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GELFER FUNCTIONS, INTEGRAL MEANS, BOUNDED MEAN OSCILLATION, AND UNIVALENCY

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ABSTRACT. A Gelfer function f is a holomorphic function in $D = \{|z| < 1\}$ such that $f(0) = 1$ and $f(z) \neq -f(w)$ for all z, w in D. The family G of Gelfer functions contains the family P of holomorphic functions f in D with $f(0) = 1$ and $\operatorname{Re} f > 0$ in D. If f is holomorphic in D and if the L^2 mean of f' on the circle $\{|z| = r\}$ is dominated by that of a function of G as $r \to 1 - 0$, then $f \in BMOA$. This has two recent and seemingly different results as corollaries. A core of the proof is the fact that $\log f \in BMOA$ if $f \in G$. Besides the properties obtained concerning $f \in G$ itself, we shall investigate some families of functions where the roles played by P in Univalent Function Theory are replaced by those of G. Some exact estimates are obtained.

1. INTRODUCTION

Let Γ be the family of functions f holomorphic in the disk $D = \{|z| < 1\}$ having the Gelfer property that

$$f(z) + f(w) \neq 0 \quad \text{for all } z, w \in D. \tag{1.1}$$

In particular, f never vanishes in D. We call a member of $G = \{g/g(0); g \in \Gamma\}$ a Gelfer function in honor of S. A. Gelfer [10]. We shall use the following notation in [8] for f holomorphic in D:

$$M_p(r, f) = \begin{cases} \left\{ \dfrac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right\}^{1/p}, & \text{if } 0 < p < \infty; \\ \max\{|f(z)|; |z| = r\}, & \text{if } p = \infty, \end{cases}$$

where $0 \leq r < 1$ and $\|f\|_p = \lim_{r \to 1} M_p(r, f)$ for $0 < p \leq \infty$.

Let $BMOA$ be the family of functions f holomorphic in D with finite $BMOA$ norm:

$$\|f\|_* = \sup_{w \in D} \|f_w\|_2 + |f(0)| < \infty,$$

where $f_w(z) = f((z + w)/(1 + \overline{w}z)) - f(w)$. Then $BMOA$ is a Banach space. We shall investigate the $BMOA$ property and univalency in conjunction with the Gelfer property. A typical result, among others, is the following.

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Theorem 1. *Let f be holomorphic in D and let $g \in \Gamma$. Suppose that .*

$$\limsup_{r \to 1} M_2(r, f')/M_2(r, g) < \infty.$$

Then $f \in BMOA$.

Although this theorem is a weak form of J. A. Cima and K. E. Petersen's [5, Theorem 2.1], it reveals the mechanism by which the following is derived:

Corollary A (see [3, Theorem]). *Suppose that a holomorphic function f in D satisfies*

$$\int_0^{2\pi} |\operatorname{Re} f'(re^{it})|\, dt = O(1) \quad \text{as } r \to 1. \tag{1.2}$$

Then $f \in BMOA$.

Note that [5] is not referred to in [3] and the proof is different from the present one. Less obvious is the following

Corollary B (see [4, p. 357]). *Let f be holomorphic in D and close-to-convex of order $\beta \geq 0$. Then $\log f' \in BMOA$.*

It should be emphasized that even under the strong condition of univalency of f holomorphic in D, the boundary behavior of $\log f'$ may be very pathological; see [17; 22, Theorem 2]. The statement $\log f' \in BMOA$ for each univalent f in D is therefore false.

In §2, emphasis is placed on the similarity of the family Π of holomorphic functions f with real part $\operatorname{Re} f > 0$ in D and the subfamily $P = \{f \in \Pi;\, f(0) = 1\}$ to Γ and G, respectively. Clearly, $\Pi \subset \Gamma$ and $P \subset G$. We shall prove, for example, $\Gamma \subset H^p$, the Hardy class, for all p, $0 < p < 1$. A role in the derivation of Corollary B will be played by the fact that $\log f \in BMOA$ if $f \in \Gamma$. Proofs of Theorem 1 and Corollary A will be given in §3. As is known, P is important in Univalent Function Theory. We can generalize some families of functions by replacing P by G. Therefore, for instance, a normalized f is called Gelfer-convex if $zf''(z)/f'(z) + 1 \in G$. Theorem 4 in §4 is a corresponding generalization of Corollary B. In §5 we shall give a short theory of univalent functions in terms of Gelfer functions. One of our tools is an improvement of Gelfer's theorem, in a sharp form, on the positiveness of the real part of Gelfer functions. Some problems are summarized in §6.

2. Gelfer functions

We summarize here some known properties of $f \in G$, most of which are due to Gelfer [10]. (See [9, pp. 266–267; 13, II, pp. 73–76 and 82–83].) Let N be the family of functions f holomorphic in D with the normalization $f(0) = 0$, $f'(0) = 1$, and let S be the family of $f \in N$ univalent in D [9, p. 9].

We suppose that $f \in G$ and $z \in D$ in the following properties (G1)–(G8). The function $\lambda(z) = (1+z)/(1-z)$, or its rotation $\lambda(e^{i\theta} z)$, θ a real constant, shows the sharpness in the estimates. Note that

$$\lambda(z) - 1 = 2z/(1-z), \qquad \lambda'(z)/\lambda(z) = 2/(1-z^2),$$
$$|\lambda(z)| \le \lambda(|z|), \qquad |\lambda(z) - 1| \le \lambda(|z|) - 1,$$
$$|\lambda'(z)/\lambda(z)| \le \lambda'(|z|)/\lambda(|z|).$$

(G1) f never assumes 0 and -1 in D. Furthermore, $1/f \in D$.

(G2) We may find a univalent $F \in G$ such that f is subordinate to F.

This is [10, Theorem 1]. Here, g is subordinate to h in D if there exists a holomorphic function φ with $|\varphi| < 1$, $\varphi(0) = 0$, and $g = h \circ \varphi$ in D.

(G3) If f is univalent, then so is f^2. Furthermore, $(f^2 - 1)/\{2f'(0)\} \in S$.

(G4) If φ is holomorphic and $|\varphi| < 1$ in D, then $f \circ \varphi / f(\varphi(0)) \in G$.

$$\text{(G5)} \qquad |f(z)| \le \lambda(|z|).$$

$$\text{(G5}'\text{)} \qquad |\arg f(z)| \le \log \lambda(|z|) \qquad (\arg f(0) = 0).$$

$$\text{(G6)} \qquad |f'(z)/f(z)| \le \lambda'(|z|)/\lambda(|z|).$$

$$\text{(G7)} \qquad |f(z) - 1| \le \lambda(|z|) - 1.$$

$$\text{(G8)} \qquad |f'(0)| \le \lambda'(0) = 2.$$

In particular, (G7) is observed in [10, (13), p. 37]; see Lemma 5.1 in §5 for an extension.

We denote the Hardy class by H^p; this is the family of holomorphic f in D with $\|f\|_p < \infty$, $0 < p \le \infty$. It is familiar that

$$\Pi \subset \bigcap_{0<p<1} H^p;$$

see [8, p. 13]. We shall show that Π can be actually replaced by a larger family Γ. This follows from

Theorem 2. *If* $f \in G$, *then* $f \in H^p$ *for all* p, $0 < p < 1$. *Furthermore,* f *is outer* [8, p. 24],

$$f(z) = \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \lambda(e^{-it} z) \log |f(e^{it})| \, dt \right\}, \tag{2.1}$$

and

$$\|f\|_p \le 2^{(1/p)-1} + \{2p\Gamma(p)\Gamma(1-p)\}^{1/p}, \qquad 0 < p < 1. \tag{2.2}$$

We note that $f(e^{it}) = \lim_{r \to 1-0} f(re^{it})$ in (2.1) is the radial limit of f finite at almost every point e^{it} and $\Gamma(\cdot)$ in (2.2) is the gamma function. Obviously, $\lambda \in G$ is not in H^1.

Proof of Theorem 2. We remember the Prawitz inequality [9, p. 61] for $g \in S$:

$$M_p^p(r, g) \le p \int_0^r t^{-1} M_\infty^p(t, g)\, dt, \tag{2.3}$$

where $0 < p < \infty$ and $0 \le r < 1$.

Assume first that f is univalent in D. Then, $(f-1)/f'(0) \in S$, which, together with (2.3) and (G7), shows that

$$\begin{aligned} M_p^p(r, f-1) &\le p \int_0^r t^{-1} M_\infty^p(t, f-1)\, dt \le 2^p p \int_0^1 t^{p-1}(1-t)^{-p}\, dt \\ &= 2^p p \Gamma(p)(1-p) \qquad \text{for } 0 < p < 1. \end{aligned}$$

Therefore, we obtain, in view of $0 < p < 1$, that

$$\|f\|_p^p \le 1 + \|f-1\|_p^p \le 1 + 2^p p \Gamma(p)\Gamma(1-p),$$

whence (2.2); see [8, pp. 37 and 57] for the calculation.

In the general case we consider (G2), together with the Littlewood subordination theorem [8, p. 10; 9, p. 191; 13, II, pp. 178–179], to obtain (2.2).

Finally, if $f \in G$, then $1/f \in G$, so that f has the singular factor $\equiv 1$ by the familiar argument [8, p. 51]. This completes the proof. Q.E.D.

The celebrated Fefferman-Stein criterion for BMO functions yields that if f is holomorphic in D, then $f \in BMOA$ if and only if $f = g + ih$, where g and h are holomorphic with bounded $\operatorname{Re} g$ and $\operatorname{Re} h$ in D; see [6, Theorem A′] for example. A version of this is, therefore, that $f \in BMOA$ if and only if there exist a constant $k > 0$ and functions g, $h \in \Pi$ such that $f = k(\log g + i \log h)$. In view of the right-hand side a problem arises: $\log f \in BMOA$ if $f \in \Gamma$? We can restrict the problem, without loss of generality, to $f \in G$, and the answer is in the affirmative.

Theorem 3. *If $f \in G$, then both $\log f$ and $\log(1+f)$ are in $BMOA$. More precisely,*

$$\|\log f\|_* \le \pi/\sqrt{2} = 2.22\ldots; \tag{2.4}$$

$$\|\log(1+f)\|_* \le \sqrt{2}\pi + \log 2 = 5.13\ldots. \tag{2.5}$$

As will be soon observed, the equality in (2.4) is attained by $f = \lambda$ or $1 - z$; we have no answer for the sharpness of (2.5).

For the proof of Theorem 3 we recall the identity

$$\|\log(1-z)\|_* = \pi\sqrt{2}, \tag{2.6}$$

due to N. Danikas [7] and the one

$$\|\log(\chi(z)/z)\|_* = 2\|\log\lambda\|_* = 2\|\log\lambda\|_2, \tag{2.7}$$

where $\chi = (\lambda^2 - 1)/4$ is the Koebe function, due to D. Girela [11, p. 119] (see [12] also); actually, Girela obtained the results in terms of $BMOA_p$ norm. Combination of (2.6) and (2.7) yields

$$2\|\log\lambda\|_* = \|\log(\chi(z)/z)\|_* = \sqrt{2}\pi.$$

Girela [11, Theorems 4 and 5] found some quantitative versions of A. Baernstein's results [1] (see [6] also) which we express in our norm:

(i) If $f \in S$, then

$$\|\log(f(z)/z)\|_* \le \|\log(\chi(z)/z)\|_* .$$

(ii) If f is univalent and zero-free in D with $f(0) = 1$, then

$$\|\log f\|_* \le 2\|\log \lambda\|_* .$$

We may replace the right-hand sides of the estimates in (i) and (ii) by the constant $\sqrt{2}\pi$. The equality in (i) (in (ii)) is attained by χ (by λ^2)

Lemma 2.1. *Let f, g, and φ be holomorphic in D. Suppose that $|\varphi| < 1$ and $f = g \circ \varphi$ in D. Then*

$$\|f_w\|_2 \le \|g_{\varphi(w)}\|_2 \quad \textit{for each } w \in D. \tag{2.8}$$

In particular, if $\varphi(0) = 0$ further, or if f is subordinate to g, then we have $\|f\|_* \le \|g\|_*$ from (2.8), together with $f(0) = g(0)$.

Proof of Lemma 2.1. We remember [21, pp. 106–107] that for $F \in H^2$ and for $\zeta \in D$,

$$(|F|^2)_P(\zeta) - |F(\zeta)|^2 = \frac{2}{\pi} \iint_D \left(\log \left| \frac{1 - \bar{\zeta} z}{z - \zeta} \right| \right) |F'(z)|^2 \, dx\, dy, \tag{2.9}$$

where $z = x + iy$ and $(|F|^2)_P(\zeta)$ is the value at ζ of the Poisson integral of $|F(e^{it})|^2$, or the value at ζ of the least harmonic majorant of the subharmonic function $|F|^2$ in D. Applying (2.9) to F_ξ and $\zeta = 0$, and then making a change of variable in the right-hand side, we have

$$\|F_\xi\|_2^2 = (|F|^2)_P(\xi) - |F(\xi)|^2 . \tag{2.10}$$

Now,

$$(|f|^2)_P(w) = (|g \circ \varphi|^2)_P(w) \le (|g|^2)_P(\varphi(w))$$

by the subharmonicity of $|g|^2$. Applying (2.10) to $F = g$ and $\xi = \varphi(w)$, we then have

$$\|g_{\varphi(w)}\|_2^2 \ge (|f|^2)_P(w) - |f(w)|^2 = \|f_w\|_2^2 .$$

This completes the proof. Q.E.D.

Proof of Theorem 3. As we have observed, subordination decreases the $BMOA$ norm. We may therefore suppose, in view of (G2) that f is univalent. Then f^2 is univalent and zero-free in D with $f^2(0) = 1$. Thus (2.4) is a consequence of (ii) applied to f^2. To consider $\log(1 + f)$ again for univalent $f \in G$ we note that

$$g = 2(f - 1)/\{f'(0)(f + 1)\} \in S,$$

and further, by (G3),

$$h = (f^2 - 1)/\{2f'(0)\} \in S .$$

Therefore,

$$\log(1+f) = (1/2)\{\log(h(z)/z) - \log(g(z)/z)\} + \log 2,$$

which, combined with (i) for g and h, yields (2.5). Q.E.D.

We emphasize that each $f \in P$ is subordinate to λ, so that the estimate (2.4) for $f \in P$ is a direct consequence of Lemma 2.1.

Remark 2.1. We shall prove that if f is holomorphic and zero-free in D and if $\log f \in BMOA$, then we have a constant $k > 0$ and functions g, $h \in P$ such that

$$f = f(0) g^k h^{ki} \quad \text{in } D. \tag{2.11}$$

In particular, if $f \in G$, then we have (2.11) with $f(0) = 1$ by Theorem 3. For the proof of (2.11) we first observe that

$$\log f = g_1 + ih_1 + \log f(0),$$

where g_1 and h_1 are holomorphic with bounded $\operatorname{Re} g_1$ and $\operatorname{Re} h_1$ in D, and further, $g_1(0) = h_1(0) = 0$. Then, there exist $k > 0$ and g, $h \in P$ such that $g_1 + ih_1 = k(\log g + i \log h)$. We thus have (2.11). See problems (8) and (9) in §6.

3. Proofs of Theorem 1 and Corollary A

Lemma 3.1. *If $f \in G$, then*

$$M_2(r, f) \le \sqrt{2}/(1-r)^{1/2}, \qquad 0 \le r < 1.$$

Proof. By subordination, we may suppose that f is univalent. For $g = (f^2 - 1)/\{2f'(0)\} \in S$ we have [9, p. 38] that

$$M_1(r, g) \le r/(1-r^2),$$

which, together with (G8), yields that

$$M_2^2(r, f) \le 4M_1(r, g) + 1 \le 2/(1-r), \qquad 0 \le r < 1,$$

from which follows the estimate. Q.E.D.

Lemma 3.2 [5, Theorem 2.1]. *If f is holomorphic in D with*

$$M_2(r, f') = O(1/(1-r)^{1/2}) \quad \textit{as } r \to 1,$$

then $f \in BMOA$.

Theorem 1 now follows from Lemmata 3.1 and 3.2.

Proof of Corollary A. If g is holomorphic in D with $\operatorname{Re} g' \ge 0$, then either $g' \in \Pi$ or g' is an imaginary constant. Therefore, $g \in BMOA$ by Theorem 1. Since $\operatorname{Re} f'$ can be expressed as the difference of nonnegative harmonic functions in D by [8, p. 2], we have $f' = h_1 - h_2$ in D with $\operatorname{Re} h_k \ge 0$ in D,

$k = 1, 2$. Choosing g_k with $g_k' = h_k$ we know that $g_k \in BMOA$ $(k = 1, 2)$, so that $f = g_1 - g_2 + \text{constant}$, is in $BMOA$. Q.E.D.

We can extend Corollary A, for example, in the following form in terms of G. Let $L(G)$ be the family of linear combinations of functions of G, that is, $a_1 f_1 + \cdots + a_n f_n$, where a_k are complex constants, and $f_k \in G$, $1 \le k \le n$, and $n \ge 1$ is arbitrary. The extension is:

If $f' \in L(G)$, then $f \in BMOA$.

In particular, if f satisfies (1.2), then $f' \in L(G)$. Actually, $1 \in G$ and each $g \in \Pi$ can be expressed as

$$g = (\operatorname{Re} g(0)) g_1 + i \operatorname{Im} g(0),$$

with $g_1 = \{g - i \operatorname{Im} g(0)\}/\operatorname{Re} g(0) \in P$.

4. From P to G in Univalent Function Theory

Following [9, pp. 40 and 46], we denote by S^*, C and K the families of $f \in S$ which are starlike, convex, and close-to-convex in D, respectively.

We call $f \in N$ close-to-convex of order $\beta \ge 0$, $f \in K(\beta)$ $(K(\beta) = C(\beta)$ in [4]) in notation, if there exist a real constant c and $g \in C$ depending on f such that

$$|\arg\{e^{ic} f'(z)/g'(z)\}| \le \pi\beta/2, \qquad z \in D. \tag{4.1}$$

Note that $K = K(1)$.

To extend the above notion, we define G^α to be the family of g^α with $g \in G$, where α is a real constant and $g^\alpha(0) = 1$. Obviously, $G^0 = \{1\}$.

(a) *We call $f \in N$ Gelfer-starlike of exponential order α, $f \in S_G^*(\alpha)$ in notation, if $zf'(z)/f(z) \in G^\alpha$.*

(b) *We call $f \in N$ Gelfer-convex of exponential order α, $f \in C_G(\alpha)$ in notation, if $zf''(z)/f'(z) + 1 \in G^\alpha$.*

We then have $S^* \subset S_G^*(1)$ and $C \subset C_G(1)$ by $P \subset G$. The Alexander-type theorem [9, p. 43; 13, I, p. 115] holds: *$f \in N$ is in $C_G(\alpha)$ if and only if $zf'(z)$* is in $S_G^*(\alpha)$.

If $f \in S_G^*(\alpha)$, then f never vanishes in $\{0 < |z| < 1\}$, and if $f \in S_G^*(\alpha) \cup C_G(\alpha)$, then f' never vanishes in D. Trivially, $S_G^*(0) = C_G(0) = \{z\}$. See Problem (5) in §6.

(c) *We call $f \in N$ Gelfer-close-to-convex of exponential order (α, β), $f \in K_G(\alpha, \beta)$ in notation, if there exists $g \in C_G(\alpha)$ such that $f'/g' \in G^\beta$.*

The derivative f' of $f \in K_G(\alpha, \beta)$ thus never vanishes in D. Since $G = G^{-1}$, it follows that

$$S_G^*(\alpha) = S_G^*(|\alpha|), \quad C_G(\alpha) = C_G(|\alpha|), \quad K_G(\alpha, \beta) = K_G(|\alpha|, |\beta|).$$

Henceforth we shall always assume that $\alpha \ge 0$ and $\beta \ge 0$ whenever constants α and β are considered.

We note that $C_G(\alpha) = K_G(\alpha, 0)$ and $S_G^*(\alpha) \subset K_G(\alpha, \alpha)$. Actually, for $f \in S_G^*(\alpha)$ we have $g \in C_G(\alpha)$ and $h \in G$ such that

$$f(z) = zg'(z) \quad \text{and} \quad zf'(z)/f(z) = h(z)^\alpha .$$

We thus have $f'/g' = h^\alpha$ or $f \in K_G(\alpha, \alpha)$.

Most interesting for our purpose in the present section would be that

$$K(\beta) \subset K_G(1, \beta). \tag{4.2}$$

If the equality in (4.1) holds at a point $z \in D$, then $e^{ic}f'/g'$ is a constant. Since f and g are normalized, it follows that $f = g$. Thus, in particular,

$$K(0) = C \subset C_G(1) = K_G(1, 0).$$

Proof of (4.2). We may suppose therefore that $\beta > 0$ and the inequality in (4.1) is strict everywhere. Then, for $f \in K(\beta)$ we have c, $g \in C$ and $h \in \Pi$ such that

$$e^{ic}f'/g' = h^\beta, \qquad h(0)^\beta = e^{ic}.$$

Now, $g \in C_G(1)$, and for $\varphi = h/h(0) \in G$, we have $f'/g' = \varphi^\beta$, whence $f \in K_G(1, \beta)$. Q.E.D.

In view of (4.2) we observe that Corollary B is contained in the following theorem which is a consequence of Theorems 1 and 3.

Theorem 4. *If* $f \in K_G(\alpha, \beta)$ *for* $\alpha \le 1$, *then* $\log f' \in BMOA$.

Proof. We first consider $g \in C_G(\alpha)$. Then, there exists $\varphi \in G$ such that

$$(\log g'(z))' = (\varphi(z)^\alpha - 1)/z.$$

Since $M_2(r, \varphi^\alpha)/M_2(r, \varphi) = O(1)$ by $\alpha \le 1$, we have then

$$M_2(r, (\log g')')/M_2(r, \varphi) = O(1) \quad \text{as } r \to 1.$$

It follows from Theorem 1 that $\log g' \in BMOA$. Next, for $f \in K_G(\alpha, \beta)$ we choose $g \in C_G(\alpha)$ and $h \in G$ such that $f' = g'h^\beta$. Then,

$$\log f' = \log g' + \beta \log h,$$

together with Theorem 3, shows that $\log f' \in BMOA$, and this completes the proof of the theorem. Q.E.D.

Remark 4.1. Suppose that $f \in N$ satisfies $f' = g'\varphi^\beta$ for $g \in C$ and $\varphi \in G$ in D. As is seen, this is the case for $f \in K(\beta)$ in particular. Since g' is subordinate to $\chi(z)/z$ by [9, Problem 13, p. 213; 13, II, p. 187], together with the Alexander theorem, it follows that

$$\| \log g' \|_* \le \| \log(\chi(z)/z) \|_* = \sqrt{2}\pi .$$

In view of (2.4) for φ, it is now easy to have

$$\| \log f' \|_* = \| \log g' + \beta \log \varphi \|_* \le (2 + \beta)\pi/\sqrt{2}.$$

The equality holds for $f(z) = \{(1-z)^{-\beta-1}-1\}/(\beta+1)$, where $g(z) = z/(1-z)$ and $\varphi(z) = 1/(1-z)$. See the same estimate [11, Theorem 6] for the specified case $K(1) = K$.

We call $f \in N$ typically real, $f \in T$ in notation, if f has real values on the real axis and nonreal values elsewhere. Each $f \in T$ never vanishes in $\{0 < |z| < 1\}$. Actually,

$$(1 - z^2)f(z)/z \in P \quad \text{if } f \in T; \tag{4.3}$$

see [9, p. 56; 13, I, p. 185]. There exists a nonunivalent $f \in T$ [9, p. 57]. We can, however, prove that $\log(f(z)/z) \in BMOA$ if $f \in T$.

Theorem 4a. *Suppose that* $f \in N$. *If there exist* $\alpha \geq 0$, *a univalent* $g \in G$, *and* $h \in \Gamma$ *such that*

$$fg/(g^2 - 1) = h^\alpha \quad \text{in } D, \tag{4.4}$$

then $\log(f(z)/z) \in BMOA$.

If $f \in T$, then (4.3) shows that (4.4) with $g = \lambda$ and $\alpha = 1$ holds.

Proof of Theorem 4a. Since $\varphi = (g^2 - 1)/\{2g'(0)\} \in S$ and $\psi = 2g'(0)/g \in \Gamma$, it follows from (i), $\log \psi \in BMOA$, and $\log h \in BMOA$, that

$$\log(f(z)/z) = \alpha \log h(z) + \log(\varphi(z)/z) + \log \psi(z)$$

is in $BMOA$. Q.E.D.

In particular, if $f \in N$, and $(1 - z^2)f(z)/z \in G$, then

$$\| \log(f(z)/z)\|_* \leq \sqrt{2}\pi = 4.44\ldots. \tag{4.5}$$

For example, (4.5) is true for $f \in T$ by (4.3). It follows from $f(z)/z = h(z)/(1 - z^2)$, $h \in G$, that

$$\log(f(z)/z) = \log h(z) - \log(1 - z^2).$$

Since $\log(1 - z^2)$ is subordinate to $\log(1 - z)$, we have $\| \log(1 - z^2)\|_* \leq \pi/\sqrt{2}$ by (2.6). We now have (4.5) from (2.4) for h. The equality in (4.5) holds for $\chi \in T$.

Remark 4.2. We call $f \in N$ spiral-like if there exists a constant b with $|b| = 1$ and $|\arg b| < \pi/2$ such that $bzf'(z)/f(z) \in \Pi$ (see [9, p. 52; 13, I, p. 149]; we note that $\alpha \neq \pm\pi/2$ for $f \in N$ satisfying [13, I, (40), p. 148]). The family S_p of spiral-like functions contains S^* and is contained in S, yet there is no inclusion relation between S_p and K (see [9, pp. 54-55]). However, we can show that $S_p \subset S_G^*(1)$. For we set $\varphi(z) = bzf'(z)/f(z)$. Then $\varphi \in \Gamma$ and $\varphi(0) = b$, so that $zf'(z)/f(z) = \varphi/\varphi(0) \in G$, whence $f \in S_G^*(1)$.

5. Properties of S_G^*, C_G, and K_G

For a complex number b, $|b| = 1$, we set

$$\Omega(b) = \{z \neq 0; |\arg(z/b)| < \pi/2\},$$

the half-plane with the boundary $\{z\,;\,\mathrm{Re}(\bar{b}z)=0\}$. Set

$$\Delta(w\,,\,r)=\{z\,;\,|z-w|/|1-\overline{w}z|<r\}$$

for $w\in D$ and $0<r\leq 1$. This is a non-Euclidean disk with non-Euclidean center w and non-Euclidean radius $\tanh^{-1}r$, on the one hand, and a (Euclidean) disk with center $w(1-r^2)/(1-|w|^2r^2)$ and radius $r(1-|w|^2)/(1-|w|^2r^2)$, on the other hand. We begin with

Theorem 5. *For each* $f\in G$ *and for each* $w\in D$,

$$f(\Delta(w\,,\,1/\sqrt{2}))\subset\Omega(f(w)/|f(w)|)\,. \tag{5.1}$$

In particular, $\mathrm{Re}\,f>0$ *in* $\{|z|<1\sqrt{2}\}$ *and the constant* $1/\sqrt{2}$ *is sharp in this case.*

Gelfer [10, Theorem 6, 1°] proved that $\mathrm{Re}\,f>0$ in the disk $\{|z|<r_G\}$ for $f\in G$, where

$$r_G=\tanh(\pi/4)=0.65\ldots<1/\sqrt{2}=0.70\ldots\,.$$

Proof of Theorem 5. The function $g=(1-f)/(1+f)$ is a Bieberbach-Eilenberg function [9, p. 265; 13, II, p. 61] in the sense that $g(z)g(w)\neq 1$ for $z\,,\,w\in D$, and $g(0)=0$. It is known that $|g(z)|\leq |z|/(1-|z|^2)^{1/2}$ in D [16, Theorem 1]; see [9, p. 265; 13, II, p. 81]. Since

$$\mathrm{Re}\,f=(1-|g|^2)/|1+g|^2\,,$$

it follows that $\mathrm{Re}\,f(z)>0$ if and only if $|g(z)|<1$ or if $|z|<1/\sqrt{2}$. Fix $w\in D$ and then consider $f\circ\varphi/f(w)$, where $\varphi(z)=(z+w)/(1+\overline{w}z)$. Then this is in G by (G4) and hence its real part is positive for $|z|<1/\sqrt{2}$. We thus have (5.1). For the sharpness at $w=0$, we note that the Möbius transformation

$$f_1(z)=(1-\bar{a}z)/(1+az)\,,\qquad a=e^{\pi i/4}\,, \tag{5.2}$$

maps D onto $\Omega(\bar{a})$, so that $f_1\in G$. Note that a Möbius transformation ψ is in Γ if and only if ψ is pole-free in D and the image of D by ψ does not contain 0. A simple calculation now shows that $\mathrm{Re}\,f_1(i/\sqrt{2})=0$. Q.E.D.

Remark 5.1. Gelfer obtained his constant r_G by making use of the estimate (G5′). It is now easy to obtain $|\arg f(z)|\leq\alpha\log\lambda(|z|)$, $z\in D$, for $f\in G^\alpha$. The G^α version of Theorem 5 is that, *if* $f\in G^\alpha$, *then*

$$f(\Delta(w\,,\,\tanh\{\pi/(4\alpha)\}))\subset\Omega(f(w)/|f(w)|) \tag{5.1′}$$

at each $w\in D$. It is easy to show that if $\alpha\leq 1$, then $z^\alpha\in\Omega(b^\alpha)$ for each $z\in\Omega(b)$. Therefore, if $f\in G^\alpha$, $\alpha\leq 1$, then (5.1) holds again. We now observe that $\tanh\{\pi/(4\alpha)\}<1/\sqrt{2}$ if $\alpha>\alpha_0\equiv\pi/\{4\tanh^{-1}(1/\sqrt{2})\}=0.891\ldots$. Thus, (5.1) is better than (5.1′) for $\alpha_0<\alpha\leq 1$.

We call $f\in N$ starlike, convex, and close-to-convex in $\{|z|<r\}$ $(0<r\leq 1)$ if $r^{-1}f(rz)$ is in S^*, C, and K, respectively. Obviously, f is starlike (or convex) in $\{|z|<r\}$ if and only if $\mathrm{Re}\{zf'(z)/f(z)\}$ (or $\mathrm{Re}\{zf''(z)/f'(z)\}+1)>0$ there. We now have the following

Corollary. *Functions of* $S_G^*(1)$, $C_G(1)$, *and* $K_G(1, 1)$ *are starlike, convex, and close-to-convex in* $\{|z| < 1/\sqrt{2}\}$, *respectively. The constant* $1/\sqrt{2}$ *for starlikeness and convexity is sharp.*

The solution $f \in N$ of the equation

$$zf'(z)/f(z) = f_1(z) \quad \text{in } D,$$

where f_1 is in (5.2), shows the sharpness of $1/\sqrt{2}$ for starlikeness. The solution $f \in N$ of the equation

$$zf''(z)/f'(z) + 1 = f_1(z) \quad \text{in } D,$$

on the other hand, shows the sharpness of $1/\sqrt{2}$ for convexity.

Remark 5.2. In view of Remark 5.1 we have the obvious results for $S_G^*(\alpha)$, $C_G(\alpha)$, and $K_G(\alpha, \alpha)$. For example, $f \in C_G(\alpha)$ is convex in $\{|z| < \tanh[\pi/(4\alpha)]\}$. Again, in case $\alpha_0 < \alpha \leq 1$, we can replace the disk by the larger one $\{|z| < 1/\sqrt{2}\}$.

Lemma 5.1. *If* $f \in G^\alpha$, *then*

$$|f(z) - 1| \leq \lambda(|z|)^\alpha - 1, \qquad z \in D. \tag{5.3}$$

This is a generalization of (G7). The equality is attained by $f = \lambda^\alpha$ at each $z = r$, $0 \leq r < 1$.

Proof of Lemma 5.1. Differentiating $f = h^\alpha$, where $h \in G$, and then combining (G5) and (G6) for h, we have

$$|f'(z)| = |\alpha h(z)^\alpha (h'(z)/h(z))| \leq (\lambda^\alpha)'(|z|),$$

whence, by $(f-1)' = f'$, we have

$$|f(z) - 1| \leq \int_0^{|z|} (\lambda^\alpha)'(t)\,dt = \lambda(|z|)^\alpha - 1. \quad \text{Q.E.D.}$$

Corollary to Lemma 5.1. *If* $f \in K_G(\alpha, \beta)$, $\alpha \leq 1$, *then there exists* r, $0 < r \leq 1$, *such that* f *is univalent in each* $\Delta(w, r)$, $w \in D$.

Proof. If $g \in C_G(\alpha)$, $\alpha \leq 1$, then

$$g''(z)/g'(z) = (\varphi(z) - 1)/z, \qquad \varphi \in G^\alpha,$$

so that

$$(1 - |z|^2)|g''(z)/g'(z)| \leq (1 - |z|^2)\{\lambda(|z|)^\alpha - 1\}/|z|,$$

from Lemma 5.1, and the right-hand side is bounded because $\alpha \leq 1$. For $f \in K_G(\alpha, \beta)$ we have $g \in C_G(\alpha)$ and $h \in G$ such that $f' = g'h^\beta$, whence $f''/f' = g''/g' + \beta h'/h$. Therefore, (G6) for h shows that $(1-|z|^2)|f''(z)/f'(z)|$ is bounded in D. By the well-known fact (see [19; 20, Theorem 2], for example), we have the corollary.

Lemma 5.1 will be of use to consider the convexity of $f \in K_G(\alpha, \beta)$. We need a second preparation. The function

$$\sigma(x) = \lambda(x)^\alpha + 2\beta x/(1-x^2)$$

of x, $0 \leq x < 1$, increases from 1 to $+\infty$ as x increases from 0 to 1, except for the trivial case $\alpha = \beta = 0$, namely, $\sigma(x) \equiv 1$. Therefore we have only one value $c = c(\alpha, \beta) > 0$ such that $\sigma(c) = 2$ for $(\alpha, \beta) \neq (0, 0)$; we set $c(0, 0) = 1$. A calculation yields that

$$3c(1, \beta) = (\beta^2 + 2\beta + 4)^{1/2} - \beta - 1,$$

for example.

Theorem 6. *Each* $f \in K_G(\alpha, \beta)$ *is convex in* $\{|z| < c(\alpha, \beta)\}$.

We do not know the sharpness of $c(\alpha, \beta)$ except for the trivial case $\alpha = \beta = 0$. However, there is a reason that $c(\alpha, \beta)$ is not so bad. The radius of convexity of S is the same as that of K and is $2 - \sqrt{3} = 0.267\ldots$; see [13, II, p. 89]. On the other hand, $K \subset K_G(1, 1)$, and the radius of convexity for $K_G(1, 1)$ is not less than $c(1, 1) = (\sqrt{7} - 2)/3 = 0.215\ldots$.

Proof of Theorem 6. We have $f'/g' = h^\beta$, where $g \in C_G(\alpha)$ and $h \in G$. It follows from (G6) for h that

$$\begin{aligned} |zf''(z)/f'(z) - zg''(z)/g'(z)| &= \beta|zh'(z)/h(z)| \\ &\leq 2\beta|z|/(1-|z|^2), \qquad z \in D. \end{aligned}$$

On the other hand, (5.3) yields that

$$|zg''(z)/g'(z)| = |(zg''(z)/g'(z) + 1) - 1| \leq \lambda(|z|)^\alpha - 1, \qquad z \in D.$$

We thus obtain $\mathrm{Re}\{zf''(z)/f'(z)\} + 1 > 0$ for $|z| < c(\alpha, \beta)$ from $|zf''(z)/f'(z)| \leq \sigma(|z|) - 1 < 1$ for $|z| < c(\alpha, \beta)$. Q.E.D.

An important example of $f \in S$ is $f \in N$ such that $f' \in \Pi$; in particular, $f \in K$ by $f'(z)/z' = f' \in \Pi$. We can now easily extend T. H. MacGregor's results [18, Theorems 2 and 3].

(I) *If* $f \in N$ *and* $f' \in G^\alpha$, *then* f *is convex in* $\{|z| < r_\alpha\}$, $r_\alpha = (\alpha^2 + 1)^{1/2} - \alpha$. *If* $f \in N$ *and* $f(z)/z \in G^\alpha$, *then* f *is starlike in* $\{|z| < r_\alpha\}$.

Setting $\alpha = 1$ and replacing G^α by P, we have the MacGregor theorems. In each case in (I) r_α is the best possible. Let $f_2 \in N$ satisfy $f_2' = 1/\lambda^\alpha$ in D, and set $f_3(z) = zf_2'(z)$ in D. Then, $f_3(z)/z = 1/\lambda(z)^\alpha$ and

$$\mathrm{Re}\{zf_2''(z)/f_2'(z)\} + 1 = \mathrm{Re}\{zf_3'(z)/f_3(z)\} = 0 \quad \text{at } z = r_\alpha.$$

For the proof of the first part of (I) we have from (G6) that

$$|zf''(z)/f'(z)| \leq 2\alpha|z|/(1-|z|^2) < 1 \quad \text{for } |z| < r_\alpha.$$

For the second part we let $g \in N$ satisfy $g'(z) = f(z)/z$ in D. Then g is convex in $|z| < r_\alpha$ by the first part, and then $f(z) = zg'(z)$ is starlike in $|z| < r_\alpha$.

Remark 5.3. We can show that if $f \in N$ and $f' \in G^\alpha$, then $f \in K_G(0, \alpha)$. Actually, $z \in C_G(0)$ and $f'(z)/z' = f'(z) \in G^\alpha$.

We next show

(II) *If f is holomorphic in D and $f' \in \Gamma$, then f is univalent in each $\Delta(w, 1/\sqrt{2})$, $w \in D$.*

The sharpness of $1/\sqrt{2}$ is open. It is well known that if g is holomorphic in a convex domain Δ in the plane, and if there exists $\Omega(b)$ such that $g'(\Delta) \subset \Omega(b)$, then g is univalent in Δ; see [9, p. 47; 13, I, p. 88]. since $f'/f'(0) \in G$ it follows from Theorem 5 that

$$f'(\Delta(w, 1/\sqrt{2})) \subset \Omega(f'(w)/|f'(w)|),$$

whence (II).

The obvious version of (II) for $f' \in G^\alpha$ in view of Remark 5.1 is left as an exercise. Finally we note

(III) *If f is holomorphic in D and $f' \in G^\alpha$ for $\alpha \le 1/2$, then f is univalent in D.*

Since $f' = h^\alpha$ for an $h \in G$, it follows from (G6) for h that

$$(1 - |z|^2)|f''(z)/f'(z)| \le 2\alpha \le 1, \qquad z \in D.$$

It then follows from J. Becker's theorem [2, Theorem 4.1, p. 35] that f is univalent in the whole D. Again, the sharpness of $1/2$ is open.

6. Problems

Some open problems are summarized here.

(1) *It is true that*

$$\|F_\xi\|_q^q = (|F|^q)_P(\xi) - |F(\xi)|^q, \qquad \xi \in D,$$

for $F \in H^q$, $0 < q < \infty$? Again $(\cdot)_P$ denotes the Poisson integral of $|F(e^{it})|^q$. The case $q = 2$ is observed in (2.10).

(2) *What is the exact value of $\|\log(1 - z)\|_{BMOA_p}$?* Here $\|f\|_{BMOA_p}$ is defined similarly as $\|f\|_*$ by replacing $\|f_w\|_2$ by $\|f_w\|_p$, $0 < p < \infty$. Danikas's result is $\pi/\sqrt{2}$ for $p = 2$.

(3) *Is it true that $G^\alpha \subset G^\beta$ if $\alpha \le \beta$?* Or, equivalently, *is it true that $G^\alpha \subset G$ if $\alpha \le 1$?* The latter problem has the positive answer if $\alpha = 1/n$ for natural numbers n [13, II, p. 83].

(4) *What are geometric meanings of*

$$zf'(z)/f(z) \in G^\alpha, \qquad zf''(z)/f'(z) + 1 \in G^\alpha, \quad \textit{and} \quad f'/g' \in G^\alpha$$

for $g \in C_G(\alpha)$, respectively?

(5) *Is $C_G(\alpha)$ a subfamily of $S_G^*(\alpha)$?* This is trivial for $\alpha = 0$.

(6) *Are the coefficients of*

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n \in G$$

bounded? The answer is in the positive if f is univalent further [10, Theorem 9]. Gelfer's result is now improved as follows.

$$(6.1) \qquad |a_n| \le 2.54\dots \qquad (n \ge 1).$$

First $|a_1| \le 2$ is obvious. J. A. Hummel [15] proved that $|a_2| \le 2.0001\dots$. A. Z. Grinshpan [14] proved that

$$|a_n| \le 2e^{\delta/2}e^{1/(4n)} \qquad (n \ge 1),$$

where δ is the Milin constant [9, pp. 153 and 154]; see the first inequality in [14, p. 13] in Russian. The constant δ is defined by

$$2\delta = \sum_{m=1}^{\infty} (\log 2)^m/(m!m) - \log\log 2 - \gamma = 0.6237\dots,$$

where γ is the Euler constant. Therefore, (6.1).

(7) *Find the exact value* r *in Corollary to Lemma* 5.1.

(8) *Find a condition for a trio* g, h, k, *where* g, $h \in P$ *and* $k > 0$ *is a constant, such that* $f = g^k h^{ki} \in G$. Obviously, for $g \in P$, $h \equiv 1$, $k \le 1$, we have $f \in G$.

If $f \in S$, then $\log(f(z)/z) \in BMOA$, so that, by Remark 2.1, we have $k > 0$ and g, $h \in P$ such that

$$(6.2) \qquad f(z) = zg(z)^k h(z)^{ki}.$$

The problem is on the converse.

(9) *Given* $k > 0$, $g \in P$, *and* $h \in P$ *is* f *defined by* (6.2) *a member of* S? Some trials are added: For $k > 0$, $g = \lambda$ and $h \equiv 1$, the function f in (6.2) is not in S because

$$f'(k - (k^2 + 1)^{1/2}) = 0.$$

Apparently, $\chi(z) = z\mu(z)^2$, where $\mu(z) = 1/(1-z) \in P$. However, for $k > 1$ and $g = h = \mu$ we have for f in (6.2) that

$$f'(-1/(k - 1 + ki)) = 0,$$

so that f is not in S. If $k = 1$, then $k + ki = 2e^{i\pi/4}\cos(\pi/4)$, so that $f(z) = z\mu(z)^{1+i}$ is $(\pi/4)$-spirallike [9, p. 55]. In particular, f is in S.

(10) Suppose that $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ are in G. *Is*

$$h(z) = 1 + \frac{1}{2}\sum_{n=1}^{\infty} a_n b_n z^n$$

in G? The corresponding problem to P instead of G is positively answered [9, p. 273; 13, I, pp. 135–136].

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