be an n-manifold with nonempty boundary such that $\partial N = N_0 \cup N_1$, where $N_0 \cap N_1 = \varnothing$, both N_0 and N_1 are closed, and there exists a homeomorphism $g_3: N_1 \to N_0$. Let $g_4: B \times N_1 \to B \times N_0$ be such that $g_4(b, s) = (f_2^{(3)}(b), g_3(s))$, where $f_2^{(3)}$ is the third iteration of f_2. Define Z_N as the quotient space $(B \times N)/g_4$.

Points in Z_N will be denoted in the same way as the corresponding points in the Cartesian products $B \times N$ or $A \times M \times N$. Specifically, if $p \in Z_N$, then $p = (b, s)$, where $b \in B$ and $s \in N - N_1$, or $p = (a, m, s)$, where $a \in A$, $m \in M - M_1$, and $s \in N - N_1$, or $p = (r, \theta, z, \overline{x}, \overline{y}, \overline{z}, s)$, where $(r, \theta, z) \in A$, $(\overline{x}, \overline{y}, \overline{z}) \in M - M_1$, and $s \in N - N_1$.

Lemma 4. *Z_N is homogeneous.*

Proof. Let $p = (b_p, s_p)$ and $q = (b_q, s_q)$ be two points in Z_N. To show that there exists a homeomorphism $h: Z_N \to Z_N$ taking p onto q, it is enough to show that there are homeomorphisms $h_1, h_2: Z_N \to Z_N$ such that $h_1(p) = (b_p, s_q)$ and $h_2(b_p, s_q) = q$.

Let U be a neighborhood of b_p in B such that U, $f_2^{(3)}(U)$, and $f_2^{(6)}(U)$ are pairwise disjoint. The set $W = \{(b, s) \in Z_N: b \in \bigcup_{i=3,6,9} f_2^{(i)}(U)\}$ is homeomorphic to $U \times Q$, where Q is an n-manifold; in fact, Q is a union of three copies of N. For any two points d_1 and d_2 in Q, there exists a homeomorphism $k: Q \to Q$ isotopic to the identity such that $k(d_1) = d_2$. Using the isotopy and the Cartesian product structure of W, it is easy to obtain the homeomorphism $h_1: Z_N \to Z_N$ with $h_1(b_p, s_p) = (b_p, s_q)$ and $h_1(v) = v$ for $v \notin W$.

By Lemma 3, there exists a homeomorphism $\overline{h}: B \to B$ such that $\overline{h}(b_p) = b_q$ and $\overline{h} \circ f_2 = f_2 \circ \overline{h}$. In particular, $\overline{h} \circ f_2^{(3)} = f_2^{(3)} \circ \overline{h}$. Hence, $h_2: Z_N \to Z_N$, where $h_2(b, s) = (\overline{h}(b), s)$, is well defined and $h_2(b_p, s_q) = q$. □

Let $p = (a_0, m_0)$ be a point in B. The sets A_p and M_p are defined by

$$A_p = \{(a, m) \in B: m = m_0\},$$

$$M_p = \{(a, m) \in B: a = f_1^{(i)}(a_0), \text{ where } i = 1, \dots, 9\}.$$

Similarly, if $p = (b_0, s_0) \in Z_N$, then the sets B_p and N_p are defined by

$$B_p = \{(b, s) \in Z_N: s = s_0\},$$

$$N_p = \{(b, s) \in Z_N: b = f_2^{(i)}(b_0), \text{ where } i = 3, 6, 9\}.$$

Each of the sets A_p, M_p, B_p, and N_p will be called a fiber.

Lemma 5. *If $h: B \to B$ is a homeomorphism, then either* (1) $h(A_p) = A_{h(p)}$ *and* $h(M_p) = M_{h(p)}$ *for all* $p \in B$, *or* (2) $h(A_p) = M_{h(p)}$ *and* $h(M_p) = A_{h(p)}$ *for all* $p \in B$.

Proof. Every point in B has a closed neighborhood in the form of a Cartesian product $X_1 \times X_2$, where X_1 is a subset of A homeomorphic to M, and by means of a homeomorphism similar to the homeomorphism Ψ_α of Lemma 3, X_2 is a subset of M/g_1 homeomorphic to M. Moreover, for every $x_1 \in X_1$ and $x_2 \in X_2$, each of the sets $\{x_1\} \times X_2$ and $X_1 \times \{x_2\}$ is contained in a fiber A_p or M_p. Since B is compact, there exists a finite collection $\{V_1, \dots, V_k\}$ of these neighborhoods such that $B = \bigcup_{i=1}^k \operatorname{Int}(V_i)$. Similarly, every point in B has arbitrarily small open neighborhoods in the form $U_1 \times U_2$, where for $i = 1, 2$, U_i is homeomorphic to a connected subset of M. Let $\{W_1, \dots, W_l\}$ be a finite collection of these neighborhoods covering B, and such that for each $j = 1, \dots, l$, there is an i such that $h(W_j) \subset V_i$. By Lemma 1, if $p \in B$, then for each $j = 1, \dots, l$, there is an i such that $h(A_p \cap W_j) \subset A_{h(p)} \cap V_i \subset A_{h(p)}$ or $h(A_p \cap W_j) \subset M_{h(p)} \cap V_i \subset M_{h(p)}$. Hence $h(A_p) \subset A_{h(p)}$ or $h(A_p) \subset M_{h(p)}$. Since B is connected, then if $h(A_p) \subset A_{h(p)}$ $[h(A_p) \subset M_{h(p)}]$ for one point $p \in B$, then $h(A_p) \subset A_{h(p)}$ $[h(A_p) \subset M_{h(p)}]$ for every $p \in B$. A similar statement holds for M_p. Since h is one-to-one, we have $h(A_p) = A_{h(p)}$ and $h(M_p) = M_{h(p)}$ for $p \in B$, or we have $h(A_p) = M_{h(p)}$ and $h(M_p) = A_{h(p)}$ for $p \in B$. □

Let $p \in B$ be a point. Denote by O_p the orbit of p under f_2, i.e., $O_p = A_p \cap M_p = \{p, f_2(p), \dots, f_2^{(8)}(p)\}$. The following lemma is an immediate consequence of Lemma 5.

Lemma 6. *If $h: B \to B$ is a homeomorphism, then $h(O_p) = O_{h(p)}$ for every $p \in B$.*

Lemma 7. *Let $p_i = (1, \frac{2\pi i}{9}, 0, 0, 0, 0) \in B$ for $i = 0, \dots, 8$. If $h: B \to B$ is a homeomorphism such that $h(p_0) = p_{i_0}$ and $h(p_1) = p_{i_1}$, then $h(p_j) = p_{[i_0 + j(i_1 - i_0)] \bmod 9}$.*

Proof. Let L_i be the arc $\{(1, \theta, 0, 0, 0, 0) \in B: \frac{2\pi i}{9} \le \theta \le \frac{2\pi(i+1)}{9}\}$, where $i = 0, \dots, 8$. The set $L = \bigcup_{i=0}^8 L_i$ is a simple closed curve invariant under

f_2. By Lemma 6, $\bigcup_{i=1}^{9} f_2^{(i)} \circ h(L_0) \subset h(L)$. Furthermore, for $i = 0, \dots, 8$, the end points of the arc $f_2^{(i)} \circ h(L_0)$ are the points $p_{(i_0+i) \bmod 9}$ and $p_{(i_1+i) \bmod 9}$. Therefore, since $h(L)$ is a simple closed curve, we have $\bigcup_{i=1}^{9} f_2^{(i)} \circ h(L_0) = h(L)$. Hence, the ends of the arc $h(L_j)$ are $p_{[i_0+j(i_1-i_0)] \bmod 9}$ and $p_{[i_0+(j+1)(i_1-i_0)] \bmod 9}$. Thus $h(p_j) = p_{[i_0+j(i_1-i_0)] \bmod 9}$. □

Lemma 8. *If $h: Z_N \to Z_N$ is a homeomorphism, then $h(N_p) = N_{h(p)}$ for every $p \in Z_N$.*

Proof. Every point in Z_N has a closed neighborhood in the form of a Cartesian product $X_1 \times X_2$, where X_1 is homeomorphic to $M \times M$ and X_2 is homeomorphic to a closed ball $\overline{B}^n$. We may assume that if $p(x_1, x_2) \in X_1 \times X_2$, then $\{x_1\} \times X_2 \subset N_p$ and $X_1 \times \{x_2\} \subset B_p$. Since Z_N is compact, there exists a finite collection $\{V_1, \dots, V_k\}$ of these neighborhoods such that $Z_N = \bigcup_{i=1}^{k} \operatorname{Int}(V_i)$. Similarly, every point in Z_N has arbitrarily small neighborhoods in the form $U_1 \times U_2$, where U_1 is homeomorphic to an open subset in $M \times M$, and U_2 is homeomorphic to an open ball B^n. Let $\{W_1, \dots, W_l\}$ be a finite collection of these neighborhoods covering Z_N such that for every $j = 1, \dots, l$, there is an i such that $h(W_j) \subset V_i$. By Lemma 2, if $p \in Z_N$, then for every $j = 1, \dots, l$, there is an i such that $h(N_p \cap W_j) \subset N_{h(p)} \cap V_i \subset N_{h(p)}$. Since N_p is connected, then $h(N_p) \subset N_{h(p)}$. □

Now, we will define a continuum C by putting $C = Z_N$, where $N = [0, 1]$, $N_1 = \{1\}$, $N_0 = \{0\}$, and $g_3(1) = 0$. Notice that $C = \{(b, s): b \in B$ and $s \in [0, 1)\}$. Consider B to be the subset $\{(b, s) \in C: s = 0\}$ of C.

Let (ρ, α) denote the polar coordinates in the plane. Assume that $S^1 = \{(\rho, \alpha) \in E^2: \rho = 1$ and $\alpha \in [0, 2\pi)\}$. Let $\Gamma: C \to S^1$ be defined by

$$\Gamma(r, \theta, z, \overline{x}, \overline{y}, \overline{z}, s) = \left(1, \left(\theta + \frac{2\pi}{9}\overline{z} + \frac{2\pi}{3}s\right) \bmod 2\pi\right).$$

Clearly, Γ is continuous.

Lemma 9. *For every point $p \in C$, $\Gamma|_{N_p}: N_p \to S^1$ is a homeomorphism.*

Proof. If $p = (r_p, \theta_p, z_p, \overline{x}_p, \overline{y}_p, \overline{z}_p, s_p)$, then

$$N_p = \{(r_p, \theta_p + \tfrac{2\pi}{3}\varepsilon, z_p, \overline{x}_p, \overline{y}_p, \overline{z}_p, s): \varepsilon \in \{0, 1, 2\} \text{ and } s \in [0, 1)\}.$$

$\Gamma((r_p, \theta_p + \frac{2\pi}{3}\varepsilon, z_p, \overline{x}_p, \overline{y}_p, \overline{z}_p, s)) = (1, (\theta_p + \frac{2\pi}{3}\varepsilon + \frac{2\pi}{9}\overline{z}_p + \frac{2\pi}{3}s) \bmod 2\pi)$. Clearly, Γ is one-to-one and therefore Γ is a homeomorphism. □

Consider $H_1(S^1)$ to be the additive group of integers. For every $p \in C$, denote by a_p the generator of $H_1(N_p)$ such that $(\Gamma|_{N_p})_*(a_p) = 1$. Just the first homology group determines an orientation on S^1 and on each fiber N_p.

Definition. A homeomorphism $h: C \to C$ is said to be *orientation preserving* [*reversing*] if for every $p \in C$, $h|_{N_p}$ is orientation preserving [reversing].

Lemma 10. *If $k: C \to C$ is a map and if for every $p \in C$ there exists a $p' \in C$ such that $k(N_p) \subset N_{p'}$, then for any two points p_1 and p_2 in C, we have* $(\Gamma \circ k|_{N_{p_1}})_*(a_{p_1}) = (\Gamma \circ k|_{N_{p_2}})_*(a_{p_2})$.

Proof. There exists a finite open cover $\{V_i\}$ of C such that each V_i is homeomorphic to $(V_i \cap B) \times S^1$, and such that if $V_i \cap V_j \neq \varnothing$, then the two Cartesian product structures coming from V_i and V_j are compatible. Hence, any two simple closed curves N_{p_1} and N_{p_2} bound a singular annulus. Therefore $(\Gamma \circ k|_{N_{p_1}})_*(a_{p_1}) = (\Gamma \circ k|_{N_{p_2}})_*(a_{p_2})$. □

Lemma 11 follows immediately from Lemma 10.

Lemma 11. *If $h: C \to C$ is a homeomorphism, then h is orientation preserving or h is orientation reversing.*

Lemma 12. *Let $p_i = (1, \frac{2\pi i}{9}, 0, 0, 0, 0, 0) \in C$, where $i = 0, \dots, 8$. Let $h: C \to C$ be a homeomorphism such that $h(B) = B$. If $h(N_{p_0}) = N_{p_1}$ and $h(N_{p_1}) = N_{p_0}$, then h is orientation reversing.*

Proof. Since $h(\{p_0, p_3, p_6\}) = \{p_1, p_4, p_7\}$, and $h(\{p_1, p_4, p_7\}) = \{p_0, p_3, p_6\}$, we have $h(p_0) = p_{i_0}$, where $i_0 \bmod 3 = 1$, and $h(p_1) = p_{i_1}$, where $i_1 \bmod 3 = 0$. By Lemma 7, $h(p_j) = p_{[i_0 + j(i_1 - i_0)] \bmod 9}$ for $j = 0, \dots, 8$. Therefore $h(p_3) = p_{(i_0+6) \bmod 9}$ and $h(p_6) = p_{(i_0+3) \bmod 9}$ which implies that h is orientation reversing. □

3. The example

Assume the following notation:

$$J^n = \{(x_1, \dots, x_n) \in E^n : x_1 \in [0, 1]\},$$
$$J_0^{n-1} = \{(x_1, \dots, x_n) \in E^n : x_1 = 0\},$$
$$J_1^{n-1} = \{(x_1, \dots, x_n) \in E^n : x_1 = 1\}.$$

For $i < n$, consider E^i to be the subset of E^n for which $x_{i+1} = \cdots = x_n = 0$.

Let T be the Möbius strip with $\partial T = T_1$, and let T_0 be the middle simple closed curve of T. Consider T to be the mapping cylinder of $\gamma: T_1 \to T_0$, where γ is a map of degree 2. Since there is a piecewise linear embedding of T in E^3, we may assume that T is a piecewise linear subset of J^4 with $T_i = T \cap J_i^3$ for $i = 0, 1$. Let ΣT_i be the suspension of T_i. Denote by ΣT the mapping cylinder of the suspension of γ. Again, assume that ΣT is a piecewise linear subset of J^5 with $\Sigma T_i = \Sigma T \cap J_i^4$ for $i = 0, 1$. Let V be a regular neighborhood of ΣT in J^5 such that for $i = 0, 1$, $V_i = V \cap J_i^4$ is a regular neighborhood of ΣT_i. Let V', V'', V_0', V_0'', and V_1', V_1'' be two copies of V, V_0, and V_1 respectively. Denote by $\sigma: \partial V' - \mathrm{Int}(V_0' \cup V_1') \to \partial V'' - \mathrm{Int}(V_0'' \cup V_1'')$, $\sigma_0: \partial V_0' \to \partial V_0''$, and $\sigma_1: \partial V_1' \to \partial V_1''$ the homeomorphisms corresponding to the identity homeomorphisms. Assume that V has an orientation compatible with the orientation of E^5, and assume that each V_i has an orientation induced by the orientation of V. Let $\overline{\gamma}: V_1 \to V_0$ be an orientation reversing homeomorphism, and let $\overline{\gamma}': V_1' \to V_0'$ and $\overline{\gamma}'': V_1'' \to V_0''$ be the homeomorphisms corresponding to $\overline{\gamma}$. Let $G = (V' \cup V'')/\sigma$. Note that $\partial G = G_0 \cup G_1$, where $G_0 = (V_0' \cup V_0'')/\sigma_0$ and $G_1 = (V_1' \cup V_1'')/\sigma_1$ are disjoint sets, each homeomorphic to $S^2 \times S^2$. The homeomorphisms $\overline{\gamma}'$ and $\overline{\gamma}''$ yield a homeomorphism $\hat{\gamma}: G_1 \to G_0$.

Denote by D the continuum obtained by putting $D = Z_N$, with $N = G$ and $g_3 = \hat{\gamma}$, where g_4 is the map appearing in the definition of Z_N given in §2. Each fiber N_p of D is an orientable 5-manifold F which is a union of three copies of G intersecting along the boundary components.

Let L be a properly embedded arc in G with end points q and $g_3(q)$ on G_1 and G_0, respectively. There exists a retraction $r: G \to L$ such that $r^{-1}(q) = G_1$, $r^{-1}(g_3(q)) = G_0$. We can write $F = G^0 \cup G^1 \cup G^2$ with $G_1^i = G_0^{(i+1) \bmod 3}$ for $i = 0, 1, 2$, where G^i, G_0^i, and G_1^i are copies of G, G_0, and G_1, respectively. Let $L_i \subset G^i$ be an arc corresponding to the arc $L \subset G$. Put $K = L_0 \cup L_1 \cup L_2$. Note that K is a retract of F. Let $\overline{r}: F \to K$ be a retraction such that for $i = 0, 1, 2$, $\overline{r}|_{G^i}: G^i \to L_i$ is the retraction corresponding to r. Notice that $\overline{r}$ induces an isomorphism of the first homology groups $\overline{r}_*: H_1(F) \to H_1(K)$.

Let $\tau: \widetilde{F} \to F$ be a covering map such that $\tau^{-1}(K)$ is homeomorphic to E^1. Clearly, for $s_0 \in \widetilde{F}$, $\pi_1(\widetilde{F}, s_0) \approx 0$. For $i = 0, \pm 1, \pm 2, \dots$, denote by F_i a subset of $\widetilde{F}$ homeomorphic to G such that $\widetilde{F} = \bigcup_{i=-\infty}^{\infty} F_{i-1} \cap F_i \neq \varnothing$ and for $k = 0, 1, 2$, $\tau^{-1}(G^k) = \bigcup_{j=-\infty}^{\infty} F_{3j+k}$.

Definition. Let m be an integer. A homeomorphism $h: \widetilde{F} \to \widetilde{F}$ is said to be an *m-shift homeomorphism* if $h(F_i) = F_{i+m}$ for $i = 0, \pm 1, \pm 2, \dots$.

Definition. Let $m = 0, 1, 2$. A homeomorphism $h: F \to F$ is said to be an *m-shift homeomorphism* if $h(G^i) = G^{(i+m) \bmod 3}$ for $i = 0, 1, 2$.

Observe that for $i = 0, \pm 1, \pm 2, \dots$, $F_{i-1} \cap F_i$ is homeomorphic to $S^2 \times S^2$.

From the properties of mapping cylinders and regular neighborhoods, it follows that the fourth homology group $H_4(\widetilde{F})$ is generated by $\{b_i\}_{i=-\infty}^{\infty}$, with relations $b_i = 2b_{i-1}$, where b_i is obtained from the 4-manifold $F_{i-1} \cap F_i$ for $i = 0, \pm 1, \pm 2, \dots$. Moreover, by choosing an appropriate orientation of $F_{i-1} \cap F_i$, we may assume that the cycle representing b_i has coefficient 1 on every simplex of $F_{i-1} \cap F_i$. Then, b_i cannot be represented by a cycle with its carrier contained in $\bigcup_{j=i+1}^{\infty} F_j$ for $i = 0, \pm 1, \pm 2, \dots$. Also, note that if $h_1, h_2: \widetilde{F} \to \widetilde{F}$ ($h_1, h_2: F \to F$) are two isotopic m_1-shift and m_2-shift homeomorphisms, respectively, then $m_1 = m_2$.

Let a be a generator of $H_1(K)$.

Lemma 13. *If $h: F \to F$ is a homeomorphism, then $(\bar{r} \circ h|_K)_*(a) = a$.*

Proof. By [4, pp. 90–91] there exists a homeomorphism $\tilde{h}: \widetilde{F} \to \widetilde{F}$, and there exists a retraction $\tilde{r}: \widetilde{F} \to \tau^{-1}(K)$ such that the diagram

$$\begin{array}{ccccc} \widetilde{F} & \xrightarrow{\tilde{h}} & \widetilde{F} & \xrightarrow{\tilde{r}} & \tau^{-1}(K) \\ \tau\downarrow & & \downarrow\tau & & \downarrow \tau|_{\tau^{-1}(K)} \\ F & \xrightarrow{h} & F & \xrightarrow{\bar{r}} & K \end{array}$$

commutes.

Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of points in $\widetilde{F}$. We will say that $\lim_{n\to\infty} p_n = \infty$ $[\lim_{n\to-\infty} p_n = -\infty]$ if for every integer n_0 almost all of the points p_n belong to $\bigcup_{i=n_0}^{\infty} F_i$ $[\bigcup_{i=-\infty}^{n_0} F_i]$. To prove Lemma 13, it is enough to show that if $\lim_{n\to\infty} p_n = \infty$, then $\lim_{n\to\infty} \tilde{h}(p_n) = \infty$ (hence $\lim_{n\to\infty} \tilde{r} \circ \tilde{h}(p_n) = \infty$) for every sequence $\{p_n\}_{n=1}^{\infty}$ contained in $\tau^{-1}(K)$.

There exist two sequences of positive integers $\{i_n\}_{n=1}^{\infty}$ and $\{j_n\}_{n=1}^{\infty}$ such that for every $n = 1, 2, \ldots$,

$$\bigcup_{k=-i_n}^{i_n} F_k \subset \bigcup_{k=-j_n}^{j_n} \tilde{h}(F_k) \subset \bigcup_{k=-i_{n+1}}^{i_{n+1}} F_k.$$

Note that if $n_0 \geq 0$, then $\bigcup_{k=-n_0}^{n_0} F_k$ separates $\widetilde{F}$ between $\bigcup_{k=-\infty}^{-n_0-2} F_k$ and $\bigcup_{k=n_0+2}^{\infty} F_k$, and $\bigcup_{k=-n_0}^{n_0} \tilde{h}(F_k)$ separates $\widetilde{F}$ between $\bigcup_{k=-\infty}^{-n_0-2} \tilde{h}(F_k)$ and $\bigcup_{k=n_0+2}^{\infty} \tilde{h}(F_k)$. Hence, there exists a strictly increasing sequence $\{i'_m\}_{m=1}^{\infty}$, and a strictly increasing or decreasing sequence $\{j'_m\}_{m=1}^{\infty}$ such that $\tilde{h}(F_{j'_m}) \subset \bigcup_{k=i'_m}^{\infty} F_k$. If $\{j'_m\}_{m=1}^{\infty}$ is strictly decreasing, then $\tilde{h}_*(b_{j'_0})$ can be represented by a cycle with its carrier in $\bigcup_{k=k_0}^{\infty} \tilde{h}(F_k)$ for some $k_0 > j'_0$, which is a contradiction. Hence, $\{j'_m\}_{m=1}^{\infty}$ is strictly increasing, and if $\{p_n\}_{n=1}^{\infty}$ is a sequence with $\lim_{n\to\infty} p_n = \infty$, then $\lim_{n\to\infty} \tilde{h}(p_n) = \infty$. □

Lemma 14. *Let $X = S^1 \times F$. Let $h: X \to X$ be a homeomorphism such that for every $\alpha \in S^1$ there exists an $\alpha' \in S^1$ with $h(\{\alpha\} \times F) = \{\alpha'\} \times F$. Let $\rho: \widetilde{X} \to X$ be a covering map defined by setting $\widetilde{X} = S^1 \times \widetilde{F}$ and $\rho = \mathrm{id}_{S^1} \times \tau$. Then there exists a map $\tilde{h}: \widetilde{X} \to \widetilde{X}$ such that the diagram*

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{h}} & \widetilde{X} \\ \rho\downarrow & & \downarrow\rho \\ X & \xrightarrow{h} & X \end{array}$$

commutes.

Proof. As defined before, $S^1 = \{\alpha: 0 \leq \alpha < 2\pi\}$. We may assume that both copies of S^1 appearing in $h: S^1 \times F \to S^1 \times F$ are parametrized in such a way that $h(\{\alpha\} \times F) = \{\alpha\} \times F$. Let $x_0 = (0, s_0) \in X$, $y_0 = (0, r_0) \in X$, $\tilde{x}_0 = (0, \tilde{s}_0) \in \widetilde{X}$, and $\tilde{y}_0 = (0, \tilde{r}_0) \in \widetilde{X}$ be points such that $\rho(\tilde{x}_0) = x_0$, $\rho(\tilde{y}_0) = y_0$, and $h(x_0) = y_0$. By [4, p. 90], it is enough to show that $(h \circ \rho)_\#(\pi_1(\widetilde{X}, \tilde{x}_0)) \subset \rho_\#(\pi_1(\widetilde{X}, \tilde{y}_0))$. Let $f: [0, 1] \to \widetilde{X}$ be a loop defined by $f(\alpha) = (2\pi\alpha \bmod 2\pi, \tilde{s}_0)$; the loop f represents a generator of $\pi_1(\widetilde{X}, \tilde{x}_0)$.

For $0 \leq a \leq b \leq 2\pi$, put

$$X_{[a,b]} = \begin{cases} \{(\alpha, s) \in X: a \leq \alpha \leq b\} & \text{if } b \neq 2\pi, \\ \{(\alpha, s) \in X: a \leq \alpha < b \text{ or } \alpha = 0\} & \text{if } b = 2\pi. \end{cases}$$

Let $\widetilde{X}_{[a,b]} = \rho^{-1}(X_{[a,b]})$. Let $\tilde{h}_1: \widetilde{X}_{[0,\pi]} \to \widetilde{X}_{[0,\pi]}$ be the unique lifting of $(h \circ \rho)|_{\widetilde{X}_{[0,\pi]}}$ with $\tilde{h}_1(\tilde{x}_0) = \tilde{y}_0$, and let $\tilde{h}_2: \widetilde{X}_{[\pi,2\pi]} \to \widetilde{X}_{[\pi,2\pi]}$ be the unique lifting of $(h \circ \rho)|_{\widetilde{X}_{[\pi,2\pi]}}$ such that $\tilde{h}_2|_{\widetilde{X}_{[\pi,\pi]}} = \tilde{h}_1|_{\widetilde{X}_{[\pi,\pi]}}$. By [4, p. 86], there exists a path $\tilde{f}: [0, 1] \to \widetilde{X}$ with $\tilde{f}(0) = \tilde{y}_0$ such that $\rho \circ \tilde{f} = h \circ \rho \circ f$. Let $i_\alpha: \widetilde{F} \to \{\alpha\} \times \widetilde{F}$ be the inclusion defined by $i_\alpha(s) = (\alpha, s)$, and let $\pi_{\widetilde{F}}: S^1 \times \widetilde{F} \to \widetilde{F}$ be the projection. If $\tilde{f}(1) \neq \tilde{y}_0$, then $k = \pi_{\widetilde{F}} \circ \tilde{h}_2^{-1} \circ \tilde{h}_1 \circ i_0: \widetilde{F} \to \widetilde{F}$ is a $3n$-shift homeomorphism with $n \neq 0$. However, $\tilde{h}_1$ and $\tilde{h}_2$ yield an isotopy $H_t: \widetilde{F} \to \widetilde{F}$ defined by

$$H_t = \pi_{\widetilde{F}} \circ \tilde{h}_j^{-1} \circ i_{2\pi t} \circ \pi_{\widetilde{F}} \circ \tilde{h}_1 \circ i_0,$$

where $j = 1$ if $t \in [0, \frac{1}{2})$ and $j = 2$ if $t \in [\frac{1}{2}, 1]$. Hence, $H_0 = \mathrm{id}_{\widetilde{F}}$ and $H_1 = k$ are isotopic, which is a contradiction. Therefore, $\tilde{f}(1) = \tilde{y}_0$ and $(h \circ \rho)_\#(\pi_1(\widetilde{X}, \tilde{x}_0)) \subset \rho_\#(\pi_1(\widetilde{X}, \tilde{y}_0))$. □

Lemma 15. *Let U and V be open connected subsets of B. Let $X = U \times F$, $Y = V \times F$, $\widetilde{X} = U \times \widetilde{F}$, and $\widetilde{Y} = V \times \widetilde{F}$. Let $\rho_X: \widetilde{X} \to X$ and $\rho_Y: \widetilde{Y} \to Y$ be covering maps defined by $\rho_X = \mathrm{id}_U \times \tau$ and $\rho_Y = \mathrm{id}_V \times \tau$ respectively. If $h: X \to Y$ is a homeomorphism, then there exists a map $\tilde{h}: \widetilde{X} \to \widetilde{Y}$ such that the diagram*

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{h}} & \widetilde{Y} \\ \rho_X \downarrow & & \downarrow \rho_Y \\ X & \xrightarrow{h} & Y \end{array}$$

commutes.

Proof. By Lemma 2, for every $u \in U$ there exists a $v \in V$ such that $h(\{u\} \times F) = \{v\} \times F$.

Let $x_0 \in X$, $y_0 \in Y$, $\tilde{x}_0 \in \widetilde{X}$, and $\tilde{y}_0 \in \widetilde{Y}$ be points such that $\rho_X(\tilde{x}_0) = x_0$, $\rho_Y(\tilde{y}_0) = y_0$, and $h(x_0) = y_0$. It is enough to show that if $f: [0, 1] \to \widetilde{X}$ is a loop with $f(0) = f(1) = \tilde{x}_0$, then there exists a loop $\tilde{f}: [0, 1] \to \widetilde{Y}$ with $\tilde{f}(0) = \tilde{f}(1) = \tilde{y}_0$ such that $h \circ \rho_X \circ f = \rho_Y \circ \tilde{f}$. Without loss of generality, we may assume that $f([0, 1]) \subset U \times \{\tilde{s}_0\}$, where $\tilde{x}_0 = (u_0, \tilde{s}_0)$.

Let $t_0 = 0 < t_1 < \cdots < t_m = 1$ be a sequence of points in $[0, 1]$, and let $f': [0, 1] \to \widetilde{X}$ be a loop such that $f'([0, 1]) \subset U \times \{\tilde{s}_0\}$, $f'(t_i) = f(t_i)$ for $i = 0, \ldots, m$, $f'|_{[0,1]-\{t_0,\ldots,t_m\}}: [0, 1] - \{t_0, \ldots, t_m\} \to f'([0, 1] - \{t_0, \ldots, t_m\})$

is one-to-one, and for each $i = 1, \dots, m$, there is a $j = 0, 1, 2$ such that $h \circ \rho_X \circ f'([t_{i-1}, t_i]) \cup h \circ \rho_X \circ f([t_{i-1}, t_i]) \subset V \times (G^j \cup G^{(j+1) \bmod 3})$.

If P is a simple closed curve in $f'([0, 1])$, then $h \circ \rho_X(P)$ is a simple closed curve in $h \circ \rho_X \circ f'([0, 1])$. Let $\pi_V: V \times F \to V$ be the projection. Since for each $u \in U$, $h(\{u\}) \times F = \{v\} \times F$ for some $v \in V$, then $\pi_V \circ h \circ \rho_X(P)$ is a simple closed curve in V, and $h \circ (\rho_X(P) \times F) = (\pi_V \circ h \circ \rho_X(P)) \times F$. By Lemma 14, for every $z \in \rho_Y^{-1} \circ h \circ \rho_X(P)$ there exists a simple closed curve $\widetilde{P} \subset \widetilde{Y}$ containing z such that $\rho_Y(\widetilde{P}) = h \circ \rho_X(P)$. Also, for $i = 0, \dots, m$, every loop whose image is the set $h \circ \rho_X \circ f'([t_{i-1}, t_i]) \cup h \circ \rho_X \circ f([t_{i-1}, t_i])$ lifts to a loop in $\widetilde{F}$. Hence, there exists a loop $\tilde{f}: [0, 1] \to \widetilde{X}$ such that $h \circ \rho_X \circ f = \rho_Y \circ \tilde{f}$ and $\tilde{f}(0) = \tilde{f}(1) = \tilde{y}_0$. □

Consider C to be a subset of D, $C = \{(b, s) \in D: s \in L\}$. Let s_0 be the point of $L \cap G_0$. Consider B to be a subset of D, $B = \{(b, s) \in D: s = s_0\}$. Let $R: D \to C$ be a retraction defined by $R(b, s) = (b, r(s))$, where $s \in G - G_1$. In this section, the notation N_p is used for fibers of D (homeomorphic to F). Hence, if $p = (b_1, s_1)$, then $N_p = \{(b, s) \in D: b = b_1,\ b = f_2^{(3)}(b_1)$, or $b = f_2^{(6)}(b_1)\}$, where f_2 is the homeomorphism defined in §2. If $p \in C$, then the fiber of C containing p, homeomorphic to S^1, is denoted by K_p, i.e., $K_p = N_p \cap C$.

Lemma 16. *If $h: D \to D$ is a homeomorphism, then there exists an orientation preserving homeomorphism $\overline{h}: C \to C$ such that $\overline{h}(B) = B$ and such that for every p and q in C, if $h(N_p) = N_q$ then $\overline{h}(K_p) = K_q$.*

Proof. Define a homeomorphism $\varphi: D \to D$ by $\varphi(b, s) = (f_2^{(3)}(b), s)$. Let $X = \{(r, \theta, z, \overline{x}, \overline{y}, \overline{z}, s) \in D: 0 < \theta < \frac{2\pi}{3},\ \frac{2\pi}{3} < \theta < \frac{4\pi}{3}$, or $\frac{4\pi}{3} < \theta < 2\pi\}$. Let U be a component of $X \cap B$. Notice that X is homeomorphic to $U \times F$ and the set $h(X) \cap B$ is not empty. Let V be a component of $h(X) \cap B$.

We claim that $h(X)$ is homeomorphic to $V \times F$. Since V is connected, if $V \cap \varphi(V) \neq \varnothing$, then V is invariant under φ. Then, there exist a point $p_0 \in V$ and an arc $P_0 \subset V$ joining p_0 and $\varphi(p_0)$ such that $P_0 \cap \varphi(P_0) = \{p_0\}$. Hence $P_0 \cup \varphi(P_0) \cup \varphi^{(2)}(P_0)$ is a simple closed curve contained in V. Consider the Cartesian product $P_0 \times F$. Let $\pi_F: P_0 \times F \to F$ be the projection, and let $i_p: F \to P_0 \times F$ be the inclusion defined by $i_p(s) = (p, s)$ for $p \in P_0$. The set $Y = \bigcup_{p \in P_0} N_p$ is homeomorphic to $(P_0 \times F)/k$, where $k: \{p_0\} \times F \to \{\varphi(p_0)\} \times F$ is a homeomorphism such that $\pi_F \circ k \circ i_{p_0}$ is a 1-shift homeomorphism.

The embedding $h^{-1}|_Y: Y \to X$ preserves fibers, i.e., for every $p \in Y$, we have $(h^{-1}|_Y)(N_p) = N_q$ for some $q \in X$. By an argument similar to that of the proof of Lemma 14, $\pi_F \circ k \circ i_{p_0}$ is isotopic to the identity, which is a contradiction. Hence, $V \cap \varphi(V) = \varnothing$, and $\bigcup_{p \in V} N_p$ is homeomorphic to $V \times F$. Since $h(X)$ is connected, $h(X) = \bigcup_{p \in V} N_p$. Notice that for every $p \in V$, V intersects each fiber N_p at exactly one point, and there is a homeomorphism of $\bigcup_{p \in V} N_p$

onto $V \times F$ which takes each fiber N_p onto some fiber $\{q\} \times F$.

Let $\omega: B \times \widetilde{F} \to D$ be a covering map such that $\omega(b, s) = (b, \tau(s))$ for $(b, s) \in B \times (F_0 - F_1)$. By Lemma 15, there exists a map $\tilde{h}_U: U \times \widetilde{F} \to B \times \widetilde{F}$ with $\tilde{h}(U \times \widetilde{F}) = V \times \widetilde{F}$ and such that the diagram

$$\begin{array}{ccc} U \times \widetilde{F} & \xrightarrow{\tilde{h}_U} & B \times \widetilde{F} \\ \omega|_{U \times \widetilde{F}} \downarrow & & \downarrow \omega \\ X & \xrightarrow{h|_X} & D \end{array}$$

commutes.

For every $p \in \overline{U}$, the closure of U, there exists a neighborhood W_p of p in B such that the set $\bigcup_{q \in W_p} N_q$ is homeomorphic to the Cartesian product $W_p \times F$. Furthermore, for every $p \in \overline{U}$, there exists a unique map $\tilde{h}_p: W_p \times \widetilde{F} \to B \times \widetilde{F}$ such that the diagram

$$\begin{array}{ccc} W_p \times \widetilde{F} & \xrightarrow{\tilde{h}_p} & B \times \widetilde{F} \\ \omega|_{W_p \times \widetilde{F}} \downarrow & & \downarrow \omega \\ \bigcup_{q \in W_p} N_q & \xrightarrow{h|_{\bigcup_{q \in W_p} N_q}} & D \end{array}$$

commutes, and $\tilde{h}_p|_{(U \cap W_p) \times \widetilde{F}} = \tilde{h}_U|_{(U \cap W_p) \times \widetilde{F}}$. Therefore there exists a map $\tilde{h}: \overline{U} \times \widetilde{F} \to B \times \widetilde{F}$ such that $\tilde{h}|_{U \times \widetilde{F}} = \tilde{h}_U$. Note that $\tilde{h}(\overline{U} \times \widetilde{F}) = \overline{V} \times \widetilde{F}$.

The set $\omega^{-1}(C)$ is homeomorphic to $B \times E^1$. Assume that $\omega^{-1}(C) = B \times E^1$ and $\omega^{-1}(C) \cap (\overline{U} \times \widetilde{F}) = \overline{U} \times E^1$. Let $(b, s) \in B \times E^1$, where $b = (r, \theta, z, \overline{x}, \overline{y}, \overline{z})$. Let $i = 0, 1, 2$. Assume that if $i \leq s \bmod 3 < i + 1$, then $\omega(b, s) = (b_i, s_i)$, where $b_i = (r, \theta + \frac{2\pi i}{3}, z, \overline{x}, \overline{y}, \overline{z})$, and $s_i = (s \bmod 3) - i$.

Let $\widetilde{R}: B \times \widetilde{F} \to B \times E^1$ be a retraction such that the diagram

$$\begin{array}{ccc} B \times \widetilde{F} & \xrightarrow{\widetilde{R}} & B \times E^1 \\ \omega \downarrow & & \downarrow \omega|_{B \times E^1} \\ D & \xrightarrow{R} & C \end{array}$$

commutes. Note that if $p, \varphi(p) \in \overline{U}$ and $s \in E^1$, then

$$\omega(p, s) = \omega(\varphi(p), s - 1).$$

Hence, if $p, \varphi(p) \in \overline{U}$, $s \in E^1$, and $\widetilde{R} \circ \tilde{h}(p, s) = (q, r)$, then $\widetilde{R} \circ \tilde{h}(\varphi(p), s-1) = (\varphi(q), r - 1)$ or $\widetilde{R} \circ \tilde{h}(\varphi(p), s - 1) = (\varphi^{(2)}(q), r + 1)$. By Lemma 13, if $\widetilde{R} \circ \tilde{h}(p, s) = (q, r)$, then $\widetilde{R} \circ \tilde{h}(\varphi(p), s - 1) = (\varphi(q), r - 1)$ for $p, \varphi(p) \in \overline{U}$ and $s \in E^1$. Finally, if $(p, s) \in \overline{U} \times E^1$ is a point and $\widetilde{R} \circ \tilde{h}(p, s) = (q, r)$, then define $h': \overline{U} \times E^1 \to \overline{V} \times E^1$ by $h'(p, s) = (q, s)$. Let $\bar{h}: C \to C$ be such that the diagram

$$\begin{array}{ccc} \overline{U} \times E^1 & \xrightarrow{h'} & \overline{V} \times E^1 \\ \omega|_{\overline{U} \times E^1} \downarrow & & \downarrow \omega|_{\overline{V} \times E^1} \\ C & \xrightarrow{\bar{h}} & C \end{array}$$

commutes. $\overline{h}$ is an orientation preserving homeomorphism, $\overline{h}(B) = B$, and if $h(N_p) = N_q$, then $\overline{h}(K_p) = K_q$ for $p, q \in C$. □

Theorem. *The continuum D is homogeneous but not bihomogeneous.*

Proof. By Lemma 4, D is homogeneous. Let $p_0 = (1, 0, 0, 0, 0, 0, s_0)$ and $p_1 = (1, \frac{2\pi}{9}, 0, 0, 0, 0, s_0)$. Suppose that there exists a homeomorphism $h: D \to D$ such that $h(p_0) = p_1$ and $h(p_1) = p_0$. Then, by Lemma 16, there exists an orientation preserving homeomorphism $\overline{h}: C \to C$ such that $\overline{h}(B) = B$, $\overline{h}(K_{p_0}) = K_{p_1}$, and $\overline{h}(K_{p_1}) = K_{p_0}$. However, by Lemma 12, $\overline{h}$ is orientation reversing. Therefore D is not bihomogeneous. □

4. Problems

Definition. A space X is said to be *semilocally bihomogeneous* if for every p there exists a neighborhood U of p such that for every $q \in U$ there exists a homeomorphism $h: X \to X$ with $h(p) = q$ and $h(q) = p$.

Remark 1. The continuum D constructed in this paper is semilocally bihomogeneous.

Problem 1. Does there exist a homogeneous continuum (locally connected continuum) which is not semilocally bihomogeneous?

Problem 2. Does there exist a homogenous, locally compact metric space (continuum) X such that for no two points p and q in X there exists a homeomorphism $h: X \to X$ with $h(p) = q$ and $h(q) = p$?

Remark 2. Cook's example (see [3]) of a homogeneous, nonbihomogeneous, locally compact metric space is of dimension 2. The example constructed in this paper is of dimension 7.

Problem 3. What is the lowest dimension of a homogeneous, nonbihomogeneous, locally compact metric space (continuum)?

Remark 3. W. R. R. Transue points out that by his result of [8], Cook's example is embeddable in E^3.

Problem 4. Does there exist a homogeneous, nonbihomogeneous continuum embeddable in E^3?

Problem 5. Is every homogeneous, metric, absolute neighborhood retract bihomogeneous?

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