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AN ASYMPTOTIC FORMULA FOR HYPO-ANALYTIC PSEUDODIFFERENTIAL OPERATORS

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ABSTRACT. An asymptotic expansion formula for hypo-analytic pseudodifferential operators is proved and applications are given.

INTRODUCTION

In [2] we introduced hypo-analytic pseudodifferential operators that are naturally associated with the hypo-analytic structures of [1]. In this paper we establish an asymptotic formula for these operators. Such an expansion is essential in several applications. It allows us to define, in a natural way, the symbol of a hypo-analytic pseudodifferential operator, as well as the symbols of the adjoint, transpose and composition of operators. The paper is organized as follows. In Chapter I we discuss and develop the asymptotic formula. Chapter II applies this formula to two results.

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1. ASYMPTOTIC EXPANSION

1. Hypo-analytic structures. We will deal with structures which are a special case of the hypo-analytic structures introduced by Baouendi, Chang and Treves in [1]. We shall summarize the relevant concepts here. Let Ω be a C^∞ manifold of dimension m. A hypo-analytic structure of maximal dimension on Ω is the data of an open covering (U_α) of Ω and for each index α, of m C^∞ functions $Z_\alpha^1, \dots, Z_\alpha^m$ satisfying the following two conditions:

(i) $dZ_\alpha^1, \dots, dZ_\alpha^m$ are linearly independent at each point of U_α;
(ii) if $U_\alpha \cap U_\beta \neq \emptyset$, there are open neighborhoods $\mathscr{O}_\alpha$ of $Z_\alpha(U_\alpha \cap U_\beta)$ and $\mathscr{O}_\beta$ of $Z_\beta(U_\alpha \cap U_\beta)$ and a holomorphic map F_β^α of $\mathscr{O}_\alpha$ onto $\mathscr{O}_\beta$

such that

$$Z_\beta = F_\beta^\alpha \circ Z_\alpha \quad \text{on } U_\alpha \cap U_\beta .$$

We will use the notation $Z_\alpha = (Z_\alpha^1, \dots, Z_\alpha^m)\colon U_\alpha \to C^m$. A distribution h defined in an open neighborhood of a point p_0 of Ω is hypo-analytic at p_0

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if there is a chart (U_α, Z_α) of the above type whose domain contains p_0 and a holomorphic function $\tilde{h}$ defined on an open neighborhood of $Z_\alpha(p_0)$ in C^m such that $h = \tilde{h} \circ Z_\alpha$ in a neighborhood of p_0. By a hypo-analytic local chart we mean an $m+1$-tuple $(U, Z^1, \dots, Z^m)$ [abbreviated (U, Z)] consisting of an open subset U of Ω and of m hypo-analytic functions whose differentials are linearly independent at every point of U.

We will now reason in a hypo-analytic local chart (U, Z) of Ω. Assume that the open set U has been contracted sufficiently so that the mapping $Z = (Z^1, \dots, Z^m) : U \to C^m$ is a diffeomorphism of U onto $Z(U)$ and that U is the domain of local coordinates x_j $(1 \le j \le m)$ all vanishing at a "central point" which will be denoted by 0. We will suppose $Z(0) = 0$ and denote by Z_x the Jacobian matrix of the Z^j with respect to the x^k. Substitution of $Z_x(0)^{-1}Z(x)$ for $Z(x)$ will allow us to assume that $Z_x(0) =$ the identity matrix. Therefore the real part of the Z^j $(j = 1, \dots, m)$ can serve as coordinates and in these new coordinates

$$Z^j = x^j + \sqrt{-1}\phi^j(x), \qquad j = 1, \dots, m,$$

where $\phi = (\phi^1, \dots, \phi^m)$ is real valued with 0 differential at the origin.

Moreover, the functions Z^j are selected so that all the derivatives of order two of the ϕ^j vanish at the origin. Indeed if this is not already so it suffices to replace each Z^j by

$$Z^j - \frac{\sqrt{-1}}{2}\sum_k\sum_l \frac{\partial^2\phi^j}{\partial x^k \partial x^l}(0)Z^k Z^l.$$

We will use $\check{Z}_x$ to denote the transpose of the inverse of the matrix Z_x. Since the first and second derivatives of all the ϕ^j are zero at the origin, after contracting U if necessary, we can find a number c, $0 < c < 1$ such that for all x, y in U and for all ξ in R_m

$$\begin{aligned} &|\Im \check{Z}_x(x)\xi| \le c|\Re \check{Z}_x(x)\xi| \qquad \text{and} \\ &\Re\{\sqrt{-1}\check{Z}_x(x)\xi \cdot (Z(x) - Z(y)) - \langle \check{Z}_x(x)\xi\rangle (Z(x) - Z(y))^2\} \\ &\qquad \le -c|\xi||Z(x) - Z(y)|^2, \end{aligned} \tag{1.1}$$

where $\langle\zeta\rangle = (\zeta_1^2 + \cdots + \zeta_m^2)^{\frac{1}{2}}$ for $|\Im\zeta| < |\Re\zeta|$.

2. Hypo-analytic pseudodifferential operators. We will continue to work in the chart (U, Z) of §1. Our aim now is to briefly describe the hypo-analytic pseudodifferential operators.

Definition 2.1. *Let d be a real number. We denote by $\tilde{S}^d(U, U)$ the space of holomorphic functions $\tilde{a}(z, w, \theta)$ in a product set $\mathscr{O} \times \mathscr{O} \times \mathscr{C}$ with $\mathscr{O}$ an open neighborhood of $Z(U)$, and $\mathscr{C}$ an open cone in $C_m\backslash\{0\}$ containing $R_m\backslash\{0\}$ which have the following property :*

Given any compact subset K of $\mathscr{O}$ and any closed cone $\mathscr{C}' \subset \mathscr{C}$ whose interior contains $R_m\backslash\{0\}$, there is a constant $r > 0$ such that for all z, w in K and all θ in $\mathscr{C}'$, we have

$$|\tilde{a}(z, w, \theta)| \le r(1+|\theta|)^d.$$

Definition 2.2. *We say that a* C^∞ *function* $a(x, y, \theta)$ *in* $U \times U \times R_m$ *is a hypo-analytic amplitude of degree* d *and we write* $a \in S^d(U, U)$ *if there is* $\tilde{a} \in \tilde{S}^d(U, U)$ *such that*

$$a(x, y, \theta) = \tilde{a}(Z(x), Z(y), \theta), \quad \textit{for all } x \textit{ in } U, y \textit{ in } U, 0 \neq \theta \in R_m.$$

Let $a(x, y, \theta) = \tilde{a}(Z(x), Z(y), \theta)$ be a hypo-analytic amplitude of degree $d \in R$ in $U \times U$. For any $\varepsilon > 0$ and $u \in C_c^0(U)$ we define the linear operator

$$(2.3) \quad A^\varepsilon u(x) = \left(\frac{1}{4\pi^3}\right)^{\frac{m}{2}} \int_U \int_{R_m} \exp(\sqrt{-1}\xi \cdot (Z(x) - Z(y) - \varepsilon|\xi|^2)) \cdot a(x, y, \xi) u(y)\, dZ(y)\, d\xi$$

We contract U sufficiently so that for every $x, y \in U$ and $\xi \in R_m$ the point $\check{Z}_x(x)\xi + \sqrt{-1}\langle \check{Z}_x(x)\xi\rangle(Z(x) - Z(y))$ will remain in the cone in which $a(x, y, .)$ is defined. We observe that each $A^\varepsilon u$ is a hypo-analytic function. The results of [2] may be consolidated into:

Theorem 2.1. *When* $\varepsilon \to 0$, A^ε *converges to a continuous linear operator* A: $E'(U) \to \mathscr{D}(U)$ *which maps* $C_c^\infty(U)$ *into* $C_c^\infty(U)$ *continuously. If* u *is hypo-analytic at* 0 *then* Au *is hypo-analytic at* 0.

The first part of the theorem is proved by first deforming the path of ξ-integration from R_m to the image of R_m under the map

$$\xi \to \zeta(\xi) = \check{Z}_x(x)(\xi) + \sqrt{-1}\langle \check{Z}_x(x)\xi\rangle(Z(x) - Z(y)).$$

The second inequality in (1.1) will then force the exponential term in (2.3) to be bounded. The integral can then be treated as an oscillatory integral.

Following [2] we will call A a hypo-analytic pseudodifferential operator. When $Z(x) = x$ this specializes to the usual analytic pseudodifferential operator.

3. Formal hypo-analytic amplitudes. In this section (U, Z) will be as in §2. Our aim is to establish an asymptotic expansion formula for hypo-analytic amplitudes.

Fix a neighborhood $\mathscr{O}$ of $Z(U)$ in C^m, a cone $\mathscr{C}$ in $C^m\backslash\{0\}$ and let $R_0(z, w)$ be a positive continuous function on $\mathscr{O} \times \mathscr{O}$. For each $j = 0, 1, 2, \ldots$ let $k_j(z, w, \theta)$ be a holomorphic function in the set

$$\{(z, w, \theta) \in \mathscr{O} \times \mathscr{O} \times \mathscr{C};\ |\theta| > R_0(z, w)\sup(j, 1)\}.$$

Set $k_j(x, y, \theta) = \tilde{k}_j(Z(x), Z(y), \theta)$.

Definition 3.1. *We will say that the series* $\sum_{j=0}^{\infty} k_j(x, y, \theta)$ *defines a formal hypo-analytic amplitude of degree* d *if there exists a continuous function* $c_0(z, w) > 0$ *on* $\mathscr{O} \times \mathscr{O}$ *such that for all* (z, w) *in* $\mathscr{O} \times \mathscr{O}$ *and all* θ *in* $\mathscr{C}$, $|\theta| > R_0(z, w)\sup(j, 1)$,

$$|\tilde{k}_j(z, w, \theta)| \leq C_0(z, w)^{j+1} j! |\theta|^{d-j}.$$

We now show how to construct a true hypo-analytic amplitude from the formal one given above. We will work in a compact set $K \subseteq U$ and a relatively compact neighborhood $\mathscr{O}_K$ of $Z(K)$ in $\mathscr{O}$. This enables us to replace the functions $C_0(z, w)$ and $R_0(z, w)$ of the above definition by constants C_0 and R_0. We will also assume that the cone $\mathscr{C}$ has been shrunk to satisfy: for some $\delta > 0$, whenever $\theta = \xi + \sqrt{-1}\eta \in \mathscr{C}$, then $\delta|\theta| \leq |\xi|$. Let $R > \max(R_0, C_0)$.

We will use a sequence of smooth cutoff functions $\phi_j(\xi)$ having the following properties:

$$0 \leq \phi_j(\xi) \text{ for all } \xi, \quad \text{and} \quad \phi_j(\xi) = 0 \text{ in } |\xi| < 2R\sup(j, 1),$$

$$\phi_j(\xi) = 1 \quad \text{if } |\xi| > 3R\sup(j, 1); \quad |D^\alpha \phi_j| \leq \left(\frac{C}{R}\right)^{|\alpha|} \quad \text{if } |\alpha| \leq 2j.$$

See [8] for the construction of such cutoffs. Define

$$\tilde{k}(z, w, \theta) = \sum_{j=0}^{\infty} \phi_j(\xi)\tilde{k}_j(z, w, \theta)$$

for $(z, w) \in \mathscr{O}_K \times \mathscr{O}_K$ and $\theta = \xi + \sqrt{-1}\eta \in \mathscr{C}$. $\tilde{k}$ is a C^∞ function of (z, w, θ) holomorphic in (z, w). $\tilde{k}$ satisfies the following estimates:

$$\begin{aligned}
|\tilde{k}(z, w, \theta)| &\leq \sum_{0\leq j<d} |\tilde{k}_j(z, w, \theta)| + \sum_{j\geq d} \phi_j(\xi)|k_j(z, w, \theta)| \\
&\leq \sum_{0\leq j<d} |\tilde{k}_j(z, w, \theta)| + \sum_{j\geq d} \phi_j(\xi) c_0^{j+1} j! |\theta|^{d-j} \\
&\leq \sum_{0\leq j<d} |\tilde{k}_j(z, w, \theta)| + \sum_{j\geq d} \phi_j(\xi) c_0^{j+1} j! |\xi|^{d-j}
\end{aligned}$$

Since for $j \geq d$ the jth term lives on the set $\{\xi : |\xi| \geq 2Rj\}$, the latter

$$\begin{aligned}
&\leq \sum_{0\leq<j<d} |\tilde{k}_j(z, w, \theta)| + |\xi|^d \sum_{j\geq d} c_0^{j+1} j! \left(\frac{1}{2Rj}\right)^j \\
&\leq \sum_{0\leq j<d} |\tilde{k}_j(z, w, \theta)| + \text{ constant } |\xi|^d \\
&\leq \text{ constant } |\theta|^d
\end{aligned}$$

$$\begin{aligned}\bar{\partial}_\theta \tilde{k}(z, w, \theta) &\le \sum_{j=0}^{\infty} |\bar{\partial}_\theta \phi_j(\xi) \tilde{k}_j(z, w, \theta)| \\ &\le \left(\sum_{j=0}^{\infty} |\bar{\partial}_\theta \phi_j(\xi)| c_0^{j+1} \frac{j!}{|\xi|^j} \right) \\ &\le \delta^d |\xi|^d \left(\sum_{j=0}^{\infty} |\bar{\partial}_\theta \phi_j(\xi)| c_0^{j+1} \frac{j!}{|\xi|^j} \right)\end{aligned}$$

We now use the fact that $\bar{\partial}_\theta \phi_j(\xi)$ lives in the set $\{\xi : 2Rj \le |\xi| \le 3Rj\}$;

$$\le \text{ constant } |\xi|^d \left(\sum_{j=0}^{\infty} c_0^{j+1} j! (\frac{1}{2Rj})^j \right)$$

Since $j!/j^j \le e^{-j}$, the latter $\le$ constant $|\xi|^d \sum_{j=0}^{\infty} (\frac{c_0}{2R})^j e^{-j}$.
Recalling that $2Rj \le |\xi| \le 3Rj$, we get

$$\begin{aligned}&\le \text{ constant } |\xi|^d \sum_{j=0}^{\infty} \left(\frac{c_0}{2R} \right)^j e^{-\frac{|\xi|}{3R}} \\ &\le \text{ constant } e^{-\frac{|\xi|}{4R}} \\ &\le \text{ constant } e^{-\frac{\delta}{4R}|\theta|}\end{aligned}$$

Thus for $(z, w, \theta) \in \mathscr{O}_K \times \mathscr{O}_K \times \mathscr{C}$, we have: $|\tilde{k}(z, w, \theta)| \le \text{const.}\ |\theta|^d$ and $|\bar{\partial}_\theta \tilde{k}(z, w, \theta)| \le \text{const.}\ e^{-\frac{\delta}{2R}|\theta|}$.

We may assume that the shape of $\mathscr{C}$ has been modified to allow us to solve the Cauchy-Riemann equations in $\mathscr{C}$ (see [5]) $\bar{\partial}_\theta \tilde{k}_1 = \bar{\partial}_\theta \tilde{k}$ in such a way that the solution $\tilde{k}_1$ is holomorphic with respect to (z, w) in $\mathscr{O}_K \times \mathscr{O}_K$ and the following estimate holds on sets of the kind $K_1 \times K_2 \times \mathscr{C}_1 (K_1, K_2 \subset\subset \mathscr{O}_K)$ and $\mathscr{C}_1$ a cone whose closure is contained in $\mathscr{C}$:

$$|\tilde{k}_1(z, w, \theta)| \le \text{const.}\ e^{-\frac{\delta}{4R}|\theta|}.$$

Define then $h = \tilde{k} - \tilde{k}_1$. We now have, in $\mathscr{O}_K \times \mathscr{O}_K \times \mathscr{C}_1$, $\bar{\partial}_\theta \tilde{h} = 0$ and $\tilde{k} - \tilde{h}$ decays exponentially as $|\theta| \to \infty$ (uniformly, provided (z, w, θ) stays in sets like $K_1 \times K_2 \times \mathscr{C}_1$ as above).

This decay together with Theorem 2.1 of §2 imply that if for $u \in \mathscr{E}'(U)$, U sufficiently small, we define

$$\operatorname{op} \tilde{k}^\varepsilon u(x) = \left(\frac{1}{4\pi^3} \right)^{\frac{m}{2}} \int_U \int_{R_m} e^{\sqrt{-1}\xi \cdot (Z(x) - Z(y)) - \varepsilon |\xi|^2} \cdot \tilde{k}(Z(x), Z(y), \xi) u(y)\, dZ(y)\, d\xi$$

then as $\varepsilon \to 0^+$, $\operatorname{op} \tilde{k}^\varepsilon$ will converge to an operator $\operatorname{op} \tilde{k}$ having the properties in Theorem 2.1, §2. Moreover, for any $u \in \mathscr{E}'(U)$, $\operatorname{op} \tilde{k} u - \operatorname{op} \tilde{h} u$ is a hypo-analytic function. We will therefore replace $\tilde{k}$ by the hypo-analytic amplitude

$\tilde{h}$ and think of $\tilde{h}$ as being the true amplitude constructed from the formal one given by $\sum_{j=0}^{\infty} k_j(x, y, \theta)$.

4. Asymptotic expansion. Let $k(x, y, \theta)$ be a hypo-analytic amplitude of degree d say $k(x, y, \theta) = \tilde{k}(Z(x), Z(y), \theta)$ where $\tilde{k}$ is holomorphic in $\mathscr{O} \times \mathscr{O} \times \mathscr{C}$, $\mathscr{O}$ and $\mathscr{C}$ are as in §1. For each $j = 1, \dots, m$, let N_j denote the vector field $N_j Z^k = -\sqrt{-1}\delta_j^k$.

If $K \subset U$ is any compact subset, by Cauchy's inequality we have $c > 0$ such that:

$$\left|\frac{1}{\alpha!}\partial_\xi^\alpha N_y^\alpha k(x, x, \xi)\right| \le c^{|\alpha|+1}\alpha!(1+|\xi|)^{d-|\alpha|}$$

for $x \in K$, $\xi \in R_m$.

Thus if we define

$$k_j(x, \xi) = \sum_{|\alpha|=j} \frac{1}{\alpha!}\partial_\xi^\alpha N_y^\alpha k(x, x, \xi)$$

then $\sum_{j=0}^{\infty} k_j(x, \xi)$ can be thought of as a formal hypo-analytic symbol. Let $(\phi_j)_j$ be the cutoff functions of the previous section. If U' is any relatively compact subset of U, we can form a true symbol by setting

$$\tilde{k}(x, \xi) = \sum_{j=0}^{\infty} k_j(x, \xi)\phi_j(\xi)$$

We then have two operators $\operatorname{op} k(x, y, \xi)$ and $\operatorname{op} \tilde{k}(x, \xi)\colon \mathscr{E}'(U') \to D'(U')$ where for $u \in \mathscr{E}'(U')$,

$$\operatorname{op} \tilde{k} u(x) = \lim_{\varepsilon\to 0^+} \left(\frac{1}{4\pi^3}\right)^{\frac{m}{2}} \int\int e^{\sqrt{-1}(Z(x)-Z(y))\cdot\xi - \varepsilon|\xi|^2} \tilde{k}(x, \xi)u(y)dZ(y)d\xi$$

and

$$\operatorname{op} k u(x) = \lim_{\varepsilon\to 0^+} \left(\frac{1}{4\pi^3}\right)^{\frac{m}{2}} \int\int e^{\sqrt{-1}(Z(x)-Z(y))\cdot\xi - \varepsilon|\xi|^2} k(x, y, \xi)u(y)dZ(y)d\xi$$

The next theorem proves that if U' is small enough, modulo a hypo-analytic regularizing operator, $\operatorname{op} k = \operatorname{op} \tilde{k}$.

Theorem 4.1. *If the neighborhood U' is sufficiently small, $\operatorname{op} k \equiv \operatorname{op} \tilde{k}$ in the sense that for any $u \in \mathscr{D}'(U')$, $\operatorname{op} k u - \operatorname{op} \tilde{k}$ is a hypo-analytic function.*

Proof. Assume U' is an open ball centered at 0, its size to be determined later. We first take $u \in C_c^0(U')$. The theorem will be proved by first establishing:

(i) $(\operatorname{op} k - \operatorname{op} \tilde{k})u$ is in $C^\infty(U')$, and

(ii) There exists $c > 0$ such that for all $\alpha \in Z_m^+$,

$$|M^\alpha(\operatorname{op} k - \operatorname{op} \tilde{k})u(x)| \le c^{|\alpha|+1}\alpha! \quad \text{where } M_j = \sqrt{-1}N_j \text{ for each } j = 1, \dots, m.$$

Taylor expansion in U' gives

$$k(x, y, \xi) = \sum_{|\alpha| \le N} \frac{(Z(y) - Z(x))^\alpha}{\alpha!} M_y^\alpha k(x, x, \xi) + \sum_{|\alpha| = N+1} (Z(y) - Z(x))^\alpha k_\alpha(x, y, \xi)$$

where $k_\alpha(x, y, \xi) = (N+1) \int_0^1 M_y^\alpha k(x, x + t(y-x), \xi)(1-t)^N dt$.

For each $N = 1, 2, \ldots$ we define the amplitudes

$$k_N(x, y, \xi) = \phi_{N+1}(\xi) k(x, y, \xi), \quad \tilde{k}_N(x, y, \xi) = \sum_{j \le N} \phi_j(\xi) k_j(x, \xi),$$

$$r_N(x, \xi) = \sum_{j \le N} (\phi_{N+1}(\xi) - \phi_j(\xi)) k_j(x, \xi),$$

$$s_N(x, y, \xi) = \left(\sum_{|\alpha| = N+1} \frac{1}{\alpha!} D_\xi^\alpha k_\alpha(x, y, \xi) \right) \phi_{N+1}(\xi), \quad \text{and}$$

$$t_N(x, y, \xi) = \sum_{|\alpha| \le N+1} \frac{1}{\alpha!} \{ D_\xi^\alpha (\phi_{N+1}(\xi) k_\alpha(x, y, \xi)) - \phi_{N+1}(\xi) D_\xi^\alpha k_\alpha \}.$$

Let K_N, $\tilde{K}_N$, R_N, S_N and T_N denote the respective operators that are defined in the same fashion as $\operatorname{op} k$. We have

$$(\operatorname{op} k - \operatorname{op} \tilde{k}) u = (\tilde{K}_N - \operatorname{op} \tilde{k}) u + (\operatorname{op} k - K_N) u + R_N u + S_N u + T_N u.$$

Our estimates will show that given any positive integer l, there exists a positive integer N such that each term on the right-hand side of the above equation is in C^l—thus establishing that $(\operatorname{op} k - \operatorname{op} \tilde{k}) u \in C^\infty(U')$.

(A) Estimate of $M^\alpha(\operatorname{op} k - K_N)u$. Since the ξ-support of

$$(1 - \phi_{N+1}(\xi)) k(x, y, \xi)$$

is compact, $(\operatorname{op} k - K_N)u$ is hypo-analytic and therefore in particular, C^∞.

Suppose $|Z(x) - Z(y)| \le A$ for all x, y in U'.

$$|(\operatorname{op} k - K_N) u(x)| = \left(\frac{1}{4\pi^3} \right)^{\frac{m}{2}} \left| \int_y \int_{|\xi| \le 3R(N+1)} e^{\sqrt{-1}(Z(x) - Z(y)) \cdot \xi} \cdot k(x, y, \xi)(1 - \phi_N(\xi)) dZ(y) d\xi \right|$$

$$\le \text{const.} \int_{|\xi| \le 3R(N+1)} e^{A|\xi|} (1 + |\xi|)^d d\xi$$

(the constant is independent of N)

$$\le \text{const.} (e^{3RA})^{N+1} (N+1)^{d+m}$$

$$\le c_1^{N+1} \quad \text{for some } c_1 > 0 \text{ independent of } N.$$

Moreover, since each $(\operatorname{op} k - K_N)u$ is hypo-analytic in a common domain, for example some neighborhood of the compact set $\bar{U}'$, we can find a constant $\tilde{c}_1 > 0$ independent of N such that for all $\alpha \in Z_m^+$,

$$|M^\alpha(\operatorname{op} k - K_N)u(x)| \leq \tilde{c}_1^{|\alpha|+1} c_1^{N+1} \alpha!$$

(B) Estimate of $M^\alpha(S_N u)$. Write

$$s_N(x, y, \xi) = \phi_{N+1}(\xi) \sum_{|\alpha|=N+1} D_\xi^\alpha k_\alpha(x, y, \xi) = \phi_{N+1}(\xi)\tilde{s}_N(x, y, \xi).$$

For $|\alpha| = N + 1$, we have

$$\left|\frac{D_\xi^\alpha k_\alpha(x, y, \xi)}{\alpha!}\right| \leq c^{|\alpha|} \alpha!(1 + |\xi|)^{d-N-1}.$$

It follows that $|\tilde{s}_N(x, y, \xi)| \leq c_2^{N+1} N!(1 + |\xi|)^{d-N-1}$ for some $c_2 > 0$.

Let

$$I_N^\varepsilon(x) = \left(\frac{1}{4\pi^3}\right)^{\frac{m}{2}} \int\int e^{\sqrt{-1}(Z(x)-Z(y))\cdot\xi - \varepsilon|\xi|^2} \phi_{N+1}(\xi)\tilde{s}_N(x, y, \xi)u(y)dZ(y)d\xi.$$

We note that $s_N u(x) = \lim_{\varepsilon\to 0^+} I_N^\varepsilon(x)$.

We will deform the path of ξ-integration from R_m to the image of R_m under the map

$$\xi \to \theta(\xi) = \phi_{2N}(\xi)\zeta(\xi) + (1 - \phi_{2N}(\xi))\xi$$

where

$$\zeta(\xi) = \check{Z}_x(x)\xi + \sqrt{-1}\langle\check{Z}_x(x)\xi\rangle(Z(x) - Z(y)).$$

The deformation is allowed since it takes place in a region where $\phi_{N+1}(\xi)$ is analytic.

We have

$$\theta(\xi) = \begin{cases} \xi, & \text{for } |\xi| \leq 4RN, \\ \zeta(\xi), & \text{for } |\xi| \geq 6RN. \end{cases}$$

$$|M^\alpha(I_N^\varepsilon(x))| \leq \left(\frac{1}{4\pi^3}\right)^{\frac{m}{2}} \sum_{\beta<\alpha} \binom{\alpha}{\beta} \left| \int\int \xi^{\alpha-\beta} e^{\sqrt{-1}(Z(x)-Z(y))\cdot\xi - \varepsilon|\xi|^2} \cdot \phi_{N+1}(\xi)M^\beta \tilde{s}_N(x, y, \xi)u(y)dZ(y)d\xi \right|$$

We use the above contour and pass to the limit to get:

$$\begin{aligned} |(M^\alpha s_N u)(x)| \leq \Bigg| &\left(\frac{1}{4\pi^3}\right)^{\frac{m}{2}} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \int_{2R(N+1)\leq|\xi|\leq 6RN} \int (\theta(\xi))^{\alpha-\beta} \\ &\cdot e^{\sqrt{-1}(Z(x)-Z(y))\cdot\theta(\xi)} \phi_{N+1}(\xi)M^\beta \tilde{s}_N(x, y, \xi)u(y)d\theta dZ(y) \\ &+ \int_{|\xi|\geq 6RN} \int (\zeta(\xi))^{\alpha-\beta} e^{\sqrt{-1}(Z(x)-Z(y))\cdot\zeta(\xi)} \phi_{N+1}(\xi) \\ &\cdot M^\beta \tilde{s}_N(x, y, \zeta(\xi))u(y)dZ(y)d\xi \Bigg| \end{aligned}$$

We recall that the exponential in the second integral is bounded (§1, (1.1)). By hypo-analyticity we get $\tilde{c}_3 > 0$ such that

$$\forall \beta, \ |M^\beta \tilde{s}_N(x, y, \xi)| \le c_3^{|\beta|+1} \beta! c_2^{N+1} N!(1+|\xi|)^{d-N-1}.$$

These observations imply that

$$|M^\alpha s_N u(x)| \le \text{const.} \left(\sum_{\beta \le \alpha} \binom{\alpha}{\beta} \int_{2R(N+1) \le |\xi| \le 6RN} |\xi|^{\alpha-\beta} e^{A|\xi|} c_3^{N+|\beta|+2} \cdot \beta! N!(1+|\xi|)^{d-N-1} \, d\xi + \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \cdot \int_{6RN \le |\xi|} |\xi|^{\alpha-\beta} c_3^{N+|\beta|+2} \beta! N!(1+|\xi|)^{d-N-1} d\xi \right)$$

for some $c_3 \ge \max(\tilde{c}_3, c_2)$. Hence, after modifying c_3 if necessary, we get

$$\begin{aligned}
|M^\alpha s_N u(x)| &\le \alpha! \left(\sum_{\beta \le \alpha} \frac{1}{(\alpha-\beta)!} \int_{2R(N+1) \le |\xi|} (1+|\xi|)^{|\alpha-\beta|+d-N-1} N! d\xi \right) c_3^N \\
&\le \alpha! c_3^N \left(\sum_{\beta \le \alpha} \frac{1}{(\alpha-\beta)!} \left(\frac{1}{1+2RN}\right)^{N-|\alpha-\beta|-d-m+1} N! \right) \\
&\le \alpha! c_3^N \left(\sum_{\beta \le \alpha} \frac{1}{(\alpha-\beta)!} (1+2RN)^{|\alpha-\beta|+d+m-1} \right) \left(\frac{1}{2R}\right)^N \frac{N!}{N^N} \\
&\le \alpha! c_3^N \left(\sum_{\beta \le \alpha} \frac{1}{(\alpha-\beta)!} (1+2RN)^{|\alpha-\beta|+d+m-1} \right) \left(\frac{1}{2Re}\right)^N Ne \\
&\le \alpha! c_3^N \left(\sum_{\beta \le \alpha} \frac{1}{(\alpha-\beta)!} (|\alpha-\beta|+d+m-1)! e^{1+2RN} \right) \left(\frac{1}{2Re}\right)^N Ne \\
&\le \alpha! c_3^N \left(\sum_{\beta \le \alpha} \frac{1}{(\alpha-\beta)!} (|\alpha-\beta|+d+m-1)! \right) \left(\frac{e^{2R}}{2Re}\right)^N Ne^2.
\end{aligned}$$

Using the inequality: $(k+l)! \le 2^{k+l} k! l!$ for any positive integers k and l, the latter is dominated by

$$\alpha! c_3^N \left(\sum_{\beta \le \alpha} \frac{1}{(\alpha-\beta)!} |\alpha-\beta|! \right) 2^{|\alpha|+d+m-1} \left(\frac{e^{2R}}{2Re}\right)^N Ne^2.$$

For $|\alpha| \le N$, we can find another constant which we will still call c_3 such that the above quantity $\le \alpha! c_3^N$.

(C) Estimate of $M^\alpha(\operatorname{op}\tilde{k} - \tilde{K}_N)u$**.** Let

$$J^\varepsilon u(x) = \left(\frac{1}{4\pi^3}\right)^{\frac{m}{2}} \int\int e^{\sqrt{-1}(Z(x)-Z(y))\cdot\xi - \varepsilon|\xi|^2} \cdot \left(\sum_{j>N} \phi_j(\xi) k_j(x, \xi)\right) u(y)\, dZ(y)\, d\xi.$$

For each $j > N$, we will use the contour

$$\theta_j(\xi) = \phi_{2j}(\xi)\zeta(\xi) + (1 - \phi_{2j}(\xi))\xi = \begin{cases} \xi, & \text{when } |\xi| \le 4Rj, \\ \zeta(\xi), & \text{when } |\xi| \ge 6Rj. \end{cases}$$

In the quantity

$$M^\alpha(J^\varepsilon u)(x) = \left(\frac{1}{4\pi^3}\right)^{\frac{m}{2}} \sum_{j>N}\left[\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\int\int \xi^{\alpha-\beta} e^{\sqrt{-1}(Z(x)-Z(y))\cdot\xi - \varepsilon|\xi|^2} \cdot \phi_j(\xi) M^\beta k_j(x, \xi) u(y) dZ d\xi\right]$$

we use the contours θ^j in each term of the sum and take limits to get

$$M^\alpha(\operatorname{op}\tilde{k} - \tilde{K}_N)u(x) = \sum_{j>N}(I_1^j(x) + I_2^j(x))$$

where

$$I_1^j(x) = \left(\frac{1}{4\pi^3}\right)^{\frac{m}{2}} \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\int_{2Rj\le|\xi|\le 6Rj}\int \theta^j(\xi) e^{\sqrt{-1}(Z(x)-Z(y))\cdot\theta^j(\xi)} \cdot \phi_j(\xi) M^\beta k_j(x, \theta^j(\xi)) u(y) dZ d\theta^j$$

while $I_2^j(x)$ is a similiar expression except that the integration in ξ is carried out over the region $\{\xi : |\xi| \ge 6Rj\}$.

Assuming that $|\alpha| \le N - d - m$, we have

$$|I_1^j(x)| \le \text{const.} \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\int_{2Rj\le|\xi|\le 6Rj} (1 + |\xi|)^{d-j+|\alpha|-|\beta|} (e^{6RA})^j c_0^{|\beta|+j+1} j!\beta! d\xi$$

$$\le \text{const.}(c_0 e^{6RA})^j \alpha! \left|\sum_{\beta\le\alpha}\frac{1}{(\alpha-\beta)!}\int_{2Rj\le|\xi|\le 6Rj}(1 + |\xi|)^{d-j+|\alpha|-|\beta|} j!\right| c_0^N$$

$$\le \text{const.}(c_0 e^{6RA})^j \alpha! \left|\sum_{\beta\le\alpha}\frac{1}{(\alpha-\beta)!}\int_{0\le\rho\le 6Rj}\rho^{d-j+|\alpha|-|\beta|+m-1} j! d\rho\right| c_0^N$$

(We have used the fact that $d - j + |\alpha| \le 0$.)

$$\le \text{const.}\ \alpha! \left(\frac{c_0 e^{6RA}}{6R}\right)^j (6R)^{d+m+N} c_0^N \left(\sum_{\beta\le\alpha}\frac{1}{(\alpha-\beta)!}\frac{j!}{j^{j-d-m-|\alpha|+|\beta|}}\right).$$

Therefore, for some $\tilde{c}_4 > 0$ independent of j and N,

$$|I_1^j(x)| \leq \alpha! c_4^{N+1} \left(\frac{c_0 e^{6RA}}{6R}\right)^j$$

Similarly, after modifying the constant $\tilde{c}_4$ if necessary,

$$\begin{aligned}|I_2^j(x)| &\leq \text{const}.\ \alpha! \sum_{\beta \leq \alpha} \frac{1}{(\alpha-\beta)!} \left(\frac{1}{1+6Rj}\right)^{j-d-m-|\alpha|+|\beta|} c_0^{|\beta|+j+1} j! \\ &\leq \alpha! c_4^{N+1} \left(\frac{c_0}{6R}\right)^j\end{aligned}$$

We recall that $c_0 \leq R$. At this point we choose U' so small that if $A = \sup_{x,y \in U'} |Z(x) - Z(y)|$, then $c_0 e^{6RA} < 6R$.

We then get a constant $c_4 > 0$ such that: $|M^\alpha(\operatorname{op}\tilde{k} - \tilde{K}_N)u(x)| \leq \alpha! c_4^{N+1}$ for $|\alpha| \leq N - d - m$.

(D) Estimate of $M^\alpha(R_N u)$.

$$R_N u(x) = \left(\frac{1}{4\pi^3}\right)^{\frac{m}{2}} \int\int e^{\sqrt{-1}(Z(x)-Z(y))\cdot\xi} u(y) \cdot \left(\sum_{j \leq N} (\phi_{N+1}(\xi) - \phi_j(\xi)) k_j(x, \xi)\right) dZ(y)\, d\xi$$

is hypo-analytic since each $\phi_{N+1} - \phi_j$ is supported in $2Rj \leq |\xi| \leq 3R(N+1)$.

We estimate

$$\begin{aligned}\left|\sum_{j \leq N} (\phi_{N+1}(\xi) - \phi_j(\xi)) k_j(x, \xi)\right| &\leq \left(\sum_{j \leq N} c_0^{j+1} j! |\xi|^{-j}\right) |\xi|^d \\ &\leq \left(\sum_{j \leq N} c_0^{j+1} j! \left(\frac{1}{2Rj}\right)^j\right) |\xi|^d \quad (\text{since } 2Rj \leq |\xi|) \\ &\leq \left(\sum_{j \leq N} \left(\frac{c_0}{2Re}\right)^j j\right) c_0 e |\xi|^d \quad \text{since } \frac{j!}{j^j} \leq j e^{-j+1}.\end{aligned}$$

It follows that

$$|R_N u(x)| \leq \text{ constant} \int_{|\xi| \leq 3R(N+1)} |\xi|^d d\xi \leq \text{const}.\ 3R(N+1)^{d+m}$$

which in turn implies that there is a constant $\tilde{c}_5 > 0$ such that $|R_N u(x)| \leq \tilde{c}_5^{N+1}$. Moreover, by hypo-analyticity, we get $c_5 > 0$ satisfying $|M^\alpha R_N u(x)| \leq \alpha! c_5^{N+1}$ for all $|\alpha| \leq N$.

(E) Estimate of $M^\alpha(T_N u)$.

$$T_N u(x) = \lim_{\varepsilon \to 0} \left(\frac{1}{4\pi^3}\right)^{\frac{m}{2}} \int\int e^{\sqrt{-1}(Z(x)-Z(y))\cdot\xi - \varepsilon|\xi|^2} t_N(x, y, \xi) u(y) dZ(y) d\xi$$

where

$$t_N(x, y, \xi) = \sum_{|\alpha|\le N+1} \frac{1}{\alpha!}\{(D_\xi^\alpha(\phi_{N+1}(\xi)k_\alpha(x, y, \xi)) - \phi_{N+1}(\xi)D_\xi^\alpha k_\alpha(x, y, \xi)\}.$$

We can therefore take the limit under the integral sign and write

$$T_N u(x) = \sum_{|\alpha|\le N+1} A_\alpha(x)$$

where for each α, $|\alpha| \le N+1$,

$$A_\alpha(x) = \left(\frac{1}{4\pi^3}\right)^{\frac{m}{2}} \sum_{0\ne\beta\le N} \int_{2R(N+1)\le|\xi|\le 3R(N+1)} \int e^{\sqrt{-1}Z((x)-Z(y))\cdot\xi} \frac{1}{\beta!} \cdot (D_\xi^\beta \phi_{N+1}(\xi)) \frac{D_\xi^{\alpha-\beta} k_\alpha(x, y, \xi)}{(\alpha-\beta)!} u(y)\, dZ(y)\, d\xi$$

Therefore

$$|A_\alpha(x)| \le \text{const.}\ \alpha! c_0^{|\alpha|+1} (e^{3RA})^{N+1} \left[\sum_{0\ne\beta\le\alpha} \frac{1}{\beta!} \left(\frac{[3R(N+1)]^{d+m+1}}{[2R(N+1)]^{|\alpha-\beta|}}\right) \left(\frac{c_0}{R}\right)^{|\beta|}\right]$$

$$\le \text{const.} \frac{\alpha!}{[2R(N+1)]^{|\alpha|}} (e^{3RA})^{N+1} c_0^{|\alpha|+1} \cdot [3R(N+1)]^{d+m+1} \left(\sum_{0\le\beta\le\alpha} \frac{[2(N+1)c_0]^{|\beta|}}{\beta!}\right).$$

Since $|\alpha| \le N$ and R may be taken to be larger than 1, we know that the factor $\frac{\alpha!}{[2R(N+1)]^{|\alpha|}} \le 1$. Therefore, we conclude that there is a constant $c_6 \ge 0$ for which $|M^\alpha(T_N u)| \le c_6^{N+1} N!$ whenever $|\alpha| \le N$.

From (a)–(e) we conclude that there is a positive number c such that

$$|M^\alpha(\operatorname{op} k - \operatorname{op}\tilde{k})u(x)| \le c^{N+1} N!$$

for all α, $|\alpha| \le N - m - d$.

If we take $|\alpha| = N - m - d$, we can get a constant $\tilde{c} \ge c$ satisfying:

$$\forall\alpha,\ |M^\alpha(\operatorname{op} k - \operatorname{op}\tilde{k})u(x)| \le \tilde{c}^{|\alpha|+1}\alpha! \quad \text{for every } x \in U'.$$

By using integration by parts we also reach the same conclusion for $u \in \mathscr{E}'(U')$. Indeed all we need is a representation of the form $u = \sum_{|\alpha|\le N} M^\alpha u_\alpha$ where each $u_\alpha \in C_c^0(U')$ which is always possible. We have thus shown that $(\operatorname{op} k - \operatorname{op}\tilde{k})u$ is in $C^\infty(U')$ and that there is $c > 0$ such that for all $\alpha \in Z_m^+$,

$$|M^\alpha(\operatorname{op} k - \operatorname{op}\tilde{k})u(x)| \le c^{|\alpha|+1}\alpha!.$$

By Theorem 3.1 of [1] it follows that $\operatorname{op} ku - \operatorname{op}\tilde{k}u$ is a hypo-analytic function.

II. Applications

1. Parametrix for an elliptic operator. As an application of Theorem 4.1 we consider here the construction of a parametrix for an elliptic hypo-analytic differential operator. We will begin by composing a hypo-analytic differential operator A with a hypo-analytic pseudodifferential operator B. In [3] we introduced hypo-analytic differential operators. In the local chart (U, Z), the operator A is given by $A = \sum_{|\alpha|\leq n} a_\alpha(x)N^\alpha$ where each $a_\alpha(x)$ is a hypo-analytic function and $N_j = -\sqrt{-1}M_j$ for $j = 1, \dots, m$.

Theorem 4.1 of the previous chapter allows us to represent the operator B by a symbol $b(x, \theta)$. From §2, Theorem 2.1 we know that both $B \circ A$ and $A \circ B$ are continuous linear maps from $\mathscr{E}'(U)$ to $\mathscr{D}'(U)$. We first assume that the operator $A = a(x)N^\beta$ for some hypo-analytic function $a(x)$ and some index β. Then $B(Au)(x)$ is by definition the limit as $\varepsilon \to +0$ of

$$B^\varepsilon(Au)(x) = \left(\frac{1}{4\pi^3}\right)^{\frac{m}{2}} \int\int e^{\sqrt{-1}(Z(x)-Z(y))\cdot\xi-\varepsilon|\xi|^2} b(x, \xi)a(y)N^\beta u(y)dZ(y)d\xi.$$

On the other hand, $\lim_{\varepsilon\to 0^+} B^\varepsilon(Au)(x) = C \circ (N^\beta u)(x)$ where C is a hypo-analytic pseudodifferential operator with amplitude given by $b(x, \xi)a(y)$. Therefore, Theorem 4.1 tells us that C can be represented by the symbol $c(x, \xi) = \sum_\alpha \frac{\partial_\xi^\alpha bN^\alpha a(x)}{\alpha!}$. It follows that modulo a hypo-analytic function, we can write

$$B(Au)(x) = \lim_{\varepsilon\to 0^+} \left(\frac{1}{4\pi^3}\right)^{\frac{m}{2}} \int\int e^{\sqrt{-1}(Z(x)-Z(y))\cdot\xi-\varepsilon|\xi|^2} \xi^\beta c(x, \xi)u(y)\, dZ(y)\, d\xi.$$

The latter says that a symbol of $B \circ A$ is given by

$$\xi^\beta c(x, \xi) = \sum_\alpha \frac{\partial_\xi^\alpha b(x, \xi)N^\alpha(a(x)\xi^\beta)}{\alpha!}.$$

On the other hand, applying the operator A to

$$B^\varepsilon u(x) = \left(\frac{1}{4\pi^3}\right)^{\frac{m}{2}} \int\int e^{\sqrt{-1}(Z(x)-Z(y))\cdot\xi-\varepsilon|\xi|^2} b(x, \xi)u(y)dZ(y)d\xi$$

gives

$$\begin{aligned} A(B^\varepsilon u(x)) &= \left(\frac{1}{4\pi^3}\right)^{\frac{m}{2}} \int\int e^{\sqrt{-1}(Z(x)-Z(y))\cdot\xi-\varepsilon|\xi|^2} \\ &\qquad \cdot \left(\sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \xi^{\beta-\gamma} a(x)N^\beta b(x, \xi)\right) u(y)\, dZ\, d\xi \\ &= \left(\frac{1}{4\pi^3}\right)^{\frac{m}{2}} \iint e^{\sqrt{-1}(Z(x)-(Z(y)\cdot\xi-\varepsilon|\xi|^2} \\ &\qquad \cdot \left(\sum_\alpha \left[\frac{\partial_\xi^\alpha(a(x)\xi^\beta)N^\alpha b(x, \xi)}{\alpha!}\right]\right) u(y)\, dZ\, d\xi. \end{aligned}$$

This means that $A \circ B$ has a symbol given by

$$\sum_{\alpha} \frac{\partial_\xi^\alpha (a(x)\xi^\beta) N^\alpha b(x, \xi)}{\alpha!}.$$

By linearity, we will have the same formulas for the symbol of $B \circ A$ and $A \circ B$ when A is also given by $A = \sum_{|\alpha| \le n} a_\alpha(x) N^\alpha$.

We have thus shown that if either A or B is hypo-analytic differential operator, the composition $A \circ B$ is hypo-analytic pseudodifferential operator with symbol

$$\sum_{\alpha} \frac{\partial_\xi^\alpha a(x, \xi) N^\alpha b(x, \xi)}{\alpha!}.$$

Definition 1.1. *Let* $P = \sum_{|\alpha| \le k} a_\alpha(Z(x)) M^\alpha$ *where the* $a_\alpha(z)$ *are holomorphic in a neighborhood of* $Z(U)$ *in* C^m. *We say a point* $(x, \xi) \in T^*U \backslash \{0\}$ *is in the characteristic set of P if the point* $(Z(x), \check{Z}_x(x)\xi)$ *is in the characteristic set of* $P^Z = \sum_{|\alpha| \le k} a_\alpha(z) (\frac{\partial}{\partial z})^\alpha$.

Notation. $\operatorname{Char} P =$ the characteristic set of P as given by Definition 1.1.

Definition 1.2. *A hypo-analytic differential operator P is said to be elliptic at a point* $x \in U$ *if for every* $(x, \xi) \in T^*U$, $(x, \xi) \notin \operatorname{Char} P$.

Now suppose $P = \sum_{|\alpha| \le k} a_\alpha(Z(x)) M^\alpha$ is a hypo-analytic differential operator that is elliptic at our central point $0 \in U$. Since $Z(0) = 0$ and $dZ(0) = \mathrm{Id}$, we can find a neighborhood $\mathscr{O}$ of 0 in C^m, a cone $\mathscr{C}$ in C_m containing $R_m \backslash \{0\}$ and constants c, $R > 0$ such that: when $z \in \mathscr{O}$ and $\zeta \in \mathscr{C}$, $|\zeta| \ge R$ we have $|\sum_{|\alpha| \le k} a_\alpha(z) \zeta^\alpha| \ge c|\zeta|^k$.

We now have all the ingredients we need to state

Theorem 1.1. *Let* A *be hypo-analytic differential operator in* Ω *that is elliptic of order* d. *Given any relatively compact open subset* $\tilde{\Omega}$ *of* Ω, *there is a hypo-analytic pseudodifferential operator B in* $\tilde{\Omega}$ *of order* $-d$ *such that* $AB - I$ *and* $BA - I$ *are hypo-analytic regularizing in* $\tilde{\Omega}$.

The proof of this theorem is a simple adaptation of that of the corresponding theorem for analytic pseudodifferential operators as given by Treves [8]. Therefore we omit it.

2. Propagation of hypo-analyticity. In [3] it was shown that hypo-analytic singularities for solutions propagate along the bicharacteristics of hypo-analytic differential operators. Here we extend this result to what may be called classical hypo-analytic pseudodifferential operator. This result may also be viewed as an extension of a theorem of Hanges [4].

We will work in the hypo-analytic local chart (U, Z) of Chapter I. Let P be a classical hypo-analytic pseudodifferential operator with principal symbol p. Let $t \to (x(t), \xi(t)) = \gamma(t)$ be a curve in $T^*U \backslash \{0\}$ and set $\tilde{\gamma}(t) = (\tilde{x}(t), \tilde{\xi}(t)) = (Z(x(t)), \check{Z}_x(x(t))\xi(t))$.

Definition 2.1. *The curve $\gamma(t)$ is said to be a bicharacteristic for P if the equations*

$$\frac{d\tilde{x}}{dt} = \frac{\partial p}{\partial \xi}(\tilde{x}(t), \tilde{x}(x)), \quad \frac{d\tilde{\xi}}{dt} = \frac{-\partial p}{\partial z}(\tilde{x}(t), \tilde{\xi}(t))$$

hold.

We can now state the theorem of this section.

Theorem 2.1. *Assume $p(0, \xi_0) = 0$ and P is of principal type at $(0, \xi_0)$. Suppose $\gamma = \{(x(t), \xi(t))\}$ is a bicharacteristic for P through $(x(0), \xi(0)) = (0, \xi_0)$ and that Pu is hypo-analytic on γ. Then either u is hypo-analytic at every point of γ or u is not hypo-analytic at any point of γ.*

The proof will use a version of the FBI transform as developed by Sjöstrand in [7]. We will therefore first discuss Sjöstrand's FBI transformations adapted to our situation here.

Let H be a totally real submanifold of C^m of maximal dimension with defining functions $h_1, \dots, h_m$.

Define

$$\Lambda_H = \left\{ \left(x, \frac{2}{i}\partial h(x)\right) : h \in C^\infty(C^m, R), h \equiv 0 \text{ on } H \right\}.$$

Note that if $x_0 \in H$, then $(x_0, \xi_0) \in \Lambda_H$ iff $\exists$ real numbers $t_1, \dots, t_m \ni$

$$\xi_0 = \frac{2}{i} \sum_{j=1}^{m} t_j \partial h_j(x_0).$$

Fix a point $(y_0, \eta_0) \in \Lambda_H$. Let φ be a holomorphic function defined near $(x_0, y_0) \ni$

(2.1) $\frac{\partial \varphi}{\partial y}(x_0, y_0) = -\eta_0$,

(2.2) $\det \frac{\partial^2 \varphi}{\partial x \partial y}(x_0, y_0) \neq 0$,

(2.3) $\Im \varphi_{yy}(x_0, y_0)|_{T_{y_0}H \times T_{y_0}H} > 0$.

Here $\Im \varphi$ is considered as a function on $C^n \times H$, defined locally.

Set

$$\varphi_1(x, y) = -\Im \varphi(x, y).$$

Condition (2.1) implies that $H \ni y \mapsto \varphi_1(x_0, y)$ has a critical point at y_0 since $\frac{2}{i}\frac{\partial \varphi_1}{\partial y}(x_0, y_0) = \frac{\partial \varphi}{\partial y}(x_0, y_0) = -\eta_0$ and that therefore $d_y \varphi_1(x_0, y_0) = dh(y_0)$ for some h vanishing on H. This together with condition (2.3) and the implicit function theorem give us neighborhoods $N(x_0)$ of x_0 in C^m, $N(y_0)$ of y_0 in H and a unique C^∞ function $y = y(x) : N(x_0) \to N(y_0)$ such that $y(x)$ is the unique critical point for $H \ni y \mapsto \varphi_1(x, y)$, $x \in N(x_0)$. We next note that for $x \in N(x_0)$, $(y(x), \frac{-2}{i}\frac{\partial \varphi_1}{\partial y}(x, y(x))) \in \Lambda_H$. Indeed, this follows from the fact that $H \ni y \mapsto \varphi_1(x, y)$ has a critical point at $y(x)$ and that $h_1, \dots, h_m$ are the defining functions for H.

For $x \in N(x_0)$, let $\eta(x) = \frac{-2}{i}\frac{\partial \varphi_1}{\partial y}(x, y(x))$. Then

$$(y(x), \eta(x)) = \left(y(x), \frac{-2}{i}\frac{\partial \varphi_1}{\partial y}(x, y(x))\right) \in \Lambda_H .$$

Moreover, for x in $N(x_0)$, $y(x)$ is the unique point in $N(y_0)$ such that

$$\frac{-\partial \varphi}{\partial y}(x, y(x)) = \frac{-2}{i}\frac{\partial \varphi_1}{\partial y}(x, y(x)) \in (\Lambda_H)_{y(x)} .$$

This is due to the uniqueness of the critical point.

Let $\Phi(x) = \varphi_1(x, y(x))$. Let $a(x, y, \lambda)$ be a classical analytic symbol defined near (x_0, y_0) and elliptic at this point. For Ψ a real-valued function defined on an open set W in C^m, we define the space $H_\Psi^{\text{loc}}(W) = \{v : W \times R_+ \to C : v(z, \lambda)$ is holomorphic in z and for any $K \subset\subset W$ and $\varepsilon > 0$ $\exists c \ni |v(z, \lambda)| \le ce^{\lambda(\psi(z)+\varepsilon)}$ for all $z \in K, \lambda \ge 1\}$.

Let $u \in D'(N(y_0))$, and for z in $N(x_0)$ set

$$Tu(z, \lambda) = \int_H e^{i\lambda\varphi(z, y)} a(z, y, \lambda)\chi(y)u(y)dy$$

where $\chi \in C_0^\infty(N(y_0))$, $\chi \equiv 1$ near y_0.

Here we are assuming that the neighborhoods $N(y_0)$ and $N(x_0)$ have been contracted so that the symbol a and the phase function φ are defined. It is easily checked that

$$T : D'(N(y_0)) \longrightarrow H_\Phi^{\text{loc}}(N(x_0)).$$

In the sequel, $WF_{ha}u$ denotes the hypo-analytic wave front set of Baouendi-Chang-Treves [1]. Our proof of Theorem 2.1 will use the following proposition of Sjöstrand [7].

Proposition 2.1. *Let $z_1 \in N(y_0)$. Then $(y(z_1), \eta(z_1)) \notin WF_{ha}u$ iff $Tu \in H_{\Phi-\varepsilon_0}^{\text{loc}}(W)$ for some $\varepsilon_0 > 0$ and some neighborhood W of z_1.*

Proof of Theorem 2.1. In order to obtain a suitable phase function, we will need the following two lemmas from [6]. For notational convenience we will use y_0 for $0 \in Z(U) = H$.

Lemma 2.1. *Set $z_0 = (y_0' - i\xi_0', 0) \in C^{n-1} \times C$. There exists a holomorphic function φ defined near (z_0, y_0) which solves*

$$\frac{\partial \varphi}{\partial z_n}(z, y) = p\left(y, \frac{-\partial \varphi}{\partial y}(z, y)\right)$$

and satisfies (2.1)–(2.3) *with $\eta_0 = \xi_0$.*

We remark that the lemma is proved by using the Cauchy-Kovalevska theorem, which guarantees the existence of a holomorphic φ that solves the initial value problem

$$\frac{\partial \varphi}{\partial z_n} = p\left(y, \frac{-\partial \varphi}{\partial y}\right)$$

and

$$\varphi(z, 0, y) = \frac{i}{2}\sum_{j=1}^{n-1}(z_j - y_j)^2 - (\xi_0)_n y_n + iC(y_n - (y_0)_n)^2$$

where $\Re C$ is chosen sufficiently large. In the sequel, the neighborhoods $N(z_0)$, $N(y_0)$ and the function Φ are related to the φ of Lemma 2.1 as before.

Lemma 2.2. *There is an elliptic analytic symbol $a(z, y, \lambda)$ such that the FBI transformation T with phase φ and symbol a satisfies $D_{z_n} T = TP$ in H_{Φ, z_0}.*

That is , if $Y \subseteq Z(U) = H$ is a small neighborhood of y_0, then for z in $W \subseteq C^m$ a small neighborhood of $z_0 = (y_0' - i\xi_0', 0)$ and $u \in \mathscr{E}'(Y)$ we have

$$D_{z_n} Tu - TPu \in H^{\text{loc}}_{\Phi-\varepsilon}(W)$$

for some $\varepsilon > 0$.

The symbol $a(z, y, \lambda)$ is constructed by solving the transport equations at each degree of homogeneity.

We recall now that

$$\tilde{\gamma}(t) = (\tilde{x}(t), \tilde{\xi}(t))$$

and

$$\tilde{\gamma}(0) = (y_0, \xi_0) = (Z(x(0)), \check{Z}_x(x(0))\xi_0).$$

Write $y_0 = (y_0', (y_0)_n)$ and $\xi_0 = (\xi_0', (\xi_0)_n)$.

We will use the equations

$$(2.4) \qquad \begin{cases} \frac{\partial\varphi}{\partial z_n}(z, y) = p\left(y, \frac{-\partial\varphi}{\partial y}(z, y)\right), \\ \frac{\partial\varphi}{\partial y}(z_0, y_0) = -\xi^0 \end{cases}$$

to prove that $\tilde{\xi}(t) = -\frac{\partial\varphi}{\partial y}(y_0' - i\xi_0', t, \tilde{x}(t))$.

We recall that

$$\begin{cases} \frac{d\tilde{x}}{dt} = \frac{\partial p}{\partial \zeta}(\tilde{x}(t), \tilde{\xi}(t)) \quad \text{and} \\ \frac{d\tilde{\xi}}{dt} = \frac{-\partial p}{\partial z}(\tilde{x}(t), \tilde{\xi}(t)). \end{cases}$$

Hence

$$\begin{aligned} &\frac{d}{dt}\left[\frac{\partial\varphi}{\partial y}(y_0' - i\xi_0', t, \tilde{x}(t))\right] \\ &\quad = \varphi_{yz_n}(y_0' - i\xi_0', t, \tilde{x}(t)) + \varphi_{yy}(y_0' - i\xi_0, t, \tilde{x}(t))\frac{d\tilde{x}}{dt} \\ &\quad = \varphi_{yz_n}(y_0' - i\xi_0', t, \tilde{x}(t)) + \varphi_{yy}(y_0' - i\xi_0', t, \tilde{x}(t))\frac{\partial p}{\partial\zeta}(\tilde{x}(t), \tilde{\xi}(t)) \end{aligned}$$

Now (2.4) implies that

$$\varphi_{yz_n}(z, y) = \frac{\partial p}{\partial y}\left(y, \frac{-\partial\varphi}{\partial y}\right) - \frac{\partial p}{\partial\zeta}\left(y, \frac{-\partial\varphi}{\partial y}\right)\varphi_{yy}(z, y).$$

It follows that

$$\frac{d}{dt}\left[\frac{-\partial\varphi}{\partial y}(y_0' - i\xi_0', t, \tilde{x}(t))\right]$$

$$(2.5)\qquad = \frac{-\partial p}{\partial y}\left(\tilde{x}(t), -\frac{\partial\varphi}{\partial y}(y_0' - \xi_0', t, \tilde{x}(t))\right). + \frac{\partial p}{\partial\zeta}\left(\tilde{x}(t), -\frac{\partial\varphi}{\partial y}(y_0' - i\xi_0', t, \tilde{x}(t))\right)\varphi_{yy}(y_0' - i\xi_0', t, \tilde{x}(t)) - \varphi_{yy}(y_0' - i\xi_0', t, \tilde{x}(t))\frac{\partial p}{\partial\zeta}(\tilde{x}(t), \tilde{\xi}(t)).$$

But $\tilde{\xi}(t)$ also satisfies (2.5) since

$$\frac{d\tilde{\xi}}{dt} = -\frac{\partial p}{\partial y}(\tilde{x}(t), \tilde{\xi}(t)) + \frac{\partial p}{\partial\zeta}(\tilde{x}(t), \tilde{\xi}(t))\varphi_{yy}(y_0' - i\xi_0', t, \tilde{x}(t)) - \varphi_{yy}(y_0' - i\xi_0', t, \tilde{x}(t))\frac{\partial p}{\partial\zeta}(\tilde{x}(t), \tilde{\xi}(t)) = -\frac{\partial p}{\partial y}(\tilde{x}(t), \tilde{\xi}(t)).$$

Moreover, by 2.4, $\frac{-\partial p}{\partial y}(y_0' - i\xi_0', 0, y_0) = \xi_0 = \tilde{\xi}(0)$.
We conclude that

$$(2.6)\qquad \tilde{\xi}(t) = \frac{-\partial\varphi}{\partial y}(y_0' - i\xi_0', t, \tilde{x}(t)).$$

For $t \in [0, 1]$, let

$$z(t) = z_0 + (0', t) = (y_0' - i\xi_0', t) \in C^{n-1} \times R.$$

We now recall that for z near z_0, $y(z)$ is the unique point in $N(y_0) \subseteq H$ such that

$$y(z_0) = y_0 \quad\text{and}\quad \frac{-\partial\varphi}{\partial y}(z, y(z)) \in (\Lambda_H)_{y(z)}.$$

But by (2.6), $\tilde{\xi}(t) = \frac{-\partial\varphi}{\partial y}(z(t), \tilde{x}(t))$ and since the forms $\frac{2}{i}\partial h_1, \dots, \frac{2}{i}\partial h_n$ are real on $H = Z(U)$ and span all of T^*H, we know that

$$\tilde{\xi}(t) = \check{Z}_x(x(t))\xi(t) \in (\Lambda_H)_{\tilde{x}(t)}.$$

It therefore follows that

$$y(z(t)) = \tilde{x}(t).$$

In our previous notation,

$$\eta(z(t)) = \frac{-\partial\varphi}{\partial y}\big(z(t), y(z(t))\big) = \frac{-\partial\varphi}{\partial y}\big(z(t), \tilde{x}(t)\big) = \tilde{\xi}(t).$$

Thus

$$(2.7)\qquad (\tilde{x}(t), \tilde{\xi}(t)) = \big(y(z(t)), \eta(z(t))\big).$$

Since $WF_{ha}(Pu) \cap \gamma = \varnothing$ and γ is compact, (2.7) and Proposition (2.1) tell us that

$$T(Pu) \in H^{\text{loc}}_{\Phi-\varepsilon_0}(N)$$

for some $\varepsilon_0 > 0$ and a neighborhood N of $\{z(t) = 0 \le t \le 1\}$ in C^m. If W is chosen as in Lemma 2.2, then

$$D_{z_n} Tu \in H^{\text{loc}}_{\Phi-\varepsilon_0}(N \cap W).$$

This may require a modification of ε_0.

Now $z(0) = z_0 \in N \cap W$. Therefore, $\exists t_1 > 0$ such that $N \cap W$ is a neighborhood of $\{z(t) : 0 \le t \le t_1\}$. It is crucial to note that the size of t_1 is independent of the distribution u.

If now K is a compact neighborhood of $\{z(t) : 0 \le t \le t_1\}$, then $\exists c > 0$ such that

$$|D_{z_n} Tu(z, \lambda)| \le ce^{\lambda(\Phi(z) - \frac{\varepsilon_0}{2})} \qquad \forall z \in K \text{ and } \lambda \ge 1. \tag{2.8}$$

If $(y_0, \xi_0) = (y(z(0)), \eta(z(0))) \notin WF_{ha}u$, we know that, after modifying c and ε_0,

$$|Tu(z, \lambda)| \le ce^{\lambda(\Phi(z) - \frac{\varepsilon_0}{2})} \qquad \forall \lambda \ge 1 \text{ and } \forall z \text{ near } z_0. \tag{2.9}$$

From (2.7,), (2.8) and (2.9), it follows that

$$WF_{ha}(u) \cap \{(y(t), \xi(t)) : 0 \le t \le t_1\} = \varnothing.$$

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