

HYPERBOLICITY PROPERTIES OF C^2 MULTIMODAL COLLET-ECKMANN MAPS WITHOUT SCHWARZIAN DERIVATIVE ASSUMPTIONS

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ABSTRACT. In this paper we study the dynamical properties of general C^2 maps $f: [0, 1] \rightarrow [0, 1]$ with quadratic critical points (and not necessarily unimodal). We will show that if such maps satisfy the well-known Collet-Eckmann conditions then one has

- (a) hyperbolicity on the set of periodic points;
- (b) nonexistence of wandering intervals;
- (c) sensitivity on initial conditions; and
- (d) exponential decay of branches (intervals of monotonicity) of f^n as $n \rightarrow \infty$;

For these results we will not make any assumptions on the Schwarzian derivative f . We will also give an estimate of the return-time of points that start near critical points.

1. INTRODUCTION AND NOTATION

In the last decade quite a number of results were obtained for iterations of one-dimensional maps $f: [0, 1] \rightarrow [0, 1]$. Most of these results were only proved under the assumption that f has *negative Schwarzian derivative*, $Sf(x) < 0$ for all $x \in [0, 1]$, where

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \cdot \left(\frac{f''(x)}{f'(x)} \right)^2,$$

and that f is *unimodal*, i.e., has only one extremum. For example, P. Collet and J.-P. Eckmann showed that such maps which have positive Liapounov exponents on the forward and backward orbit of the critical point (see conditions (CE1) and (CE2) below) have an absolutely continuous invariant measure (see [CE1, CE2]). Moreover the first author showed that such maps have many hyperbolicity properties. In fact he showed that condition (CE1) implies condition (CE2) (see [No1–No3]).

However the condition on the Schwarzian derivative of f is quite strong (it implies that $|f'|$ cannot have positive local minima) and also it is unnatural (it is not invariant under coordinate changes). In this and a companion paper

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it will be shown that these conditions on the Schwarzian derivative are entirely superfluous. In this paper we will show that C^2 maps with quadratic critical points (this notion is defined below) and an arbitrary number of critical points satisfying conditions (CE1) and (CE2) have many hyperbolicity properties. In [Str2] the second author has developed similar properties for Misiurewicz maps. For these maps one strengthens condition (CE1) and drops condition (CE2). Condition (CE2) gives good control on all backward iterates of the critical points and we do not know whether the results presented here also hold if we just assume condition (CE1).

The results from this paper were proved before for unimodal maps with $Sf < 0$ in [No1, Gu]. For maps without conditions on the Schwarzian derivative of f and for multimodal maps (even when $Sf < 0$) these results are new.

In a companion paper we use the results from this paper to construct invariant measures for general unimodal C^2 maps satisfying the Collet-Eckmann conditions (see [NS]).

For later use we will introduce the following notation. We denote $f^n(z)$ by z_n . In order not to have to worry about the ordering of x and y in $[0, 1]$ we let $[x, y]$ denote the smallest interval containing x and y . Similarly (x, y) denotes the interior of $[x, y]$. The length of an interval J is denoted by $|J|$.

2. STATEMENT OF RESULTS

Let $f: [0, 1] \rightarrow [0, 1]$ be a C^1 map. We say that c is a critical point of f if $f'(c) = 0$. We denote the set of critical points by $C(f)$. We say that f is *quadratic* in a critical point c if f is even C^3 near c and $D^2f(c) \neq 0$. Notice that if f is quadratic at every critical point then $C(f)$ consists of at most a finite number of points.

We say that f satisfies the Collet-Eckmann conditions if there exists $\lambda > 1$ and $K > 0$ such that for all $n \geq 1$ one has the following conditions for each $c \in C(f)$:

$$(CE1) \quad |Df^n(c)| \geq K \cdot \lambda^n,$$

$$(CE2) \quad f^n(x) \in C(f) \Rightarrow |Df^n(x)| \geq K \cdot \lambda^n.$$

We say that an interval I_n is a *branch* of f^n if I_n is a maximal interval such that $f^n|_{I_n}$ is a diffeomorphism. A periodic point p of period n is *hyperbolically repelling* (resp. *hyperbolically attracting*) if $|Df^n(p)| > 1$ (resp. $|Df^n(p)| < 1$). If $|Df^n(p)| = 1$ then p is called *nonhyperbolic*.

Theorem A. *Let $f: [0, 1] \rightarrow [0, 1]$ be a C^2 map with all critical points quadratic and satisfying (CE1) and (CE2). Then there exists a constant $K' > 0$ and $\lambda' > 1$ such that:*

(a) *For any periodic point of (minimal) period s*

$$|Df^s(p)| \geq K' \cdot \lambda'^s.$$

In particular, there exists N such that all periodic orbits with (minimal) period $n \geq N$ are repelling.

(b) f has no wandering intervals, i.e., there is no interval T such that $f^i(T) \cap f^j(T) \neq \emptyset$ for $i \neq j$ and such that no point of T is contained in the basin of wandering intervals.¹

(c) f is globally expanding, i.e., for any $n \geq 1$ and any branch I of f^n such that every fixed point of $f^n|_{I_n}$ is repelling one has

$$|I| \leq \frac{1}{K'} \cdot \left(\frac{1}{\lambda'} \right)^n.$$

If $f^n(I) \cap C(f) \neq \emptyset$ then one even has

$$\frac{|f^n(I)|}{|I|} \geq K' \cdot (\lambda')^n.$$

(d) If all periodic orbits of f are hyperbolically repelling orbits then f has sensitivity on initial conditions, i.e., there exists $\varepsilon > 0$ such that for each interval I there exists $n < \infty$ such that $|f^n(I)| \geq \varepsilon$.

In [Str2] the second author has shown similar results for C^2 maps satisfying the so-called Misiurewicz conditions

$$(Mis) \quad \bigcup_{j \geq 1} \overline{f^j(C(f))} \cap C(f) = \emptyset,$$

such that f has no periodic attractors. In fact from the proof given there it follows that these maps satisfy the Collet-Eckmann conditions (CE1) and (CE2). That these maps satisfy (CE1) follows already from Mané's result in [Ma]. That (CE2) also holds is more difficult to prove since for this one needs to consider points whose orbit may come close to $C(f)$ where one has big nonlinearity. For maps for which $Sf < 0$ these results were already proved in [CE1 and Mi] (they are much easier to prove if $Sf < 0$ since then $Sf^n < 0$ and one gets a good control for f^n).

For unimodal maps with $Sf < 0$ the first author has shown that (CE1) implies (CE2) (see [No2, No3]). For general maps we do not know whether this implication is true.

From Theorem A one immediately gets the following corollary.

Corollary. *Let $f: [0, 1] \rightarrow [0, 1]$ be a C^2 map with all critical points quadratic and satisfying (CE1) and (CE2). If all periodic points of f are hyperbolic and repelling, then there exists $M < \infty$ such that for any interval such that $f^n|_{I_n}$ is a diffeomorphism one has*

$$\sum_{i=0}^{n-1} |f^i(I_n)| \leq M.$$

¹Since this paper was written this has been proved for general C^2 -maps without flat critical points, see Lyubich and Blokh [Ergodic Theory and Dynamical Systems 9 (1989), 737–758] and Martens, de Melo and van Strien [preprint].

In a companion paper we use this corollary to prove the existence of invariant measures of positive entropy for unimodal Collet-Eckmann maps (see [NS]).

Moreover we will prove the following result about the return-time near the critical point. This theorem generalises Proposition (6.1) of [No1].

Theorem B. *Let $f: [0, 1] \rightarrow [0, 1]$ be some C^2 map with a finite number of critical points, $\#C(f) < \infty$. Then there exists $K_1 > 0$ such that for any $\varepsilon > 0$, if there exists $x \in [0, 1]$ and $c \in C(f)$ with $|x - c|, |x_n - c| < \varepsilon$, then $n < K_1 \cdot \log(1/\varepsilon)$ implies that x and c are contained in the basin of some periodic attractor in $[c - 2\varepsilon, c + 2\varepsilon]$.*

3. CROSS-RATIO OPERATORS

3.1 Lemma. *Let f be a C^2 map with only quadratic critical points. Then there exists $C < \infty$ such that for each $n > 0$ and each interval $I = (x, y)$ so that $f^n|I$ is a diffeomorphism one has*

$$(3.1) \quad |x_n - y_n| \geq (Df^n(x)Df^n(y))^{1/2} \cdot \exp\left(-C \cdot \sum_{i=0}^{n-1} |f^i(I)|\right) \cdot |x - y|.$$

Moreover for each $w \in (x, y)$

$$(3.2) \quad |Df^n(w)| \geq \min(|Df^n(x)|, |Df^n(y)|) \cdot \exp\left(-3 \cdot C \cdot \sum_{i=0}^{n-1} |f^i(I)|\right).$$

Finally for each $\tau > 0$ there exists $\kappa > 0$ and $\rho < \infty$ such that if

$$\frac{|f^n(x, w)|}{|f^n(x, y)|}, \frac{|f^n(w, y)|}{|f^n(x, y)|} \geq \tau,$$

then

$$(3.3) \quad |Df^n(w)| \geq \kappa \cdot \max(|Df^n(x)|, |Df^n(y)|) \cdot \exp\left(-\rho \cdot C \cdot \sum_{i=0}^{n-1} |f^i(I)|\right).$$

Proof. The proof of the first two inequalities in the lemma can be found in [MS]. The proof is a simplified version of the proof of Lemma 1.4 of [MS]. Notice that in [MS] f is assumed to be C^3 and in that case we get an improved statement: the summation terms in (3.1) and (3.2) are, in that case, of the form $\sum_{i=0}^{n-1} |f^i(I)|^2$ (this gives a sharper estimate). In this paper we do not need this improvement. Equation (3.3) follows from the Koebe inequality proved in [Str]. \square

Equation (3.1) is called the ‘square-root’ lemma and equation (3.3) we shall refer to as the ‘Koebe inequality’.

4. (CE2) IMPLIES THE UNIFORM HYPERBOLICITY ON PERIODIC POINTS

In this section we will prove that uniform hyperbolicity on the set of periodic points follows from the second condition of Collet and Eckmann.

4.1 Lemma. Let $f: [0, 1] \rightarrow [0, 1]$ be an arbitrary C^2 mapping with all critical points quadratic. Assume that condition (CE2) is satisfied, i.e., that there exists $K > 0$ and $\lambda > 1$ such that for any $n \geq 1$

$$(CE2) \quad z_n \in C(f) \Rightarrow |Df^n(z)| \geq K \cdot \lambda^n.$$

Then there exists $K' > 0$ such that for any periodic point of (minimal) period s one has

$$(4.1) \quad |Df^s(p)| \geq K' \cdot \lambda^s.$$

Corollary. Under the assumptions of Lemma 4.1 the (minimal) period of periodic points which are attracting or nonhyperbolic is uniformly bounded.

Proof. The proof follows immediately from equation (4.1). \square

Proof of Lemma 4.1. Notice that (CE2) implies that none of the critical points are periodic. Since f has only a finite number of critical points it suffices to consider only periodic points which do not contain some critical point in their immediate basin. (If condition (CE1) is satisfied then no periodic attractor can contain a critical point in its basin.)

So let p be a periodic point of (minimal) period s not having a critical point in its immediate basin. Let $k = s$ if f^s is orientation preserving near p and $k = 2s$ otherwise (so that f^k is orientation preserving in any case). Let (α, β) be the maximal interval containing p for which f^k is a diffeomorphism.

Case 1. First assume that either (α, p) or (p, β) (or both) contains no fixed points of f^k . We will show that

$$(4.2) \quad |Df^k(p)| \geq \lambda^k.$$

For definiteness assume the former holds. Then there exists $r \geq 0$ with $\alpha_r \in C(f)$. Since p does not contain $\alpha_r \in C(f)$ in its basin, we can choose a sequence of points x_{-i} in the backward orbit of α , such that $x_{-i} \rightarrow p$ as $i \rightarrow \infty$, by taking recursively $x_{-(i+1)} = f^{-k}(x_i) \cap (\alpha, p)$, $x_0 = \alpha$, and $f^k(x_{-(i+1)}) = x_i$.

Assume by contradiction that $|Df^k(p)|^{1/k} \leq \lambda_1 < \lambda$. Then there exists a neighbourhood V of p such that

$$|Df^k(x)|^{1/k} \leq (\lambda + \lambda_1)/2$$

for all $x \in V$. For some $N < \infty$, $x_{-j} \in V$ for $j \geq N$. Hence, for all $i \geq 0$,

$$\begin{aligned} |Df^{ki+r}(x_{-i})| &= |Df^r(\alpha)| \cdot |Df^{ki}(x_{-i})| \\ &= |Df^r(\alpha)| \cdot |Df^{kN}(x_{-N})| \cdot \left(\prod_{j=N+1}^i |Df^k(x_{-j})| \right) \\ &\leq |Df^r(\alpha)| \cdot |Df^{kN}(x_{-N})| \cdot ((\lambda + \lambda_1)/2)^{k(i-N)}. \end{aligned}$$

It follows that there exists $K_0 > 0$ such that for all $i \geq 0$,

$$(4.3) \quad |Df^{ki+r}(x_{-i})| \leq K_0 \cdot ((\lambda + \lambda_1)/2)^{ki+r}.$$

Since $f^{ki+r}(x_{-i}) = f^r(x_0) = f^r(\alpha) \in C(f)$ one gets from (CE2)

$$(4.4) \quad |Df^{ki+r}(x_{-i})| \geq K \cdot \lambda^{ki+r}.$$

Since $\lambda_1 < \lambda$ statements (4.3) and (4.4) contradict each other. The result follows in this special case. (Notice that in this case we only used that f is C^1 .)

Case 2. If we are not in Case 1 then both (α, p) and (p, β) contain fixed points of f^k . Therefore we can choose fixed points p_1 and p_2 of f^k such that (α, p_1) and (p_2, β) do not contain fixed points of f^k . From Case 1 one has

$$(4.5) \quad |Df^k(p_1)|, |Df^k(p_2)| \geq \lambda^k.$$

Denote $I = (p_1, p_2)$. Since k is at most twice the period of p one has each point of $[0, 1]$ is contained in at most two of the intervals $f^i(I)$, $0 \leq i \leq k-1$, and therefore $\sum_{i=0}^{k-1} |f^i(I)| \leq 2$. From inequality (3.2), in Lemma 3.1, one gets a constant $C < \infty$ (which is independent of k) such that

$$(4.6) \quad |Df^k(p)| \geq \min(|Df^k(p_1)|, |Df^k(p_2)|) \cdot \exp(-3 \cdot C \cdot 2).$$

Inequalities (4.5) and (4.6) finish the proof of (4.1) and the lemma. \square

5. SUBEXPONENTIAL GROWTH OF ORBITS OF BRANCHES

Definition. As usual we say that T is a *wandering interval* if (i) $f^i(T) \cap f^j(T) = \emptyset$ for $i \neq j$ and $i, j \geq 0$ and (ii) no point of T is contained in basins of periodic attractors. Similarly we say that T is a *homterval* if $f^n|_T$ is a homeomorphism for all $f \geq 0$.

It is well known that every homterval I is either a wandering interval or each point of I is either eventually periodic or contained in the basin of a periodic attractor (see, for example, [MS or Str1]). Vice versa, if I is a wandering interval of f , then from the disjointness of $f^i(I)$ for some $i_0 \geq 0$, $f^i(I) \cap C(f) = \emptyset \forall i \geq i_0$. So some iterate of a wandering interval is a homterval.

5.1 Lemma. Let $f: [0, 1] \rightarrow [0, 1]$ be some continuous map. Then for any $\delta > 0$ there exists a constant $M_\delta < \infty$ such that for any interval I_n such that $f^n|_{I_n}$ is a diffeomorphism and such that $f^n(I_n)$ does not contain a periodic attractor or a nonhyperbolic periodic point one has

$$(5.1) \quad \sum_{i=0}^{n-1} |f^i(I_n)| \leq M_\delta + n \cdot \delta.$$

Proof. Step 1. First notice that if T is a wandering interval then the disjointness of the forward orbit of T implies $\sum_{i=0}^{n-1} |f^i(T)| \leq 1$. Now let $\tilde{N} < \infty$ be such that $\delta \cdot \tilde{N} > 3$. Then choose $\tilde{\delta} > 0$ so that if J is an interval with $|J| \leq \tilde{\delta}$ then $|f^i(J)| \leq \frac{1}{6}\delta$ for all $i = 0, \dots, \tilde{N} - 1$.

Step 2. We will show that there exists $N < \infty$ such that if $0 \leq k \leq n - N$ and $|f^k(I_n)| \geq \frac{1}{3}\delta$ then there exists a wandering interval $T \subset f^k(I_n)$ such that each of the two components of $f^k(I_n) \setminus T$ has length $\leq \tilde{\delta}$.

Let us prove this claim by contradiction. Let $I_{n(i)}$ be a sequence of branches for $f^{n(i)}$, $n(i) \rightarrow \infty$, $k(i) \leq n(i) - N(i)$, and $N(i) \rightarrow \infty$ such that $f^{N(i)}|_{f^{k(i)}(I_{n(i)})}$ is a diffeomorphism, $f^{n(i)}(I_{n(i)})$ contains no periodic attractors, and such that $f^{k(i)}(I_{n(i)})$ has length $\geq \frac{1}{3} \cdot \delta$ and contains no wandering interval of length $\geq |f^{k(i)}(I_{n(i)})| - \tilde{\delta} > \frac{1}{3} \cdot \delta - \tilde{\delta} \geq \tilde{\delta} > 0$. Then one can find an interval L contained in a subsequence of $f^{k(i)}(I_{n(i)})$ of length $\geq \frac{1}{2}\tilde{\delta}$ which is not a wandering interval and such that $f^k|_L$ is a diffeomorphism for all $k \geq 0$. Hence L is a homterval and therefore, since L is not a wandering interval (using the observation above the statement of the lemma), all points of L are either eventually periodic or contained in the basin of periodic attractors. Since $L \subset f^{k(i)}(I_{n(i)})$ for infinitely many i 's, $n(i) - k(i) \rightarrow \infty$, and $f^{n(i)}(I_{n(i)})$ contains no attracting periodic orbits or nonhyperbolic periodic points, the claim in Step 2 follows by contradiction.

Step 3. Let $N_1 = \min\{N, \tilde{N}\}$. By choosing $M_\delta = N_1$, inequality (5.1) follows from

$$(5.2) \quad \sum_{i=0}^{n-N_1-1} |f^i(I_n)| \leq (n - N_1) \cdot \delta.$$

So it suffices to prove (5.2). Let $k(1)$ be the smallest k , $0 \leq k \leq n - N_1$, such that $|f^k(I_n)| \geq \frac{1}{3}\delta$, or if no such $k(1)$ exists set $k(1) = n - N_1$. Clearly

$$(5.3) \quad \sum_{i=0}^{k(1)-1} |f^i(I_n)| \leq k(1) \cdot \frac{1}{3}\delta < k(1) \cdot \delta.$$

So if $k(1) = n - N_1$ then we have proved (5.2) and we are finished.

If $k(1) < n - N_1$ then we continue. As in Step 2, let T be the homterval contained in $I_n^1 = f^{k(1)}(I_n)$ such that each of the components of $I_n^1 \setminus T$ has length $\leq \tilde{\delta}$. Let $k(2)$ be the smallest number k , $k(1) \leq k \leq n - N_1$, such that

$$(5.4) \quad \text{the length of each of the two components of } f^{k-k(1)}(I_n^1 \setminus T) \text{ is } \geq \frac{1}{3}\delta.$$

If no such $k(2)$ exists let $k(2) = n - N_1$. From $I_n^1 = f^{k(1)}(I_n)$, statement (5.4),

and the disjointness of the orbit of T one has

$$\begin{aligned}
 \sum_{i=k(1)}^{k(2)-1} |f^i(I_n)| &\leq \sum_{i=0}^{k(2)-k(1)-1} |f^i(I_n^1)| \\
 (5.5) \qquad &\leq \sum_{i=0}^{k(2)-k(1)-1} (|f^i(T)| + |f^i(I_n^1 \setminus T)|) \\
 &\leq 1 + 2 \cdot (k(2) - k(1)) \cdot \frac{1}{3}\delta \\
 &\leq \frac{1}{3}\delta \cdot \tilde{N} + 2 \cdot (k(2) - k(1)) \cdot \frac{1}{3}\delta.
 \end{aligned}$$

From Step 1 and from the choice of $\tilde{\delta}$ and N_1 we get $k(2) - k(1) > \tilde{N}$ and we have that

$$(5.6) \qquad \sum_{i=k(1)}^{k(2)-1} |f^i(I_n)| \leq (k(2) - k(1)) \cdot \delta.$$

So if $k(2) = n - N_1$ then (5.2) follows and we are finished.

Otherwise let $I_n^2 = f^{k(2)}(I_n)$ and we can find a homterval T in I_n^2 as before. Then define $k(3)$ as before. In this way we prove (5.2) by induction and complete the proof of Lemma 5.1. \square

6. THE EXPONENTIAL DECAY OF BRANCHES OF f^n

In this section we will prove that the length of branches of f^n goes down exponentially to zero as $n \rightarrow \infty$. We will do this step by step in several lemmas.

6.1 Lemma. *Let $f: [0, 1] \rightarrow [0, 1]$ be a C^2 map whose critical points are all quadratic and satisfy (CE1) and (CE2). (We do not assume that f is unimodal.) Then there exist constants $A_1 > 0$ and $\lambda' > 1$ such that for each $r \geq 0$ and each interval (a, b) such that (a_r, b_r) does not contain nonrepelling points, and so that $f^r|(a, b)$ is a diffeomorphism, $a \in C(f)$ and $b_r \in C(f)$, one has*

$$(6.1) \qquad \frac{|b_r - a_r|}{|a - b|} \geq A_1 \cdot (\lambda')^r.$$

Proof. Let $0 < \theta_1 < \theta_2 < \infty$ be so that for every $c \in C(f)$ and all $x \in [0, 1]$ with $f|(z, c)$ monotone

$$(6.2) \qquad \theta_1 \cdot |z - c|^2 \leq |f(z) - f(c)| \leq \theta_2 \cdot |z - c|^2.$$

Choose $\tau = \min\{\theta_1/(4 \cdot \theta_2), \frac{1}{2}\}$. Let $\kappa > 0$ and $\rho < \infty$ be the numbers corresponding to this choice of τ from Lemma 3.1.

Let $\chi < 1$ be so close to 1 that

$$(6.3) \qquad \chi^2 \cdot \lambda > \lambda^{1/2}.$$

From Lemma 5.1 it follows that there exists a constant $m_0 > 0$ such that

$$(6.4) \quad \exp \left(-\rho \cdot \sum_{j=0}^n |f^j(I_n)| \right) \geq m_0 \cdot \chi^n$$

for any $n \geq 0$ and any f^n branch I_n such that $f^n(I_n)$ does not contain nonrepelling periodic points.

Let $u \in (a, b)$ be so that

$$(6.5) \quad |a_r - u_r| = |u_r - b_r|.$$

Then, since $|(a_r - u_r)|/|(a_r - b_r)| = \frac{1}{2} \geq \tau$ we can apply the Koebe inequality (3.3) to $f^r|_{[a, b]}$ and using (6.4) one gets

$$|Df^r(u)| \geq \kappa \cdot m_0 \cdot \chi^r \cdot |Df^r(b)|.$$

Using the ‘square-root’ lemma and this last inequality one gets

$$(6.6) \quad \begin{aligned} \frac{|u_r - b_r|}{|u - b|} &\geq m_0 \cdot \chi^r \cdot (Df^r(u) \cdot Df^r(b)) \\ &\geq m_0^{3/2} \cdot \kappa^{1/2} \cdot \chi^{(3/2)r} \cdot |Df^r(b)|. \end{aligned}$$

Using $b_r \in C(f)$, (CE2), and (6.3) gives a constant $A_2 > 0$ such that

$$(6.7) \quad \frac{|u_r - b_r|}{|u - b|} \geq A_2 \cdot (\lambda^{1/2})^r.$$

Moreover, since $b_r \in C(f)$ and $a \in C(f)$ one has from equation (6.2) and the ‘square-root’ lemma

$$(6.8) \quad \begin{aligned} |a_r - b_r|^2 &\geq \frac{1}{\theta_2} \cdot |a_{r+1} - b_{r+1}| \geq \frac{1}{\theta_2} \cdot |a_{r+1} - u_{r+1}| \\ &\geq \frac{1}{\theta_2} \cdot m_0 \cdot \chi^r \cdot (Df^r(a_1) Df^r(u_1))^{1/2} \cdot |a_1 - u_1| \\ &\geq \frac{\theta_1}{\theta_2} \cdot m_0 \cdot \chi^r \cdot (Df^r(a_1) Df^r(u_1))^{1/2} \cdot |a - u|^2. \end{aligned}$$

Now, from (6.5) and (6.2),

$$\frac{|b_{r+1} - u_{r+1}|}{|b_{r+1} - a_{r+1}|} \geq \frac{\theta_1}{4\theta_2}.$$

So we can apply the Koebe inequality (3.3) on $f^r|_{(a_1, b_1)}$ with τ as above and using (6.4) we get

$$(6.9) \quad |Df^r(u_1)| \geq \kappa \cdot m_0 \cdot \chi^r \cdot |Df^r(a_1)|.$$

Together with (6.8) this gives

$$(6.10) \quad |a_r - b_r|^2 \geq \frac{\theta_1}{\theta_2} \cdot m_0 \cdot \chi^r \cdot (\kappa \cdot m_0 \cdot \chi^r)^{1/2} \cdot |Df^r(a_1)| \cdot |a - u|^2.$$

This, $a_1 \in f(C(f))$, (CE1), and (6.3) give a constant $A_3 > 0$ such that

$$(6.11) \quad \frac{|a_r - b_r|}{|a - u|} \geq A_3(\lambda^{1/4})^r.$$

Statements (6.7) and (6.11) together prove Lemma 6.1. \square

In the next lemma we will drop the assumption that $a \in C(f)$. (But get a weaker conclusion.)

6.2 Lemma. *Let $f: [0, 1] \rightarrow [0, 1]$ be a C^2 map whose critical points are all quadratic and satisfy (CE1) and (CE2). (We do not assume that f is unimodal.) Then there exist constants $A_4 > 0$ and $\lambda' > 1$ such that for each $r \geq 0$ and each interval (a, b) such that (a_r, b_r) does not contain nonrepelling points, and so that $f^r|(a, b)$ is a diffeomorphism and $b_r \in C(f)$, one has*

$$(6.12) \quad \frac{|b_r - a_r|}{|b - a|} \geq A_4.$$

Proof. Let $(u, b) \supset (a, b)$ be the maximal interval such that $f^r|(a, u)$ is a diffeomorphism. Then $u_s \in C(f)$ for some $0 \leq s < r$. First we prove the following.

Claim. For each $n_0 < \infty$ there exists a constant $A_5 > 0$ such that for any intervals (a, b) as above with either $|r - s| \leq n_0$ or $|b_r - a_r| \leq \frac{1}{2}|a_r - u_r|$ one has

$$(6.13) \quad \frac{|b_r - a_r|}{|b - a|} \geq A_5.$$

Proof of the claim. In this case $u_r \in f^{r-s}(C(f))$ and since $a_r \in C(f)$, $a_r \neq u_r$, there exists a constant $A_6 > 0$ depending only on $|r - s|$ such that $|a_r - u_r| \geq A_6$ (for all choices of a, b as above). So if $|b_r - a_r| \geq \frac{1}{2}|a_r - u_r|$ then

$$(6.14) \quad \frac{|b_r - a_r|}{|b - a|} \geq \frac{\frac{1}{2}|a_r - u_r|}{|b - a|} \geq \frac{\frac{1}{2}A_6}{1}.$$

On the other hand if $|b_r - a_r| \leq \frac{1}{2}|a_r - u_r|$, i.e., if $|b_r - u_r| \geq 3 \cdot |b_r - a_r|$, then we apply the Koebe lemma and get constants $\kappa > 0$, and $C, \rho < \infty$ such that

$$\frac{|b_r - a_r|}{|b - a|} \geq \kappa \cdot |Df^r(b)| \cdot \exp \left(-\rho \cdot C \cdot \sum_{i=0}^{r-1} |f^i(u, b)| \right).$$

Choose $\chi \cdot \lambda > \lambda^{1/2}$. As in (6.4) there exists $m_0 > 0$ such that

$$\exp \left(-\rho \cdot C \cdot \sum_{i=0}^{r-1} |f^i(u, b)| \right) \geq m_0 \cdot \chi^r.$$

Using $b_r \in C(f)$ and (CE2) this gives

$$(6.15) \quad \frac{|b_r - a_r|}{|b - a|} \geq \kappa \cdot |Df^r(b)| \cdot m_0 \cdot \chi^r \geq \kappa \cdot m_0 \cdot \lambda^{r/2}$$

The claim follows.

Now we continue with the proof of Lemma 6.2. Let A_1 and λ' be the constants from Lemma 6.1. Now let n_0 be so big that

$$(6.16) \quad A_1 \cdot (\lambda')^k > 3 \quad \forall k \geq n_0,$$

and A_5 the number from the claim above corresponding to this choice of n_0 . Choose $A_4 \in (0, A_5)$ so small that the conclusion of the lemma, i.e., equation (6.12), is satisfied for all $r \geq n_0$ and for all possible intervals (a, b) . Let us prove by induction that (6.12) is true for all r . So assume that it is satisfied for all $r \leq r'$. Then we will prove (6.12) for $r' + 1$. From the way A_5 is chosen we may assume that $|b_r - a_r| \geq \frac{1}{2}|u_r - a_r|$ and $|r - s| \geq n_0$. Using this and the induction assumption one has

$$(6.17) \quad \begin{aligned} \frac{|b_r - u_r|}{|b - u|} &\geq \frac{1}{3} \cdot \frac{|b_r - u_r|}{|b - u|} = \frac{1}{3} \cdot \frac{|b_s - u_s|}{|b - u|} \frac{|b_r - u_r|}{|b_s - u_s|} \\ &\geq \frac{1}{3} \cdot A_4 \cdot \frac{|b_r - u_r|}{|b_s - u_s|}. \end{aligned}$$

Since $b_r \in C(f)$ and $u_s \in C(f)$ we can use Lemma 6.1 and get, using $|r - s| \geq n_0$,

$$\frac{|b_r - u_r|}{|b_s - u_s|} \geq A_1 \cdot (\lambda')^{r-s} > 3.$$

Using this in (6.17) gives $|b_r - a_r|/|b - a| \geq A_4$, which proves Lemma 6.2 by induction. \square

6.3 Proposition. *Let $f: [0, 1] \rightarrow [0, 1]$ be a C^2 map whose critical points are all quadratic satisfying (CE1) and (CE2). (We do not assume that f is unimodal.) We assume there exists $K' < \infty$, $K'' > 0$, $\lambda' > 1$, and any interval I of f^n such that with $f^n|I$, I is a diffeomorphism for any $n \geq 0$ and such that I does not contain nonrepelling periodic points. Then the following hold.*

(i) *For any such interval one has*

$$(6.18) \quad |I| < K' \cdot (\lambda')^{-n}.$$

(ii) *If $f^n(I) \cap C(f) \neq \emptyset$, then*

$$(6.19) \quad |f^n(I)|/|I| \geq K'' \cdot (\lambda')^n.$$

(iii) *f does not have wandering intervals.*

Remark. From §3 we know that there is an upper bound for the period of nonrepelling periodic points of f . So this theorem says that, except possibly for intervals containing periodic points of low periods, (6.18) and (6.19) are always satisfied.

Proof. Let us first prove (i). Let $C > 0$ be the constant from §3. Let I_n be a branch of f^n such that $f^n(I_n)$ does not contain nonrepelling periodic points. Let $\chi < 1$ be so close to 1 that

$$(6.20) \quad \chi \cdot \lambda^{1/2} > \lambda^{1/4}.$$

From Lemma 5.1 it follows that there exists a constant $m_0 > 0$ such that

$$(6.21) \quad \exp \left(-C \cdot \sum_{j=0}^n |f^j(I_n)| \right) \geq m_0 \cdot \chi^n$$

for any $n \geq 0$ and any f^n branch I_n such that $f^n(I_n)$ does not contain nonrepelling periodic points. Let $\theta > 0$ be such that for every $c \in C(f)$ and all $z \in [0, 1]$ with $f|(z, c)$ monotone

$$(6.22) \quad \frac{|f(z) - f(c)|}{|z - c|} \geq \theta \cdot |Df(z)|.$$

Since f is quadratic at each critical point c , such θ exists.

Let $\gamma > 0$ be so small that

$$\left(\frac{\lambda}{\max\{|Df(x)|; x \in [0, 1]\}} \right)^\gamma \cdot \lambda \geq \sqrt{\lambda}.$$

Let $(a, b) \supset I$ be a branch of f^n not containing nonrepelling periodic points. Since (a, b) is a branch of f^n , $a_r, b_k \in C(f)$ for some $0 \leq r, k < n$. For definiteness assume that $r \leq k$. We want to estimate $|b - a|$.

Let us first deal with the case

$$(*) \quad |k - r| < \gamma \cdot n.$$

By the “square-root” estimate (3.1) and by (6.21) we get

$$(6.23) \quad |b_r - a_r| \geq (Df^r(b) \cdot Df^r(a))^{1/2} \cdot m_0 \cdot \chi^r \cdot |b - a|.$$

From $a_r \in C(f)$ and (6.22) one gets

$$(6.24) \quad |b_{r+1} - a_{r+1}| \geq \theta \cdot |Df(b_r)| \cdot |b_r - a_r|.$$

Then

$$(6.25) \quad \begin{aligned} |b_k - a_k| &\geq (Df^{k-r-1}(a_{r+1}) \cdot Df^{k-r-1}(b_{r+1}))^{1/2} \\ &\quad \cdot m_0 \cdot \chi^{k-r-1} \cdot |b_{r+1} - a_{r+1}|. \end{aligned}$$

From $b_k \in C(f)$ and (6.22) one then gets

$$(6.26) \quad |b_{k+1} - a_{k+1}| \geq \theta \cdot |Df(a_k)| \cdot |b_k - a_k|.$$

Finally

$$(6.27) \quad \begin{aligned} |b_n - a_n| &\geq (Df^{n-k-1}(b_{k+1}) \cdot Df^{n-k-1}(a_{k+1}))^{1/2} \\ &\quad \cdot m_0 \cdot \chi^{n-k-1} \cdot |b_{k+1} - a_{k+1}|. \end{aligned}$$

Combining (6.23)–(6.27) and using the chain rule one gets

$$(6.28) \quad \begin{aligned} |b_n - a_n| &\geq (m_0)^3 \cdot \theta^2 \cdot \chi^{n-2} \\ &\quad \times (|Df^r(a)| \cdot |Df^k(b)| \cdot |Df^{n-r-1}(a_{r+1})| \cdot |Df^{n-k-1}(b_{k+1})|)^{1/2} \\ &\quad \times (|Df(a_k)| \cdot |Df(b_r)|)^{1/2} \times |b - a|. \end{aligned}$$

Since $b_k \in C(f)$, one gets from the chainrule, and from (CE2),

$$\begin{aligned} (\max |Df|)^{k-r-1} \cdot |Df(b_r)| &\geq |Df^{k-r-1}(b_{r+1})| \cdot |Df(b_r)| \\ &= |Df^{k-r}(b_r)| \geq K \cdot \lambda^{k-r}. \end{aligned}$$

Hence

$$(6.29) \quad |Df(b_r)| \geq K \cdot \left(\frac{\lambda}{\max |Df|} \right)^{k-r}.$$

Similarly using $a_r \in C(f)$, (CE1) and $(\max |Df|)^{k-r-1} \cdot |Df(a_k)| \geq |Df(a_k)| \times |Df^{k-r-1}(a_{r+1})| = |Df^{k-r}(a_{r+1})| \geq K \cdot \lambda^{k-r}$, we get

$$(6.30) \quad |Df(a_k)| \geq K \cdot \left(\frac{\lambda}{\max |Df|} \right)^{k-r}.$$

Using $a_r, b_k \in C(f)$, $a_{r+1}, b_{k+1} \in f(C(f))$ and (CE1) and (CE2) one gets from (6.28), (6.29), and (6.30) and the choice of γ ,

$$\begin{aligned} |b_n - a_n| &\geq (m_0^3) \cdot \chi^{n-2} \cdot \theta^2 \cdot \left(K^4 \cdot \lambda^{(2n-2)} \cdot K^2 \cdot \left(\frac{\lambda}{\max |Df|} \right)^{2(k-r)} K^2 \right)^{1/2} \cdot |b - a| \\ &\geq (m_0^3) \cdot \chi^{n-2} \cdot \theta^2 \cdot K^3 \cdot \frac{1}{\lambda} \cdot \lambda^n \cdot \left(\frac{\lambda}{\max |Df|} \right)^{k-r} \cdot |b - a| \\ &\geq (m_0^3) \cdot \chi^{n-2} \cdot \theta^2 \cdot K^3 \cdot \frac{1}{\lambda} \cdot \left(\lambda \cdot \frac{\lambda}{\max |Df|} \right)^\gamma \cdot |b - a| \\ &\geq (m_0^3) \cdot \chi^{n-2} \cdot \theta^2 \cdot K^3 \cdot \frac{1}{\lambda} \cdot \lambda^{n/2} \cdot |b - a|. \end{aligned}$$

Since $\chi \cdot \lambda^{1/2} \geq \lambda^{1/4}$, this gives

$$(6.31) \quad \frac{|b_n - a_n|}{|b - a|} \geq A_1 \lambda^{n/4},$$

where

$$(6.32) \quad A_1 = \frac{m_0^3 \cdot \theta^2 \cdot K^3}{\chi^2 \cdot \lambda}.$$

This implies (6.18) for intervals such that $|k - r| < \gamma \cdot n$.

Let us now assume that

$$(**) \quad |k - r| \geq \gamma \cdot n.$$

From Lemmas 6.2 and 6.1 one gets

$$\begin{aligned} |b_r - a_r| &\geq A_4 \cdot |b - a|, \\ |b_k - a_k| &\geq A_1 \cdot \lambda^{k-r} \cdot |b_r - a_r|. \end{aligned}$$

All this gives

$$\begin{aligned} (6.33) \quad |b_k - a_k| &\geq A_1 \cdot A_4 \cdot \lambda^{k-r} \cdot |b - a| \\ &\geq A_1 \cdot A_4 \cdot \lambda^{\gamma n} \cdot |b - a|. \end{aligned}$$

So in this case (6.18) follows by choosing $\lambda' = \lambda^\gamma$ and $K' = A_1 \cdot A_4$.

Let us now prove statement (ii) of the proposition. Let $I = (u, b)$. First assume that $\partial(f^n(I)) \cap C(f) \neq \emptyset$, say $b_n \in C(f)$. Let $(a, b) \supset (u, b)$ be the maximal interval such that $f^n|_{(a, b)}$ is a diffeomorphism. Then $a_r \in C(f)$ for some $r < n$ and $b_k \in C(f)$. So using the notation from above one has $k = n$. Combining cases (*), (**) gives that there exists $A_7 > 0$ and $\tilde{\lambda} > 1$ such that

$$\frac{|b_n - a_n|}{|b - a|} \geq A_7 \cdot (\tilde{\lambda})^n.$$

If $|I| = |b_n - u_n| \geq \frac{1}{2}|b_n - a_n|$ then this gives

$$\frac{|f^n(I)|}{|I|} \geq \frac{1}{2} \cdot A_7 \cdot (\tilde{\lambda})^n,$$

which proves the statement in this case. Otherwise, when $|b_n - u_n| \leq \frac{1}{2}|b_n - a_n|$, we apply the Koebe inequality and, as in Lemma 6.1, we get some $m_0 > 0$ and $\kappa > 0$ such that

$$(6.34) \quad \frac{|f^n(I)|}{|I|} = \frac{|b_n - u_n|}{|b - u|} \geq \kappa \cdot m_0 \cdot \chi^n |Df^n(b)|.$$

Since $b_n \in C(f)$ equation (6.19) follows in the case that $\partial(f^n(I_n)) \cap C(f) \neq \emptyset$ from (CE2) and (6.20). Otherwise $\text{int}(f^n(I)) \cap C(f) \neq \emptyset$, and then write I as the union of two adjacent intervals I_1 and I_2 such that $\partial(f^n(I_i)) \cap C(f) \neq \emptyset$. Since we have proved (6.19) for I_i we get

$$\frac{|f^n(I)|}{|I|} \geq \min_{i=1,2} \frac{|f^n(I_i)|}{|I_i|} \geq K' \cdot (\lambda')^n.$$

This finishes the proof of statement (ii).

Statement (iii) follows immediately from statement (i). Indeed, if f had a wandering interval then some iterate I of this wandering interval would have the property that $f^n|_I$ is a diffeomorphism for all $n \geq 0$. Then let $I_n \supset I$ be the branch of f^n . From (i) one has that $|I| \leq |I_n| \rightarrow 0$ as $n \rightarrow \infty$. This contradiction finishes the proof of the proposition. \square

Corollary. *Let $f: [0, 1] \rightarrow [0, 1]$ be a C^2 map whose critical points are all quadratic and which satisfies (CE1) and (CE2). (We do not assume that f is unimodal.) Then there exists $K''' > 0$, $\lambda'' > 1$, and for any $n \geq 0$ an interval I of f^n which contains a critical point of f such that $f^n|_I$ has precisely one critical point, I does not contain nonrepelling periodic points, and $f^n(I) \cap C(f) \neq \emptyset$. Then*

$$(6.35) \quad \frac{|f^n(I)|}{|I|} \geq K''' \cdot (\lambda'')^n.$$

Proof. Let c be the (unique) critical point of f in I and let I^i be the two components of $I \setminus \{c\}$. Assume for example that $|f(I^1)| \geq |f(I^2)|$. Since c is an extremum of f , $f(I^1) \supset f(I^2)$ and since c is a quadratic critical point,

there exists $k > 0$ such that $k \cdot |I| \geq k \cdot |I^1|$. Applying Proposition 6.3 to I^1 one has

$$\frac{|f^n(I^1)|}{|I^1|} \geq K' \cdot (\lambda')^n.$$

Hence

$$\frac{|f^n(I)|}{|I|} \geq k \cdot \frac{|f^n(I^1)|}{|I^1|} \geq k \cdot K' \cdot (\lambda')^n.$$

The corollary follows. \square

7. SENSITIVITY ON INITIAL CONDITIONS

We say that I is a restrictive interval of period n if $f^n(I) \subset I$. The *minimal period* of I is the smallest number s such that $f^s(I) \subset I$.

7.1 Lemma. *Let $f: [0, 1] \rightarrow [0, 1]$ be a C^2 map whose critical points are all quadratic satisfying (CE1) and (CE2). (We do not assume that f is unimodal.) Then there exists $N < \infty$ and $\delta > 0$ such that*

- (i) *If I is a restrictive interval such that (a) I is not contained in the basin of one single attractor and (b) I is not a homterval, then I has length $\geq \delta$;*
- (ii) *f has no restrictive intervals of (minimal) period $s \geq N$.*

Proof. From the corollary to Proposition 4.1 there exists a number $N < \infty$ such that any nonrepelling orbit has period less than N . Moreover, every restrictive homterval contains a periodic attractor.

Let I be a restrictive interval with minimal period s and $I' = \bigcap_{n \geq 0} f^{sn}(I)$. I' is a nonempty connected set since $f^s(I) \subset I$. If I' is just a point or if $f^s|_{I'}$ is a homeomorphism, then I contains a periodic point of period s which is a periodic attractor. Hence according to the corollary to Lemma 4.1, $s \leq N$. From this (i) and (ii) follow easily by contradiction in this case.

So we may assume that I' is a nontrivial interval and that $f^s|_{I'}$ has critical points. From the definition of I' one has $f^s(I') = I'$. Let $I'_j = f^j(I')$. Since $f^s|_{I'}$ has critical points, the set $J = \{j \in \mathbb{N}; I'_j \cap C(f) \neq \emptyset\}$ is nonempty and $J + s \subset J$.

Let us first prove (i). Choose $m \in J$ such that $|I'_m| = \max_{j \in J} |I'_j|$ and let $j > m$ be the smallest integer such that $j \in J$, $I'_j \cap C(f) \neq \emptyset$. From the corollary at the end of §6 one has

$$|I'_j| = |f^{j-m}(I'_m)| \geq K'' \cdot (\lambda')^{j-m} \cdot |I'_m|.$$

Since m corresponds to the integer such that $|I'_j|$ is maximal for $j = m$, it follows that

$$K'' \cdot (\lambda')^{j-m} \leq 1.$$

Hence there exists $N < \infty$ such that, for any interval I as above, $j - m \leq N$. Now let there exist $\delta' > 0$ such that any interval \tilde{I} has length at least δ' ,

where $\tilde{I} \cap C(f) \neq \emptyset$ and $f^k(\tilde{I}) \cap C(f) \neq \emptyset$ for some $k \leq N$. Since none of the critical points is mapped onto another critical point it follows that there exists such a $\delta' > 0$. Furthermore let there exist $\delta > 0$ such that any interval \tilde{I} has length at least δ , where $|f^k(\tilde{I})| \geq \delta'$ for some $k \leq N$. Because I'_m and $f^{j-m}(I'_m) = I'_j$ both contain critical points and $j - m \leq N$ it follows that $|I'_m| \geq \delta'$. Hence, from the choice of δ , $|I| \geq |I'| = |f^s(I')| = |f^{s-m}(I'_m)| \geq \delta$. (i) follows.

Let us now prove (ii). Since any nonrepelling periodic orbit has period less than N and every restrictive homterval contains a periodic attractor, (ii) follows from (i) and the following claim.

Claim. For every $\delta > 0$ there exists $N_0 < \infty$ such that every restrictive interval I with $|I| \geq \delta$ has a (minimal) period at most N_0 .

Proof of the claim. Let I be a restrictive interval I with $|I| \geq \delta$ and with period s . Since f has no wandering intervals, there exists $N_0 < \infty$ such that for every such interval I there exist $i, k < N_0$ with $f^i(I) \cap f^{i+k}(I) \neq \emptyset$. But then $L = \bigcup_{j \geq 0} f^{jk+i}(I)$ is an interval and f^k maps L into itself. Hence L contains a periodic point p of period $k \leq N_0$. But since $f^s(I) \subset I$ it follows that I also contains a periodic point p' in the orbit of p and therefore has period $k \leq N$. But since $f^s(I) \subset I$, and therefore $f^{ks}(I) \subset I$, and since $f^k(p') = p'$, it follows that $f^{2k}(I) \subset I$ and therefore $s \leq 2k \leq 2N_0$. \square

7.2 Corollary. Let $f: [0, 1] \rightarrow [0, 1]$ be a C^2 map whose critical points are all quadratic satisfying (CE1) and (CE2). Assume that f has no attracting periodic or nonhyperbolic periodic orbits. There exists $\varepsilon > 0$ such that for any interval I there exists $n_0 \geq 0$ such that $|f^n(I)| \geq \varepsilon$ for all $n \geq n_0$.

In particular, f has sensitive dependence on initial conditions.

Proof. Let $N < \infty$ be the number from Lemma 7.1. Let $\varepsilon' > 0$ be such that for each periodic point p of period k with $k \leq N$, one has $f^k|[p - \varepsilon', p + \varepsilon']|$ is a diffeomorphism. (Since the critical points are not periodic this is possible.) Moreover let $\varepsilon > 0$ be such that any interval \tilde{I} has length at least ε such that $|f^k(\tilde{I})| \geq \varepsilon'$ for some $k \leq N$.

Let I be an interval. Since f has no wandering intervals, there exists $i, k > 0$ such that $f^i(I) \cap f^{i+k}(I) \neq \emptyset$. Then write $I_0 = f^i(I)$. Since $f^k(I_0) \cap I_0 \neq \emptyset$, one has that $f^{(j+1)k}(I_0) \cap f^{jk}(I_0) \neq \emptyset$ for all $j \geq 0$. Hence $L = \bigcup_{j \geq 0} f^{jk}(I_0)$ is an interval and f^k maps L into itself. So L is a restrictive interval and from Lemma 7.1 the minimal period s of L is at most N . Let p be a fixed point of $f^s: L \rightarrow L$. One of the iterates of I_0 (and therefore of I) contains p . Since $f^s|[p - \varepsilon', p + \varepsilon']|$ is a diffeomorphism, there exists $i(0) \geq 0$ and $r \geq 0$ such that for all $i \geq i(0)$, either $f^{is+r}(I) \supset [p, p + \varepsilon']$ or $f^{is+r}(I) \supset [p - \varepsilon', p]$. Hence $|f^{is+r}(I)| \geq \varepsilon'$ for all $i \geq i(0)$. Therefore, from the choice of ε , $|f^{i+r}(I)| \geq \varepsilon$ for all $i \geq i(0)$. The corollary follows. \square

8. RETURN-TIME OF ORBITS NEAR THE CRITICAL POINTS

Let $f: [0, 1] \rightarrow [0, 1]$ be some C^2 map with a finite number of critical points, $\#C(f) < \infty$. Let

$$(8.1) \quad L = \sup\{|Df(x)|; x \in [0, 1]\}.$$

Since f is C^2 in c there exists $K < \infty$ and $l \geq 2$ with

$$(8.2) \quad |Df(z)| \leq K \cdot |z - c|^{l-1}$$

for all $z \in [0, 1]$.

8.1 Theorem. *Let f be as above. Then there exists $K_1 > 0$ such that for any $\varepsilon > 0$, if there exists $x \in [0, 1]$, $c \in C(f)$ with $|x - c|, |x_n - c| < \varepsilon$ then*

$$n < K_1 \cdot \log(1/\varepsilon)$$

implies $f^n[c - 2\varepsilon, c + 2\varepsilon] \subset [c - 2\varepsilon, c + 2\varepsilon]$ and $|f'(x)| < 1$ for all $x \in [c - 2\varepsilon, c + 2\varepsilon]$. (So all points in $[c - 2\varepsilon, c + 2\varepsilon]$ are contained in the basin of a unique attractor in this interval.)

Proof. We may assume that $\varepsilon < 1$. By the chain rule, (8.1), and (8.2) one gets

$$(8.3) \quad |Df^n(z)| = |Df(z)| \cdot |Df^{n-1}(f(z))| \leq K \cdot |z - c|^{l-1} \cdot L^{n-1}.$$

Assume that

$$(8.4) \quad K \cdot (2 \cdot \varepsilon)^{l-1} \cdot L^{n-1} < \frac{1}{8}.$$

Equations (8.3) and (8.4) imply that for all $z \in [c - 2\varepsilon, c + 2\varepsilon]$,

$$(8.5) \quad |Df^n(z)| \leq \frac{1}{8}.$$

If (8.4) holds, then for all $z_1, z_2 \in [c - 2\varepsilon, c + 2\varepsilon]$,

$$(8.6) \quad |f^n(z_1) - f^n(z_2)| \leq \int_{z_1}^{z_2} |Df^n(z)| dz \leq \int_{c-2\varepsilon}^{c+2\varepsilon} \frac{1}{8} dz \leq \frac{1}{2}\varepsilon.$$

$|x_n - c| < \varepsilon$ and (8.6) imply that for all $z \in [c - 2\varepsilon, c + 2\varepsilon]$,

$$(8.7) \quad \begin{aligned} |f^n(z) - c| &\leq |f^n(z) - f^n(x)| + |f^n(x) - c| \\ &\leq \frac{1}{2}\varepsilon + |f^n(x) - c| \leq \frac{3}{2}\varepsilon. \end{aligned}$$

Hence from (8.5) and (8.7), f^n maps $[c - 2\varepsilon, c + 2\varepsilon]$ into itself and there is a unique attracting fixed point of $f^n|_{[c - 2\varepsilon, c + 2\varepsilon]}$. In particular c is attracted to this periodic point. This finishes the proof of the theorem. \square

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