TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 322, Number 1, November 1990

INTERPRETATIONS OF EUCLIDEAN GEOMETRY

S. ŚWIERCZKOWSKI

ABSTRACT. Following Tarski, we view n-dimensional Euclidean geometry as a first-order theory E_n with an infinite set of axioms about the relations of *betweenness* (among points on a line) and *equidistance* (among pairs of points). We show that for $k < n$, E_n does not admit a k-dimensional interpretation in the theory RCF of real closed fields, and we deduce that E_n cannot be interpreted r-dimensionally in E_s, when $r \cdot s < n$.

1. INTRODUCTION

We know since Descartes that points of the Euclidean n-dimensional space can be identified with n-tuples of real numbers. Moreover, each statement of n-dimensional geometry that involves a variable x representing a point has its counterpart in algebra that is a statement involving n variables $x_1, \dots, x_n$ (the "coordinates" of x). In such a situation, we say (to be made precise later in Definition 2.2) that we have an *n-dimensional interpretation* of the language of geometry in the language of algebra.

In the sequel E_n will be the first-order theory with equality that has been proposed by Alfred Tarski [T2] for describing the n-dimensional Euclidean geometry. We shall denote by $L_{\beta\delta}$ the language of E_n; this is the first-order language whose vocabulary consists of three predicate symbols: $=$ (binary), β (ternary), and δ (quaternary). If $\mathscr{E}_n$ is the Euclidean n-dimensional space, i.e., the usual model of E_n whose universe is $\mathbf{R}^n$, and $a, b, c, d \in \mathbf{R}^n$, then the meaning of the relations β and δ is as follows:

$\beta(a, b, c)$ iff b lies between a and c (possibly $b = a$ or $b = c$),

$\delta(a, b, c, d)$ iff a is as distant from b as c is from d.

As Tarski has shown, his axiomatization of E_n yields a complete theory. Thus, the theorems of E_n are just those sentences in $L_{\beta\delta}$ that are true in $\mathscr{E}_n$.

The preceding Cartesian n-dimensional interpretation of the language $L_{\beta\delta}$ in the language of algebra (see the end of §2, where e stands for equality) is also an *interpretation of theories* in the sense that the algebraic counterparts of *theorems* of E_n are *theorems* of RCF, i.e., of the theory of real closed fields,

Received by the editors November 28, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 03F25; Secondary 51M05.
Key words and phrases. Interpretation, definable set, semialgebraic, Cartesian coordinates.

of which the field of real numbers is one. The language of RCF is the language L_{OF} of ordered fields, i.e., the f.o. language whose vocabulary consists of the symbols: $+$, $-$, $\cdot$ (binary functions), $<$, $=$ (binary predicates), and 0, 1 (constants). Thus, we have the Cartesian n-dimensional interpretation of $L_{\beta\delta}$ in L_{OF}, which is an interpretation of the *theory* E_n in the *theory* RCF.

Mycielski [M] has conjectured that an interpretation of E_n in RCF cannot be achieved more economically than in the preceding Cartesian way, that is, using k-tuples of variables for some $k < n$ instead of n-tuples. In 1980, Boffa [B] proved that this is so for the case when $n = 2$ and $k = 1$. The general conjecture is confirmed by the main result of this paper:

Theorem 1.1. *For every $n \geq 2$, there exists a theorem φ_n of E_n such that for every $k < n$, the image of φ_n under any k-dimensional interpretation of the language $L_{\beta\delta}$ in the language L_{OF} is not a theorem of RCF.*

Corollary 1.1. *If $r \cdot s < n$, there does not exist an r-dimensional interpretation of the theory E_n in the theory E_s.*

Indeed, if such an interpretation were to exist, one could superimpose it with the Cartesian s-dimensional interpretation of E_s in RCF, thus obtaining an rs-dimensional interpretation of E_n in RCF, contrary to Theorem 1.1.

Our following proof combines the basic idea of Boffa in [B, p. 123] with a result of Łojasiewicz [Ł] on triangulation of semialgebraic sets (proved also in [C, VdD1, and H]).

In 1984, Szczerba announced in [Sz, p. 677] that he had found the affirmative answer to Mycielski's conjecture; however, his work is not known to us and does not seem to have been published.

The author wishes to express his gratitude to Jan Mycielski for fruitful discussions that led to many improvements of this paper.

2. Interpretations

Given a formula α, we say that a variable x_j is *substitutable* for a variable x_i in α if, by replacing x_i with x_j at any of its free occurrences, we get a free occurrence of x_j, or, in other words, if no free occurrence of x_i is within the scope of the quantifier $\exists x_j$.

Suppose all the free variables of α (i.e., variables that have at least one free occurrence) are among $x_{i_1}, \dots, x_{i_n}$, where $i_1 < i_2 < \cdots < i_n$, and let $x_{j_1}, \dots, x_{j_n}$ be such that x_{j_s} is substitutable for x_{i_s} in α for each $s = 1, \dots, n$. Then the result of replacing each x_{i_s} at every free occurrence by x_{j_s} will be denoted by:

$$\alpha \begin{bmatrix} x_{i_1} & \cdots & x_{i_n} \\ x_{j_1} & \cdots & x_{j_n} \end{bmatrix}.$$

If the n-tuple $(x_{i_1}, \dots, x_{i_n})$ coincides with $(x_1, \dots, x_n)$ or with $(x_0, \dots, x_{n-1})$, we shall write the above as $\alpha[x_{j_1}, \dots, x_{j_n}]$.

By a *variant* of α we shall mean any formula obtained from α by a sequence of replacements as follows:

$$\text{replace } \exists x_k \beta \quad \text{by } \exists x_s \beta \begin{bmatrix} x_k \\ x_s \end{bmatrix},$$

where $\exists x_k \beta$ is any subformula of α such that x_s is substitutable for x_k in β. Obviously, such replacements do not affect the free occurrences of variables and they do not change the scopes of quantifiers.

We assume until the end of this section that L is a relational language (i.e., L has no function symbols) and that $x_0, x_1, \dots$ is the list of all variables of L. Let L' be any language that contains the variables $x_0, x_1, \dots$.

Definition 2.1. By a one-dimensional *interpretation* of L in L', we mean a map $\alpha \mapsto \alpha^0$, which assigns to every formula α of L a formula α^0 of L' so that

(i) every variable free in α^0 is also free in α;

(ii) for any n-place predicate symbol p of L and any $x_{i_1}, \dots, x_{i_n}$, $(p(x_{i_1}, \dots, x_{i_n}))^0$ is $((p(x_1, \dots, x_n))^0)'[x_{i_1}, \dots, x_{i_n}]$, where $(p(x_1, \dots, x_n)^0)'$ is a variant of $(p(x_1, \dots, x_n))^0$ in which x_{i_s} is substitutable for x_s for all $s = 1, \dots, n$;

(iii) if α is of the form $\neg\mu$, $\mu \vee \eta$, $\mu \to \eta$, $\mu \,\&\, \eta$, $\forall x_i \mu$ or $\exists x_i \mu$, then α^0 is correspondingly $\neg\mu^0$, $\mu^0 \vee \eta^0$, $\mu^0 \to \eta^0$, $\beta^0 \,\&\, \eta^0$, $\forall x_i \mu^0$ or $\exists x_i \mu^0$.

Obviously, an interpretation $\alpha \mapsto \alpha^0$ is uniquely determined by its restriction to the atomic formulas $p(x_{i_1}, \dots, x_{i_n})$, and thus, up to variants, by its restriction to the formulas $p(x_1, \dots, x_n)$. The images of the latter can be chosen arbitrarily, provided (i) is satisfied.

For example, there is exactly one interpretation of $L_{\beta\delta}$ in L_{OF} such that:

$$\begin{aligned} &(\beta(x_1, x_2, x_3))^0 && \text{is } (x_3 - x_2)(x_2 - x_1) \geq 0, \\ &(\delta(x_1, x_2, x_3, x_4))^0 && \text{is } (x_2 - x_1)^2 = (x_4 - x_3)^2, \text{ and} \\ &(x_1 = x_2)^0 && \text{is } x_1 = x_2. \end{aligned}$$

This is also the obvious interpretation of E_1 in RCF.

In the sequel, we shall treat equality just as any other binary predicate symbol. In particular, it will not necessarily be assumed that $(x_1 = x_2)^0$ is $x_1 = x_2$. To prevent misunderstandings, we shall use henceforth the letter e to denote the binary predicate symbol for equality in any language L. Thus, for example, to define an L-structure $\mathscr{A}$, we shall have to specify also the binary relation on $\mathscr{A}$ corresponding to e. (This will not, in general, be the relation of equality in the structure $\mathscr{A}$.)

Remarks. (1) Interpretations can be superimposed, that is, from an interpretation of L in L' and another from L' to L'', we get an interpretation of L in L''.

(2) For interpreting L in L' it is irrelevant which variables are in these languages, provided all variables of L are also in L'.

Given a positive integer k and L as just described, let us denote by $L^{(k)}$ the language such that:

(a) the only variables of $L^{(k)}$ are x_{ij}, $i, j, = 0, 1, \dots$,

(b) there is a one-to-one correspondence $p \mapsto p^{(k)}$ between the predicate symbols p in L and $p^{(k)}$ in $L^{(k)}$ such that if p is n-place then $p^{(k)}$ is kn-place.

We define a mapping $\alpha \mapsto \alpha^{(k)}$, called the *k-spread*, from formulas of L to formulas of $L^{(k)}$, as follows. Given α, one obtains $\alpha^{(k)}$ by replacing each occurrence of Qx_i in α (where Q is $\exists$ or $\forall$) by $Qx_{i1}Qx_{i2}\cdots Qx_{ik}$, and every occurrence of x_i, other than in Qx_i, by $x_{i1}, x_{i2}, \dots, x_{ik}$.

Suppose that L'' contains the variables x_{ij}; $i, j = 0, 1, 2, \dots$. Then one has interpretations of $L^{(k)}$ in L''.

Definition 2.2. By a *k-dimensional interpretation* of L in L'', we mean a map $\alpha \mapsto \alpha^*$ from the formulas of L to the formulas of L'' that is the composite of the k-spread $\alpha \mapsto \alpha^{(k)}$ with a one-dimensional interpretation $\beta \mapsto \beta^0$ of $L^{(k)}$ in L'' (i.e., α^* is $(\alpha^{(k)})^0$).

Obviously, the k-spread is the simplest example of a k-dimensional interpretation. To describe the Cartesian k-dimensional interpretation $\alpha \mapsto \alpha^*$ of $L_{\beta d}$ in L_{OF}, assume that L_{OF} has all x_{ij}; $i, j = 0, 1, \dots$, among its variables. Then $\alpha \mapsto \alpha^*$ is uniquely determined by the following conditions:

(i_1) $(\beta(x_1, x_2, x_3))^*$ is the conjunction of the $k + k^2$ formulas

$$(x_{3i} - x_{2i})(x_{2i} - x_{1i}) \geq 0; \qquad 1 \leq i \leq k,$$
$$(x_{3i} - x_{2i})(x_{2j} - x_{1i}) = (x_{3j} - x_{2j})(x_{2i} - x_{1j}); \qquad 1 \leq i, j \leq k,$$

(i_2) $(\delta(x_1, x_2, x_3, x_4))^*$ is $\displaystyle\sum_{i=1}^{k}(x_{2i} - x_{1i})^2 = \sum_{i=1}^{k}(x_{4i} - x_{3i})^2$,

(i_3) $(e(x_1, x_2))^*$ is $e(x_{11}, x_{21}) \& e(x_{12}, x_{22}) \& \cdots \& e(x_{1k}, x_{2k})$.

3. Interpretations with parameters

Let L, L' be first-order languages where L is relational and all variables of L are also in L'. We denote by L'_p the language obtained by adding to L' *new* variables $p_0, p_1, \dots$, which we shall call *parameter* or *p-variables*. Let $\alpha \mapsto \exists p\alpha$ be the map that assigns to every α in L'_p the existential closure of α with respect to all the p-variables. We shall call this map the *existential p-closure*.

Definition 3.1. A *one-dimensional interpretation with parameters* of L in L' is a map $\alpha \mapsto \alpha^{(0)}$ from formulas of L to formulas of L' obtained by combining a one-dimensional interpretation $\alpha \mapsto \alpha^0$ of L in L'_p with the existential p-closure and next taking a variant that has no p-variables. (Thus $\alpha^{(0)}$ is a variant without p-variables of $\exists p \alpha^0$.)

Definition 3.2 [M, M-P-S]. Suppose L has only the variables $x_0, x_1, \ldots$ and L'' contains all the variables x_{ij}; $i, j = 0, 1, \ldots$. Then, by a *k-dimensional interpretation with parameters* of L in L'', we mean the superposition of the k-spread of L in $L^{(k)}$ with a one-dimensional interpretation with parameters of $L^{(k)}$ in L''.

Our main result (Theorem 1.1) is valid for interpretations *with parameters.* However, it will suffice to prove this theorem for interpretations without parameters, because of the following known fact. (See a remark in [M, §3, p. 299] concerning theories with selectors [M-V].)

Theorem. *Given a k-dimensional interpretation with parameters $\alpha \mapsto \alpha^{(*)}$ of a first-order language L in L_{OF} and a closed formula φ of L such that $RCF \vdash \varphi^{(*)}$, there exists a k-dimensional interpretation without parameters $\alpha \mapsto \alpha^*$ of L in L_{OF} such that $RCF \vdash \varphi^*$.*

In view of this, by an interpretation we shall mean henceforth an interpretation *without* parameters.

4. Structures and definability

Given an L-structure $\mathscr{A}$, we shall denote by $L_{|\mathscr{A}|}$ the diagram language of L, i.e., the language obtained by adding to L the elements of the universe $|\mathscr{A}|$ (or rather, their names) as constants. If α is a formula in L with no free variables other than $x_1, \ldots, x_n$ and $a_1, \ldots, a_n \in |\mathscr{A}|$, then the formula $\alpha[a_1 \cdots a_n]$ is closed in $L_{|\mathscr{A}|}$, and thus it has a logical value in $\mathscr{A}$. We shall indicate the value *true* by prefixing the formula with $\mathscr{A} \vDash$.

Definition 4.1. Given an L-structure $\mathscr{A}$, a *set* S will be called *$\mathscr{A}$-definable* if $S \subset |\mathscr{A}|^n$ for some $n \geq 1$ and there are $a_0, \ldots, a_m \in |\mathscr{A}|$ and a formula α in L with no free variables other than $x_0, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}$ such that

$$(s_1, \ldots, s_n) \in S \Leftrightarrow \mathscr{A} \vDash \alpha[a_0, \ldots, a_m, s_1, \ldots, s_n]$$

for all $s_1, \ldots, s_n \in |\mathscr{A}|$. We shall call $x_0, \ldots, x_m$ the *parameter variables,* and we shall denote S by $\{\alpha[a_0, \ldots, a_m, \cdot]\}$. An r-place *relation* on an $\mathscr{A}$-definable subset of $|\mathscr{A}|^n$ will be called *$\mathscr{A}$-definable* if its graph is an $\mathscr{A}$-definable subset of $|\mathscr{A}|^{nr}$.

Now let L be a relational language such that $x_0, x_1, \ldots$ are all the variables of L. We shall need the following theorem.

Theorem 4.1. *Given a k-dimensional interpretation $\alpha \mapsto \alpha^*$ of L in L'', there corresponds to every L''-structure $\mathscr{A}$ an L-structure $\mathscr{A}^*$ with $|\mathscr{A}^*| = |\mathscr{A}|^k$ such that*

(i) $\mathscr{A}^* \models \alpha \Leftrightarrow \mathscr{A} \models \alpha^*$ *for every closed α in L,*
(ii) *every $\mathscr{A}^*$-definable set (relation) is $\mathscr{A}$-definable.*

We note that it makes sense to talk of $\mathscr{A}$-definability of $\mathscr{A}^*$-definable sets because $|\mathscr{A}^*| = |\mathscr{A}|^k$. We deal first with the case $k = 1$:

Lemma 4.1. *Given a one-dimensional interpretation $\alpha \mapsto \alpha^0$ of L in L', there corresponds to every L'-structure $\mathscr{A}$ an L-structure $\mathscr{A}^0$ such that $|\mathscr{A}^0| = |\mathscr{A}|$ and*

(i) $\mathscr{A}^0 \models \alpha \Leftrightarrow \mathscr{A} \models \alpha^0$ *for every closed α in L,*
(ii) *every $\mathscr{A}^0$-definable set (relation) is $\mathscr{A}$-definable.*

Proof. We define the L-structure $\mathscr{A}^0$ by requiring that for any n-place predicate symbol p of L, the corresponding n-place relation on $\mathscr{A}^0$, *also denoted by* p, satisfies

$$\mathscr{A}^0 \models p(a_1, \dots, a_n) \Leftrightarrow \mathscr{A} \models (p(x_1, \dots, x_n))^0[a_1, \dots, a_n] \tag{4.1}$$

for every $a_1, \dots, a_n \in |\mathscr{A}^0| = |\mathscr{A}|$. It is easy to see that (i) and (ii) are consequences of the following fact: for every formula α in L that has no free variables other than $x_{i_1}, \dots, x_{i_n}$,

$$\mathscr{A}^0 \models \alpha \begin{bmatrix} x_{i_1} & \cdots & x_{i_n} \\ a_1 & \cdots & a_n \end{bmatrix} \Leftrightarrow \mathscr{A} \models \alpha^0 \begin{bmatrix} x_{i_1} & \cdots & x_{i_n} \\ a_1 & \cdots & a_n \end{bmatrix} \tag{4.2}$$

for arbitrary $a_1, \dots, a_n \in |\mathscr{A}|$. To show (4.2), note that when α is $p(x_1, \dots, x_n)$, (4.2) coincides with (4.1). For other atomic formulas (i.e., α of the form $p(x_{i_1}, \dots, x_{i_n})$), (4.2) is a consequence of (4.1) and (ii) in Definition 2.1. Finally, to extend (4.2) to nonatomic formulas, we use (iii), Definition 2.1.

Lemma 4.2. *If $k \geq 1$ and $\alpha \mapsto \alpha^{(k)}$ is the k-spread of L in $L^{(k)}$, then to every $L^{(k)}$-structure $\mathscr{A}$ there corresponds an L-structure $\mathscr{A}^{(k)}$ such that $|\mathscr{A}^{(k)}| = |\mathscr{A}|^k$ and*

(i) $\mathscr{A}^{(k)} \models \alpha \Leftrightarrow \mathscr{A} \models \alpha^{(k)}$ *for every closed α in L,*
(ii) *every $\mathscr{A}^{(k)}$-definable set (relation) is $\mathscr{A}$-definable.*

The proof is analogous to the proof of Lemma 4.1. For example, the L-structure $\mathscr{A}^{(k)}$ is defined by the requirement that for any n-place predicate symbol p of L, the corresponding n-place relation on $\mathscr{A}^{(k)}$, also denoted by p, satisfies:

$$\mathscr{A}^{(k)} \models p\,(a_1, \dots, a_n) \Leftrightarrow \mathscr{A} \models p^{(k)}(a_{11}, \dots, a_{1k}, \dots, a_{n1}, \dots, a_{nk})$$

for any $a_i = (a_{i1}, \dots, a_{ik}) \in |\mathscr{A}^{(k)}| = |\mathscr{A}|^k$, $i = 1, \dots, n$.

Combining Lemmas 4.1 and 4.2, we obtain Theorem 4.1 with $\mathscr{A}^* = (\mathscr{A}^0)^{(k)}$. For a first application, let us denote by $\mathscr{R}$ the standard model of RCF, i.e., the field of real numbers viewed as an L_{OF}-structure with universe $|\mathscr{R}| = \mathbf{R}$. Let $\alpha \mapsto \alpha^*$ be the k-dimensional Cartesian interpretation of $L_{\beta\delta}$ in L_{OF} (see end of §2). Then Theorem 4.1 taken with $\mathscr{A} = \mathscr{R}$, yields at place of $\mathscr{A}^*$ the usual model $\mathscr{E}_k$ of E_k.

Definition 4.2. A structure $\mathscr{A}$ will be called *semialgebraic* if

(s_1) $|\mathscr{A}| = \mathbf{R}^k$ for some $k \geq 1$,
(s_2) every $\mathscr{A}$-definable set (relation) is $\mathscr{R}$-definable.

Thus, by the preceding construction from Theorem 4.1, $\mathscr{E}_k$ is semialgebraic. Now consider an *arbitrary* k-dimensional interpretation $\alpha \mapsto \alpha^*$ of $L_{\beta\delta}$ in L_{OF}, and let φ be a sentence in $L_{\beta\delta}$ such that $RCF \vdash \varphi^*$. Then $\mathscr{R} \models \varphi^*$ and Theorem 4.1 gives us a semialgebraic $L_{\beta\delta}$-structure $\mathscr{R}^*$ such that $\mathscr{R}^* \models \varphi$. It follows from this that to establish the main result of this paper, i.e., Theorem 1.1, we only need to prove the following:

Theorem 4.2. *For every* $n \geq 2$, *there exists a theorem* φ_n *of* E_n *such that the sentence* φ_n *is false in every semialgebraic* $L_{\beta\delta}$*-structure* $\mathscr{A}$ *with* $|\mathscr{A}| = \mathbf{R}^k$ *and* $1 \leq k < n$.

The rest of this paper is devoted to establishing this result. We proceed as follows. In §5 we collect various known facts about $\mathscr{R}$-definable (also called *semialgebraic*) sets and relations, and we introduce the concept of an *n-tower* in $\mathbf{R}^k$ (a system of subsets of $\mathbf{R}^k$, Definition 5.1). We also prove that if $1 \leq k < n$ then *there does not exist an* *n-tower* in $\mathbf{R}^k$ (Theorem 5.1). In §6, we define the concept of an *n-grid* in a structure (Definition 6.1) and, if the structure $\mathscr{A}$ is semialgebraic, we *associate with every* *n-grid in* $\mathscr{A}$ *an* *n-tower in* $|\mathscr{A}|$ (Theorem 6.1). Finally, in §7 we construct a theorem φ_n of E_n such that for every semialgebraic $L_{\beta\delta}$-structure $\mathscr{A}$, $\mathscr{A} \models \varphi_n$ *implies the existence of an* *n-grid* in $\mathscr{A}$ (Theorem 7.1). Obviously Theorems 5.1, 6.1, and 7.1 show that $\mathscr{A} \models \varphi_n$ cannot hold in a semialgebraic $L_{\beta\delta}$-structure $\mathscr{A}$ with $|\mathscr{A}| = \mathbf{R}^k$ and $1 \leq k < n$.

5. Semialgebraic sets and n-towers

For every $n \geq 1$, the class of *semialgebraic* subsets of $\mathbf{R}^n$ is the smallest class of sets closed under forming complements, finite unions, and finite intersections, and containing all sets of the form $\{s \in \mathbf{R}^n : f(s) \geq 0\}$, where $s = (s_1, \ldots, s_n)$ and f is a polynomial in n variables with real coefficients. The well-known Seidenberg-Tarski theorem states that *a set is semialgebraic if it is* $\mathscr{R}$*-definable.* To prove this, one views $\mathscr{R}$ as a model of the theory RCF and applies Tarski's theorem on the elimination of quantifiers from formulas of RCF. See [B, K-K, Se, T1], and a recent discussion in [VdD3]. Thus, without further reference to this result, we shall accept from now on *$\mathscr{R}$-definable* as equivalent to *semialgebraic*.

A *continuous* map between semialgebraic subsets of $\mathbf{R}^m$ and $\mathbf{R}^n$ is called *semialgebraic* if its graph is a semialgebraic subset of $\mathbf{R}^m \times \mathbf{R}^n$. According to Definition 4.1, an r-place relation on a semialgebraic subset of $\mathbf{R}^n$ is *semialgebraic* if its graph is a semialgebraic subset of $\mathbf{R}^{nr}$. The following properties of semialgebraic sets, maps, and relations will be needed later:

I. The intersection of two semialgebraic sets is semialgebraic.

II. The composite of semialgebraic maps (wherever defined) is semialgebraic.

III. The direct and inverse image of a semialgebraic set under a semialgebraic map is semialgebraic.

IV. Each semialgebraic equivalence relation on a semialgebraic set has a semialgebraic set of representatives.

V. Each semialgebraic set has a finite semialgebraic triangulation.

Properties I, II, and III follow without difficulty from Definition 4.1 of $\mathscr{R}$-definability. Property IV is shown in [VdD1, Proposition 4.3]. To state property V precisely, let us use Δ as a generic name for an open simplex in $\mathbf{R}^m$:

$$\Delta = \left\{ \sum_{i=0}^{d} t_i v_i \colon \sum_{i=0}^{d} t_i = 1 ; \text{ all } t_i > 0 \right\},$$

where the vertices $v_0, \dots, v_d \in \mathbf{R}^m$ are not all contained in some hyperplane of dimension $d-1$. We call d the *dimension* of Δ. Then a *finite simplicial complex* in $\mathbf{R}^m$ is a set $K = \{\Delta_j \colon j \in J\}$, where J is finite and Δ_j are disjoint open simplexes in $\mathbf{R}^m$, such that all faces of each Δ_j also belong to K. In 1964, Łojasiewicz [L] proved the following:

Triangulation theorem. *Given a semialgebraic set S (in some $\mathbf{R}^n$), there exists a finite simplicial complex $K = \{\Delta_j \colon j \in J\}$ (in some $\mathbf{R}^m$) and, for some subset $J' \subset J$, a semialgebraic map*

$$h \colon S \to \bigcup \{\Delta_j \colon j \in J'\},$$

which is also a topological homeomorphism.

Other proofs can be found in [C and H]. (The usually made assumption that S is bounded is not relevant for the previous version; see [H, Remark 1.10].) In a more general setting of 0-minimal Tarski systems, this result is stated in [VdD2] and proved in [VdD1]. As a corollary we obtain

Lemma 5.1. *An infinite semialgebraic set contains an arc (homeomorphic image of a nonzero interval in $\mathbf{R}$).*

We define now by recursion on n the n-towers under semialgebraic sets.

Definition 5.1. (1) The 1-towers under S are precisely the uncountable semialgebraic subsets of S.

(2) If $n \geq 2$, then an n-tower $\mathbf{T}$ under S is a system $(\mathbf{\Sigma}, \mathbf{\Pi}, \mathbf{T}')$, where
(a) $\mathbf{\Sigma}$ is any uncountable set,
(b) $\mathbf{\Pi}$ is a mapping that assigns to every $b \in \mathbf{\Sigma}$ a semialgebraic set $\mathbf{\Pi}(b) \subset S$ so that $\mathbf{\Pi}(b) \cap \mathbf{\Pi}(b') = \varnothing$ for every two distinct b, b' in $\mathbf{\Sigma}$,
(c) $\mathbf{T}'$ is a mapping that assigns to every $b \in \mathbf{\Sigma}$ an $(n-1)$-tower $\mathbf{T}'(b)$ under $\mathbf{\Pi}(b)$.

An n-tower under a subset S of $\mathbf{R}^k$ will also be called simply an *n-tower in* $\mathbf{R}^k$. It is easily seen that a set S under which there exists an n-tower is uncountable. Thus, for $n \geq 2$, each $\mathbf{\Pi}(b)$ is uncountable and we can regard $\mathbf{\Pi}(b)$ as a 1-tower under itself. We conclude that if there is an n-tower in $\mathbf{R}^k$ with $n \geq 2$, then there is also a 2-tower in $\mathbf{R}^k$.

Lemma 5.2. *If $n \geq 2$, then there does not exist an n-tower in $\mathbf{R}$.*

Proof. By the preceding remarks, it is enough to show that there does not exist a 2-tower in $\mathbf{R}$. Now, if there were a 2-tower $(\mathbf{\Sigma}, \mathbf{\Pi}, \mathbf{T}')$ in $\mathbf{R}$, we would have an uncountable family $\{\mathbf{\Pi}(b): b \in \mathbf{\Sigma}\}$ of uncountable, mutually disjoint, semialgebraic subsets of $\mathbf{R}$. By Lemma 5.1, each of these sets would have to contain an arc, thus also a rational number. However, we do not have uncountably many rational numbers.

Lemma 5.3. *If S, S_1, S_2 are semialgebraic such that $S \subset S_1 \cup S_2$ and there exists an n-tower under S, then there exists an n-tower under at least one of the intersections $S \cap S_1$, $S \cap S_2$.*

Proof. We use induction on n. If $n = 1$, we are given a 1-tower under S, i.e., an uncountable semialgebraic subset S' of S. Then at least one of the sets $S' \cap S_1$, $S' \cap S_2$ is uncountable, and is thus a 1-tower under itself.

We assume now that $n > 1$ and that the assertion holds for all $(n-1)$-towers. Let $\mathbf{T} = (\mathbf{\Sigma}, \mathbf{\Pi}, \mathbf{T}')$ be an n-tower under $S \subset S_1 \cup S_2$. Then for each $b \in \mathbf{\Sigma}$, $\mathbf{T}'(b)$ is an $(n-1)$-tower under $\mathbf{\Pi}(b) \subset S_1 \cup S_2$, so by the inductive assumption, there is an $(n-1)$-tower under $\mathbf{\Pi}(b) \cap S_1$ or under $\mathbf{\Pi}(b) \cap S_2$. Renaming the S_i, if need be, we may assume that for each $b \in \mathbf{\Sigma}'$, where $\mathbf{\Sigma}'$ is some *uncountable* subset of $\mathbf{\Sigma}$, there is an $(n-1)$-tower $\mathbf{T}''(b)$ under $\mathbf{\Pi}(b) \cap S_1$. Denoting $\mathbf{\Pi}(b) \cap S_1$ by $\mathbf{\Pi}'(b)$, we obtain an n-tower $(\mathbf{\Sigma}', \mathbf{\Pi}', \mathbf{T}'')$ under $S \cap S_1$. □

Given a semialgebraic injective map $h: S \to S'$, one can associate in an obvious fashion with every n-tower $\mathbf{T}$ under S an n-tower $h(\mathbf{T})$ under S': the image of $\mathbf{T}$ by h. We shall use this observation several times in the subsequent proof.

Lemma 5.4. *If $k, n \geq 2$ and there exists an n-tower in $\mathbf{R}^k$, then there exists an $(n-1)$-tower in $\mathbf{R}^{k-1}$.*

Proof. Let $\mathbf{T} = (\mathbf{\Sigma}, \mathbf{\Pi}, \mathbf{T}')$ be an n-tower in $\mathbf{R}^k$, where $k, n \geq 2$. As we observed earlier, each $\mathbf{\Pi}(b)$ is uncountable. Thus, we have an uncountable family $\{\mathbf{\Pi}(b)\colon b \in \mathbf{\Sigma}\}$ of nonempty, mutually disjoint, semialgebraic subsets of $\mathbf{R}^k$. Obviously at least one of them *has an empty interior* (contains no open subset). Let this be $\mathbf{\Pi}(b)$. By the triangulation theorem, we have a semialgebraic map and topological homeomorphism

$$h\colon \mathbf{\Pi}(b) \to \bigcup\{\Delta_j\colon j \in J'\},$$

where Δ_j are open simplexes in some $\mathbf{R}^m$. Consider the $(n-1)$-tower $\mathbf{T}'(b)$ under $\mathbf{\Pi}(b)$. Since $\mathbf{\Pi}(b) = \bigcup\{h^{-1}(\Delta_j)\colon j \in J'\}$ and Δ_j and hence $h^{-1}(\Delta_j)$ are semialgebraic sets, we infer from Lemma 5.3 that there exists an $(n-1)$-tower under one of the sets $h^{-1}(\Delta_j)$. Applying h, which is injective, we get an $(n-1)$-tower under Δ_j. The restriction of h^{-1} to Δ_j is a topological homeomorphism onto $h^{-1}(\Delta_j) \subset \mathbf{\Pi}(b) \subset \mathbf{R}^k$, and since $\mathbf{\Pi}(b)$ does not contain an open subset of $\mathbf{R}^k$, the dimension of Δ_j is less than k, by the "invariance of domain theorem." See [E-S, p. 303]. Let $d = \dim \Delta_j \le k-1$. We have $\Delta_j \subset \mathbf{R}^m$, so there is a linear map $g\colon \mathbf{R}^m \to \mathbf{R}^d$ such that the restriction $g|\Delta_j$ is injective. Clearly g is semialgebraic and maps the just-obtained $(n-1)$-tower under Δ_j onto an $(n-1)$-tower under $g(\Delta_j) \subset \mathbf{R}^d$. Taking the image of this tower under the linear embedding $\mathbf{R}^d \to \mathbf{R}^{k-1}$, we obtain an $(n-1)$-tower in $\mathbf{R}^{k-1}$.

Theorem 5.1. *If $n > k$, then there does not exist an n-tower in $\mathbf{R}^k$.*

Proof. We use induction on k. The case $k = 1$ is Lemma 5.2. If $k \ge 2$, and there exists an n-tower in $\mathbf{R}^k$ for some $n > k$, then by Lemma 5.4, there also exists an $(n-1)$-tower in $\mathbf{R}^{k-1}$. This provides the inductive step.

6. n-GRIDS IN STRUCTURES

If we choose a rectangular coordinate system in $\mathscr{E}_n$, then the set of hyperplanes perpendicular to the coordinate axes is an example of an n-grid in $\mathscr{E}_n$ (see the example in this section). In general, we have:

Definition 6.1. Let $\mathscr{A}$ be a structure for a first-order language. A system $(S, \Sigma_1, \dots, \Sigma_n, P_1, \dots, P_n)$, where $n \ge 2$, will be called an *n-grid under S in $\mathscr{A}$* if

(1) S is an $\mathscr{A}$-definable subset of $|\mathscr{A}|$;

(2) $\Sigma_1, \dots, \Sigma_n$ are any uncountable sets;

(3) each P_i is a map from Σ_i into the set of $\mathscr{A}$-definable subsets of S, such that $P_i(b) \cap P_i(b') = \varnothing$ for every two distinct b, $b' \in \Sigma_i$ and every $i = 1, \dots, n$;

(4) for arbitrary $b_i \in \Sigma_i$ $(i = 1, \dots, n)$,

$$P_1(b_1) \cap P_2(b_2) \cap \cdots \cap P_n(b_n) \neq \varnothing.$$

Let us observe that in every n-grid all the sets $P_i(b)$ are *uncountable.* Indeed, for any $j \neq i$, $P_i(b)$ contains all the disjoint sets $P_i(b) \cap P_j(a)$, where $a \in \Sigma_j$.

Definition 6.2. Given an n-grid as above and a fixed $a \in \Sigma_n$, we shall call $(P_n(a), \Sigma_1, \dots, \Sigma_{n-1}, P'_1, \dots, P'_{n-1})$, where $P'_i(b) = P_i(b) \cap P_n(a)$ for each $b \in \Sigma_i$; $i \le n-1$, the *associated* $(n-1)$-grid under $P_n(a)$.

Example. For $\mathscr{A} = \mathscr{E}_n$, let us define $\Sigma_1, \dots, \Sigma_n$ and $P_i(b)$ as follows: Σ_i is the ith coordinate axis

$$\Sigma_i = \{(x_1, \dots, x_n) \in \mathbf{R}^n : x_j = 0 \text{ for all } j \neq i\},$$

and for each $b \in \Sigma_i$, $P_i(b)$ is the $(n-1)$-dimensional hyperplane passing through b and perpendicular to Σ_i:

$$P_i(b) = \{(x_1, \dots, x_n) \in \mathbf{R}^n : (0, \dots, 0, x_i, 0, \dots, 0) = b\}.$$

Putting $S = |\mathscr{E}_n| = \mathbf{R}^n$, we check easily that the conditions of Definition 6.1 are satisfied, except perhaps for $\mathscr{E}_n$-definability; however, that will be obtained subsequently as a by-product of the proof of Theorem 7.1. Henceforth, we shall consider n-grids in semialgebraic $L_{\beta\delta}$-structures (see Definition 4.2).

Theorem 6.1. *Given a semialgebraic $L_{\beta\delta}$-structure $\mathscr{A}$, there is for each $n \ge 2$ a mapping that assigns to every n-grid in $\mathscr{A}$ an n-tower in $|\mathscr{A}|$, so that to a grid under S there corresponds a tower under S.*

Proof. We define the mappings by recursion on n. Let us consider an n-grid $(S, \Sigma_1, \dots, \Sigma_n, P_1, \dots, P_n)$ in $\mathscr{A}$, where $n \ge 2$.

We assign to this n-grid an n-tower $\mathbf{T} = (\boldsymbol{\Sigma}, \boldsymbol{\Pi}, \mathbf{T}')$ under S, where $\boldsymbol{\Sigma} = \Sigma_n$, $\boldsymbol{\Pi}(b) = P_n(b)$ for each $b \in \boldsymbol{\Sigma}$, and $\mathbf{T}'$ is yet to be defined. Obviously $\boldsymbol{\Sigma}$ is uncountable and $\boldsymbol{\Pi}(b)$ is semialgebraic and also uncountable, as we observed just after Definition 6.1. Moreover:

$$\boldsymbol{\Pi}(b) \cap \boldsymbol{\Pi}(b') = P_n(b) \cap P_n(b') = \varnothing \quad \text{for } b \neq b'.$$

Hence, if $n = 2$, we may take $\mathbf{T}'(b) = \boldsymbol{\Pi}(b)$, which is a 1-tower under itself, and we are done. If $n > 2$ then, by recursive assumption, there is already assigned to every $(n-1)$-grid under S' in $\mathscr{A}$ an $(n-1)$-tower under S'. In particular, consider for each $b \in \Sigma_n = \boldsymbol{\Sigma}$, the *associated* (see Definition 6.2) $(n-1)$-grid under $P_n(b)$, i.e., under $\boldsymbol{\Pi}(b)$. By assumption, to this $(n-1)$-grid there is assigned an $(n-1)$-tower under $\boldsymbol{\Pi}(b)$. We denote this tower by $\mathbf{T}'(b)$, thus completing the description of the n-tower $\mathbf{T}$.

7. Existence of n-grids

To complete the proof of Theorem 4.2 (and hence of Theorem 1.1; see end of §4) we still need to show the following:

Theorem 7.1. *For each $n \ge 2$, there is a theorem φ_n of E_n such that if $\mathscr{A}$ is a semialgebraic $L_{\beta\delta}$-structure and $\mathscr{A} \models \varphi_n$, then there exists an n-grid under $S = |\mathscr{A}|$ in $\mathscr{A}$.*

Proof. Let $\mathscr{A}$ be a semialgebraic $L_{\beta\delta}$-structure (fixed for the rest of this proof). Let us associate with any $a_0, a_1, a_2 \in |\mathscr{A}|$ certain $\mathscr{A}$-definable sets, which we shall call by the names "line" and "hyperplane," since such names are normally used if a_0, a_1, a_2 are distinct and $\mathscr{A}$ is $\mathscr{E}_n$. For this, we need two formulas of $L_{\beta\delta}$ which we shall denote by λ and π.

λ is $\beta(x_0, x_1, x_2) \vee \beta(x_1, x_2, x_0) \vee \beta(x_2, x_0, x_1)$ and, if $a_0 \neq a_1$, the set $\{\lambda[a_0, a_1, \cdot]\}$ (see Definition 4.1, with parameter variables x_0, x_1) gets the name "the line through a_0, a_1."

π is chosen so that for $a_0 \neq a_1$, $\{\pi[a_0, a_1, a_2, \cdot]\}$ should be called "the $(n-1)$-dimensional hyperplane containing a_2 and perpendicular to the line $\{\lambda[a_0, a_1, \cdot]\}$"; explicitly, we may take for π the formula:

$$\exists x_4 \exists x_5 (\neg e(x_4, x_5) \,\&\, \lambda[x_0, x_1, x_4] \,\&\, \lambda[x_0, x_1, x_5] \\ \&\, \delta(x_2, x_4, x_2, x_5) \,\&\, \delta(x_3, x_4, x_3, x_5)) .$$

Further, let $\overline{\beta}$ be the formula expressing "strict betweeness":

$$\beta(x_1, x_2, x_3) \,\&\, \neg e(x_1, x_2) \,\&\, \neg e(x_2, x_3) ,$$

and let σ be the conjunction of these five formulas:

(1) $\forall x_0 e(x_0, x_0)$,
(2) $\forall x_0 \forall x_1 (e(x_0, x_1) \rightarrow e(x_1, x_0))$,
(3) $\forall x_0 \forall x_1 \forall x_2 (e(x_0, x_1) \,\&\, e(x_1, x_2) \rightarrow e(x_0, x_2))$,
(4) $\forall x_0 \forall x_1 \exists x_2 (\neg e(x_0, x_1) \rightarrow \overline{\beta}[x_0, x_2, x_1])$,
(5) $\forall x_0 \forall x_1 \forall x_2 \forall x_3 (\overline{\beta}[x_0, x_2, x_1] \,\&\, \overline{\beta}[x_0, x_3, x_2] \rightarrow \overline{\beta}[x_0, x_3, x_1])$.

Lemma 7.1. *Suppose $\mathscr{A} \models \sigma$ and let a_0, a_1 be such that $\mathscr{A} \models \neg e(a_0, a_1)$. Then there exists an uncountable set $\Sigma \subset \{\lambda[a_0, a_1, \cdot]\}$ such that $\mathscr{A} \models \neg e(b, b')$ whenever b, $b' \in \Sigma$ and $b \neq b'$.*

Proof. It follows from $\mathscr{A} \models \sigma$ that e is an equivalence relation on $|\mathscr{A}|$. We show first that $\{\lambda[a_0, a_1, \cdot]\}$ contains infinitely many pairwise not e-equivalent elements. Let $a_0, a_1, a_2, \ldots$ be obtained from (4), so that $\mathscr{A} \models \overline{\beta}[a_0, a_{i+1}, a_i]$ for $i = 1, 2, \ldots$. By the transitivity (5), we conclude that $\mathscr{A} \models \overline{\beta}[a_0, a_{i+1}, a_j]$ for all $j = 1, 2, \ldots, i$, which implies $\mathscr{A} \models \neg e(a_{i+1}, a_j)$, by the definition of $\overline{\beta}$. Also, putting $j = 1$, we get $\mathscr{A} \models \overline{\beta}[a_0, a_{i+1}, a_1]$ whence all a_i are on the required line.

Obviously e is an $\mathscr{A}$-definable relation, so it is semialgebraic. Restricting e to $\{\lambda[a_0, a_1, \cdot]\}$ we get a semialgebraic equivalence relation on a semialgebraic set. Thus, property IV in §5 yields a semialgebraic set of representatives $\Sigma \subset \{\lambda[a_0, a_1, \cdot]\}$ of the e-equivalence classes. As we have already shown, each of the $a_0, a_1, \ldots$ is in a different equivalence class. Hence, Σ is infinite. By Lemma 5.1, Σ is uncountable. □

Evidently there exists a sentence τ_n in $L_{\beta\delta}$ such that the meaning of $\mathscr{A} \models \tau_n$ is as follows:

$$\begin{array}{ll} & \text{THERE EXIST } a_0, a_1, \ldots, a_n \text{ such that:} \\ & \neg e(a_i, a_j) \text{ for all distinct } i, j = 0, \ldots, n \\ \text{AND} & \text{for any } 1 \le i \le n \text{ and every } b, b' \in \{\lambda[a_0, a_i, \cdot]\} \text{ such that} \\ & \neg e(b, b') \text{ one has } \{\pi[a_0, a_i, b, \cdot]\} \cap \{\pi[a_0, a_i, b', \cdot]\} = \varnothing \\ \text{AND} & \text{for every } b_1, \ldots, b_n, \text{ if } b_i \in \{\lambda[a_0, a_i, \cdot]\} \text{ for } 1 \le i \le n \\ & \text{then } \{\pi[a_0, a_1, b_1, \cdot]\} \cap \cdots \cap \{\pi[a_0, a_n, b_n, \cdot]\} \ne \varnothing. \end{array}$$

Let φ_n be σ & τ_n. We have $\mathscr{E}_n \models \tau_n$ (taking a_0 above to be the origin of a rectangular coordinate system in $\mathscr{E}_n$ and a_i on the ith coordinate axis). Hence, $E_n \vdash \tau_n$, and since also $E_n \vdash \sigma$, we get $E_n \vdash \varphi_n$, as required.

Suppose $\mathscr{A} \models \varphi_n$, and let $a_0, a_1, \ldots, a_n \in |\mathscr{A}|$ have the properties assured by $\mathscr{A} \models \tau_n$. For each $i = 1, \ldots, n$ we denote by Σ_i an uncountable subset of $\{\lambda[a_0, a_i, \cdot]\}$ such that $\mathscr{A} \models \neg e(b, b')$ for every two distinct b, $b' \in \Sigma_i$ (see Lemma 7.1). Further, let $S = |\mathscr{A}|$ and let $P_i(b) = \{\pi[a_0, a_i, b, \cdot]\}$ for every $b \in \Sigma_i$ and $i = 1, \ldots, n$. Then, by $\mathscr{A} \models \tau_n$, the system $(S, \Sigma_1, \ldots, \Sigma_n, P_1, \ldots, P_n)$ satisfies all the conditions of Definition 6.1. Thus we have obtained an n-grid in $\mathscr{A}$.

References

[B] M. Boffa, *Elimination des quantificateurs en algèbre*, Bull. Soc. Math. Belg. **32** (1980), 107–133.

[C-K] C. C. Chang and H. J. Keisler, *Model theory*, North-Holland, Amsterdam, 1973.

[C] M. Coste, *Ensembles semi-algébriques* (Proc. Conf. on Semi-algebraic Geometry and Quadratic Forms, Rennes, 1981), Lecture Notes in Math., vol. 959, Springer-Verlag, Berlin and New York, 1982.

[VdD1] L. van den Dries, *Definable sets in* 0*-minimal structures*, Lectures at University of Konstanz, spring 1985.

[VdD2] ——, *A generalization of the Tarski-Seidenberg theorem, and some nondefinability results*, Bull. Amer. Math. Soc. **15** (1986), 189–193.

[VdD3] ——, *Alfred Tarski's elimination theory for real closed fields*, J. Symbolic Logic **53** (1988), 7–19.

[E-S] S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton Univ. Press, Princeton, N. J., 1952.

[K-K] G. Kreisel and J. L. Krivine, *Elements of mathematical logic*, North-Holland, Amsterdam, 1967.

[H] H. Hironaka, *Triangulation of algebraic sets*, Proc. Sympos. Pure Math., vol. 29, Amer. Math. Soc., Providence, R.I., 1975, pp. 165–185.

[L] S. Lojasiewicz, *Triangulation of semialgebraic sets*, Ann. Sci. École Norm. Sup. (3) **18** (1964), 449–474.

[M-V] R. Montague and R. L. Vaught, *A note on theories with selectors*, Fund. Math. **47** (1959), 243–247.

[M] J. Mycielski, *A lattice of interpretability types of theories*, J. Symbolic Logic **42** (1977), 297–305.

[M-P-S] J. Mycielski, P. Pudlák, and A. S. Stern, *A lattice of chapters of mathematics*, Mem. Amer. Math. Soc. (to appear).

[Se] A. Seidenberg, *A new decision method for elementary algebra*, Ann. of Math. (2) **60** (1954), 365–374.

[Sz] L. W. Szczerba, *Dimensions of interpretations* (*abstract*), J. Symbolic Logic **49** (1984), 677.

[T1] A. Tarski, *Decision method for elementary algebra and geometry*, 2nd ed., Berkeley, 1951.

[T2] ——, *What is elementary geometry*?, Symposium on the Axiomatic Method with Special Reference to Geometry and Physics, North-Holland, Amsterdam, 1959, pp. 16–29.

Department of Mathematics, Sultan Qaboos University, Sultanate of Oman