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ON GELFAND PAIRS ASSOCIATED WITH SOLVABLE LIE GROUPS

CHAL BENSON, JOE JENKINS AND GAIL RATCLIFF

ABSTRACT. Let G be a locally compact group, and let K be a compact subgroup of $\mathrm{Aut}(G)$, the group of automorphisms of G. There is a natural action of K on the convolution algebra $L^1(G)$, and we denote by $L^1_K(G)$ the subalgebra of those elements in $L^1(G)$ that are invariant under this action. The pair (K, G) is called a Gelfand pair if $L^1_K(G)$ is commutative. In this paper we consider the case where G is a connected, simply connected solvable Lie group and $K \subseteq \mathrm{Aut}(G)$ is a compact, connected group. We characterize such Gelfand pairs (K, G), and determine a moduli space for the associated K-spherical functions.

INTRODUCTION

Let G be a locally compact group, and let K be a compact subgroup of $\mathrm{Aut}(G)$, the group of automorphisms of G. There is a natural action of K on the convolution algebra $L^1(G)$, and we denote by $L^1_K(G)$ the subalgebra of those elements in $L^1(G)$ that are invariant under this action. The pair (K, G) is called a Gelfand pair if $L^1_K(G)$ is commutative. A more general and more usual definition of Gelfand pairs assumes that K is a compact subgroup of G. One then defines (K, G) to be a Gelfand pair if the subalgebra of K-bi-invariant elements in $L^1(G)$ is commutative. This is the case, for example, if (G, K) is a Riemannian symmetric pair, as was shown by Gelfand in 1950, [Ge]. In this paper we consider the case where G is a connected, simply connected solvable Lie group and $K \subseteq \mathrm{Aut}(G)$ is a compact, connected group.

For the remainder of the paper, unless otherwise stated, S will denote a connected, simply connected solvable Lie group and N will denote a connected, simply connected nilpotent Lie group, with corresponding Lie algebras $\mathscr{S}$, $\mathscr{N}$, and K will denote a compact, connected subgroup of the appropriate automorphism group.

The classification of Gelfand pairs involving solvable groups presupposes a classification for such pairs involving nilpotent groups, which is the subject we

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first consider. An important reduction is given by

Theorem A. *If* (K, N) *is a Gelfand pair then* N *is at most two step.*

The proof is based on the observation that (K, G) is a Gelfand pair if, and only if, products (as sets) of K-orbits in G commute, i.e. for each $x, y \in G$, $(K \cdot x)(K \cdot y) = (K \cdot y)(K \cdot x)$.

The criterion that we generally use to determine if (K, N) is a Gelfand pair is contained in a theorem due to Carcano, [Ca], which we now recall. Let $\pi \in \widehat{N}$, and denote by K_π the set of all elements $k \in K$ such that $\pi_k \simeq \pi$ where π_k is the element of $\widehat{N}$ defined by $\pi_k(x) = \pi(k \cdot x)$ for all $x \in N$. Then there is a projective representation W_π of K_π on $\mathbf{H}_\pi$, the representation space of π. W_π is called the *intertwining representation* for π. If σ is the cocycle of W_π there is a decomposition

$$W_\pi = \sum_{T \in \widehat{K}_\pi^\sigma} c(T, W_\pi) T,$$

where $c(T, W_\pi)$ denotes the multiplicity of T in W_π. Carcano's theorem states that (K, N) is a Gelfand pair if $c(T, W_\pi) \leq 1$ for all π in a set of full Plancherel measure, and that, conversely, if (K, N) is a Gelfand pair then $c(T, W_\pi) \leq 1$ for every $\pi \in \widehat{N}$.

Since the representations of 2-step nilpotent groups factor through tensor products of representations of Heisenberg $\times$ abelian groups, the classification of Gelfand pairs (K, N) reduces to classification of Gelfand pairs (K, H_n), where H_n is the $2n + 1$-dimensional Heisenberg group. We realize H_n as $\mathbf{C^n} \times \mathbf{R}$ with multiplication given by $(z, t)(z', t') = (z + z', t + t' + 2\Im z\bar{z}')$. If $K \subseteq \mathrm{Aut}(H_n)$, then, after conjugating by an element of $\mathrm{Aut}(H_n)$ if necessary, we may assume that $K \subseteq U(n)$, the group of $n \times n$ unitary matrices acting on $\mathbf{C^n}$ in the usual fashion. Given such a K, we denote by $K_{\mathbf{C}}$ its complexification, which may be considered as a subgroup of $Gl(n, \mathbf{C})$. We denote by $\mathbf{C}[\mathbf{C^n}]$ the polynomial ring over $\mathbf{C^n}$. There is a natural action of $K_{\mathbf{C}}$ on $\mathbf{C}[\mathbf{C^n}]$.

Theorem B. *Suppose that* K *acts irreducibly on* $\mathbf{C^n}$. (K, H_n) *is a Gelfand pair if, and only if,* $K_{\mathbf{C}}$ *acts without multiciplicity on* $\mathbf{C}[\mathbf{C^n}]$.

Victor Kac, [Ka], has given a complete list of all such groups $K_{\mathbf{C}}$ acting without multiplicity on $\mathbf{C}[\mathbf{C^n}]$. If the action of K on $\mathbf{C^n}$ is not irreducible, consider the irreducible decomposition $\mathbf{C^n} = \sum_{\mathbf{j}=1}^{\mathbf{p}} \mathbf{V_j}$, and let K_j denote the subgroup of $U(V_j)$ given by the (irreducible) action of K on V_j. The subset of H_n given by $V_j \times \mathbf{R}$ is isomorphic to H_{m_j}, where $m_j = \dim(V_j)$. For $n_1, \ldots, n_p \in \mathbf{Z}^+$ let $\mathbf{P}^{\mathbf{n}_1, \ldots, \mathbf{n}_\mathbf{p}} = \bigotimes_{\mathbf{j}=1}^{\mathbf{p}} \mathbf{P}_{\mathbf{j}, \mathbf{n}_\mathbf{j}}$, where $\mathbf{P}_{\mathbf{j}, \mathbf{n}_\mathbf{j}}$ is a K_j-irreducible subspace of $\mathbf{C}[V_j]$.

Theorem C. (K, N) *is a Gelfand pair if, and only if, the subrepresentations of* K *on the various* $\mathbf{P}^{\mathbf{n}_1, \ldots, \mathbf{n}_\mathbf{p}}$ *are all distinct.*

We next consider the free, two-step nilpotent Lie group on n-generators, $F(n)$. We identify its Lie algebra $\mathscr{F}(n)$ with $\mathbf{R}^n \oplus \Sigma_n$, where $\mathbf{R}^n$ is viewed as $1 \times n$ real matrices, Σ_n is the set of $n \times n$ skew symmetric matrices, and the bracket is defined by $[(u, U), (v, V)] = (0, u^t v - v^t u)$. The automorphism group of $\mathscr{F}(n)$ is identified with $Gl(n, \mathbf{R}) \times \mathrm{Hom}(\mathbf{R}^n, \Sigma_n)$ with the action of (A, ν) on (u, U) given by $(A, \nu) \cdot (u, U) = (uA, A^t UA + \nu(u))$. Thus, $O(n)$, the group of $n \times n$ orthogonal matrices is a maximal compact subgroup of $\mathrm{Aut}(\mathscr{F}(n))$. We denote by $SO(n)$ the subgroup of matrices of determinant one.

Theorem D. *Let K be a closed (not necessarily connected) subgroup of $SO(n)$. $(K, F(n))$ is a Gelfand pair if, and only if $K = SO(n)$.*

Suppose now that a two-step N is given with $[\mathscr{N}, \mathscr{N}] = \mathscr{Z}$, where $\mathscr{Z}$ is the center of $\mathscr{N}$. (If this condition is not satisfied, then N has an abelian direct product factor that does not play a role in the current considerations.) Given a compact, connected $K \subseteq \mathrm{Aut}(N)$, we fix a K-invariant inner product, $\langle \cdot, \cdot \rangle$, on $\mathscr{N}$, and denote by $\mathscr{N}_1$, the orthogonal complement of $\mathscr{Z}$ in $\mathscr{N}$. Let $X_1, \ldots, X_n$ be an orthonormal basis for $\mathscr{N}_1$. Define the homomorphism $\lambda: \mathscr{F}(n) \to \mathscr{N}$ by setting $\lambda(e_i) = X_i$ (where $e_1, \ldots, e_n$ is the standard basis for $\mathbf{R}^n$), and $\lambda(E_{i,j}) = [X_i, X_j]$, (where $E_{i,j} = [(e_i, 0), (e_j, 0)] \in \mathscr{F}(n)$). Let $\mathscr{K}$ denote the kernel of λ $(\subseteq \Sigma_n)$. Note that $\lambda: \mathbf{R}^n \to \mathscr{N}_1$ is an isometry (where $\mathscr{F}(n)$ is equipped with the (standard) inner product $\langle (u, U), (v, V) \rangle = uv^t + \frac{1}{2}\mathrm{tr}(UV^t)$). Given $k \in K$, we define $\tilde{k} \in \mathrm{Aut}(\mathscr{F}(n))$ by $\tilde{k}(e_i) = \lambda^{-1}(k \cdot (\lambda(e_i)))$ and $\tilde{k}(E_{i,j}) = [\tilde{k} \cdot e_i, \tilde{k} \cdot e_j]$, and set $\widetilde{K} = \{\tilde{k} | k \in K\}$. Then $\widetilde{K} \subseteq O(n)$, and one has that K is maximal compact if, and only if, $\widetilde{K} = O_{\mathscr{K}}(n) := \{A \in O(n) | A \cdot \mathscr{K} \ (:= A^t \mathscr{K} A) = \mathscr{K}\}$.

Let $\mathscr{Z}$ denote the orthogonal complement in Σ_n of $\mathscr{K}$, and set $\mathscr{N}_{\mathscr{Z}} = \mathbf{R}^n \oplus \mathscr{Z}$ with Lie bracket defined by $[(u, U), (v, V)]_{\mathscr{Z}} = P_{\mathscr{Z}}(u^t v - v^t u)$, where $P_{\mathscr{Z}}$ is the orthogonal projection of Σ_n onto $\mathscr{Z}$. Then $\mathscr{N}_{\mathscr{Z}} \simeq \mathscr{N}$ and $\widetilde{K} \subseteq \mathrm{Aut}(\mathscr{N}_{\mathscr{Z}})$.

For nonzero $B \in \mathscr{Z}$, let $\mathscr{H}_B$ denote the subset of $\mathscr{N}_{\mathscr{Z}}$ given by $\mathbf{R}^n B \oplus \mathbf{R}B$, i.e. the range of B in $\mathbf{R}^n$ plus the line through B, and define a Lie bracket similar to the above by following the bracket in $\mathscr{F}(n)$ with the orthogonal projection onto $\mathbf{R}B$. The quotient Lie algebra $\mathscr{N}_{\mathscr{Z}}/\mathscr{Z}_0$, where $\mathscr{Z}_0$ is the orthogonal complement in $\mathscr{Z}$ of $\mathbf{R}B$ is isomorphic to the direct sum of ideals $\mathscr{N}_B$ and $(\mathbf{R}^n B)^\perp$, the latter being commutative. Let H_B denote the simply connected Lie group corresponding to $\mathscr{N}_B$, and given $b \in (\mathbf{R}^n B)^\perp$, let $\widetilde{K}_{(b,B)} = \{\tilde{k} \in \widetilde{K} | \tilde{k} \cdot (b, B) = (b, B)\}$.

Theorem E. *(K, N) is a Gelfand pair if $(\widetilde{K}_{(b,B)}, H_B)$ is a Gelfand pair for all (b, B) in a set of full Plancherel measure, and conversely, if (K, N) is a Gelfand pair, then $(\widetilde{K}_{(b,B)}, H_B)$ is a Gelfand pair for all $B \in \mathscr{Z}$, $b \in (\mathbf{R}^n B)^\perp$.*

We demonstrate the use of Theorem E in two examples. In the first, let N be the group whose Lie algebra has a basis X, Y_1, Y_2, Z_1, Z_2, and with all non-trivial commutators determined by $[X, Y_1] = Z_1$ and $[X, Y_2] = Z_2$. We show that there is no compact subgroup $K \subseteq \mathrm{Aut}(N)$ for which (K, N) is a Gelfand pair.

In the second example, we give a short proof of a theorem due to H. Leptin [Le] which states that if K is the n-dimensional torus (and N is a two-step group with $[\mathscr{N}, \mathscr{N}] = \mathscr{Z}$, the center of $\mathscr{N}$) then (K, N) is a Gelfand pair if, and only if, N is the quotient of the direct product of n-copies of H_1, with K lifting to a $U(1)$ action on each factor H_1.

We turn now to solvable groups. The essential new ingredient is another theorem due to H. Leptin, which was privately communicatated to the authors. Since a proof has not appeared in the literature, we include his proof here.

Theorem (Leptin). *Let $\mathscr{S}$ be a solvable Lie algebra with nilradical $\mathscr{N}$. Let K be a compact, connected subgroup of* $\mathrm{Aut}(\mathscr{S})$*, and let $\mathscr{S}_0 = \{X \in \mathscr{S} | k \cdot X = X, \ \forall\, k \in K\}$. Then $\mathscr{S} = \mathscr{S}_0 + \mathscr{N}$.*

For $X \in \mathscr{S}$, let i_X denote the inner-automorphism of S determined by $\exp X$, and denote by $\mathrm{rad}(S)$ the simply connected nilpotent Lie group whose Lie algebra is the nilradical of $\mathscr{S}$. Using Leptin's theorem we can prove

Theorem F. *(K, S) is a Gelfand pair if, and only if, $(K, \mathrm{rad}(S))$ is a Gelfand pair, and for each $X \in \mathscr{S}_0$, $y \in S$ there is a $k \in K$ such that $i_X(y) = k \cdot y$.*

Finally, we consider the K-spherical functions associated to a Gelfand pair (K, S). Recall that a K-spherical function ϕ is a continuous, complex valued function defined on S satisfying $\phi(e) = 1$ and $\int_K \phi(xk \cdot y)\, dk = \phi(x)\phi(y)$ for each $x, y \in S$. It is well known that integration against a K-spherical function, ϕ, defines a complex homomorphism on $L^1_K(S)$, that this homomorphism is continuous if ϕ is bounded, and that each continuous homomorphism of $L^1_K(S)$ is obtained in this manner. We denote by $\Delta(K, S)$ the set of continuous homomorphisms on $L^1_K(S)$. It follows from Theorem F, that if (K, S) is a Gelfand pair then S has polynomial growth, [Je], and hence that $L^1(S)$ is a symmetric Banach $*$-algebra, [Lu]. From this one can show that the bounded K-spherical functions are positive definite, in sharp contrast to the case when (G, K) is a Riemannian symmetric pair (cf. [He]).

We first consider Gelfand pairs (K, N). One shows that if $\pi \in \widehat{N}$ and $\pi' = \pi_k$, then the intertwining representations W_π and $W_{\pi'}$ have the same irreducible subspaces.

Theorem G. *Let (K, N) be a Gelfand pair. Then ϕ is a bounded K-spherical function if, and only if, there is a $\pi \in \widehat{N}$ and a $\xi \in V_\alpha \subseteq \mathbf{H}_\pi$, $\|\xi\| = 1$, such that for each $x \in N$,*

$$\phi(x) = \phi_{\pi,\xi}(x) := \int_K \langle \pi(k \cdot x)\xi, \xi\rangle\, dk,$$

where V_α *is an irreducible subspace for the intertwining representation* W_π. *Furthermore, bounded* K*-spherical functions* $\phi_{\pi,\xi} = \phi_{\pi',\xi'}$ *if, and only if,* $\pi' = \pi_k$ *for some* $k \in K$ *and* ξ, ξ' *belong to the same* V_α.

Theorem G states that there is a 1-1 corespondence between $\Delta(K, N)$ and the fibered product $\widehat{N}/K \times_\pi \sigma(W_\pi, \mathbf{H}_\pi)$, where $\widehat{N}/K$ denotes the K-orbits in $\widehat{N}$, and $\sigma(W_\pi, \mathbf{H}_\pi)$ denotes the irreducible components of W_π in $\mathbf{H}_\pi$.

Suppose now that (K, S) is a Gelfand pair. Let $X_1, \ldots, X_p$ be a basis for a complement of $\mathscr{N}$, the nilradical of $\mathscr{S}$, in $\mathscr{S}_0$. For each $y \in S$, there exist unique $n(y) \in N$ $(=\exp(\mathscr{N}))$ and $\mathbf{t}(y) \in \mathbf{R}^p$ such that $y = n(y)\Pi_i \exp(t_i(y)X_i)$.

Theorem H. ϕ *is a bounded* K*-spherical function on* S *if, and only if,* $\phi|_N$ *is a bounded* K*-spherical function on* N *and there exists* $\mathbf{a} \in \mathbf{R}^\mathbf{p}$ *such that* $\phi(y) = \phi(n(y))e^{i\langle \mathbf{a}, \mathbf{t}(y)\rangle}$. *Thus,*

$$\Delta(K, S) = \Delta(K, N) \times \mathbf{R}^p.$$

Remarks. A number of authors, in addition to those already mentioned, have considered Gelfand pairs of the form (K, N), and the associated K-spherical functions. In [HR] it is shown that the usual action of a maximal torus in $U(n)$ on H_n provides an example of a Gelfand pair, and the K-spherical functions are expressed in terms of Laguerre polynomials. The paper [KR] exhibits examples (K, N), where N is an irreducible group of Heisenberg type and K is either $\mathrm{Spin}(n)$ or a maximal connected compact subgroup of $\mathrm{Aut}(n)$. In [Ca], examples are presented where N arises as the Šilov boundary of a Siegel domain of type II and $K = SU(p) \times U(q)$. The generalized Laguerre polynomials introduced in [Hz] are shown in [Di] to be associated to certain Gelfand pairs $(U(n), H_n)$.

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Preliminaries

Consider a unimodular group G with $K \subseteq G$ a compact subgroup. We denote the L^1-functions that are invariant under both the left and right actions of K on G by $L^1(G//K)$. These form a subalgebra of the group algebra $L^1(G)$ with respect to the convolution product

$$f * g(x) = \int_G f(y)g(y^{-1}x)\,dy = \int_G f(xy^{-1})g(y)\,dy. \tag{1.1}$$

According to the traditional definition, one says that $K \subseteq G$ is a Gelfand pair if $L^1(G//K)$ is commutative.

Suppose now that K is a compact group acting on G by automorphisms via some homomorphism $\phi: K \to \mathrm{Aut}(G)$. One can form the semidirect product $K \propto G$, with group law

$$(k_1, x_1)(k_2, x_2) = (k_1k_2, x_1k_1 \cdot x_2), \tag{1.2}$$

where we write $k \cdot x$ for $\phi(k)(x)$. Right K-invariance of a function $f: K \propto G \to \mathbf{C}$ means that $f(k, x)$ depends only on x. Accordingly, if one defines $f_G: G \to \mathbf{C}$ by $f_G(x) = f(e, x)$, then one obtains a bijection $L^1(K \propto G//K) \simeq L^1_K(G)$ given by $f \leftrightarrow f_G$. Here $L^1_K(G)$ denotes the K-invariant functions on G, i.e. those $f \in L^1(G)$ such that $f(k \cdot x) = f(x)$ for all $x \in G$ and $k \in K$. One verifies easily that this map respects the convolution product and we see that $K \subseteq K \propto G$ is a Gelfand pair if, and only if, the convolution algebra $L^1_K(G)$ is commutative. Thus, the definition given in the introduction agrees with the more standard one.

Note that if (K_1, G) is a Gelfand pair and $K_1 \subseteq K_2$, then (K_2, G) is also a Gelfand pair. Also note that we can assume that K acts faithfully on G since we can always replace K by $K/\ker(\phi)$. In this way we can regard K as a compact subgroup of $\mathrm{Aut}(G)$. It is a useful fact that the Gelfand pair property depends only on the conjugacy class of K in $\mathrm{Aut}(G)$.

Lemma 1.3. *Let K, L be compact groups acting on G which are conjugate inside $\mathrm{Aut}(G)$. Then (K, G) is a Gelfand pair if, and only if, (L, G) is a Gelfand pair.*

Proof. For $f \in L^1(G)$, define $f^L \in L^1_L(G)$ by

$$f^L(x) = \int_L f(l \cdot x)\, dl. \tag{1.4}$$

The map $f \mapsto f^L$ is onto $L^1_L(G)$. Suppose that $L = uKu^{-1}$ for some $u \in \mathrm{Aut}(G)$. Then

$$\begin{aligned} f^L(x) &= \int_K f((uku^{-1}) \cdot x)\, dk \\ &= \int_K (f \circ u)(k \cdot (u^{-1}(x)))\, dk \\ &= (f \circ u)^K(u^{-1}(x)). \end{aligned}$$

It follows that $f^L(u(x)) = (f \circ u)^K(x)$ and that $L^1_L(G) \to L^1_K(G): f \mapsto f \circ u := \Phi(f)$ is a vector space isomorphism.

Let dx denote Haar measure on G. Then $u^*(dx) = \Delta(u)\, dx$ for some nonzero real number $\Delta(u)$. We will show that $\Phi(f) * \Phi(g) = \Delta(u)\Phi(f * g)$. It follows that $f * g = g * f \Leftrightarrow \Phi(f) * \Phi(g) = \Phi(g) * \Phi(f)$. We compute

$$\begin{aligned}
(\Phi(f) * \Phi(g))(x) &= \int_G \Phi(f)(y)\Phi(g)(y^{-1}x)\,dy \\
&= \int_G (f \circ u)(y)(g \circ u)(y^{-1}x)\,dy \\
&= \int_G f(u(y))g(u(y^{-1})u(x))\,dy \\
&= \int_G f(y)g(y^{-1}u(x))u^*(dy) \\
&= \Delta(u)\int_G f(y)g(y^{-1}u(x))\,dy \\
&= \Delta(u)(f * g)(u(x)) \\
&= \Delta(u)\Phi(f * g)(x). \quad \square
\end{aligned}$$

Suppose now that G is a Lie group. For $D \in \mathscr{E}'(G)$, the space of compactly supported distributions, define the K-average D^K by

$$\langle D^K, f\rangle = \langle D, f^K\rangle, \tag{1.5}$$

for each $f \in C_c^\infty(G)$, where f^K is defined by (1.4). The space of K-invariant, compactly supported distributions is

$$\mathscr{E}_K'(G) = \{D \in \mathscr{E}' | D^K = D\} = \{D^K | D \in \mathscr{E}'(G)\}. \tag{1.6}$$

If δ_x is the delta function at $x \in G$ then $\delta_x^K \in \mathscr{E}_K'(G)$ has compact support $K \cdot x$. One has

$$\langle \delta_x^K, f\rangle = \int_K f(k \cdot x)\,dk. \tag{1.7}$$

Lemma 1.8. *The K-invariant test functions are dense in $\mathscr{E}_K'(G)$.*

Proof. Merely note that if $\{u_n\} \subseteq \mathscr{E}(G)$, and $u_n \to D \in \mathscr{E}'(G)$, then $u_n^K \to D^K = D$, for each $D \in \mathscr{E}_K'(G)$. $\square$

The convolution of distributions $D_1, D_2 \in \mathscr{E}'(G)$ is defined by

$$\langle D_1 * D_2, f\rangle = \langle D_1(x), \langle D_2, l_{x^{-1}} f\rangle\rangle, \tag{1.9}$$

where $l_x f(y) = f(x^{-1}y)$. In particular, one has

$$\langle \delta_x^K * \delta_y^K, f\rangle = \int_K \int_K f((k_1 \cdot x)(k_2 \cdot y))\,dk_1\,dk_2. \tag{1.10}$$

Lemma 1.11. *If (K, G) is a Gelfand pair then convolution in $\mathscr{E}_K'(G)$ is commutative.*

Proof. This follows immediately from commutativity of $L_K^1(G)$ and Lemma 1.8. $\square$

Theorem 1.12. *(K, G) is a Gelfand pair if, and only if, for all $x, y \in G$, $xy \in (K\cdot y)(K\cdot x)$.*

Proof. Suppose that $xy \notin (K\cdot y)(K\cdot x)$. We will show that $\delta_x^K * \delta_y^K \neq \delta_y^K * \delta_x^K$, so (K, G) fails to be a Gelfand pair by Lemma 1.11. Indeed, one can find a non-negative test function $f: G \to \mathbf{R}$ with $f(xy) = 1$ and $f((K\cdot y)(K\cdot x)) = \{0\}$ by compactness of $(K\cdot y)(K\cdot x)$. But then (1.10) shows that $\langle \delta_x^K * \delta_y^K, f\rangle$ is positive, whereas $\langle \delta_y^K * \delta_x^K, f\rangle = 0$.

Conversely, suppose $xy \in (K\cdot y)(K\cdot x)$ for all $x, y \in G$, and let $f, g \in L_K^1(G)$. Then

$$f * g(x) = \int_G f(xy)g(y^{-1})\,dy = \int_G f((k_3\cdot y)x)g(y^{-1})\,dy,$$

where $xy = (k_1\cdot y)(k_2\cdot x) = k_2((k_3\cdot y)x)$. Note that k_1, k_2, and k_3 depend on the integration variable y. Using K-invariance of f we write

$$f * g(x) = \int_G\int_K f(k\cdot((k_3\cdot y)x))g(y^{-1})\,dk\,dy$$

$$= \int_G\int_K f((k\cdot y)(kk_3^{-1}\cdot x))g(y^{-1})\,dk\,dy$$

via $k \mapsto kk_3^{-1}$

$$= \int_K\int_G f(y(kk_3^{-1}\cdot x))g(k^{-1}\cdot y^{-1})\,dy\,dk$$

via $y \mapsto k^{-1}\cdot y$

$$= \int_G\int_K f(y(kk_3^{-1}\cdot x))g(y^{-1})\,dk\,dx$$

using K-invariance

$$= \int_G\int_K f(y(k\cdot x))g(y^{-1})\,dk\,dy$$

via $k \mapsto kk_3$

$$= \int_K g * f(k\cdot x)\,dk$$

changing the order of integration

$$= g * f(x)$$

using K-invariance. □

It is not difficult to check that the condition in Theorem 1.12 is equivalent to the more symmetrical condition that $(K\cdot x)(K\cdot y) = (K\cdot y)(K\cdot x)$.

Three-step groups

We now begin our consideration of Gelfand pairs that involve nilpotent groups. Let N be a connected, simply connected nilpotent Lie group with Lie algebra $\mathcal{N}$. Recall the descending central series for $\mathcal{N}$,

$$\mathcal{N} = \mathcal{N}^{(1)} \supset \mathcal{N}^{(2)} \supset \cdots \supset \mathcal{N}^{(n)} \supset \mathcal{N}^{(n+1)} = \{0\}, \tag{2.1}$$

where $\mathcal{N}^{(k)} = [\mathcal{N}, \mathcal{N}^{(k-1)}]$ for $k > 1$. We say that N is an n-step group if $\mathcal{N}^{(n)} \neq \{0\}$.

Fix any inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{N}$, and let $\mathcal{N}_k$ denote the orthogonal complement to $\mathcal{N}^{(k+1)}$ inside $\mathcal{N}^{(k)}$ for $1 \leq k \leq n-1$. Also, set $\mathcal{N}_n = \mathcal{N}^{(n)}$ so that

$$\mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2 \oplus \cdots \oplus \mathcal{N}_n \quad \text{and} \quad \mathcal{N}^{(k)} = \mathcal{N}_k \oplus \cdots \oplus \mathcal{N}_n \tag{2.2}$$

for $1 \leq k \leq n$.

Lemma 2.3. *Let N be an n-step group with $n \geq 3$. Then*

$$[\mathcal{N}_1, \mathcal{N}^{(n-1)}] \neq \{0\}.$$

Proof. Suppose $[\mathcal{N}_1, \mathcal{N}^{(n-1)}] = \{0\}$, and choose any n elements $X_1, X_1, \ldots, X_{n-1}, Y \in \mathcal{N}$. Then $W = [X_1, [X_2, [\cdots [X_{n-2}, X_{n-1}] \cdots]]]$ is an element of $\mathcal{N}^{(n-1)}$, and writing $Y = U + V$ where $U \in \mathcal{N}_1$, $V \in \mathcal{N}^{(2)}$, we see that

$$[Y, W] = [U, W] + [V, W] = [V, W] = 0$$

since $[\mathcal{N}_1, \mathcal{N}^{(n-1)}] = 0$ and any n-fold bracket of terms in $\mathcal{N}^{(2)}$ must vanish. However, this shows that $\mathcal{N}$ cannot be n-step since all n-fold brackets in $\mathcal{N}$ are zero. □

The main result of this section is

Theorem 2.4. *If N is an n-step group with $n \geq 3$ then there are no Gelfand pairs (K, N).*

Proof. Since K is compact, there is a K-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{N}$. Indeed, such an inner product can be obtained by averaging an arbitrary one with respect to the K-action. Form the decomposition (2.2) using this inner product and choose any $X \in \mathcal{N}_1$, $Y \in \mathcal{N}_{n-1}$ with $[X, Y] \neq 0$. This is possible by Lemma 2.3, and the observations that $\mathcal{N}^{(n-1)} = \mathcal{N}_{n-1} \oplus \mathcal{N}_n$ and $\mathcal{N}_n$ is contained in the center.

Let exp denote the exponential map from $\mathcal{N}$ to N. We will show that for $x = \exp(X)$, $y = \exp(Y)$ one has $xy \notin (K \cdot y)(K \cdot x)$. Suppose otherwise, and pick $k_1, k_2 \in K$ so that $xy = (k_1 \cdot y)(k_2 \cdot x)$. By the Baker-Campbell-Hausdorf formula one has

$$X + Y + \tfrac{1}{2}[X, Y] = k_2 \cdot X + k_1 \cdot Y + \tfrac{1}{2}[k_1 \cdot Y, k_2 \cdot X], \tag{2.5}$$

where $(k, X) \mapsto k \cdot X$ is the derived action of K on $\mathcal{N}$.

Since any automorphism of $\mathcal{N}$ must preserve each $\mathcal{N}^{(k)}$, we have $k_1 \cdot Y \in \mathcal{N}^{(n-1)}$. Thus X and $k_2 \cdot X$ differ by an element $W \in \mathcal{N}^{(n-1)}$, so that $k_2 \cdot X = X + W$. As $\mathcal{N}_1$ and $\mathcal{N}^{(n-1)}$ are orthogonal subspaces in $\mathcal{N}$ and the K-action preserves orthogonality, we see that $W = 0$. That is $k_2 \cdot X = X$, and (2.5) becomes

$$Y + \tfrac{1}{2}[X, Y] = k_1 \cdot Y + \tfrac{1}{2}[k_1 \cdot Y, X]. \tag{2.6}$$

The same trick now shows that $k_1 \cdot Y = Y$, since the two differ by an element of $\mathscr{N}_n$. Finally, (2.6) becomes $[X, Y] = [Y, X]$, which is impossible since $[X, Y] \neq 0$. □

Some representation theory

This section will serve to introduce some notation and to describe a result due to G. Carcano. Since this result is of primary importance to our analysis, we will include a sketch of the proof.

If π and π' are irreducible unitary representations of N, we write $\pi \simeq \pi'$ to indicate that π and π' are unitarily equivalent. We denote by $\widehat{N}$ the equivalence classes of irreducible unitary representations of N. Given $k \in K$ and $\pi \in \widehat{N}$ we denote by π_k the representation defined by

$$\pi_k(x) = \pi(k \cdot x). \tag{3.1}$$

The stabilizer of π under this action is

$$K_\pi = \{k \in K : \pi_k \simeq \pi\}. \tag{3.2}$$

We denote by $\mathscr{O}_\pi$ the coadjoint orbit in $\mathscr{N}^*$ corresponding to π according to the Kirillov theory, and note that K_π is also the stabilizer of $\mathscr{O}_\pi$ under the dual action of K on $\mathscr{N}^*$.

For each $k \in K_\pi$, one can choose an intertwining operator $W_\pi(k)$ with $\pi_k(x) = W_\pi(k)\pi(x)W_\pi(k)^{-1}$ for each $x \in N$. The map $k \mapsto W_\pi(k)$ need not be a representation of K_π. Indeed, the $W_\pi(k)$'s are only characterized up to multiplicative constants in the circle $\mathbf{T}$ by the intertwining condition. In fact, there will be a map

$$\sigma \ (= \sigma_\pi) \colon K_\pi \times K_\pi \to \mathbf{T} \tag{3.3}$$

for which $W_\pi(k_1 k_2) = \sigma(k_1, k_2) W_\pi(k_1) W_\pi(k_2)$. The map σ can be made measurable and is called the multiplier for the projective representation W_π. We call W_π the *intertwining representation* for the representation π.

Many aspects of representation theory can be extended to projective representations as well (cf. [Ma]). In particular, compactness of K_π implies that W_π decomposes as a direct sum of irreducible (projective) representations. Writing $c(T, W_\pi)$ for the multiplicity of T in W_π, one has

$$W_\pi = \sum_{T \in \widehat{K}_\pi^\sigma} c(T, W_\pi) T. \tag{3.4}$$

Here, $\widehat{K}_\pi^\sigma$ denotes the set of unitary equivalence classes of projective representations of K_π with multiplier σ $(= \sigma_\pi)$. The following theorem is from [Ca].

Theorem 3.5. *If (K, N) is a Gelfand pair, then $c(T, W_\pi) \leq 1$ for all $\pi \in \widehat{N}$, and conversely, if $c(T, W_\pi) \leq 1$ for almost all (with respect to Plancherel measure) $\pi \in \widehat{N}$ then (K, N) is a Gelfand pair.*

Proof. For completness we sketch what is essentially Carcano's proof.

Let $\pi \in \widehat{N}$ and let W_π be the intertwining representation of K_π with multiplier σ. If $\overline{T}$ is any irreducible projective representation of K_π with multiplier $\overline{\sigma}$, then

$$R(k, x) = \overline{T}(k) \otimes \pi(x) W_\pi(k) \tag{3.6}$$

is an irreducible representation of $K_\pi \propto N$ whose restriction to N is a multiple of π, and the induced representation $\mathrm{Ind}_{K_\pi \propto N}^{K \propto N}(R)$ is irreducible for $K \propto N$. By considering all π and $\overline{T}$, one obtains all equivalence classes of irreducible representations of $K \propto N$ in this manner (cf. [Ma]).

It is well known that if $K \subset G$ is a Gelfand pair, then for each irreducible representation π of G, the space of K-fixed vectors has dimension $c(1_K, \pi|_K) \in \{0, 1\}$ (cf. [He]). For the representation R given by (3.6), one has

$$\mathrm{Ind}_{K_\pi \propto N}^{K \propto N}(R)|_K \simeq \mathrm{Ind}_{K_\pi}^{K}(R|_{K_\pi}) = \mathrm{Ind}_{K_\pi}^{K}(\overline{T} \otimes W_\pi),$$

and by Frobenius reciprocity for compact groups,

$$c(1_K, \mathrm{Ind}_{K_\pi}^{K}(\overline{T} \otimes W_\pi)) = c(1_K|_{K_\pi}, \overline{T} \otimes W_\pi) = c(1_{K_\pi}, \overline{T} \otimes W_\pi).$$

This last value can be written as $c(T, W_\pi)$ since 1_{K_π} has multiciplicity 1 in $\overline{T} \otimes T$ and multiplicity 0 in $\overline{T} \otimes S$ for S not equivalent to T. This shows the necessity of the condition.

Now suppose $\pi \in \widehat{N}$ satisfies the multiplicity condition. Denote the Hilbert space on which it acts by $\mathbf{H}_\pi$, and form the decomposition

$$\mathbf{H}_\pi = \sum_{\mathrm{T} \in \widehat{\mathbf{K}}_\pi^\sigma} \mathbf{H}_{\pi, \mathrm{T}} \tag{3.7}$$

into K_π-irreducible subspaces. (If T is not a subrepresentation of W_π, then $\mathbf{H}_{\pi,\mathrm{T}} = \{0\}$.) If $f \in L_K^1(N)$ then one shows that the operator $\pi(f)$ commutes with every $W_\pi(k)$. Since each factor $\mathbf{H}_{\pi,\mathrm{T}}$ in (3.7) occurs only once, $\pi(f)$ must preserve these factors and thus, acts as a scalar in each by Schur's Lemma. It follows that if $f, g \in L_K^1(N)$ then the operators $\pi(f)$ and $\pi(g)$ commute and hence $\pi(f * g) = \pi(g * f)$. When this equality holds for almost all $\pi \in \widehat{N}$, one concludes that $f * g = g * f$ by appealing to the Plancherel Theorem. □

We remark that the result holds more generally for compact actions on separable locally compact groups.

Heisenberg groups

The $(2n+1)$-dimensional Heisenberg group H_n has Lie algebra $\mathscr{H}_n$ with basis $X_1, \dots, X_n, Y_1, \dots, Y_n, Z$ and structure equations given by $[X_i, Y_i] = Z$. The group $Sp(n, \mathbf{R})$ of real $2n \times 2n$ symplectic matrices acts on $\mathrm{Span}(X_1, \dots, X_n, Y_1, \dots, Y_n)$ by automorphisms of $\mathscr{H}_n$. It is well known that $U(n) = Sp(n, \mathbf{R}) \cap O(2n) = Sp(n, \mathbf{R}) \cap SO(2n)$ is a maximal compact connected subgroup of $\mathrm{Aut}(H_n)$ (cf. [Ho]). (The full automorphism group contains inner automorphisms, dilations and an involution that sends Z to $-Z$ in addition to these symplectic automorphisms.) If one models H_n as $\mathbf{C}^n \times \mathbf{R}$, as we generally will, then $U(n)$ becomes the group of $n \times n$ unitary matrices acting on $\mathbf{C}^n$ in the usual fashion.

We recall the representation theory of H_n. A generic set of coadjoint orbits in $\mathscr{H}_n^*$ is parametrized by nonzero $\lambda \in \mathbf{R}$, where the orbit $\mathscr{O}_\lambda$ is the hyperplane in $\mathscr{H}_n^*$ of all functionals taking the value λ at Z. The action of $U(n)$ on $\mathscr{H}_n^*$ preserves each $\mathscr{O}_\lambda$. Hence, if π_λ is the element of $\widehat{H}_n$ corresponding to $\mathscr{O}_\lambda$, then $U(n)$ also preserves the equivalence class of π_λ.

One can realize π_λ in the Fock space

$$\mathbf{H}_\lambda(\mathbf{n}) = \left\{ \text{entire } f\colon \mathbf{C}^n \to \mathbf{C} \middle| \int_{\mathbf{C}^n} e^{-2|\lambda||w|^2} |f(w)|^2 \, dw < \infty \right\} \tag{4.1}$$

as

$$\pi_\lambda(z, t) f(w) = e^{-i\lambda t + \lambda(2\langle w, z\rangle - |z|^2)} f(w - z) \tag{4.2}$$

for $\lambda > 0$ and

$$\pi_\lambda(z, t) f(w) = e^{-i\lambda t - \lambda(2\langle w, \bar{z}\rangle - |z|^2)} f(w - \bar{z}) \tag{4.3}$$

for $\lambda < 0$. Here $\langle w, z\rangle$ denotes the Hermitian inner product on $\mathbf{C}^n$. We refer the reader to [Ho or Ta] for a discussion of the Fock model.

Define $W_\lambda(k)\colon \mathbf{H}_\lambda(\mathbf{n}) \to \mathbf{H}_\lambda(\mathbf{n})$ by

$$W_\lambda(k) f(z) = f(k^{-1} z). \tag{4.4}$$

Then $W_\lambda(k)$ intertwines $\pi_\lambda(z, t)$ and $(\pi_\lambda)_k(z, t) = \pi_\lambda(kz, t)$. We verify this for $\lambda > 0$. Indeed,

$$\begin{aligned} W_\lambda(k)(\pi_\lambda(k^{-1}z, t) f)(w) &= \pi_\lambda(k^{-1}z, t) f(k^{-1}w) \\ &= e^{-i\lambda t + \lambda(2\langle k^{-1}w, k^{-1}z\rangle - |k^{-1}z|^2)} f(k^{-1}w - k^{-1}z) \\ &= e^{-i\lambda t + \lambda(2\langle w, z\rangle - |z|^2)} W_\lambda(k) f(w - z) \\ &= (\pi_\lambda(z, t) W_\lambda(k) f)(w), \end{aligned}$$

and hence

$$W_\lambda(k) \pi_\lambda(z, t) W_\lambda(k)^{-1} = \pi_\lambda(kz, t) \tag{4.5}$$

as claimed. That is, $U(n)$ is the stabilizer of the equivalence class of $\pi_\lambda \in \widehat{H}_n$ under the action of $U(n)$ and $W_\lambda\colon \mathbf{H}_\lambda(\mathbf{n}) \to \mathbf{H}_\lambda(\mathbf{n})$ is the intertwining representation as in (3.4). (We remark that up to a factor of $\det(k)^{\frac{1}{2}}$, W_λ lifts to the oscillator representation on the double cover $MU(n)$ of $U(n)$ (cf. [Ta]).) It follows that for any compact subgroup $K \subseteq U(n)$, $K_{\pi_\lambda} = K$, and the intertwining representation of K is given by the restriction of W_λ to K.

Given a compact, connected subgroup $K \subseteq U(n)$, we denote its complexification by $K_{\mathbf{C}}$. The action of K on $\mathbf{C}^n$ yields a representation of $K_{\mathbf{C}}$ on

$\mathbf{C}^n$, and one can view $K_{\mathbf{C}}$ as a subgroup of $Gl(n, \mathbf{C})$. (A discussion of the complexification construction can be found in [BtD].)

A finite dimensional representation $\rho: G \to Gl(V)$ in a complex vector space V is said to be *multiplicity free* if each irreducible G-module occurs at most once in the associated representation on the polynomial ring $\mathbf{C}[V]$ (given by $(x \cdot p)(z) = p(\rho(x^{-1})z)$).

Theorem 4.6. *Let K be a compact, connected subgroup of $U(n)$ acting irreducibly on $\mathbf{C}^n$. The following are equivalent:* (i) *(K, H_n) is a Gelfand pair.* (ii) *The representation of $K_{\mathbf{C}}$ on $\mathbf{C}^n$ is multiplicity free.* (iii) *The representation of $K_{\mathbf{C}}$ on $\mathbf{C}^n$ is equivalent to one of the representations in the following table:*

Multiplicity Free Representations		
Group	Acting On	Subject To
$Sl(n, \mathbf{C})$	$\mathbf{C}^n$	$n \geq 2$
$Gl(n, \mathbf{C})$	$\mathbf{C}^n$	$n \geq 1$
$Sp(k, \mathbf{C})$	$\mathbf{C}^n$	$n = 2k$
$\mathbf{C}^* \times Sp(k, \mathbf{C})$	$\mathbf{C}^n$	$n = 2k$
$\mathbf{C}^* \times SO(n, \mathbf{C})$	$\mathbf{C}^n$	$n \geq 2$
$Gl(k, \mathbf{C})$	$S^2(\mathbf{C}^k) \simeq \mathbf{C}^n$	$n = k(k+1)/2$, $k \geq 2$
$Sl(k, \mathbf{C})$	$\Lambda^2(\mathbf{C}^k) \simeq \mathbf{C}^n$	$n = \binom{k}{2}$ and k is odd
$Gl(k, \mathbf{C})$	$\Lambda^2(\mathbf{C}^k) \simeq \mathbf{C}^n$	$n = \binom{k}{2}$
$Sl(k, \mathbf{C}) \times Sl(l, \mathbf{C})$	$\mathbf{C}^k \otimes \mathbf{C}^l \simeq \mathbf{C}^n$	$n = kl$, $k \neq l$
$Gl(k, \mathbf{C}) \times Sl(l, \mathbf{C})$	$\mathbf{C}^k \otimes \mathbf{C}^l \simeq \mathbf{C}^n$	$n = kl$
$Gl(2, \mathbf{C}) \times Sp(k, \mathbf{C})$	$\mathbf{C}^2 \otimes \mathbf{C}^{2k} \simeq \mathbf{C}^n$	$n = 4k$
$Sl(3, \mathbf{C}) \times Sp(k, \mathbf{C})$	$\mathbf{C}^3 \otimes \mathbf{C}^{2k} \simeq \mathbf{C}^n$	$n = 6k$
$Gl(3, \mathbf{C}) \times Sp(k, \mathbf{C})$	$\mathbf{C}^3 \otimes \mathbf{C}^{2k} \simeq \mathbf{C}^n$	$n = 6k$
$Gl(4, \mathbf{C}) \times Sp(4, \mathbf{C})$	$\mathbf{C}^4 \otimes \mathbf{C}^8 \simeq \mathbf{C}^n$	$n = 32$
$Sl(k, \mathbf{C}) \times Sp(4, \mathbf{C})$	$\mathbf{C}^k \otimes \mathbf{C}^8 \simeq \mathbf{C}^n$	$n = 8k$, $k > 4$
$Gl(k, \mathbf{C}) \times Sp(4, \mathbf{C})$	$\mathbf{C}^k \otimes \mathbf{C}^8 \simeq \mathbf{C}^n$	$n = 8k$, $k > 4$
$\mathbf{C}^* \times \mathrm{Spin}(7, \mathbf{C})$	$\mathbf{C}^n$	$n = 8$
$\mathbf{C}^* \times \mathrm{Spin}(9, \mathbf{C})$	$\mathbf{C}^n$	$n = 16$
$\mathrm{Spin}(10, \mathbf{C})$	$\mathbf{C}^n$	$n = 16$
$\mathbf{C}^* \times \mathrm{Spin}(10, \mathbf{C})$	$\mathbf{C}^n$	$n = 16$
$\mathbf{C}^* \times G_2$	$\mathbf{C}^n$	$n = 7$
$\mathbf{C}^* \times E_6$	$\mathbf{C}^n$	$n = 27$

Proof. The complexification $K_{\mathbf{C}}$ of K is connected, reductive, algebraic (cf. [BtD]) and acts irreducibly on $\mathbf{C}^n$. Moreover, the representation of K on $\mathbf{C}^n$ is multiplicity free if, and only if, the complexified representation of $K_{\mathbf{C}}$ on $\mathbf{C}^n$ is multiplicity free. The multiplicity free irreducible linear representations of connected, reductive, algebraic groups have been classified by V. Kac. The table given here is taken from Theorem 3 of [Ka]. This gives the equivalence of (ii) and (iii).

The equivalence of (i) and (ii) is an immediate consequence of Theorem 3.5 once one observes that for each $\lambda \neq 0$, W_λ is the completion of the associated representation of K on $\mathbf{C}[\mathbf{C}^n]$. □

Remarks. Some comments are in order regarding the table. $\mathbf{C}^*$ denotes the nonzero complex numbers, S^2 the symmetric 2-tensors and Λ^2 the alternating 2-tensors. The group $\mathbf{C}^* \times Sp(k, \mathbf{C})$ acts on $\mathbf{C}^{2k}$ via $(\lambda, A)\cdot v = \lambda v A$. We can view $\mathbf{C}^* \times Sp(k, \mathbf{C})$ as the group of $n \times n$ complex matrices that transform the standard symplectic structure on $\mathbf{C}^n$ into a scalar multiple of itself. There are similar interpretations for the other groups $\mathbf{C}^* \times G$. $\mathrm{Spin}(n, \mathbf{C}) = \mathrm{Spin}(n, \mathbf{R})_{\mathbf{C}}$ is a double cover of $SO(n, \mathbf{C})$ and acts by the complexified half-spin representation. $\mathrm{Spin}(7, \mathbf{C})$ and $\mathrm{Spin}(9, \mathbf{C})$ are simply connected and $\pi_1(\mathrm{Spin}(10, \mathbf{C})) = \mathbf{Z}_2$.

Suppose now that the action of K on $\mathbf{C}^n$ is reducible, and let

$$\mathbf{C}^n = \sum_{j=1}^{p} V_j \tag{4.7}$$

be a decomposition of $\mathbf{C}^n$ into K-irreducible (not necessarily complex) subspaces. If (K, H_n) is a Gelfand pair, then the $V_\alpha' s$ are orthogonal with respect to the skew-symmetric form on $\mathbf{C}^n$ given by Λ: $(z, w) \mapsto \Im\langle z, w\rangle$. Indeed, if $z_i \in V_{\alpha_i}$ for $i = 1, 2$ then by Theorem 1.12 there exist $k_1, k_2 \in K$ such that $(z_1, 0)(z_2, 0) = (k_2 \cdot z_2, 0)(k_1 \cdot z_1, 0)$. It follows that

$$\sum_i z_i = \sum_i k_i \cdot z_i \tag{4.8}$$

and that

$$\Lambda(z_1, z_2) = \Lambda(k_2 \cdot z_2, k_1 \cdot z_1). \tag{4.9}$$

Since the V_α's are orthogonal with respect to the usual Hermitian inner product $\langle \cdot, \cdot \rangle$ on $\mathbf{C}^n$ and are K-invariant, one concludes from (4.8) that $k_i \cdot z_i = z_i$, for $i = 1, 2$, and hence from (4.9) that $\Lambda(z_1, z_2) = 0$. It now follows that the V_α's have complex structure, i.e. $iV_\alpha = V_\alpha$. Suppose not, and let $z \in V_\alpha$ such that $iz \notin V_\alpha$. Then $iz = \sum_\beta z_\beta$, and $z_\beta \neq 0$ for some $\beta \neq \alpha$. Thus,

$$|z|^2 = -\Lambda(z, iz) = \sum_\beta -\Lambda(z, z_\beta) = -\Lambda(z, \overline{z}_\alpha) < |z|^2.$$

Finally, since the V_α's are invariant under multiplication by i, the skew-symmetric form Λ is nondegenerate on each V_α. Therefore, if $m_j = \dim(V_j)$,

$H_{m_j} \simeq V_j \times \mathbf{R}$. (This isomorphism is made explicit in the proof of Theorem 5.12.)

Let K_j denote the subgroup of $U(V_j)$, the group of unitary transformations on V_j obtained by the restriction of K to V_j, and let

$$\mathbf{C}[V_j] = \sum_{n=0}^{\infty} \mathbf{P}_{\mathbf{j},\mathbf{n}} \tag{4.10}$$

be the decomposition of the polynomial ring over V_j into K_j-irreducible subspaces, with the convention that $\mathbf{P}_{\mathbf{j},0} = \{0\}$. For each p-tuple $(n_1, \ldots, n_p) \in (\mathbf{Z}^+)^{\mathbf{p}}$, let $\mathbf{P}^{\mathbf{n}_1, \cdots, \mathbf{n}_\mathbf{p}} = \mathbf{P}_{1,\mathbf{n}_1} \otimes \cdots \otimes \mathbf{P}_{\mathbf{p},\mathbf{n}_\mathbf{p}}$. If $W_{\lambda,j}$ denotes the intertwining representation associated to the pair (K_j, H_{m_j}) as above, then for each $k \in K$, the restriction of W_λ to $\mathbf{P}^{\mathbf{n}_1, \cdots, \mathbf{n}_\mathbf{p}}$ is given by $W_{\lambda,1} \otimes \cdots \otimes W_{\lambda,p}$. Thus, if (K, H_n) is a Gelfand pair, Theorem 4.6 implies that (K_j, H_{m_j}) is a Gelfand pair for each $j = 1, \ldots, p$. But it also implies the stronger condition that the subrepresentations of K on $\mathbf{P}^{n_1, \cdots, n_p}$, as $(n_1, \ldots, n_p)$ ranges over $(\mathbf{Z}^+)^{\mathbf{p}}$, are distinct. This establishes the necessity of the condition in the following theorem. The sufficiency is an immediate consequence of Theorem 4.6 and the observation that

$$\mathbf{C}[\mathbf{C}^n] = \sum_{(n_1, \ldots, n_p) \in (\mathbf{Z}^+)^{\mathbf{p}}} \mathbf{P}^{\mathbf{n}_1, \cdots, \mathbf{n}_\mathbf{p}}.$$

Theorem 4.11. *(K, H_n) is a Gelfand pair if, and only if, the subrepresentations of W_λ on $\mathbf{P}^{\mathbf{n}_1, \cdots, \mathbf{n}_\mathbf{p}}$ are distinct as $(n_1, \ldots, n_p)$ ranges over $(\mathbf{Z}^+)^{\mathbf{p}}$.*

We consider two examples. For the first, let K be the subgroup of matrices of determinant one in $U(2) \times U(1) \subseteq U(3)$, i.e. $K = \{(A, \overline{\det(A)}) | A \in U(2)\}$. The decomposition of $\mathbf{C}^3$ corresponding to (4.7) is $\mathbf{C}^3 = \mathbf{C}^2 \oplus \mathbf{C}$, in the obvious sense, and corresponding to (4.10) one has that $\mathbf{C}[\mathbf{C}^2] = \sum_{n=1}^{\infty} \mathbf{P}_{1,\mathbf{n}}$, where $\mathbf{P}_{1,\mathbf{n}}$ is the space of homogeneous polynomials in z_1, z_2 of degree n, and $\mathbf{C}[\mathbf{C}] = \sum_{n=1}^{\infty} \mathbf{P}_{2,\mathbf{n}}$, where $\mathbf{P}_{2,\mathbf{n}} = \mathbf{C}z_3^{\mathbf{n}}$. The intertwining representation of K on $\mathbf{P}^{\mathbf{n}_1, \mathbf{n}_2}$ is equivalent to the representation $A \mapsto (\det(A))^{n_2} W_\lambda(A)$ of $U(2)$ on $\mathbf{P}_{1,\mathbf{n}}$. These representations are clearly irreducible and inequivalent for distinct (n_1, n_2). Thus (K, H_3) is a Gelfand pair.

For the second example, let K be the subgroup of $U(1) \times U(1)$ consisting of all matrices of determinant one. In this case, both (K_1, H_1) and (K_2, H_1) are Gelfand pairs, and in fact, the subrepresentations of the intertwining representations of K_1 and K_2 on $\mathbf{C}[\mathbf{C}]$ are distinct (corresponding to $\mathbf{Z}^+$ for K_1, and $\mathbf{Z}^-$ for K_2). However, the intertwining representation on $\mathbf{P}^{\mathbf{n},\mathbf{n}}$ is the identity for each n, and thus (K, H_2) is not a Gelfand pair.

We conclude this section with an immediate corollary to Theorem 4.11.

Corollary 4.12. *Let K_j be a compact subgroup of $U(n_j)$ for $1 \leq j \leq p$, $K = \prod K_j$, and let $n = \sum n_j$. Then (K, H_n) is a Gelfand pair if, and only if (K_j, H_{n_j}) is a Gelfand pair for $1 \leq j \leq p$.*

FREE GROUPS

In this section we turn our attention to the free, two-step nilpotent Lie group on n-generators, $F(n)$. We realize its Lie algebra, $\mathscr{F}(n)$, as $\mathbf{R}^n \oplus \Sigma_n$, where $\mathbf{R}^n$ is viewed as $1 \times n$ real matrices, Σ_n is the space of real $n \times n$ skew symmetric matrices, and the Lie bracket is given by

$$[(u, U), (v, V)] = (0, u^t v - v^t u). \tag{5.1}$$

The group law is thus

$$(u, U)(v, V) = (u + v, U + V + \tfrac{1}{2}(u^t v - v^t u)). \tag{5.2}$$

Lemma 5.3. *There is a bijection between* $\mathrm{Aut}(F(n)) \simeq \mathrm{Aut}(\mathscr{F}(n))$ *and the set* $Gl(n, \mathbf{R}) \times \mathrm{Hom}(\mathbf{R}^n, \Sigma_n)$.

Proof. The exponential map establishes the isomorphism

$$\mathrm{Aut}(F(n)) \simeq \mathrm{Aut}(\mathscr{F}(n)).$$

For $(A, \nu) \in Gl(n, \mathbf{R}) \times \mathrm{Hom}(\mathbf{R}^n, \Sigma_n)$, define $\phi_{(A,\nu)}: \mathscr{F}(n) \to \mathscr{F}(n)$ by

$$\phi_{(A,\nu)}(u, U) = (uA, A^t U A + \nu(u)). \tag{5.4}$$

It is easy to check that $\phi_{(A,\nu)}$ is a Lie algebra automorphism. On the other hand, if $\phi: \mathscr{F}(n) \to \mathscr{F}(n)$ is any given automorphism, then $\phi = \phi_{(A,\nu)}$, where A and ν are the composites

$$\mathbf{R}^n \hookrightarrow \mathscr{F}(n) \xrightarrow{\phi} \mathscr{F}(n) \to \mathbf{R}^n$$

and

$$\mathbf{R}^n \hookrightarrow \mathscr{F}(n) \xrightarrow{\phi} \mathscr{F}(n) \to \Sigma_n$$

respectively. $\square$

Note that the correspondence in Lemma 5.3 becomes a group isomorphism if the set $Gl(n, \mathbf{R}) \times \mathrm{Hom}(\mathbf{R}^n, \Sigma_n)$ is given the group structure

$$(A, \nu)(B, \mu) = (AB, A \cdot \mu + \nu B), \tag{5.4}$$

with $Gl(n, \mathbf{R})$ acting on Σ_n by $A \cdot V = A^t V A$. In particular, we see that a maximal compact subgroup of $\mathrm{Aut}(F(n))$ can be identified with $O(n)$, the group of real orthogonal matrices. This acts on $\mathscr{F}(n)$ by

$$A \cdot (u, U) = (uA, A \cdot U) = (uA, A^t U A), \tag{5.5}$$

and preserves the inner product

$$\langle (u, U), (v, V) \rangle = uv^t + \tfrac{1}{2}\mathrm{tr}(UV^t). \tag{5.6}$$

Suppose that $\mathscr{Z}$ is a subspace of Σ_n. We define a Lie algebra $\mathscr{N}_{\mathscr{Z}} := \mathbf{R}^n \times \mathscr{Z}$ with bracket

$$[(u, U), (v, V)]_{\mathscr{Z}} = (0, P_{\mathscr{Z}}(u^t v - v^t u)), \tag{5.7}$$

where $P_{\mathscr{Z}}$ is the orthogonal projection of Σ_n onto $\mathscr{Z}$.

We now describe the coadjoint orbits in $\mathscr{F}(n)^*$ and $\mathscr{N}_{\mathscr{Z}}^*$. First, using the inner product (5.6) we identify $\mathscr{F}(n)^*$ with $\mathscr{F}(n)$ and $\mathscr{N}_{\mathscr{Z}}^*$ with $\mathscr{N}_{\mathscr{Z}}$. This gives an inclusion $\mathscr{N}_{\mathscr{Z}}^* \hookrightarrow \mathscr{F}(n)^*$ dual to the projection $P_{\mathscr{Z}}$. For $B \in \Sigma_n$, define a map

$$J_B\colon \mathbf{R}^n \to \mathbf{R}^n \tag{5.8}$$

by $\langle J_B(u), v\rangle = \langle B, u^t v - v^t u\rangle$. Similarly, if $B \in \mathscr{Z}$ define a map

$$J_B^{\mathscr{Z}}\colon \mathbf{R}^n \to \mathbf{R}^n \tag{5.9}$$

by $\langle J_B^{\mathscr{Z}}(u), v\rangle = \langle B, [(u, 0), (v, 0)]_{\mathscr{Z}}\rangle$. In fact, though, for $B \in \mathscr{Z}$, $J_B = J_B^{\mathscr{Z}}$ since

$$\begin{aligned}\langle J_B^{\mathscr{Z}}(u), v\rangle &= \langle B, P_{\mathscr{Z}}[(u, 0), (v, 0)]\rangle \\ &= \langle B, [(u, 0), (v, 0)]\rangle = \langle J_B(u), v\rangle.\end{aligned}$$

Accordingly, we denote both maps by J_B. One computes

$$\begin{aligned}\langle J_B(u), v\rangle &= \langle B, [(u, 0), (v, 0)]\rangle \\ &= \tfrac{1}{2}\operatorname{tr}(B(u^t v - v^t u)^t) = \langle uB, v\rangle\end{aligned}$$

to conclude that

$$J_B(u) = uB. \tag{5.10}$$

The coadjoint orbit through $(b, B) \in \mathscr{F}(n)^*$ $(\cong \mathscr{F}(n))$ is

$$\mathscr{O}_{(b, B)} = \operatorname{Ad}^*(F(n))(b, B).$$

For (u, U), $(v, V) \in \mathscr{F}(n)$ one has

$$\begin{aligned}\langle \operatorname{Ad}^* \exp(u, U)(b, B), (v, V)\rangle &= \langle (b, B), (v, V) + [(u, U), (v, V)]\rangle \\ &= bv^t + \tfrac{1}{2}\operatorname{tr}(BV^t) + \tfrac{1}{2}\operatorname{tr}(B(u^t v - v^t u)^t) \\ &= \langle (b, B), (v, V)\rangle + \langle J_B(u), v\rangle \\ &= \langle (b + J_B(u), B), (v, V)\rangle.\end{aligned}$$

Thus,

$$\mathscr{O}_{(b, B)} = (b, B) + (\operatorname{Image}(J_B), 0) = (b + \mathbf{R}^n B, B). \tag{5.11}$$

The same reasoning shows that when $B \in \mathscr{Z}$ the orbit $\mathscr{O}_{(b,B)}^{\mathscr{Z}}$ through $(b, B) \in \mathscr{N}_{\mathscr{Z}}^*$ is also given by $(b + \mathbf{R}^n B, B)$, i.e. the inclusion $\mathscr{N}_{\mathscr{Z}}^* \hookrightarrow \mathscr{F}(n)^*$ maps $\mathscr{O}_{(b,B)}^{\mathscr{Z}}$ diffeomorphically to $\mathscr{O}_{(b,B)}$. Accordingly, we denote both of these orbits by $\mathscr{O}_{(b,B)}$, and will write $\mathscr{O}_B$ for $\mathscr{O}_{(0,B)}$.

For n even, the orbits $\mathscr{O}_B := \mathscr{O}_{(0,B)} = \mathbf{R}^n \times \{B\}$ with B nondegenerate provide a generic set of orbits in $\mathscr{F}(n)^*$, while for n odd, the orbits $\mathscr{O}_{(b,B)}$ with $b \in \mathbf{R}^n$ and B of rank $(n-1)$ form a generic set. (Note that these orbits are not distinct since $\mathscr{O}_{(b_1,B)} = \mathscr{O}_{(b_2,B)}$, provided $b_1 - b_2 \in \mathbf{R}^n B$.)

Theorem 5.12. *$(SO(n), F(n))$ is a Gelfand pair for all $n \geq 2$.*

Proof. The proof is an application of Theorem 3.5. Since the generic orbits in $\mathscr{F}(n)^*$ depend on the parity of n, we consider the cases separately.

Suppose first that $n = 2k$ and let $B \in \Sigma_n$ be nondegenerate. We may also assume that B has distinct eigenvalues which we denote $\pm i\lambda_1, \ldots, \pm i\lambda_k$, with $\lambda_j > 0$. The orbits $\mathscr{O}_B = \mathbf{R}^n \times \{B\}$ for such B form a generic set in $\mathscr{F}(n)^*$.

Let $\mathscr{H}_B$ denote the Lie algebra defined in (5.7) with $\mathscr{Z} = \mathbf{R}B$. B is central in $\mathscr{H}_B$ and for $u, v \in \mathbf{R}^n$ one has

$$[(u, 0)(v, 0)] = \langle J_B(u), v\rangle B = \omega_B(u, v)B, \tag{5.13}$$

where $\omega_B(u, v) = uBv^t$ is the skew symmetric bilinear form on $\mathbf{R}^n$ with matrix B. Nondegeneracy of B implies that $\mathscr{H}_B$ is isomorphic to the Heisenberg algebra $\mathscr{H}_k$. We can make this isomorphism explicit by changing the basis on $\mathbf{R}^n$. Suppose B has eigenvectors $\alpha_1, \ldots, \alpha_k$ in $\mathbf{C}^k$ corresponding to the eigenvalues $i\lambda_1, \ldots, i\lambda_k$. Writing $\alpha_j = v_j + iu_j$, one has $u_jB = \lambda_j v_j$ and $v_jB = -\lambda_j u_j$. The matrix of B in the basis $\{u_1, v_1, \ldots, u_k, v_k\}$ is

$$B = \begin{pmatrix} \lambda_1 J & 0 & \ldots & 0 \\ 0 & \lambda_2 J & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_k J \end{pmatrix} \tag{5.14}$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0. \end{pmatrix}$$

By scaling the α_j's we can ensure that $\{u_1, v_1, \ldots, u_k, v_k\}$ is an orthonormal basis. Writing $X_j' = (u_j, 0)$, $Y_j' = (v_j, 0)$, and $Z = (0, B)$ in $\mathscr{H}_B$ we obtain a basis in which the Lie bracket in (5.7) becomes $[X_j', Y_j'] = \lambda_j Z$ with other brackets vanishing. Replacing X_j' by $X_j = (1/\sqrt{\lambda_j})X_j'$, and Y_j' by $Y_j = (1/\sqrt{\lambda_j})Y_j'$ one obtains a basis $\{X_1, Y_1, \ldots, X_k, Y_k, Z\}$ for $\mathscr{H}_B$ in which the nonzero brackets are determined by $[X_j, Y_j] = Z$.

Let $Sp(\omega_B) = \{A \in Gl(n, \mathbf{R}) | ABA^t = B\}$. This is the group of linear transformations preserving the symplectic form ω_B. The stabilizer of $\mathscr{O}_B$ under the action of $SO(n)$ is

$$K_B = SO(n) \cap Sp(\omega_B) = \{A \in SO(n) | AB = BA\}. \tag{5.15}$$

K_B also acts on $\mathscr{H}_B$ and stabilizes $\mathscr{O}_B$ regarded as an orbit in $\mathscr{H}_B^*$. In view of (5.14), K_B acts on $\mathscr{H}_B$ as $U(1)^k$ on $\mathrm{Span}(X_1, Y_1, \ldots, X_k, Y_k)$. Here each factor $U(1) = SO(2) \cap Sp(1, \mathbf{R}) = \{A \in SO(2) | AJ = JA\}$ acts on $\mathrm{Span}(X_j, Y_j)$ in the usual fashion. The representations of $H_B = \exp(\mathscr{H}_B)$ and $F(n)$ given by $\mathscr{O}_B$ coincide under the orthogonal projection $\mathscr{F}(n) \to \mathscr{H}_B$ and hence have the same intertwining representations. In view of Corollary 4.12, this must satisfy the conditions of Theorem 3.5, and we conclude that $(SO(n), F(n))$ is a Gelfand pair.

Now consider the case $n = 2k+1$. Let $b \in \mathbf{R}^n$ and let $B \in \Sigma_n$ have rank $n-1 = 2k$ and distinct eigenvalues $0, \pm i\lambda_1, \dots, \pm i\lambda_k$ with $\lambda_j > 0$. We obtain a generic set of orbits $\mathscr{O}_{(b,B)}$ in $\mathscr{F}(n)^*$ from such pairs (b, B).

Let $\mathscr{N}_B$ be defined as in (5.7) with $\mathscr{Z} = \mathbf{R}B$, and let X be any nonzero vector in $\ker(B)$. From (5.10) one concludes that the center of $\mathscr{N}_B$ is given by $\mathrm{Span}(B, X)$ and that $\mathscr{N}_B = \mathscr{H}_B \times \mathbf{R}$ (as Lie algebras) where $\mathscr{H}_B = \mathscr{N}_B/\mathbf{R}X \simeq \mathscr{H}_k$.

In view of (5.5), the stabilizer of $\mathscr{O}_{(b,B)}$ under the action of $SO(n)$ is given by

$$(5.16)\qquad \begin{aligned} K_{(b,B)} &= \{A \in SO(n) | bA = b \text{ and } AB = BA\} \\ &= \{A \in SO(2k) | AB = BA\}, \end{aligned}$$

where we are regarding $SO(2k)$ as the stabilizer of $b \in \mathbf{R}^n$ under the action of $SO(n)$.

$\mathscr{O}_{(b,B)}$ can be viewed as an orbit in $\mathscr{N}_B$ and also as an orbit in $\mathscr{H}_B$. The action of $K_{(b,B)}$ on $\mathscr{N}_B$ descends to $\mathscr{H}_B$ since each $A \in K_{(b,B)}$ preserves $\ker(B)$. Just as in the case where n is even, one shows that this corresponds to the action of $U(1)^k$ on $\mathscr{H}_k$ and completes the proof using Corollary 4.12 and Theorem 3.5. □

Theorem 5.17. *If K is a proper, closed (not necessarily connected) subgroup of $SO(n)$ then $(K, F(n))$ is not a Gelfand pair.*

Proof. As in the proof of Theorem 5.12, one must consider separately the cases n even and n odd. Here we present the argument for the case $n = 2k$. We assume at first that K is connected. The stabilizer of a generic orbit $\mathscr{O}_B$ can be viewed as a compact subgroup A_B of $K_B \simeq U(1)^k$ (see equation (5.15)). We regard A_B as acting on a Heisenberg group H_k and conclude that if $(K, F(n))$ is a Gelfand pair then so is (A_B, H_k), as in the proof of Theorem 5.12.

For a suitable choice of B, A_B is a proper subgroup of K_B. Indeed, let $C \in SO(n)\backslash K$ and let $\mathbf{T}$ be a maximal torus in $SO(n)$ that contains C. Choose a basis for $\mathbf{C}^k \simeq \mathbf{R}^n$ which transforms $\mathbf{T}$ into the usual $U(1)^k$ and let B be given in this basis by

$$(5.18)\qquad \begin{pmatrix} J & 0 & \dots & 0 \\ 0 & 2J & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & kJ \end{pmatrix}.$$

One has $K_B = \mathbf{T}$ so that $A_B = K \cap K_B$ is a proper subgroup of K_B.

$A = A_B$ is a proper connected subgroup of $U(1)^k$ and hence is a torus. One can decompose $\mathbf{C}^k$ into a sum of weight spaces for the action of A,

$$(5.19)\qquad \mathbf{C}^k = \sum_{\alpha \in P} V_\alpha .$$

Here $\alpha \in \mathscr{A}^*$, where $\mathscr{A}$ is the Lie algebra of A,

$$(5.20)\qquad V_\alpha = \{v \in \mathbf{C}^k | \exp(X) \cdot v = e^{2\pi i \alpha(X)} v \text{ for all } X \in \mathscr{A}\},$$

and P denotes the set of weights: $P = \{\alpha \in \mathscr{A}^* | V_\alpha \neq \{0\}\}$. Each $\alpha \in P$ is an integral form, that is $\alpha(L) \subseteq \mathbf{Z}$, where $L = \ker(\exp\colon \mathscr{A} \to A)$. There is a corresponding decomposition of the polynomial functions on $\mathbf{C}^{\mathbf{k}}$:

$$\mathbf{C}[\mathbf{C}^{\mathbf{k}}] = \bigotimes \mathbf{C}[V_\alpha]. \tag{5.21}$$

The A-action on $\mathbf{C}[\mathbf{C}^{\mathbf{k}}]$ preserves each $\mathbf{C}[V_\alpha]$ and acts via the character

$$\chi_\alpha(\exp(X)) = e^{2\pi i\alpha(X)}. \tag{5.22}$$

There are two cases to consider:

(i) Some weight space V_α has $\dim_{\mathbf{C}}(V_\alpha) > 1$.

(ii) $\dim_{\mathbf{C}}(V_\alpha) = 1$ for all $\alpha \in P$.

Suppose (i). Any decomposition $V_\alpha = U \oplus W$ into nontrivial subspaces U and W will be preserved by the A-action. Moreover, A will act on the invariant subspaces $\mathbf{C}[U]$ and $\mathbf{C}[W]$ of $\mathbf{C}[\mathbf{C}^{\mathbf{k}}]$ via the character χ_α. This shows that the action of A on $\mathbf{C}^{\mathbf{k}}$ is not multiplicity free and hence that $(K, F(n))$ is not a Gelfand pair.

Next assume that $\dim_{\mathbf{C}}(V_\alpha) = 1$ for all $\alpha \in P$. In this case, P consists of k weights $\{\alpha_1, \ldots, \alpha_k\}$ and we obtain a basis $\{v_1, \ldots, v_k\}$ of $\mathbf{C}^{\mathbf{k}}$ by choosing $v_j \in V_{\alpha_j}$ with $v_j \neq 0$. Note that any monomial $v_1^{j_1} v_2^{j_2} \cdots v_k^{j_k}$ generates an A-invariant subspace in $\mathbf{C}[\mathbf{C}^{\mathbf{k}}]$.

As $\dim(\mathscr{A}) < k$, the weights $\alpha_1, \ldots, \alpha_k$ must satisfy some nontrivial linear dependence relation:

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_k\alpha_k = 0. \tag{5.23}$$

In fact, one can find an integer solution $(c_1, c_2, \ldots, c_k)$ to this equation, since the forms α_j are integral. Suppose $c_1, \ldots, c_l$ are nonnegative and that $c_{l+1}, \ldots, c_k$ are negative (after rearranging the weights). Consider the monomials

$$p = v_1^{c_1} \cdots v_l^{c_l} \quad \text{and} \quad q = v_{l+1}^{-c_{l+1}} \cdots v_k^{-c_k}. \tag{5.24}$$

One has

$$\exp(X)p = e^{2\pi i(c_1\alpha_1 + \cdots + c_l\alpha_l)(X)} p \quad \text{and} \quad \exp(X)q = e^{-2\pi i(c_{l+1}\alpha_{l+1} + \cdots + c_k\alpha_k)(X)} q$$

for $X \in \mathscr{A}$. One concludes that the A-irreducible subspaces of $\mathbf{C}[\mathbf{C}^{\mathbf{k}}]$ spanned by p and q are equivalent. As in case (i), the action of A on $\mathbf{C}^{\mathbf{k}}$ is not multiplicity free and $(K, F(n))$ fails to be a Gelfand pair.

Finally, consider a nonconnected, proper subgroup $K \subseteq SO(n)$. The stabilizer $A' = A'_B$ of a generic orbit $\mathscr{O}_B$ now has the form $A' = A \times F$, where A is a torus with $\dim(A) < k$ and F is a finite abelian group. As before, we decompose $\mathbf{C}^{\mathbf{k}}$ into weight spaces V_α for the action of A. Note that the action of F and A commute so that each V_α is F-invariant. As before, we consider two cases:

(i) Suppose $\dim(V_\alpha) > 1$. Choose two linearly independent vectors $u, v \in V_\alpha$. The actions of A' on the monomials $u^{|F|}$ and $v^{|F|}$ agree and hence the representation of A' on $\mathbf{C}^K$ is not multiplicity free.

(ii) Suppose $\dim(V_\alpha) = 1$ for all α. In this case, the actions of A' on $p^{|F|}$ and $q^{|F|}$ agree, where p and q are given by (5.24). □

Two-step groups

In this section we do not assume that K is a connected group. Suppose now that a two-step N is given with $[\mathscr{N}, \mathscr{N}] = \mathscr{Z}$, where $\mathscr{Z}$ is the center of $\mathscr{N}$. If this condition is not satisfied, then $\mathscr{N} = \mathscr{N}_1 \oplus \mathscr{A}$ where $\mathscr{N}_1$ is a K-invariant, nilpotent Lie algebra with $[\mathscr{N}_1, \mathscr{N}_1]$ spanning the center of $\mathscr{N}_1$, and $\mathscr{A}$ is commutative. Thus, $N = N_1 \times A$ and $L^1(N) = L^1(N_1) \otimes L^1(A)$. It is now easy to show that $L^1_K(N)$ is commutative if, and only if, $L^1_K(N_1)$ is commutative. Thus there is no loss in assuming that $[\mathscr{N}, \mathscr{N}] = \mathscr{Z}$.

Given a compact subgroup $K \subseteq \mathrm{Aut}(N)$, we fix a K-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathscr{N}$, and denote by $\mathscr{N}_1$ the orthogonal complement to $\mathscr{Z}$ in $\mathscr{N}$. Let $X_1, \ldots, X_n$ be an orthonormal basis for $\mathscr{N}_1$. Define the homomorphism $\lambda\colon \mathscr{F}(n) \to \mathscr{N}$ by setting $\lambda(e_i) = X_i$ (where $e_1, \ldots, e_n$ is the standard basis for $\mathbf{R}^n$), and $\lambda(E_{i,j}) = [X_i, X_j]$, (where $E_{i,j} = [(e_i, 0), (e_j, 0)] \in \mathscr{F}(n)$). Let $\mathscr{K}$ denote the kernel of λ ($\subseteq \Sigma_n$). Note that $\lambda\colon \mathbf{R}^n \to \mathscr{N}_1$ is an isometry (where $\mathscr{F}(n)$ is equipped with the inner product $\langle (u, U), (v, V)\rangle = (0, uv^t + \frac{1}{2}\operatorname{tr}(UV^t))$). Given $k \in K$, we define $\tilde{k} \in \mathrm{Aut}(\mathscr{F}(n))$ by $\tilde{k}(e_i) = \lambda^{-1}(k\cdot(\lambda(e_i)))$ and $\tilde{k}(E_{i,j}) = [\tilde{k}\cdot e_i, \tilde{k}\cdot e_j]$, and set $\widetilde{K} = \{\tilde{k} | k \in K\}$. Note that $\widetilde{K} \simeq K$.

Lemma 6.1. *Let K be a compact subgroup of* $\mathrm{Aut}(N)$. *For any choice of orthonormal basis of $\mathscr{N}_1$, $\widetilde{K}$ is a compact subgroup of $O(n)$. If $\widetilde{K}, \widetilde{K}'$ are constructed using different orthonormal bases of $\mathscr{N}_1$ then $\widetilde{K} = A^t\widetilde{K}'A$ for some $A \in O(n)$. K is a maximal compact subgroup of* $\mathrm{Aut}(N)$ *if, and only if, $\widetilde{K} = O_{\mathscr{K}}(n) := \{A \in O(n) | A\cdot\mathscr{K}\ (:= A^t\mathscr{K}A) = \mathscr{K}\}$.*

Proof. Given $\tilde{k} \in \widetilde{K}$, $\tilde{k}(\mathbf{R}^n) \subseteq \mathbf{R}^n$. Thus, there is an $A_k \in Gl(n, \mathbf{R})$ such that $\tilde{k}\cdot(u, U) = (uA_k, A_k\cdot U)$. Since $\lambda\colon \mathbf{R}^n \to \mathscr{N}_1$ is an isometry and the inner product on $\mathscr{N}$ is K-invariant, $A_k \in O(n)$. Finally note that $\lambda\tilde{k} = k\lambda$. It follows that $\mathscr{K} = \ker(\lambda)$ is $\tilde{k}$-invariant, and hence that $\widetilde{K} \subseteq O_{\mathscr{K}}(n)$.

Suppose that $A \in O_{\mathscr{K}}(n)$. Define $k_A \in \mathrm{Aut}(N)$ by requiring that $k_A\cdot\lambda((u, U)) = \lambda(A\cdot(u, U))$. It is clear that $A \mapsto k_A\colon O_{\mathscr{K}}(n) \to \mathrm{Aut}(N)$ is a 1-1 homomorphism, and hence, since $O(n)$ is a maximal compact subgroup of $Gl(n, \mathbf{R})$, that K is a maximal compact subgroup of $\mathrm{Aut}(N)$ if, and only if, $\widetilde{K} = O_{\mathscr{K}}(n)$. □

Let $\mathscr{Z}$ denote the orthogonal complement in Σ_n of $\mathscr{K}$, and let $\mathscr{N}_{\mathscr{Z}} = \mathbf{R}^n \times \mathscr{Z}$ be the Lie algebra defined as in (5.7), i.e. with Lie bracket defined by

$[(u, U), (v, V)]_{\mathscr{Z}} = P_{\mathscr{Z}}(u^t v - v^t u)$, where $P_{\mathscr{Z}}$ is the orthogonal projection of Σ_n onto $\mathscr{Z}$. Let $\bar{\lambda}: \mathscr{F}(n)/\mathscr{K} \to \mathscr{N}$ be the canonical isomorphism, define $i: \mathscr{N}_{\mathscr{Z}} \to \mathscr{F}(n)/\mathscr{K}$ by $i(X) = X + \mathscr{K}$, and let $\tilde{\lambda} = \bar{\lambda} \circ i$. Then $\tilde{\lambda}$ is a Lie algebra isomorphism. Since $\widetilde{K} \subseteq O_{\mathscr{K}}(n)$, by restriction we may consider $\widetilde{K} \subseteq \mathrm{Aut}(N_Z)$, where $N_Z = \exp(\mathscr{N}_{\mathscr{Z}})$. One can easily check that $k \cdot \lambda(X) = \tilde{\lambda}(\tilde{k} \cdot X)$ and thus prove

Lemma 6.2. *(K, N) is a Gelfand pair if, and only if, $(\widetilde{K}, N_Z)$ is a Gelfand pair.*

Pick a nonzero $B \in \mathscr{Z}$. Let $\mathscr{N}_B$ denote the Lie algebra defined as in (5.7) with $\mathscr{Z} = \mathbf{R}B$. $\mathscr{N}_B$ is a concrete realization of the quotient Lie algebra $\mathscr{N}_{\mathscr{Z}}/\mathscr{Z}_0$, where $\mathscr{Z}_0$ is the orthogonal complement in $\mathscr{Z}$ of $\mathbf{R}B$. Let $\mathscr{H}_B$ denote the subset of $\mathscr{N}_B$ given by $\mathbf{R}^n B \times \mathbf{R}B$, and define a Lie bracket as in (5.7). Let N_B and H_B denote the corresponding simply connected Lie groups. Since the bilinear form defined on $\mathbf{R}^n$ by B is nondegenerate on its range, one has as in the proof of Theorem 5.12 (see equation (5.13)) that H_B is isomorphic to a Heisenberg group.

Given $b \in (\mathbf{R}^n B)^{\perp}$, the orthogonal complement in $\mathbf{R}^n$ of the range of B, set

$$\widetilde{K}_{(b,B)} = \{\tilde{k} \in \widetilde{K} \mid \tilde{k} \cdot B = B, \text{ and } \tilde{k} \cdot b = b\}. \tag{6.3}$$

By restriction, we may consider $\widetilde{K}_{(b,B)}$ as a subgroup of $\mathrm{Aut}(H_B)$.

Theorem 6.4. *If (K, N) is a Gelfand pair then $(\widetilde{K}_{(b,B)}, H_B)$ is a Gelfand pair for all B in $\mathscr{Z}$, and all $b \in (\mathbf{R}^n B)^{\perp}$. Conversely, if $(\widetilde{K}_{(b,B)}, H_B)$ is a Gelfand pair for (b, B) in a set of full Plancherel measure, then (K, N) is a Gelfand pair.*

Proof. Recall that we identify Lie algebras and their duals using the selected inner products. Given $B \in \mathscr{Z}$ and $b \in (\mathbf{R}^n B)^{\perp}$ we let $\mathscr{O}_{(b,B)}$ denote the orbit in $\mathscr{N}_Z$ $(\cong \mathscr{N}_Z^*)$ through (b, B). By (5.11), $\mathscr{O}_{(b,B)} = (b + \mathbf{R}^n B, B)$. Thus, $\widetilde{K}_{(b,B)}$ is the subgroup of $\widetilde{K}$ that preserves the equivalence class of $\pi_{(b,B)}$, the representation of N_Z corresponding to $\mathscr{O}_{(b,B)}$.

As above, let $\mathscr{Z}_0$ be the orthogonal complement in $\mathscr{Z}$ of $\mathbf{R}B$. Then $\mathscr{Z}_0$ is the subset of $\mathscr{Z}$ on which the *functional* B vanishes. Thus, $\pi_{(b,B)}$ factors through a representation of $N_B = N_Z/\exp(\mathscr{Z}_0)$.

Note that for $u \in \mathbf{R}^n$ and $v \in (\mathbf{R}^n B)^{\perp}$, equation (5.10) implies that

$$\begin{aligned}[(u, 0), (v, 0)]_{\mathbf{R}B} &= P_{\mathbf{R}B}([(u, 0), (v, 0)]) = \langle B, [(u, 0), (v, 0)]\rangle B \\ &= \langle J_B(u), v\rangle B = \langle uB, v\rangle B = 0.\end{aligned}$$

Thus, $\mathscr{N}_B$ is the direct sum of the Heisenberg Lie algebra $\mathscr{H}_B = \mathbf{R}^n B \times \mathbf{R}B$ and the commutative algebra $(\mathbf{R}^n B)^\perp$ $(= (\mathbf{R}^n B)^\perp \times \{0\})$. Writing $N_B = H_B \times (\mathbf{R}^n B)^\perp$, $\pi_{(b,B)}$ factors as $\pi_B \otimes \chi_b$, where π_B is the element of $\widehat{H}_B$ corresponding to B and χ_b is the unitary character defined on $(\mathbf{R}^n B)^\perp$ by $\chi_b(v) = e^{2\pi i \langle b, v\rangle}$.

The intertwining representation of $\widetilde{K}_{(b,B)}$ fixes the factor χ_b, and thus is multiplicity free if, and only if, the representation of $\widetilde{K}_{(b,B)}$ on the space of π_B is multiplicity free. This proves the theorem. □

Remark. If K is a maximal compact, connected subgroup of $\mathrm{Aut}(N)$ then $\tilde{K}_{(b,B)} = O(\mathbf{R}^n B) \times O_b((\mathbf{R}^n B)^\perp)$, where $O_v(V)$ denotes the group of all orthogonal transformations of V that fix $v \in V$. We consider two applications of Theorem 6.4. in the first, let $\mathscr{N}$ be the Lie algebra with basis X, Y_1, Y_2, Z_1, Z_2, and with all nonzero brackets determined by $[X, Y_j] = Z_j$ for $j = 1, 2$. Let K be a maximal compact subgroup of $\mathrm{Aut}(\mathscr{N})$, and fix a K-invariant inner product on $\mathscr{N}$. Pick an orthonormal basis X_i, $i = 1, 2, 3$, for $\mathscr{Z}^\perp$, and define $\lambda\colon \mathscr{F}(3) \to \mathscr{N}$ by requiring that $\lambda(e_i) = X_i$, $i = 1, 2, 3$. Then, $\dim(\mathscr{K} = \ker\lambda) = 1$. Thus, if $\mathscr{Z}$ is the orthogonal complement to $\mathscr{K}$ in Σ_3, $\dim(\mathscr{Z}) = 2$. Hence, if $B \in \mathscr{Z}$, $B \neq 0$, and $b \in \mathbf{R}^3$, one easily sees that $\widetilde{K}_{(b,B)} = \{e\}$. Thus there are no compact subgroups K' of $\mathrm{Aut}(\mathscr{N})$ such that (K', N) is a Gelfand pair.

The next application of Theorem 6.4 will be to offer a short proof of a theorem due to H. Leptin, [Le]. We assume, as always, that $\mathscr{N}$ is the nilpotent Lie algebra of a simply connected group N with $[\mathscr{N}, \mathscr{N}] = \mathscr{Z}$, the center of $\mathscr{N}$.

Theorem (Leptin). *Suppose that K is the k-torus contained in* $\mathrm{Aut}(N)$. *Then (K, N) is a Gelfand pair if, and only if, N is the quotient of the direct product of k-copies of the 3-dimensional Heisenberg group H_1, with K acting trivially on the center of N and lifting to the product of the usual $U(1)$ action on each factor H_1.*

Proof. Let $\lambda\colon \mathscr{F}(n) \to \mathscr{N}$, and $\widetilde{K} \subseteq \mathrm{Aut}(F(n))$ be defined as above. Let

$$\mathbf{R}^n = \sum_{i=1}^{k} V_{\alpha_i}$$

be the decomposition into $\widetilde{K}$-root spaces. First note that if $X_{\alpha_i} \in V_{\alpha_i}$, $i = 1, 2$, and $\alpha_1 \neq \alpha_2$, then $[X_{\alpha_1}, X_{\alpha_2}] = 0$. Indeed, since $(\widetilde{K}, N_Z)$ is a Gelfand pair, there exist $k_i \in \widetilde{K}$, $i = 1, 2$, such that

$$X_{\alpha_1} + X_{\alpha_2} + \tfrac{1}{2}[X_{\alpha_1}, X_{\alpha_2}] = k_1 \cdot X_{\alpha_1} + k_2 \cdot X_{\alpha_2} + \tfrac{1}{2}[k_2 \cdot X_{\alpha_2}, k_1 \cdot X_{\alpha_1}].$$

From the $\widetilde{K}$-invariance of each V_α, one concludes that $k_i \cdot X_{\alpha_i} = X_{\alpha_i}$, and thus that $[X_{\alpha_1}, X_{\alpha_2}] = 0$.

Next observe that for $\alpha \in \{\alpha_i \mid 1 \le i \le k\}$, $\dim(V_\alpha) = 2$. For this note that if $\widetilde{K}_\alpha$ is the action of $\widetilde{K}$ on $\mathscr{N}_\alpha := V_\alpha \oplus \mathscr{Z}$, considered as a subalgebra of $\mathscr{N}_{\mathscr{Z}}$, then $(\widetilde{K}_\alpha, \exp(\mathscr{N}_\alpha))$ is a Gelfand pair. $\dim(V_\alpha) > 1$, since for each nonzero $X \in V_\alpha$ there is a $Y \in V_\alpha$ such that $[X, Y] \ne 0$, and since $\widetilde{K}_\alpha$ acts as a subgroup of $\mathbf{T}$ on $\mathscr{N}_\alpha$, one concludes as in the proof of Theorem 5.17 that $\dim(V_\alpha) = 2$, and so $n = 2k$.

Let $\{e_{2i-1}, e_{2i}\}$ be an orthonormal basis for V_{α_i}, and let

$$\Omega = \operatorname{span}\{E_{2i-1,2i} \mid 1 \le i \le k\}.$$

We will show that if $B \in \mathscr{Z}$, the orthogonal complement to $\mathscr{K} := \ker(\lambda)$ in Σ_{2k}, then $B \in \Omega$. Given such a B, let $\mathbf{R}^n B = \sum_{i=1}^{l} V_i$ be the decomposition corresponding to the standard form of the skew-symmetric B. Since B is nondegenerate on its range, for each nonzero $X \in \mathbf{R}^n B$ there is a $Y_X \in \mathbf{R}^n B$ such that $[X, Y_X] \ne 0$. Since $(\widetilde{K}_B, H_B)$ is a Gelfand pair, one concludes as before, that if $X \in V_i$, then $Y_X \in V_i$. It then follows that $V_i = \operatorname{span}\{\widetilde{K}_B \cdot X\}$ for any nonzero $X \in V_i$. This amounts to showing that if $\widetilde{K}_B \cdot X = X$ for some $X \in V_i$, then $X = 0$. But this is clear, for otherwise, by Theorem 1.12, there exist $k \in \widetilde{K}_B$ such that

$$X + Y_X + \tfrac{1}{2}[X, Y_X] = X + k \cdot Y_X + \tfrac{1}{2}[k \cdot Y_X, X].$$

This forces the contradiction that $[X, Y_X] = 0$. It now follows that each V_i equals some V_{α_j}, and hence that $B \in \Omega$. Therefore, $\mathscr{K}$ contains the orthogonal complement to Ω in Σ_{2k}, and $F(n)/\exp(\mathscr{K})$ is the quotient of the direct product of k-copies of H_1. Finally, since $\widetilde{K}$ fixes each element of Ω, K acts trivially on the center of N. $\square$

Solvable groups

We now consider a simply connected solvable Lie group S with Lie algebra $\mathscr{S}$. We denote by $\mathscr{N}_{\mathscr{S}}$, or more simply by $\mathscr{N}$, the nilradical of $\mathscr{S}$. Given a compact subgroup $K \subseteq \operatorname{Aut}(\mathscr{S})$, we set

$$\mathscr{S}_0 = \{X \in \mathscr{S} \mid k \cdot X = X, \ \forall\, k \in K\}.$$

The following theorem and proof was communicated to the authors by H. Leptin.

Theorem (Leptin). *If K is connected, then $\mathscr{S} = \mathscr{S}_0 + \mathscr{N}$.*

Proof. Let $\mathscr{S}_{\mathbf{C}} = \mathscr{S} \otimes_{\mathbf{R}} \mathbf{C}$ be the complexification of $\mathscr{S}$. Then $K \subseteq \operatorname{Aut}(\mathscr{S}_{\mathbf{C}})$, $(\mathscr{S}_0)_{\mathbf{C}} = (\mathscr{S}_{\mathbf{C}})_0$, and $\mathscr{N}_{\mathscr{S}_{\mathbf{C}}} = (\mathscr{N}_{\mathscr{S}})_{\mathbf{C}}$. Thus, we may assume that $\mathscr{S}$ is complex.

Now, if K is abelian and

$$\mathscr{S}_\chi = \{X \in \mathscr{S} \mid k \cdot X = \chi(k) X, \ \forall\, k \in K\},$$

then

$$\mathscr{S} = \sum_{\chi \in \widehat{K}} \mathscr{S}_\chi. \tag{7.1}$$

If $X \in \mathcal{S}_\chi$, $X \neq 0$, and λ is an eigenvalue of $\operatorname{ad} X$, then there is a nonzero $Y \in \mathcal{S}$ such that $[X, Y] = \lambda Y$. For $k \in K$,

$$k \cdot (\lambda Y) = [k \cdot X, k \cdot Y] = \chi(k)[X, k \cdot Y].$$

Thus, $\overline{\chi(k)}\lambda$ is also an eigenvalue of $\operatorname{ad} X$ for all $k \in K$. But if $\chi \neq \varepsilon$, the identity, $\chi(K) = \mathbf{T}$, and thus, λt is an eigenvalue of $\operatorname{ad} X$ for all $t \in \mathbf{T}$. It follows that $\lambda = 0$, and so $\operatorname{ad} X$ is nilpotent. Therefore, $\mathcal{S}_\chi \subseteq \mathcal{N}$ for all $\chi \neq \varepsilon$, i.e. $\mathcal{S} = \mathcal{S}_0 + \mathcal{N}$.

We turn now to the general case. Let $t \in \mathbf{T} \subseteq K$, and $X \in \mathcal{S}$. Since $\mathcal{S} = \mathcal{S}_0' + \mathcal{N}$, where $\mathcal{S}_0' = \{X \in \mathcal{S} \mid t \cdot X = X, \ \forall t \in \mathbf{T}\}$, by the argument above, $t \cdot X \equiv X \pmod{\mathcal{N}}$. But every element of K is in a torus, and so for all $k \in K$, $k \cdot X \equiv X \pmod{\mathcal{N}}$. It follows that

$$X_0 := \int_K k \cdot X dk \equiv X \pmod{\mathcal{N}}.$$

Since $X_0 \in \mathcal{S}_0$, the theorem is proven. □

Given $X \in \mathcal{S}$, we define $i_X \in \operatorname{Aut}(S)$ by $i_X(y) = \exp(X) y \exp(-X)$. Consider the following condition:

(7.2) For each $X \in \mathcal{S}_0$, $y \in S$, $\exists\, k \in K \ni i_X(y) = k \cdot y$.

Theorem 7.3. *Suppose K is connected. Then (K, S) is a Gelfand pair if, and only if, (K, N) is a Gelfand pair, and condition (7.2) is satisfied.*

Proof. Suppose (K, S) is a Gelfand pair. By Theorem 1.12, for all $x, y \in N$, $xy \in (K \cdot y)(K \cdot x)$, which implies that (K, N) is a Gelfand pair. Furthermore, if $X \in \mathcal{S}_0$ and $y \in S$, then $\exp(X)y \in (K \cdot y)(K \cdot \exp(X)) = (K \cdot y)\exp(X)$. This proves the necessity of the conditions.

Suppose now the converse. Note that $S = \exp(\mathcal{S}_0)N$. Given $X, Y \in \mathcal{S}_0$, and $x, y \in N$ we compute

$$\begin{aligned}(K \cdot \exp(X)x)(K \cdot \exp(Y)y) &= \exp(X)(K \cdot x)\exp(Y)(K \cdot y)\\ &= \exp(X)\exp(Y)(\exp(-Y)(K \cdot x)\exp(Y))(K \cdot y)\\ &= \exp(X)\exp(Y)(K \cdot x)(K \cdot y)\\ &= \exp(X)\exp(Y)(K \cdot y)(K \cdot x)\\ &= (\exp(X)(K \cdot \exp(Y)y)\exp(-X))(K \cdot (\exp(X)x)\\ &= (K \cdot \exp(Y)y)(K \cdot \exp(X)x).\end{aligned}$$

Theorem 1.12 implies that (K, S) is a Gelfand pair. □

Recall that a connected Lie group G is said to be *type-R* if the eigenvalues of $\operatorname{ad} X$, as a linear operator on $\mathcal{G}$, are pure imaginary. Note that $i_X(\exp(Y)) = \exp(\operatorname{Ad}(\exp(X)) \cdot Y) = \exp(\exp(\operatorname{ad} X) \cdot Y)$. Thus, if (7.2) is satisfied, and $\|\cdot\|$ is a K invariant norm on $\mathcal{S}$, then for all $X \in \mathcal{S}_0$, $\|\exp(\operatorname{ad} X) \cdot Y\| = \|i_X \cdot Y\| = \|Y\|$. This implies that the eigenvalues of $\operatorname{ad} X$ are pure imaginary for all $X \in \mathcal{S}_0$. The same holds true for $X \in \mathcal{N}$, since $\operatorname{ad} X$ is nilpotent as a linear operator on $\mathscr{S}$. Thus

Corollary 7.4. *If* (K, S) *is a Gelfand pair, then* S *is type-R.*

A very simple example of a Gelfand pair (K, S) involving a non-nilpotent group is given by letting $S = \mathbf{R} \propto \mathbf{C}$, with $\mathbf{R}$ acting on $\mathbf{C}$ by $t \colon z \mapsto e^{it} z$, and $K = U(1)$ acting as usual on $\mathbf{C}$.

Spherical functions

In this section we identify a moduli space for the K-spherical functions associated to a Gelfand pair (K, S). Recall that a K-spherical function associated to such a pair is a continuous, complex-valued function, ϕ, defined on S, satisfying

$$\phi(e) = 1 \quad \text{and} \quad \int_K \phi(xk \cdot y)\, dk = \phi(x)\phi(y) \tag{8.1}$$

for all $x, y \in S$. It easily follows that a K-spherical function is K-invariant. One also has that integration against a K-spherical function, ϕ, defines a complex-valued homomorphism on $L^1_K(N)$, that this homomorphism is continuous if ϕ is bounded, and that all continuous homomorphisms of $L^1_K(N)$ are given in this manner (cf. [He]). We first consider K-spherical functions associated to a Gelfand pair (K, N).

Lemma 8.2. *Suppose* ϕ *is a bounded* K*-spherical function on* N. *Then there is a* $\pi \in \widehat{N}$ *and a unit vector* $\xi \in \mathbf{H}_\pi$ *such that*

$$\phi(x) = \int_K \langle \pi(k \cdot x)\xi, \xi\rangle\, dk,$$

for each $x \in N$.

Proof. Let $\lambda_\phi \colon L^1_K(N) \to \mathbf{C}$ be given by integration against ϕ.

Since $L^1(N)$ is a symmetric Banach $*$-algebra, [Le2], there is a representation $\overline{\pi}$ of $L^1(N)$ and a one-dimensional subspace $\mathbf{H}_\phi$ of $\mathbf{H}_{\overline{\pi}}$ such that $(\overline{\pi}|_{L^1_K(N)}, \mathbf{H}_\phi)$ is equivalent to $(\lambda_\phi, \mathbf{C})$. As λ_ϕ is irreducible, the extension $\overline{\pi}$ is also irreducible (cf. [Na]). Using approximate identities at each point of N, one can show that $\overline{\pi}$ is the integrated version of some $\pi \in \widehat{N}$, with $\mathbf{H}_\pi = \mathbf{H}_{\overline{\pi}}$.

Choose $\xi \in \mathbf{H}_\phi$ with $\|\xi\| = 1$. Then for each $f \in L^1_K(N)$, $\pi(f)\xi = \lambda_\phi(f)\xi$, so that

$$\begin{aligned}
\langle \phi, f\rangle = \lambda_\phi(f) &= \langle \pi(f)\xi, \xi\rangle \\
&= \int_N f(x)\langle \pi(x)\xi, \xi\rangle\, dx \\
&= \int_K \int_N f(k^{-1} \cdot x)\langle \pi(x)\xi, \xi\rangle\, dx\, dk
\end{aligned}$$

since f is K-invariant

$$= \int_K \int_N f(x)\langle \pi(k\cdot x)\xi, \xi\rangle\, dx\, dk.$$

Since ϕ is K-invariant, we change the order of integration and obtain

$$\phi(x) = \int_K \langle \pi(k\cdot x)\xi, \xi\rangle\, dk. \quad \square \tag{8.3}$$

Notation. We denote the function defined by (8.3) as $\phi_{\pi,\xi}$.

Corollary 8.4. *If ϕ is a bounded K-spherical function on N, then ϕ is positive definite.*

Recall from §3 that for $\pi \in \widehat{N}$ we denote by K_π the subgroup of K that preserves the equivalence class of π, and that W_π denotes the intertwining representation of K_π.

Let $\mathbf{H}_\pi = \sum_\alpha V_\alpha$ be the decomposition of $\mathbf{H}_\pi$ into irreducible subspaces invariant under the action of W_π. The assumption that (K, N) is a Gelfand pair implies that as K_π-modules, the V_α's are inequivalent for different α's.

Lemma 8.5. *If $\pi' = \pi_{k_0}$, then $K_{\pi'} = k_0^{-1}K_\pi k_0$.*

Proof. If $k' \in K_{\pi'}$, then $\pi'_{k'} \simeq \pi'$. That is, $\pi'_{k'}(x) = W_{\pi'}(k')\pi'(x)W^*_{\pi'}(k')$ for each $x \in N$. Thus

$$\begin{aligned}\pi_{k_0k'k_0^{-1}}(x) &= \pi_{k_0k'}(k_0^{-1}\cdot x) = \pi'_{k'}(k_0^{-1}\cdot x)\\ &= W_{\pi'}(k')\pi'(k_0^{-1}\cdot x)W^*_{\pi'}(k') = W_{\pi'}(k')\pi(x)W^*_{\pi'}(k').\end{aligned}$$

Thus, $\pi_{k_0k'k_0^{-1}} \simeq \pi$, so $k_0k'k_0^{-1} \in K_\pi$. $\square$

Note that for $k' \in K_{\pi'}$, the above calculation shows that we could choose $W_{\pi'}$ so that $W_\pi(k_0k'k_0^{-1}) = W_{\pi'}(k')$.

Corollary 8.6. *For $\pi' = \pi_{k_0}$, $\mathbf{H}_\pi$ and $\mathbf{H}_{\pi'}$ have the some decomposition into W_π- and $W_{\pi'}$-irreducible subspaces respectively.*

Theorem 8.7. (i) *$\phi_{\pi,\xi}$ is a K-spherical function if, and only if, $\xi \in V_\alpha$ for some α, and $\|\xi\| = 1$.* (ii) *$\phi_{\pi,\xi} = \phi_{\pi',\eta}$ if, and only if, there is a $k \in K$ such that $\pi' = \pi_k$ and ξ, η belong to the same V_α.*

Proof. Let $f \in L^1_K(N)$. Since f is K_π-invariant, $\pi(f)$ commutes with the action of W_π on $\mathbf{H}_\pi$. Since W_π is multiplicity free, $\pi(f)$ preserves each V_α. Now by Schur's lemma, the irreducibility of W_π on V_α implies that $\pi(f)$ acts as a scalar multiple of the identity on each V_α. Note that this scalar is computed by the formula $\langle \pi(f)\xi, \xi\rangle$ for any $\xi \in V_\alpha$ with $\|\xi\| = 1$.

For $\xi \in V_\alpha$ with $\|\xi\| = 1$, $\phi_{\pi,\xi}$ is clearly a continuous function on N. We only need to show that λ_ϕ (with $\phi = \phi_{\pi,\xi}$) is a homomorphism on $L^1_K(N)$.

Note that for $f \in L^1_K(N)$,

$$\begin{aligned}\langle \phi_{\pi,\xi}, f\rangle &= \int_N \int_K \langle \pi(k\cdot x)\xi, \xi\rangle f(x)\, dk\, dx \\ &= \int_K \int_N \langle \pi(x)\xi, \xi\rangle f(k^{-1}\cdot x)\, dx\, dk \qquad (8.8)\\ &= \langle \pi(f)\xi, \xi\rangle .\end{aligned}$$

Thus, if $f, g \in L^1_K(N)$,

$$\begin{aligned}\lambda_\phi(f * g) &= \langle \pi(f * g)\xi, \xi\rangle = \langle \pi(f)\pi(g)\xi, \xi\rangle \\ &= \langle \pi(g)\xi, \xi\rangle\langle \pi(f)\xi, \xi\rangle = \lambda_\phi(f)\lambda_\phi(g).\end{aligned}$$

Conversely, suppose $\xi \in \mathbf{H}_\pi$, $\|\xi\| = 1$. Write $\xi = \sum t_\alpha \xi_\alpha$ with $\xi_\alpha \in V_\alpha$, $\|\xi_\alpha\| = 1$, $t_\alpha \geq 0$, and $\sum t_\alpha^2 = \|\xi\|^2 = 1$. Then

$$\langle \phi_{\pi,\xi}, f\rangle = \langle \pi(f)\xi, \xi\rangle = \sum_{\alpha,\beta} t_\alpha t_\beta \langle \pi(f)\xi_\alpha, \xi_\beta\rangle = \sum_\alpha t_\alpha^2 \langle \pi(f)\xi_\alpha, \xi_\alpha\rangle$$

since $\pi(f)$ preserves the mutually orthogonal V_α's

$$= \sum_\alpha t_\alpha^2 \langle \phi_{\pi,\xi_\alpha}, f\rangle .$$

Thus, for $\xi = \sum t_\alpha \xi_\alpha$, $t_\alpha \geq 0$, $\phi_{\pi,\xi} = \sum_\alpha t_\alpha^2 \phi_{\pi,\xi_\alpha}$, and $\|\xi\|^2 = 1$ implies that $\sum t_\alpha^2 = 1$. Note that positive definite homomorphisms are extreme points in the Gelfand space of $L^1_K(N)$, so if $\phi_{\pi,\xi}$ is a positive definite K-spherical function, it cannot be a convex sum of positive definite K-spherical functions. Thus $\xi = \xi_\alpha$ for some α.

Now suppose $\pi' = \pi_{k_0}$ and ξ, η belong to $V_\alpha \subseteq \mathbf{H}_\pi$. Then

$$\langle \phi_{\pi,\xi}, f\rangle = \langle \pi(f)\xi, \xi\rangle = \langle \pi(f)\eta, \eta\rangle$$

since $\pi(f)$ is constant on V_α

$$\begin{aligned}&= \int_N \int_K \langle \pi(k\cdot x)\eta, \eta\rangle f(x)\, dk\, dx \\ &= \int_N \int_K \langle \pi(k_0 k\cdot x)\eta, \eta\rangle f(x)\, dk\, dx \\ &= \langle \phi_{\pi',\eta}, f\rangle .\end{aligned}$$

Thus, $\phi_{\pi,\xi} = \phi_{\pi',\eta}$.

For the converse of (ii), we need to understand $\widehat{K \propto N}$ via the Mackey machine. Let $\pi \in \widehat{N}$, and suppose the intertwining representation W_π of K_π is a σ-representation, as described in §3. Let T be any $\overline{\sigma}$-representation of K_π. Then $\rho = T \otimes \pi W_\pi$ is an irreducible representation of $K_\pi \propto N$. Let $\tilde{\rho}$ be the representation of $K \propto N$ induced from ρ. Then $\tilde{\rho} \in \widehat{K \propto N}$, and any irreducible representation of $K \propto N$ is obtained in this manner. More precisely,

$\widehat{K \propto N}$ is given by pairs (π, T), where $\pi \in \widehat{N}$, and $T \in \widehat{K}_\pi^{\overline{\sigma}}$. Another pair (π', T') yields an equivalent representation if, and only if, $\pi' \simeq \pi_{k_0}$ for some k_0 and $T' \simeq T \circ i_{k_0}$, where $i_{k_0}: K_{\pi'} \to K_\pi = k_0 K_{\pi'} k_0^{-1}$.

As a function on $G = K \propto N$, any positive definite K-spherical function is given as follows: Let $\tilde{\rho} \in \widehat{G}$. If there is a K-fixed vector $v \in \mathbf{H}_{\tilde{\rho}}$ (the space of K-fixed vectors has dimension at most one), then $\phi(x) = \langle \tilde{\rho}(x)v, v\rangle$. This yields a 1-1 correspondence between the representations in $\widehat{G}$ with K-fixed vectors and positive definite K-spherical functions on G (cf. [He]).

By Frobenius reciprocity, we see that the dimension of the space of K-fixed vectors in $\mathbf{H}_{\tilde{\rho}}$ equals the dimension of the space of K_π-fixed vectors in $\mathbf{H}_\rho$. Note that $T \otimes W_\pi$ has K_π-fixed vectors if, and only if, $\overline{T}$ is a subrepresentation of W_π, i.e. $\mathbf{H}_T = V_\alpha$ for some W_π-irreducible component of $\mathbf{H}_\pi$, and $T = \overline{W}_\pi|_{V_\alpha}$. Thus there is a 1-1 correspondence between positive definite K-spherical functions and pairs (π, V_α), where $\pi \in \widehat{N}$ and $V_\alpha \subseteq \mathbf{H}_\pi$ is a W_π-irreducible component. We will see that these K-spherical functions coincide with the formulas in the statement of the theorem. Let $\{v_1, \dots, v_n\}$ be an orthonormal basis for V_α, and set

$$v = \frac{1}{\sqrt{m}} \sum v_i \otimes v_i, \tag{8.9}$$

regarded as an element of $\mathbf{H}_\rho = V_\alpha \otimes \mathbf{H}_\pi$. For $k \in K_\pi$,

$$\begin{aligned}\rho(k)v &= \frac{1}{\sqrt{m}} \sum_i \overline{W}_\pi(k)v_i \otimes W_\pi(k)v_i \\ &= \frac{1}{\sqrt{m}} \sum_{i,j,k} \bar{a}_{i,j} v_j \otimes a_{i,k} v_k,\end{aligned}$$

where $A = (a_{i,j})$ is the matrix corresponding to $W_\pi(k)|_{V_\alpha}$. But

$$\sum_i \bar{a}_{i,j} a_{i,k} = (A^* A)_{j,k} = \delta_{j,k}.$$

Thus

$$\rho(k)v = \frac{1}{\sqrt{m}} \sum_j v_j \otimes v_j, \tag{8.10}$$

so v is a K_π-fixed vector in $\mathbf{H}_\rho$.

To construct a corresponding K-fixed vector in $\mathbf{H}_{\tilde{\rho}}$, define $f: K \propto N \to V_\alpha \otimes \mathbf{H}_\pi$ by $f(k, n) = (1 \otimes \pi(n))v$. To ensure that $f \in \mathbf{H}_{\tilde{\rho}}$, we need $f(hg) = \rho(h)f(g)$, for $h \in K_\pi \propto N$, $g \in K \propto N$. (Actually it is sufficient to take $g = (k, e)$ with $k \in K$.) We have

$$f((k_\pi, n)(k, e)) = f(k_\pi k, n) = (1 \otimes \pi(n))v.$$

On the other hand,

$$\begin{aligned}\rho(k_\pi, n)f(k, e) &= \overline{W}_\pi(k_\pi) \otimes \pi(n) W_\pi(k_\pi)v \\ &= (1 \otimes \pi(n))\rho(k_\pi)v = (1 \otimes \pi(n))v,\end{aligned}$$

as required. Thus $f \in \mathbf{H}_{\tilde{\rho}}$, and for $k \in K$,

$$\tilde{\rho}(k)f(k', n) = f((k', n)(k, e)) = f(k'k, n) = (1 \otimes \pi(n))v = f(k', n),$$

so f is a K-fixed vector.

We check that f is a unit vector.

$$\begin{aligned}\|f\|^2 &= \int_{(K \propto N)/(K_\pi \propto N)} \|f(k, n)\|^2\, dk\, dn \\ &= \int_{(K \propto N)/(K_\pi \propto N)} \|(1 \otimes \pi(n))v\|^2\, dk\, dn \\ &= \int_{K/K_\pi} \|v\|^2\, dk = 1,\end{aligned}$$

since

$$\|v\|^2 = \frac{1}{m}\sum_{i=1}^{m} \|v_i \otimes v_i\|^2 = 1.$$

The K-spherical function $\tilde{\phi}$ on G associated with f is given by $\tilde{\phi}(g) = \langle \tilde{\rho}(g)f, f\rangle$. The restriction ϕ of $\tilde{\phi}$ to N is given by

$$\begin{aligned}\phi(n) &= \langle \tilde{\rho}(n)f, f\rangle \\ &= \int_{K/K_\pi} \langle \tilde{\rho}(n)f(k), f(k)\rangle\, dk \\ &= \int_{K/K_\pi} \langle f((k, e)(e, n)), f(k)\rangle\, dk \\ &= \int_{K/K_\pi} \langle f(k, k \cdot n), f(k)\rangle\, dk \\ &= \int_{K/K_\pi} \langle (1 \otimes \pi(k \cdot n))v, v\rangle\, dk.\end{aligned}$$

For $k \in K$,

$$\begin{aligned}\langle (1 \otimes \pi(k \cdot n))v, v\rangle &= \frac{1}{m}\sum_{i,j} \langle v_j \otimes \pi(k \cdot n)v_j, v_i \otimes v_i\rangle \\ &= \frac{1}{m}\sum_{i} \langle \pi(k \cdot n)v_i, v_i\rangle\end{aligned}$$

For $k \in K_\pi$,

$$\begin{aligned}\sum_i \langle \pi(k \cdot n)v_i, v_i\rangle &= \sum_i \langle W_\pi(k)\pi(n)W_\pi(k)^{-1}v_i, v_i\rangle \\ &= \sum_i \langle \pi(n)W_\pi(k)^{-1}v_i, W_\pi(k)^{-1}v_i\rangle \\ &= \sum_i \langle \pi(n)v_i, v_i\rangle,\end{aligned}$$

by an easy trace argument. Thus,

$$\phi(n) = \frac{1}{m}\int_K \sum_i \langle \pi(k\cdot n)v_i, v_i\rangle\, dk$$
$$= \frac{1}{m}\sum_i \phi_{\pi, v_i}(n) = \phi_{\pi, m^{-1/2}\sum v_i}(n).$$

Thus, $\phi = \phi_{\pi,\xi}$, where ξ is any element of V_α (since any unit vector in V_α can be written as $1/\sqrt{m}\sum v_i$ for some orthonormal basis $\{v_1, \dots, v_n\}$). □

Suppose now that (K, S) is a Gelfand pair. Note that if ϕ is a K-spherical function, $X, Y \in \mathscr{S}_0$, and $y \in S$, then by (8.1)

$$\phi(y\exp X\exp Y) = \phi(y)\phi(\exp X)\phi(\exp Y).$$

One also sees from (8.1) that the restriction of ϕ to $N := \exp(\mathscr{N})$, where $\mathscr{N}$ is the nilradical of $\mathscr{S}$, is a K-spherical function. This indicates how one constructs K-spherical functions on S.

Let $X_1, \dots, X_p$ be a basis for a complement of $\mathscr{N}$, the nilradical of $\mathscr{S}$, in $\mathscr{S}_0$. Since S is simply connected, for each $y \in S$, there exist unique $n(y) \in N$ $(= \exp(\mathscr{N}))$ and $\mathbf{t}(y) \in \mathbf{R}^p$ such that $y = n(y)\Pi_i \exp(t_i(y)X_i)$. Thus, if ϕ is a bounded K-spherical function on S then

$$\phi(y) = \phi(n(y))\Pi_i\phi(\exp(t_i(y)))$$

for each $y \in S$. Again by (8.1), for any $X \in \mathscr{S}_0$, the mapping $t \mapsto \phi(\exp(tX))$ is a homomorphism of $\mathbf{R}$ into $\mathbf{C}$. Thus, there exist an $\mathbf{a} \in \mathbf{R^p}$ such that $\phi(y) = \phi(n(y))e^{i\langle \mathbf{a}, \mathbf{t}(y)\rangle}$. Thus one has

Theorem 8.11. *ϕ is a bounded K-spherical function on S if, and only if, there is a bounded K-spherical function ψ on N and an $\mathbf{a} \in \mathbf{R^p}$ such that $\phi(y) = \psi(n(y))e^{i\langle \mathbf{a}, \mathbf{t}(y)\rangle}$. Thus $\Delta(K, S) = \Delta(K, N) \times \mathbf{R}^p$.*

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Department of Mathematics and Computer Science, University of Missouri-St. Louis, St. Louis, Missouri 63121 (C1792@UMSLVAXA)

Department of Mathematics and Statistics, The University at Albany/SUNY, Albany, New York 12222 (JWJ71@ALBNY1VX)

Department of Mathematics and Computer Science, University of Missouri-St. Louis, St. Louis, Missouri 63121