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THE DIFFEOTOPY GROUP OF THE TWISTED 2-SPHERE BUNDLE OVER THE CIRCLE

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ABSTRACT. The diffeotopy group of the nontrivial 2-sphere bundle over the circle is shown to be isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. The first generator is induced by a reflection across the base circle, while a second generator comes from rotating the 2-sphere fiber as one travels around the base circle. The technique employed also shows that homotopic diffeomorphisms are diffeotopic.

1. INTRODUCTION

Gluck proved in [G] that the diffeotopy group $\mathscr{H}$ of $S^1 \times S^2$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. In fact, for a large class of 3-dimensional manifolds, Rubinstein, Laudenbach, Waldhausen and others computed the diffeotopy groups. The methods exploited there do not seem to work for the twisted S^2-bundle over S^1, $S^1 \tilde{\times} S^2$. Therefore, we use different methods in conjunction with Gluck's arguments to compute the diffeotopy group $S^1 \tilde{\times} S^2$. In this paper, we shall prove the following:

Theorem. *The diffeotopy group $\mathscr{G}$ of the twisted 2-sphere bundle over S^1, $S^1 \tilde{\times} S^2$, is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.*

Corollary. *Each diffeomorphism homotopic to the identity is diffeotopic to the identity.*

2. NOTATION AND PROOF OF THE THEOREM

We adopt the following notation. Let I be the unit interval $[0, 1]$ and S^1 the unit circle in the plane, i.e., the set of all complex numbers whose absolute value is 1. We will use $\exp 2\pi i\theta$ as a point of S^1, where θ is a real number and i is $\sqrt{-1}$. Let S^2 be the unit sphere in the 3-dimensional Euclidean space. We will write v as a point of S^2. In $S^1 \tilde{\times} S^2$, $\sim$ means every $(\exp 2\pi i\theta, v)$ in $S^1 \times S^2$ is identified with $(-\exp 2\pi i\theta, -v)$. D^2 will be the unit disk in the

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plane which is the set of all complex numbers whose absolute value is less than or equal to 1.

To compute $\mathscr{G}$, we need the following crucial lemma.

(2.1) **Lemma.** *Let* $\mathrm{Map}^1(S^2, S^2)$ *be the set of all degree one continuous maps from* S^2 *to* S^2. *We assume that the topology is induced from the compact open topology.*

Define a $\mathbb{Z}_2$*-action on* $\mathrm{Map}^1(S^2, S^2)$ *by*

$$\mathbb{Z}_2 \times \mathrm{Map}^1(S^2, S^2) \to \mathrm{Map}^1(S^2, S^2)$$
$$\lambda \to A \circ \lambda \circ A$$

where A *is the antipodal mapping of* S^2. *Then the fundamental group of the quotient space is* $\mathbb{Z}_2$.

Proof. Put $E = \{[\alpha] | \alpha : I \to \mathrm{Map}^1(S^2, S^2),\ \alpha(0) = \text{the identity map}\}$, where $[\alpha] = [\beta]$ means that $\alpha(1) = \beta(1)$ and $\alpha * \overline{\beta}$ is homotopic to a constant path. More precisely, $\alpha * \overline{\beta}$ is the composition of α and β, i.e.,

$$\alpha * \overline{\beta}(t) = \begin{cases} \alpha(2t), & 0 \le t \le 1/2, \\ \overline{\beta}(2t-1) = \beta(2-2t), & 1/2 \le t \le 1. \end{cases}$$

Define $\pi : E \to \mathrm{Map}^1(S^2, S^2)$ by $\pi([\alpha]) = \alpha(1)$. The space E is simply connected, since $\mathrm{Map}^1(S^2, S^2)$ is path-connected, and has a universal covering space (cf. [M, p. 394]).

We define two commuting $\mathbb{Z}_2$-actions on E. The first is given by

$$(x, [\alpha]) \to [A \circ \alpha \circ A]$$

where x denotes the nontrivial element of $\mathbb{Z}_2$ in the first action. Next, we are going to use the fact, proved in [Hu], that $\Pi_1 \mathrm{Map}^1(S^2, S^2) = \mathbb{Z}_2$. Then, given any path α starting at the identity, we can find a path γ, also, starting at the identity map with $\alpha(1) = \gamma(1)$ and $\alpha * \overline{\gamma}$ not homotopic to a constant path. We define the second involution

$$(y, [\alpha]) \to [\gamma]$$

where y is the nontrivial element of $\mathbb{Z}_2$. This involution describes the generator of the group of covering transformations on E. This enables us to conclude that

$$\Pi_1(\mathrm{Map}^1(S^2, S^2)/\mathbb{Z}_2) = \mathbb{Z}_2, \quad \text{i.e.} \quad \Pi_1(E/\mathbb{Z}_2 \oplus \mathbb{Z}_2) = \mathbb{Z}_2,$$

because of M. Armstrong's result in [A]:

Let G be a discontinuous group of homeomorphism of a simply connected, locally path connected, Hausdorff space X. Then the fundamental group of the quotient X/G is G/N where N is the subgroup of G generated by those elements which have fixed points.

Therefore, we have only to show that there exists no $[\alpha]$ in E such that $xy[\alpha] = [\alpha]$ since y has no fixed point and x has a fixed point.

Suppose there exists such $[\alpha]$. Then we get

$$\mathscr{x}\mathscr{y}[\alpha] = \mathscr{x}[\gamma] = [A \circ \gamma \circ A] = [\alpha].$$

Since $\alpha(1) = A \circ \gamma(1) \circ A = \gamma(1)$, $\alpha(1)$ is in the set $\mathscr{F}$ of fixed points of the $\mathbb{Z}_2$-action on $\mathrm{Map}^1(S^2, S^2)$. Since $[\gamma] = [A \circ \alpha \circ A]$, $(A \circ \alpha \circ A) * \overline{\gamma} = 0$, and $\alpha * \overline{\gamma}$ is not homotopic to a constant path. Since $(A \circ \alpha \circ A) * \overline{\gamma} * \gamma * \overline{\alpha}$ is not homotopic to a constant path, $(A \circ \alpha \circ A) * \overline{\alpha}$ is not homotopic to a constant path. This will lead to a contradiction.

We claim that there is a path β in $\mathrm{Map}^1(S^2, S^2)$ going from identity to $\alpha(1)$ which is fixed under the involution on $\mathrm{Map}^1(S^2, S^2)$. We use the fact that the set of self-homotopy equivalences of RP^2 is path-connected (see [GK]). This set corresponds exactly to the set of maps in $\mathrm{Map}^1(S^2, S^2)$ that are fixed by the given action $\lambda \to A \circ \lambda \circ A$. So there is a map $\beta : I \to \mathrm{Map}^1(S^2, S^2)$ with $\beta(0) =$ identity and $\beta(1) = \alpha(1)$, such that $\beta(t)$ lies in the fixed points set $\mathscr{F}$. Observe that $(A \circ \alpha \circ A) * \overline{\beta} * \beta * \overline{\alpha}$ is not homotopic to a constant path.

Let $\delta = \beta * \overline{\alpha}$. Then $A \circ \delta \circ A = (A \circ \beta \circ A) * (A \circ \overline{\alpha} \circ A) = \beta * (A \circ \overline{\alpha} \circ A)$. Since $\lambda \to A \circ \lambda \circ A$ is an involution and $\Pi_1(\mathrm{Map}^1(S^2, S^2)) = \mathbb{Z}_2$, $A \circ \delta \circ A$ is homotopic to δ. Also, each element is its own inverse and $\beta * (A \circ \overline{\alpha} \circ A)$ is homotopic to $(A \circ \alpha \circ A) * \overline{\beta}$. Thus $(A \circ \alpha \circ A) * \overline{\beta} * \beta * \overline{\alpha}$ is homotopic to $\delta * \delta$, which is trivial. This is a contradiction. We have proved the lemma. □

(2.2) **Corollary.** *Let g be the self-diffeomorphism of $S^1 \tilde{\times} S^2$ defined by*

$$[\exp 2\pi i\theta, v] \to \left[\exp 2\pi i\theta, \begin{pmatrix} \cos 4\pi\theta & \sin 4\pi\theta & 0 \\ -\sin 4\pi\theta & \cos 4\pi\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} (v)\right].$$

Then g cannot be extended to a map from $(D^2 \times S^2)/\simeq$ to itself, where $\simeq$ means every point $(r \exp 2\pi i\theta, v)$ in $D^2 \times S^2$ is identified with $(-r \exp 2\pi i\theta, -v)$, $0 \le r \le 1$.

Proof. Suppose there exists an extension k of g. Since $D^2 \times S^2$ is the universal covering space of $(D^2 \times S^2)/\simeq$, we can lift k to $\tilde{k}$ on $D^2 \times S^2$. Let us examine the value of the second coordinate under the mapping $\tilde{k}$, i.e., consider the following commutative diagram

$$\begin{array}{ccc}
(r \exp 2\pi i\theta, v) & & (_, K(r \exp 2\pi i\theta, v)) \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
D^2 \times S^2 & \xrightarrow{\tilde{k}} & D^2 \times S^2 \\
\downarrow & & \downarrow \\
(D^2 \times S^2)/\simeq & \xrightarrow{k} & (D^2 \times S^2)/\simeq
\end{array}$$

Without loss of generality, since $\mathrm{Map}^1(S^2, S^2)$ is path-connected, we may assume that K is a map from $D^2 \times S^2$ to S^2 such that

$$K(\exp 2\pi i\theta, v) = \begin{pmatrix} \cos 4\pi\theta & \sin 4\pi\theta & 0 \\ -\sin 4\pi\theta & \cos 4\pi\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} (v),$$

$$K(0, v) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} (v),$$

and

$$-K(-r\exp 2\pi i\theta, -v) = K(r\exp 2\pi i\theta, v).$$

Define $\widetilde{K}$ from D^2 to $\mathrm{Map}^1(S^2, S^2)$ by

$$r\exp 2\pi i\theta \to K(r\exp 2\pi i\theta, \_).$$

From the diagram above, we get $\widetilde{K}(-r\exp 2\pi i\theta) = A \circ \widetilde{K}(r\exp 2\pi i\theta) \circ A$. Consider $p \circ \widetilde{K}$. Then $p \circ \widetilde{K}(-r\exp 2\pi i\theta) = p \circ \widetilde{K}(r\exp 2\pi i\theta)$ where p is the projection from $\mathrm{Map}^1(S^2, S^2)$ to $\mathrm{Map}^1(S^2, S^2)/\mathbb{Z}_2$.

Define $\tilde{\tilde{K}}$ from D^2 to $\mathrm{Map}^1(S^2, S^2)/\mathbb{Z}_2$ by

$$\tilde{\tilde{K}}(r\exp 2\pi i\theta) = p \circ \widetilde{K}(r\exp \pi i\theta).$$

This is well defined and

$$\tilde{\tilde{K}}(\exp 2\pi i\theta) = \left\langle \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta & 0 \\ -\sin 2\pi\theta & \cos 2\pi\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$$

in $\mathrm{Map}^1(S^2, S^2)/\mathbb{Z}_2$ where $\langle\ \rangle$ means the image under the projection p.

$$\tilde{\tilde{K}}(0) = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

This implies $\tilde{\tilde{K}}$ is a homotopy between following two maps

$$\exp 2\pi i\theta \to \left\langle \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta & 0 \\ -\sin 2\pi\theta & \cos 2\pi\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$$

and

$$\exp 2\pi i\theta \to \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Since $p_*\colon \Pi_1(\mathrm{Map}^1(S^2, S^2) \to \Pi_1(\mathrm{Map}^1(S^2, S^2)/\mathbb{Z}_2) = \mathbb{Z}_2$ is onto, and the map $\widetilde{T}$

$$\exp 2\pi i\theta \to \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta & 0 \\ -\sin 2\pi\theta & \cos 2\pi\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the nontrivial loop in $\mathrm{Map}^1(S^2, S^2)$ (which we shall show later),

$$\exp 2\pi i\theta \to \left\langle \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta & 0 \\ -\sin 2\pi\theta & \cos 2\pi\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$$

represents the nontrivial loop, we have a contradiction. It remains to show $\widetilde{T}$ is nontrivial. Consider the nontrivial S^2-fiber bundle over S^2 with structure group $SO(3)$, i.e., the space is given as follows:

$$\begin{aligned}&(D^2 \times S^2) \cup_T (D^2 \times S^2) \\ &\qquad = \text{gluing two copies of } D^2 \times S^2 \text{ along the boundary by } T.\end{aligned}$$

where T is a map from $S^1 \times S^2$ to itself given by

$$(\exp 2\pi i\theta, v) \to (\exp 2\pi i\theta, \ \widetilde{T}(\exp 2\pi i\theta)(v)).$$

Suppose $\widetilde{T}$ is homotopic to a constant path. Then we can get a homotopy equivalence from $(D^2 \times S^2) \cup_{\mathrm{id}} (D^2 \times S^2) (= S^2 \times S^2)$ to $(D^2 \times S^2) \cup_T (D^2 \times S^2)$, and we have a contradiction (see [S]). This completes the proof. □

Remark. If we use the results in [KKR], we can give a shorter proof of the corollary above. More precisely, if g can be extended, then we may construct two spaces which must be homotopy equivalent, but by the homotopy invariant in [KKR], the two spaces can not be homotopy equivalent. So we have a contradiction.

(2.3) **Theorem.** *The diffeotopy group* $\mathscr{G}$ *of* $S^1 \tilde{\times} S^2$ *is* $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Proof. Our argument is divided into 4 steps.

(1) We construct a map φ from $\mathscr{G}$ to $\mathscr{H}/\mathbb{Z}_2$ $(= \mathbb{Z}_2 \oplus \mathbb{Z}_2)$.
(2) We show that the image of φ is $\mathbb{Z}_2$.
(3) $\operatorname{Ker}\varphi$ is $\mathbb{Z}_2$.
(4) $\mathbb{Z}_2 \rightarrowtail \mathscr{G} \to \mathbb{Z}_2$ is split.

Recall that $\mathscr{H} = \mathrm{Diff}(S^1 \times S^2)/\sim$ where $\sim$ is the normal subgroup consisting of those diffeomorphisms which are diffeotopic to identity and $\mathscr{G} = \mathrm{Diff}(S^1 \tilde{\times} S^2)/\sim$, where $\sim$ means the same as above.

According to Gluck,

$$\mathscr{H} = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle T\rangle \oplus \langle f\rangle \oplus \langle h\rangle,$$

$$S^1 \times S^2 \to S^1 \times S^2,$$

$$T\colon (\exp 2\pi i\theta, v) \to \left(\exp 2\pi i\theta, \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta & 0 \\ -\sin 2\pi\theta & \cos 2\pi\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}(v)\right),$$

$$f\colon (\exp 2\pi i\theta, v) \to (\exp(-2\pi i\theta), v),$$

$$h\colon (\exp 2\pi i\theta, v) \to (\exp 2\pi i\theta, -v).$$

(i) First step. Define a $\mathbb{Z}_2$-action on $\mathscr{H}$ as follows:

$$\begin{gathered}\mathbb{Z}_2 \times \mathscr{H} \to \mathscr{H} \\ \langle \ell \rangle \to \langle \phi \circ \ell \rangle\end{gathered}$$

where $\phi\colon S^1 \times S^2 \to S^1 \times S^2$ is a diffeomorphism defined by

$$(\exp 2\pi i\theta\,,\, v) \to (-\exp 2\pi i\theta\,,\, -v)\,.$$

We want to construct a map φ from $\mathscr{G}$ to $\mathscr{H}/\langle\phi\rangle$. To do it, we have to show that, given any diffeomorphism w of $S^1 \tilde{\times} S^2$, we have a lift defined up to a covering transformation, i.e.,

$$\begin{array}{ccc} S^1 \times S^2 & \xrightarrow{\tilde{w}} & S^1 \times S^2 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ S^1 \tilde{\times} S^2 & \xrightarrow{w} & S^1 \tilde{\times} S^2 \end{array}$$

where $p\colon S^1 \times S^2 \to S^1 \tilde{\times} S^2$ is the natural projection.

To show the existence of $\tilde{w}$, consider $(w \circ p)_* \Pi_1(S^1 \times S^2)$. $S^1 \times S^2$ is a double covering of $S^1 \tilde{\times} S^2$, so $p_*\Pi_1(S^1 \times S^2)$ is $2\mathbb{Z} \subset \mathbb{Z} = \Pi_1(S^1 \tilde{\times} S^2)$. Hence, any automorphism of $\Pi_1(S^1 \tilde{\times} S^2)$ preserves $p_*\Pi_1(S^1 \times S^2)$. By the lifting lemma, there exist $\tilde{w}$ from $S^1 \times S^2$ to itself such that $p \circ \tilde{w} = w \circ p$. Note that $p \circ \phi \circ \tilde{w} = w \circ p$, since ϕ is a regular covering transformation. Furthermore, if w_1 is isotopic to w_2 by an isotopy H on $S^1 \tilde{\times} S^2$, then, as in the above argument, we have $\widetilde{H}\colon I \times S^1 \times S^2 \to S^1 \times S^2$ such that $p \circ \widetilde{H} = H \circ (\mathrm{Id} \times p)$. Now we can define a map φ from $\mathscr{G}$ to $\mathscr{H}/\mathbb{Z}_2$ by $\langle\eta\rangle \to \langle\tilde{\eta}\rangle$.

Since $f \circ \phi = \phi \circ f$, f induces a self-diffeomorphism on $S^1 \tilde{\times} S^2$. Thus $\operatorname{Im}\varphi \supset \mathbb{Z}_2$. Note that since h is isotopic to ϕ, $\langle h \rangle$ is trivial in $\mathscr{H}/\mathbb{Z}_2$.

(ii) Second step. We know that $\operatorname{Im}\varphi \supset \mathbb{Z}_2$, from First step. To demonstrate $\operatorname{Im}\varphi = \mathbb{Z}_2$, we shall show that there exist no T' in $\mathscr{H}$ which is isotopic to T and commutes with ϕ.

Suppose that there exists such T'. That means the following:

$$\begin{gathered} S^1 \times S^2 \times I \to S^1 \times S^2\,, \\ \{(\exp 2\pi i\theta\,,\, v)\,,\, t\} \to (G_t(\exp 2\pi i\theta\,,\, v)\,,\, F_t(\exp 2\pi i\theta\,,\, v)) \end{gathered}$$

where $t = 0$, $G_0 = \exp 2\pi i\theta$, and

$$F_0(\exp 2\pi i\theta)\,,\, v) = \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta & 0 \\ -\sin 2\pi\theta & \cos 2\pi\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}(v)\,.$$

Note that F_0 is the second coordinate of T. For $t = 1$, we have

$$\begin{array}{ccc} (\exp 2\pi i\theta, v) & \longrightarrow & (G_1(\exp 2\pi i\theta, F_1(\exp 2\pi i\theta, v)) \\ \Cap & & \Cap \\ S^1 \times S^2 & \longrightarrow & S^1 \times S^2 \\ \phi \downarrow & & \downarrow \phi \\ S^1 \times S^2 & \xrightarrow{T'} & S^1 \times S^2 \\ \Cup & & \Cup \\ (-\exp 2\pi i\theta, -v) & \longrightarrow & (G_1(-\exp 2\pi i\theta, -v), F_1(-\exp 2\pi i\theta, -v)). \end{array}$$

From the commutativity of the above diagram (since $T' \circ \phi = \phi \circ T'$)

$$F_1(\exp 2\pi i\theta, -v) = -F_1(\exp 2\pi i\theta, v)$$

Define F_t' from S^1 to $\mathrm{Map}^1(S^1, S^2)$ by

$$\exp 2\pi i\theta \to F_t(\exp 2\pi i\theta, \_).$$

Note that, since $\Pi_1(\mathrm{Map}^1(S^2, S^2)) = \mathbb{Z}_2$ is abelian, we need not worry about choosing a base point.

Recall the $\mathbb{Z}_2$-action on $\mathrm{Map}^1(S^2, S^2)$

$$\begin{gathered} \mathbb{Z}_2 \times \mathrm{Map}^1(S^2, S^2) \to \mathrm{Map}^1(S^2, S^2) \\ \lambda \to A \circ \lambda \circ A \end{gathered}$$

For each t,

$$\begin{gathered} [F_t'] \in \Pi_1(\mathrm{Map}^1(S^2, S^2)) \quad \text{and} \\ [p \circ F_t'] \in \Pi_1(\mathrm{Map}^1(S^2, S^2)/\mathbb{Z}_2), \end{gathered}$$

where [ ] means the equivalence class of loops. Then $[p \circ F_1']$ is trivial, since $p \circ F_1'(-\exp 2\pi i\theta) = p \circ F_1'(\exp 2\pi i\theta)$. Therefore $[p \circ F_0']$ is trivial, since $p \circ F_1'$ is homotopic to $p \circ F_0'$.

By (2.1), $\Pi_1(\mathrm{Map}^1(S^2, S^2)/\mathbb{Z}_2) = \mathbb{Z}_2$. Since $\Pi_1(\mathrm{Map}^1(S^2, S^2)) = \mathbb{Z}_2$ and the nontrivial element is represented by

$$F_0' \colon S^1 \to \mathrm{Map}^1(S^2, S^2)$$

$$\exp 2\pi i\theta \to \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta & 0 \\ -\sin 2\pi\theta & \cos 2\pi\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then, the fact that the nontrivial element in $\Pi_1(\mathrm{Map}^1(S^2, S^2)/\mathbb{Z}_2)$ is lifted as the nontrivial element in $\Pi_1(\mathrm{Map}^1(S^2, S^2))$ implies a contradiction. Thus, we can conclude that $\mathrm{Im}\,\varphi = \mathbb{Z}_2$.

(iii) Third step. Suppose $\langle q \rangle$ in $\ker \varphi$. Then $\tilde{q}$ is isotopic to the identity or $\phi \circ \tilde{q}$ is isotopic to the identity of $S^1 \times S^2$. By a straightforward argument of Gluck (see [G, pp. 315–316] and cf. [T]), we deform q so that the restriction to $\{1, -1\} \tilde{\times} S^2$ is the identity. Since $\tilde{q}$ (or $\phi \circ \tilde{q}$) is isotopic to the identity, q can be considered as a diffeomorphism $\overline{q}$ from $I \times S^2$ to $I \times S^2$ such that the restriction to $\{1, -1\} \times S^2$ of $\overline{q}$ is the identity.

By Gluck (cf. [Ha]), $\overline{q}$ is isotopic to the identity or to $\overline{d}$ while fixing the $\{1, -1\} \times S^2$, where $\overline{d}: I \times S^2 \to I \times S^2$

$$(t, v) \to \left(t, \begin{pmatrix} \cos 2\pi t, & \sin 2\pi t & 0 \\ -\sin 2\pi t, & \cos 2\pi t & 0 \\ 0 & 0 & 1 \end{pmatrix} (v)\right).$$

We claim that $\overline{d}$ is $\overline{g}$, where g is the self-diffeomorphism of $S^1 \tilde{\times} S^2$ in (2.2). Obviously, g is the identity on $\{1, -1\} \tilde{\times} S^2 \subset S^1 \tilde{\times} S^2$. Restrict g to $S^1 \tilde{\times} S^2 - \{1, -1\} \tilde{\times} S^2$, and under the following identification,

$$\begin{array}{ccc} (\theta, v) & \longrightarrow & [(\exp 2\pi i\theta, v)] \\ \in & & \in \\ (0, 1/2) \times S^2 & \longrightarrow & S^1 \tilde{\times} S^2 - \{1, -1\} \tilde{\times} S^2 \\ {\scriptstyle g'} \downarrow & & \downarrow {\scriptstyle g} \\ (0, 1/2) \times S^2 & \longrightarrow & S^1 \tilde{\times} S^2 - \{1, -1\} \tilde{\times} S^2 \\ \in & & \in \end{array}$$

$$\left(\theta, \begin{pmatrix} \cos 4\pi\theta & \sin 4\pi\theta & 0 \\ -\sin 4\pi\theta & \cos 4\pi\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} (v)\right) \to \left[\left(\exp 2\pi i\theta, \begin{pmatrix} \cos 4\pi\theta & \sin 4\pi\theta & 0 \\ -\sin 4\pi\theta & \cos 4\pi\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} (v)\right)\right]$$

we get $g': (0, 1/2) \times S^2 \to (0, 1/2) \times S^2$

$$(\theta, v) \to \left(\theta, \begin{pmatrix} \cos 4\pi\theta & \sin 4\pi\theta & 0 \\ -\sin 4\pi\theta & \cos 4\pi\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} (v)\right).$$

Identify $(0, 1/2) \times S^2$ with $(0, 1) \times S^2$ by $(\theta, v) \to (2\theta, v)$.

Then, $g'': (0, 1) \times S^2 \to (0, 1) \times S^2$

$$(2\theta, v) \to \left(2\theta, \begin{pmatrix} \cos 4\pi\theta & \sin 4\pi\theta & 0 \\ -\sin 4\pi\theta & \cos 4\pi\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} (v)\right), \qquad 0 < \theta < 1/2.$$

If we replace 2θ by θ, we get

$$g'' = l\colon (0,1)\times S^2 \to (0,1)\times S^2$$
$$(\theta, v) \to \left(\theta, \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta & 0 \\ -\sin 2\pi\theta & \cos 2\pi\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}(v)\right).$$

This proves the claim, i.e., $\overline{g} = \overline{d}$. Hence, q is isotopic to the identity or g; g is not isotopic to the identity, otherwise g could be extended to $(D^2 \times S^2)/\simeq$ (see (2.2)). To show $\operatorname{Ker}\varphi = \mathbb{Z}_2$, it remains to show that $\langle g\rangle^2 = \mathrm{id}$, i.e., g^2 is isotopic to the identity. An isotopy is constructed as follows:

$$S^1 \tilde{\times} S^2 \times I \to S^1 \tilde{\times} S^2$$
$$\{[(\exp 2\pi i\theta, v)], t\} \to [(\exp 2\pi i\theta, H(\exp 4\pi i\theta, t)(v))]$$

where $H\colon S^1 \times I \to SO(3)$ is a homotopy between the maps

$$S^1 \to SO(3) \qquad \text{and} \qquad S^1 \to SO(3)$$
$$\exp 2\pi i\theta \to \begin{pmatrix} \cos 4\pi\theta & \sin 4\pi\theta & 0 \\ -\sin 4\pi\theta & \cos 4\pi\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \exp 2\pi i\theta \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(iv) Fourth step. The splitting follows from the fact that $\langle f\rangle$ has order 2. Now we have completed the proof. □

From (2.2) and (2.3), we get the following result.

Corollary. *Any self-diffeomorphism of* $S^1 \tilde{\times} S^2$ *homotopic to the identity is diffeotopic to the identity.*

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