A CONVERGENT FRAMEWORK
FOR THE MULTICOMPONENT KP-HIERARCHY

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ABSTRACT. In this paper we describe how to construct convergent solutions of the multicomponent KP-hierarchy, starting from a certain open subset of the Grassmann manifold of a special kind of Banach space, and derive an expression of its solutions in terms of Fredholm determinants. Further we show that the simplest nonscalar reduction of the present hierarchy leads to the AKNS-hierarchy.

INTRODUCTION

In §1 we introduce the multicomponent KP-hierarchy, following [U], and we show that in order to obtain solutions of this collection of nonlinear partial differential equations it suffices to solve a set of linear equations. §2 contains a description of the Grassmann manifold of a certain Banach space, which is the starting point of our construction. The actual construction of the solutions is given in §3. The scheme according to which we obtain solutions of the multicomponent KP-hierarchy is the same as that used in [Se] to construct solutions of the KP-hierarchy. §4 expresses the wave function in terms of τ-functions. We conclude (§5) with a discussion of reductions of the present hierarchy and give the relation with the AKNS-hierarchy.

1. THE MULTICOMPONENT KP-HIERARCHY

1.1. First we describe the framework in which the above-mentioned hierarchy is formulated. Let B be an associative algebra over $\mathbb{C}$ with identity element and ∂ a $\mathbb{C}$-derivation of B. Following [Wi], we denote the algebra of "pseudo-differential operators with coefficients in B" by $B[\xi, \xi^{-1}]$. Its elements are formal expressions,

$$\sum_{j=-\infty}^{N} b_j \xi^j, \qquad b_j \in B \text{ for all } j \text{ and } N \in \mathbb{Z},$$

which are added componentwise and whose multiplication is based on the

following rules:

$$\xi^i \cdot \xi^j = \xi^{i+j} \qquad\qquad \text{for all } i \text{ and } j \text{ in } \mathbb{Z},$$

$$\xi^m \cdot b = \sum_{j=0}^{\infty} \binom{m}{j} \partial^j(b)\xi^{m-j} \quad \text{for all } m \text{ in } \mathbb{Z} \text{ and } b \in B.$$

If $P = \sum b_j \xi^j$ is any element of $B[\xi, \xi^{-1}]$, then we write P_+ for the differential operator part of P and we denote $P - P_+$ by P_-, i.e.,

$$P_+ = \sum_{i \geq 0} b_i \xi^i \quad \text{and} \quad P_- = \sum_{i < 0} b_i \xi^i.$$

The subalgebra of the differential operators in $B[\xi, \xi^{-1}]$, i.e., those P for which $P = P_+$, is denoted by $B[\xi]$.

1.1.1. *Remark.* One easily verifies that a $K = \sum_{i \leq N} k_i \xi^i$ in $B[\xi, \xi^{-1}]$, with $k_N \neq 0$, is invertible if and only if k_N is invertible in B.

1.2. The elements of $B[\xi, \xi^{-1}]$ can be interpreted as operators by letting them act through multiplication on $B[\xi, \xi^{-1}]$ itself. This leads one to the definition of the following $B[\xi, \xi^{-1}]$-module: Let $M(1)$ be the space of all power series in λ with coefficients in B which have a "pole at infinity," i.e., all sums $\sum_{i=-\infty}^{N} b_i \lambda^i$, with $b_i \in B$. Its $B[\xi, \xi^{-1}]$-module structure is an algebra extension of the following actions:

$$b \cdot \sum_i b_i \lambda^i = \sum_i b b_i \lambda^i, \quad \text{with } b \in B,$$

$$\xi \cdot \sum_i b_i \lambda^i = \sum_i \partial(b_i)\lambda^i + \sum_i b_i \lambda^{i+1},$$

$$\xi^{-1} \cdot \sum_i b_i \lambda^i = \sum_{j=0}^{\infty} \sum_i (-1)^j \partial^j(b_i)\lambda^{i-j-1}.$$

Clearly $M(1)$ is a free $B[\xi, \xi^{-1}]$-module with generator 1. The $B[\xi, \xi^{-1}]$-module structure on a space of power series in λ with another prescribed behaviour around infinity is trivially carried over from $M(1)$. To be more precise, let $g(\lambda)$ be a power series in λ with coefficients in B. Denote $\{f(\lambda)g(\lambda) | f(\lambda) \in M(1)\}$ by $M(g(\lambda))$. Then $M(g(\lambda))$ becomes a $B[\xi, \xi^{-1}]$-module by simply putting

$$P(f(\lambda)g(\lambda)) = (P \cdot f(\lambda))g(\lambda) \quad \text{for each } P \text{ in } B[\xi, \xi^{-1}].$$

Hence each $M(g(\lambda))$ is a free $B[\xi, \xi^{-1}]$-module with generator $g(\lambda)$. This fact is used to obtain equations for operators in $B[\xi, \xi^{-1}]$ by their action on elements of $M(g(\lambda))$.

1.3. To give the form of the hierarchy we now specify B. The algebra B consists of the square matrices of size m with entries from a commutative $\mathbb{C}$-algebra R. We assume that R contains $\mathbb{C}[t_{i,\alpha}]$, as a subalgebra, where $i \in \mathbb{N}$

and $1 \leq \alpha \leq m$, and that there exist for all those i and α derivations $\partial_{i,\alpha}$ of R which extend the partial derivative w.r.t. $t_{i,\alpha}$ from $\mathbb{C}[t_{i,\alpha}]$ to R. These derivations of R extend in a natural way to derivations of B and they are denoted by the same symbol. For each i and α, we let $\partial_{i,\alpha}$ act coefficientwise on $B[\xi, \xi^{-1}]$ and $M(1)$. Of the derivation ∂ of B we require for the moment only that it commutes with all the $\partial_{i,\alpha}$. We write E_α for the element of B, which has a 1 on the (α, α)-entry and zeros elsewhere. We consider operators L and U_α, $1 \leq \alpha \leq m$, in $B[\xi, \xi^{-1}]$ of the form

$$L = \xi + \sum_{j<0} l_j \xi^j \quad \text{and} \quad U_\alpha = E_\alpha + \sum_{j<0} u_{j,\alpha} \xi^j.$$

Now the multicomponent KP-hierarchy consists of an infinite set of differential equations for the coefficients l_j and $u_{j,\alpha}$, which are conveniently formulated in terms of the operators L and U_α. These equations are

$$(1.3.1) \qquad \qquad \partial_{i,\alpha}(L) = [(L^i U_\alpha)_+, L],$$

$$(1.3.2) \qquad \qquad \partial_{i,\alpha}(U_\beta) = [(L^i U_\alpha)_+, U_\beta].$$

Since the left-hand sides of (1.3.1) and (1.3.2) contain only powers of ξ with negative exponents, it would be nice if this would hold for the right-hand sides too. This is the case if L and the U_β commute with the $L^i U_\alpha$, for then we have $[(L^i U_\alpha)_+, L] = -[(L^i U_\alpha)_-, L]$ and $[(L^i U_\alpha)_+, U_\beta] = -[(L^i U_\alpha)_-, U_\beta]$. Therefore we assume that the U_α commute among each other and with L. The KP-hierarchy corresponds to the case $m = 1$ and $U_\alpha = 1$.

1.4. An essential step in the construction of solutions of (1.3.1) and (1.3.2) is to view them as compatibility conditions for a set of linear equations. Before giving them, we introduce for each $i \in \mathbb{N}$ and $1 \leq \alpha \leq m$ an endomorphism $\tilde{\partial}_{i,\alpha}$ of $M(g(\lambda))$ by

$$\tilde{\partial}_{i,\alpha}(f(\lambda)g(\lambda)) = \{\partial_{i,\alpha}(f(\lambda)) + f(\lambda)E_\alpha \lambda^i\}g(\lambda).$$

The property of $\tilde{\partial}_{i,\alpha}$ one needs is that for all P in $B[\xi, \xi^{-1}]$ and $\underline{m}$ in $M(g(\lambda))$,

$$\tilde{\partial}_{i,\alpha}(P \cdot \underline{m}) = \partial_{i,\alpha}(P) \cdot \underline{m} + P \cdot \tilde{\partial}_{i,\alpha}(\underline{m}).$$

If we choose $g(\lambda)$ such that applying $\partial_{i,\alpha}$ termwise to $g(\lambda)$ results in $E_\alpha \lambda^i g(\lambda)$, then $\tilde{\partial}_{i,\alpha}$ coincides with applying $\partial_{i,\alpha}$ termwise to the series $f(\lambda)g(\lambda)$. Such a choice is available in the present context, for if we take

$$g(\lambda) = \exp\left(\text{diag}\left(\sum_{i \geq 1} t_{i,1}\lambda^i, \ldots, \sum_{i \geq 1} t_{i,m}\lambda^i\right)\right),$$

then the entries of the coefficients of $g(\lambda)$ belong to $\mathbb{C}[t_{i,\alpha}]$. This is the $g(\lambda)$ that we choose in practice. Let L and the U_α be as above. Consider now for

w in $M(g(\lambda))$ the equations

(1.4.1) $L(w) = \lambda w\,,$

(1.4.2) $U_\alpha(w) = wE_\alpha\,,$

(1.4.3) $\tilde{\partial}_{i,\alpha}(w) = (L^i U_\alpha)_+(w)\,.$

A nonzero w satisfying these equations is called a wave function corresponding to L and the U_α. Clearly if $r_1, \ldots, r_m$ belong to $\mathbb{Z}$ and w is a wave function corresponding to L and the U_α, then so is $w \operatorname{diag}(\lambda^{r_1}, \ldots, \lambda^{r_m})$. Hence we may assume that a wave function has the form $w = \{\sum_{i \le 0} w_i \lambda^i\} g(\lambda)$, with $w_0 \ne 0$. If we associate to w a $\widehat{W} = \sum_{i \le 0} w_i \xi^i$ in $B[\xi, \xi^{-1}]$, then the equations (1.4.1), (1.4.2), and (1.4.3) amount to the following relations in $B[\xi, \xi^{-1}]$:

(1.4.4) $L\widehat{W} = \widehat{W}\xi\,,$

(1.4.5) $U_\alpha\widehat{W} = \widehat{W}E_\alpha\,,$

(1.4.6) $\partial_{i,\alpha}(\widehat{W}) + \widehat{W}E_\alpha\xi^i = (L^i U_\alpha)_+\widehat{W}\,.$

If one applies $\tilde{\partial}_{i,\beta}$ to both sides of (1.4.1) and (1.4.2) and substitutes (1.4.3) into it, then, after translating the result back to $B[\xi, \xi^{-1}]$, one obtains

$$\partial_{i,\beta}(L)\widehat{W} = (L^i U_\beta)_+\widehat{W}\xi - L(L^i U_\beta)_+\widehat{W}$$

and

$$\partial_{i,\beta}(U_\alpha)\widehat{W} = (L^i U_\beta)_+\widehat{W}E_\alpha - U_\alpha(L^i U_\beta)_+\widehat{W}\,.$$

If $\widehat{W}$ is invertible, then these relations imply that L and the U_α satisfy the equations of the multicomponent KP-hierarchy and for that it is sufficient that w_0 is invertible, see (1.1.1). In that case we see moreover that

$$L = \widehat{W}\xi\widehat{W}^{-1} \quad \text{and} \quad U_\alpha = \widehat{W}E_\alpha\widehat{W}^{-1}\,.$$

Hence L and the U_α are obtained by "dressing" the trivial solution (ξ, E_α). Note that all such defined L and U_α commute. Conversely, given an operator W of the form $W = \sum_{i \le 0} w_i \xi^i$, with $w_0 = \operatorname{Id}$, one can then define L and the U_α by these formulas and, in order that they satisfy the equations of the multicomponent KP-hierarchy, it is sufficient to show that $\sum_{i \le 0} w_i \lambda^i$ satisfies (1.4.3). This is the procedure that we will follow in §3. Solutions obtained in this way satisfy $[L, U_\alpha] = [U_\alpha, U_\beta] = 0$, $U_\alpha U_\beta = \delta_{\alpha\beta} U_\beta$, and $\sum_{\alpha=1}^m U_\alpha = \operatorname{Id}$. These are the assumptions imposed on L and the U_α in $[U]$.

1.5. The equations (1.3.1) and (1.3.2) possess a certain scaling stability. We will explain here what we mean by this. In order that the following computations make sense, we will assume that R consists of functions in the parameters $t_{i,\alpha}$ and that each $\partial_{i,\alpha}$ is differentiation w.r.t. $t_{i,\alpha}$. Since all solutions to be constructed are of this type, this is no serious restriction.

Let c be in $\mathbb{C}^*$. For each u in R we put

$$\tilde{u}(t_{i,\alpha}) = u(c^i t_{i,\alpha})$$

and likewise, for $b = (b_{ij}) \in B$, $(\tilde{b})_{ij} = \tilde{b}_{ij}$. We write $\tilde{R}$, resp. $\tilde{B}$, for $\{\tilde{u}|u \in R\}$, resp. $\{\tilde{b}|b \in B\}$. Now let (L, U_α) as in (1.3) be a solution of the multicomponent KP-hierarchy. Consider

$$\tilde{L} = \xi + \sum_{i<0} c^{-i+1}\tilde{l}_i\xi^i \quad\text{and}\quad \tilde{U}_\alpha = E_\alpha + \sum_{j<0} c^{-j}\tilde{u}_{j,\alpha}\xi^j .$$

Then $\tilde{L}$ and $\tilde{U}_\alpha$ belong to $\tilde{B}[\xi, \xi^{-1}]$ and we call them the scalings of (L, U_α) by a factor c. We claim that $(\tilde{L}, \tilde{U}_\alpha)$ are again a solution of the multicomponent KP-hierarchy. This is the stability we had in mind. This statement is most easily seen if one introduces the $\text{End}(\mathbb{C}^m)$-algebra $\underline{B}$ generated by the formal noncommutative symbols $\underline{\partial^k(l_i)}$ and $\underline{\partial^r(u_{j,\alpha})}$, with $k, r \geq 0$, $i, j < 0$, and $1 \leq \alpha \leq m$. Clearly we can speak of $\underline{B}[\xi, \xi^{-1}]$ and we put a grading on this algebra by prescribing it on its generators. We give $\underline{\partial^k(l_i)}$ degree $k - i + 1$, $\underline{\partial^r(u_{j,\alpha})}$ degree $-j + r$, and ξ^r degree r. An element $\sum_{i \leq N_i} a_i\xi^i$ in $\underline{B}[\xi, \xi^{-1}]$ is called *homogeneous* of degree n if the degree of $a_i\xi^i$ is n for all i. In particular $\underline{L} = \xi + \sum_{i<0}\underline{l_i}\xi^i$ is homogeneous of degree one and $\underline{U}_\alpha = E_\alpha + \sum_{j<0}\underline{u}_{j,\alpha}\xi^j$ is homogeneous of degree zero. Since homogeneity is preserved under multiplication, we get

$$[(\underline{L}^i\underline{U}_\alpha)_+, \underline{L}] = \sum_{j>0} A_{ij}\xi^{-j}$$

and

$$[(\underline{L}^i\underline{U}_\alpha)_+, \underline{U}_\beta] = \sum_{j>0} C_{ij}\xi^{-j} ,$$

with A_{ij}, resp. C_{ij}, homogeneous polynomial expressions in the $\underline{\partial^r(l_i)}$ and $\underline{\partial^s(u_{j,\alpha})}$ of degree $i - j + 1$, resp. $i - j$. Then (1.3.1) and (1.3.2) amount to the following relations for the l_i and $u_{j,\alpha}$:

$$\partial_{i,\alpha}(l_j) = A_{ij}(\partial^r(l_s), \partial^p(u_{t,\gamma})),$$
$$\partial_{i,\alpha}(u_{j,\beta}) = C_{ij}(\partial^r(l_s), \partial^p(u_{t,\gamma})).$$

Thus we must show the same relations for l_j replaced by $c^{-j+1}\tilde{l}_j$ and $u_{j,\beta}$ replaced by $c^{-j}\tilde{u}_{j,\beta}$. But

$$\partial_{i,\alpha}(c^{-j+1}\tilde{l}_j) = c^{i+1-j}(\partial_{i,\alpha}(l_j))^\sim = c^{\text{degree}(A_{ij})}\tilde{A}_{ij}$$
$$= A_{ij}(\partial^r(c^{-s+1}\tilde{l}_s), \partial^p(c^{-t}\tilde{u}_{t,\gamma})),$$

$$\partial_{i,\alpha}(c^{-j}\tilde{u}_j) = c^{i-j}(\partial_{i,\alpha}(u_{j,\beta}))^\sim = c^{\text{degree}(C_{ij})}\tilde{C}_{ij}$$
$$= C_{ij}(\partial^r(c^{-s+1}\tilde{l}_s), \partial^p(c^{-t}\tilde{u}_{t,\gamma})),$$

and this proves the assertion.

2. The Grassmann manifold

2.1. In this section we recall the definition and some properties of the Grassmann manifold of a certain Banach space H, such as was introduced in [He]. First of all H is a subspace of the space of all functions from $\mathbb{Z}$ to $\mathbb{C}$. Denote the map which sends i to one and all other integers to zero by v_i. Hence all elements of H have a decomposition

$$\sum_{i\in\mathbb{Z}} a_i v_i, \qquad a_i \in \mathbb{C},$$

into the basic elements v_i. For any $S \subseteq \mathbb{Z}$, we write H_s for

$$\left\{ h = \sum a_i v_i \,\middle|\, h \in H, a_i = 0 \,\forall i \notin S \right\}.$$

In particular, we combine this notation with the following two: $\geq 0 = \{n | n \in \mathbb{Z}, n \geq 0\}$ and $< 0 = \{m | m \in \mathbb{Z}, m < 0\}$. Further we make the following convention: if we use the word "natural" as an adjective for a map from a subspace W to H to some H_s, then we mean the map

$$w = \sum c_i v_i \to \sum_{i\in S} c_i v_i.$$

We assume that the normed space H has the following properties:

2.1.1. H contains all the v_i, $i \in \mathbb{Z}$, and all the natural projections $H \to H_{\{i\}}$ are continuous.

2.1.2. Let p_+ be the prescription $\sum a_i v_i \to \sum_{i\geq 0} a_i v_i$. Then p_+ maps H continuously onto $H_{\geq 0}$. In particular, we have the decomposition $H = H_{\geq 0} \oplus H_{<0}$.

2.1.3. The prescription $\Lambda: \sum_{i\in\mathbb{Z}} a_i v_i \to \sum_{i\in\mathbb{Z}} a_i v_{i+1}$ defines a bounded endomorphism of H with operator norm $\|\Lambda\|$.

2.1.4. The v_i, $i \in \mathbb{Z}$, form a topological base of H, i.e., if $v = \sum a_i v_i \in H$ then $v = \lim_{M\to\infty} \sum_{i=-M}^{M} a_i v_i$.

2.2. *Remarks.* (a) Examples of Banach spaces satisfying all these properties are the $l^p(\mu)$, $1 \leq p < \infty$ and μ a positive measure on $\mathbb{Z}$ satisfying $\mu(i+1) \leq M\mu(i)$, for some constant M and all i in $\mathbb{Z}$.

(b) Thanks to property 2.1.3, each power series $\sum_{i\geq 0} a_i z^i$, $a_i \in \mathbb{C}$, with radius of convergence greater than $\|\Lambda\|$ determines an endomorphism $\sum_{i\geq 0} a_i \Lambda^i$ of H. We will use the notation $\Gamma(\Lambda)$ for $\{\gamma | \gamma \in \text{Aut}(H), \gamma = \exp(\sum_{i\geq 1} a_i \Lambda^i)\}$ with the a_i coefficients of such a power series}.

(c) H can be realized as a certain space of complex power series in λ by mapping $\sum a_i v_i$ to $\sum a_i \lambda^i$ and simply transferring the norm. This realization

was used in the case of the KP-hierarchy. Here we need a vector version of this realization (see 3.1).

(d) Since $H_{\geq 0}$ possesses the approximation property, we can speak of the determinant for each operator in $\text{End}(H_{\geq 0})$ of the form "Id + nuclear" (see [He]).

2.3. For any subspace W of H we denote the canonical embedding of W into H by i_w. Let $\text{Gr}_k(H)$ be the collection of closed subspaces W in H such that $p_+ \circ i_w$ is a Fredholm operator of index k. As in [He] we introduce

2.3.1. Definition. The *Grassmann manifold* of H is the collection of all subspaces $\text{Gr}_k(H)$, $k \in \mathbb{Z}$.

Our object of interest is the space $\text{Gr}_0(H)$, which we also denote by Gr. Recall that the Grassmann manifold consisting of all n-dimensional subspaces of $\mathbb{C}^{n+m}$ is isomorphic to the quotient of the space of complex $(n+m) \times n$-matrices of rank n under the natural right action of $\text{Gl}_n(\mathbb{C})$ or, to say it in other words, it is the space of all embeddings of $\mathbb{C}^n$ into $\mathbb{C}^{n+m}$ divided by the group of automorphisms of $\mathbb{C}^n$. An analogue of this description exists also for Gr. Consider the space $\mathbf{P}$ of all continuous embeddings w of $H_{\geq 0}$ into H, which satisfy

2.3.2. $w_+ = p_+ \circ w$ has the form "identity + a nuclear operator."

Since nuclear operators are compact, w_+ is a Fredholm operator of index zero and, in particular, the image of w is a closed subspace of H. Clearly $\text{im}(w)$ belongs to Gr. Since finite-dimensional operators are nuclear, each W in Gr occurs as the image of an element of $\mathbf{P}$. Two elements w_1 and w_2 of $\mathbf{P}$ have the same image if and only if they differ by an element of $\mathbf{T}$, i.e., $w_2 = w_1 \circ t$, $t \in \mathbf{T}$, where $\mathbf{T}$ is the group of automorphisms of $H_{\geq 0}$ which have the form "identity + nuclear." Hence under the projection $\pi: \mathbf{P} \to \text{Gr}$, $\pi(w) = \text{im}(w)$, we get an identification of Gr with $\mathbf{P}/\mathbf{T}$.

The topology of Gr is derived from that of $\mathbf{P}$: a bases of neighbourhoods of w in $\mathbf{P}$ consists of

$$\{p \mid p \in \mathbf{P}, \; \|p_+ - w_+\|_1 < \varepsilon \text{ and } \|p_- - w_-\| < \varepsilon, \; \varepsilon > 0\},$$

where $\|-\|_1$ is the Schatten-norm on the space of nuclear operators, $\|-\|$ denotes the operator norm and $\binom{w_+}{w_-}$, resp. $\binom{p_+}{p_-}$, are the decompositions of w and p w.r.t. $H = H_{\geq 0} \oplus H_{<0}$. $\mathbf{T}$ becomes a topological group if we take $\{t \mid t \in \mathbf{T}, \|t - \text{id}\|_1 < \varepsilon\}$, $\varepsilon > 0$, as the neighbourhood base of the identity. It is not difficult to see that the natural right action of $\mathbf{T}$ on $\mathbf{P}$ is continuous. We then put the quotient topology on Gr.

Apart from the description given above of Gr, we can also see it as an orbit of a certain group of automorphisms of H under its natural action. To be more precise, let G_n be the group of bounded automorphisms of H which have, with

respect to the decomposition $H = H_{\geq 0} \oplus H_{<0}$, the form

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{array}{l} a \text{ and } d \text{ are Fredholm operators with} \\ \text{index}(a) = \text{index}(d) = 0, c \text{ is bounded} \\ \text{and } b \text{ is a nuclear operator.} \end{array} \right\}$$

G_n acts on Gr by its natural action on H and it is not difficult to see that this action is transitive. Hence we have Gr as the quotient G_n/B_n, where B_n is the stabilizer of $H_{\geq 0}$. Further G_n is easily seen to act continuously on Gr if we take the following as a neighbourhood base of the identity in G_n:

$$\left\{ g | g \in G_n, g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \|a - \text{id}\| < \varepsilon, \|d - \text{id}\| < \varepsilon, \|c\| < \varepsilon, \|b\|_1 < \varepsilon \right\}.$$

It is shown in [He] that the elements of $\Gamma(\Lambda)$ are examples of automorphisms lying in B_n and the realizations $\mathbf{P}/\mathbf{T}$ and G_n/B_n of Gr are chosen to include operators of this form.

2.4. On Gr we define the determinant bundle Det and its dual Det* as the orbit spaces corresponding respectively to the following actions of $\mathbf{T}$ on $\mathbf{P} \times \mathbb{C}$:

$$(w, \lambda) \to (w \circ t, \lambda \det(t)^{-1}),$$
$$(w, \lambda) \to (w \circ t, \lambda \det(t)).$$

Here w belongs to $\mathbf{P}$, t to $\mathbf{T}$, and λ lies in $\mathbb{C}$. From the second action it will be clear that we can define a section σ of Det* by

$$\sigma(\pi(w)) = (w, \det(w_+)).$$

Since it is not clear how to define an action of G_n on Det or Det* we pass to an extension G of G_n. First we note that for each $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in G_n there exists an automorphism q of $H_{\geq 0}$ such that $aq^{-1} - \text{id}$ is finite dimensional. For G we now take the collection of pairs (g, q) with g as above in G_n and q an automorphism of $H_{\geq 0}$ such that $aq^{-1} - \text{id}$ is nuclear. This set forms a group w.r.t. the natural composition. Since $\det(t) = \det(qtq^{-1})$ for each t in $\mathbf{T}$ and for each q in $\text{Aut}(H_{\geq 0})$, we can define an action of G on Det* by

$$(g, q) \cdot (w, \lambda) = (g \circ w \circ q^{-1}, \lambda).$$

G becomes a continuous transformation group of Det* if we equip it with the topology generated by

$$\left\{ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, q \right) \in G \middle| \begin{array}{l} \|a - \text{id}\| < \varepsilon, \|d - \text{id}\| < \varepsilon, \|q - \text{id}\| < \varepsilon, \\ \|aq^{-1} - \text{id}\|_1 < \varepsilon, \|b\|_1 < \varepsilon, \|c\| < \varepsilon \end{array} \right\}.$$

Note that there is a natural way to embed the group B_n into G, namely

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \to \left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, a \right).$$

This will be assumed from now on. We conclude this section with defining for each w in $\mathbf{P}$ a function $\tau_w : G \to \mathbb{C}$ by

$$\tau_w((g, q)^{-1}) = \det((g \circ w \circ q^{-1})_+).$$

The restriction of τ_w to B_n measures the failure of B_n-equivariance of σ at the point $\pi(w)$ of Gr. Moreover its zero-set forms the boundary of the domain to which the solutions (constructed below) of the hierarchy have an analytic continuation.

2.5. We introduce here the Banach spaces needed for the multicomponent case. Take m copies $H^{(i)}$, $1 \leq i \leq m$, of the space H from (2.1). Denote their basic elements by $v_k^{(i)}$, $k \in \mathbb{Z}$ and $1 \leq i \leq m$. Consider the space $\mathbf{K} = \bigoplus_{i=1}^m H^{(i)}$ with the sum norm and the basic elements $e_{i-k-m-1} = v_k^{(i)}$, $k \in \mathbb{Z}$ and $1 \leq i \leq m$. $\mathbf{K}$ satisfies the conditions (2.1.1), (2.1.2), (2.1.3), and (2.1.4) so that we can speak of $\mathrm{Gr}_0(\mathbf{K})$ and the dual of the determinant bundle over this space. We assume that all the $H^{(i)}$ are realized as power series in λ as indicated in 2.2(c). Hence the elements of $\mathbf{K}$ are realized as power series in λ with coefficients in $\mathbb{C}^m$ whose elements are written as rows. E.g. the elements $v_k^{(i)}$ correspond to $f_i \lambda^k$, with f_i the standard basic vector of $\mathbb{C}^m$ with a one in the ith place and zeros elsewhere. We embed $\mathrm{End}(\mathbb{C}^m)$ into $\mathrm{End}\,\mathbf{K}$ in the following way: if $A = (a_{ij}) \in \mathrm{End}(\mathbb{C}^m)$ then the operator

$$h = (k_1, \dots, k_m) \to hA$$

corresponds to A with $h_i \in H$. We write $\underline{\Lambda}$ for the endomorphism $\mathrm{diag}(\Lambda, \dots, \Lambda)$ of $\mathbf{K}$. Then we can define the operator $\sum_{i \geq 0} A_i \underline{\Lambda}^i$ for each series A_i in $\mathrm{End}(\mathbb{C}^m)$ such that $\sum_{i \geq 0} \|A_i\| |\underline{\Lambda}|^i < \infty$. The relevant flows on $\mathrm{Gr}_0(\mathbf{K})$ for the multicomponent KP-hierarchy are of this form. Namely let Γ be the m-fold product of the $\Gamma(\Lambda)$. If $\gamma = (\gamma_1, \dots, \gamma_m) \in \Gamma$, with $\gamma_\alpha = \exp(\sum_{j \geq 1} t_{j\alpha} \Lambda^j)$, then it acts on $\mathbf{K}$ by

$$(h_1, \dots, h_m) \to \left(h_1 \exp\left(\sum_{j \geq 1} t_{j1} \lambda^j \right), \dots, h_m \exp\left(\sum t_{jm} \lambda^j \right) \right).$$

In other words, γ acts as $\exp(\sum_{j \geq 1} \mathrm{diag}(t_{j1}, \dots, t_{jm}) \underline{\Lambda}^j)$. Moreover these operators belong to G_n.

Next we want to introduce a "scaling" on the space $\mathrm{Gr}_0(\mathbf{K})$. Here we restrict attention to those c in $\mathbb{C}^*$ for which

(2.5.1) $$\sum_{i \leq 0} w_i \lambda^i \to \sum_{i \leq 0} c^{-i} w_i \lambda^i, \qquad w_i \in \mathbb{C}^m,$$

defines a bounded operator from $\mathbf{K}_{\leq 0}$ to $\mathbf{K}_{\leq 0}$. Clearly such c satisfy $|c| \leq 1$. Every w in $\mathbf{P}$ can, thanks to (2.1.4), be uniquely written in the form

$$\sum_{i \geq 0} v_i \lambda^i \to \sum_{i \geq 0} \sum_{j \in \mathbb{Z}} v_i w_{ji} \lambda^i$$

with $w_{ji} \in \mathrm{End}(\mathbb{C}^m)$ and $v_i \in \mathbb{C}^m$ for all i and j. As in the one-dimensional case one shows that one may assume that the w_{ij} satisfy the additional conditions $w_{ji} = 0$ for all $j > i$ and there is a $N > 0$ such that $w_{ii} = \mathrm{Id}$ for all $i \geq N$ and $w_{ji} = 0$ for all $j \neq i$, $i, j \geq N$. Now let R_c be the "scaling" prescription

$$\sum_{i \in \mathbb{Z}} w_i \lambda^i \to \sum_{i \in \mathbb{Z}} w_i c^{-i} \lambda^i.$$

If one starts with a general power series in $\mathbf{K}$, then applying R_c does not lead to an element in $\mathbf{K}$. However if we consider elements $\sum_{i \leq N} w_i \lambda^i$ in K, then (2.5.1) tells you that $R_c(\sum w_i \lambda^i)$ is again in K. As we have seen above, every W in $\mathrm{Gr}_0(\mathbf{K})$ has a topological basis of this form. Now the scaling of W by the factor c, $R_c W$, is by definition the closure of the span of the R_c-transforms of the elements in such a basis. $R_c W$ belongs again to $\mathrm{Gr}_0(\mathbf{K})$, since it is the image of the embedding

$$\sum_{i \geq 0} v_i \lambda^i \to \sum_{i \geq 0} \sum_{j \in \mathbb{Z}} v_i (w_{ji} c^{i-j}) \lambda^j,$$

where w_{ji} satisfies the assumptions above and thus belongs to $\mathbf{P}$.

3. The construction of the solutions

3.1. The starting point of our construction is the space $\mathrm{Gr} = \mathrm{Gr}_0(\mathbf{K})$. For each W in Gr, consider the subset Γ^W of Γ, defined by

$$\Gamma^W = \{\gamma | \gamma \in \Gamma, \text{ the natural projection } W\gamma^{-1} \to \mathbf{K}_{\geq 0} \text{ is an isomorphism}\}$$
$$= \{\gamma | \gamma \in \Gamma, \ \tau_w(\gamma) \neq 0 \text{ for some } w \text{ in } \mathbf{P} \text{ with } \pi(w) = W\}.$$

From this last equality it is clear that Γ^W is an open subset of Γ. However, contrary to the case $m = 1$, for $m \geq 2$ there are always subspaces W in Gr such that Γ^W is empty. Take, e.g. for each i in $\{1, \dots, m\}$, a W_i in $\mathrm{Gr}_{k_i}(H)$, where not all k_i are zero and $\sum_{i=1}^{m} k_i = 0$. Then $W = \bigoplus_{i=1}^{m} W_i$ lies in $\mathrm{Gr} = \mathrm{Gr}_0(\mathbf{K})$ but Γ^W is empty. Nevertheless the collection $\mathbf{O}$ consisting of all W in Gr with Γ^W nonempty, is a nonvoid open subset of Gr. For each W in $\mathbf{O}$ we will construct a solution of the hierarchy, whose coefficients belong to the algebra B_W of functions from Γ^W to $M_m(\mathbb{C})$ that are analytic w.r.t. each of the parameters $t_{k,i}$. For the derivation ∂ of B_W we take $\sum_{1 \leq \alpha \leq m} \partial_{1,\alpha}$.

3.2. For each $\gamma \in \Gamma^W$, let $\hat{\psi}_W(\gamma)_i$ be the inverse image of $v_0^{(i)}$ under the natural projection $W\gamma^{-1} \to \mathbf{K}_{\geq 0}$. If the action of γ^{-1} decomposes in G_n as $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ and if $w = \begin{pmatrix} w_+ \\ w_- \end{pmatrix} \in \mathbf{P}$, then one verifies directly that

$$\hat{\psi}_W(\gamma)_i = v_0^{(i)} + dw_- a^{-1}(aw_+ a^{-1} + bw_- a^{-1})^{-1}(v_0^{(i)})$$
$$= v_0^{(i)} + \sum_{k<0} \sum_{j=1}^{m} \alpha_{(k),ij}(\gamma) v_k^{(j)}.$$

Since all the $\hat{\psi}_W(\gamma)_i$ belong to $W\gamma^{-1}$, we can define maps $\psi_{W,i} \colon \Gamma^W \to W$ by

$$\psi_{W,i}(\gamma) = (\hat{\psi}_W(\gamma)_i)\gamma = \sum_{k \in \mathbb{Z}} \sum_{j=1}^{m} \beta_{(k),ij}(\gamma) v_k^{(j)}.$$

From this formula and the form of the operators a, b, and d, it will be clear that all the maps $t_{j,i} \to \psi_{W,i}(\gamma)$, with $1 \leq i \leq m$ and $j \in \mathbb{N}$, are Fréchet-differentiable. Next we introduce the matrices $\hat{\Psi}_W(\gamma)$ and $\Psi_W(\gamma)$ whose rows are the $\hat{\psi}_{W,i}(\gamma)$ and $\psi_{W,i}(\gamma)$, resp., $1 \leq i \leq m$. An essential property of $\Psi_W(\gamma)$ is that all its derivatives w.r.t. the $t_{i,j}$ have rows that all belong to W. We write $\alpha_{(k)}(\gamma)$, resp. $\beta_{(k)}(\gamma)$, for the matrices whose (i,j)th entry is $\alpha_{(k),ij}(\gamma)$, resp. $\beta_{(k),ij}(\gamma)$; then $\alpha_{(k)}$ and $\beta_{(k)}$ belong to B_W. Following the lines set out in the first section, we introduce in $B_W[\xi, \xi^{-1}]$ the operators

$$K_W = \mathrm{Id} + \sum_{k<0} \alpha_{(k)}\xi^k, \quad L_W = K_W \xi K_W^{-1}, \quad U_\alpha = L_W E_\alpha L_W^{-1},$$

and we associate to $\hat{\Psi}_W(\gamma)$, $\Psi_W(\gamma)$, and $\gamma \in \Gamma$, the power series $\hat{\Psi}_W[\lambda]$, $\Psi_W[\lambda]$, and $\gamma[\lambda]$:

$$\hat{\Psi}_W[\lambda] = \mathrm{Id} + \sum_{k<0} \alpha_{(k)}\lambda^k, \qquad \Psi_W[\lambda] = \sum_{k \in \mathbb{Z}} \beta_{(k)}\lambda^k,$$

$$\gamma[\lambda] = \exp\left(\sum_{k \geq 1} \mathrm{diag}(t_{k,1}, \ldots, t_{k,m})\lambda^k\right).$$

The action of Γ is such that $\hat{\Psi}_W[\lambda]$, $\Psi_W[\lambda]$, and $\gamma[\lambda]$ are related through $\Psi_W[\lambda] = \hat{\Psi}_W[\lambda]\gamma[\lambda]$, where the product is the usual one for the multiplication of power series. In other words, $\Psi_W[\lambda] \in M(\gamma[\lambda])$.

3.3. We want to show that $\Psi_W[\lambda]$ is a wavefunction corresponding to L_W and the U_α. Since $\partial_{k,\alpha}\gamma[\lambda] = \lambda^k E_\alpha \gamma[\lambda]$, we have $\partial_{k,\alpha} = \tilde{\partial}_{k,\alpha}$ on $M(\gamma[\lambda])$ and according to section (1.4), it suffices to prove

$$\partial_{k,\alpha}(\Psi_W[\lambda]) = (L_W^k U_\alpha)_+(\Psi_W[\lambda]).$$

This relation is a consequence of the following

3.3.1. Proposition. *For all $1 \leq \alpha \leq m$ and all $k \geq 1$ there exists a $P_{k,\alpha} \in B_W[\xi]$ of the form $P_{k,\alpha} = \sum_{j=0}^{k} P_{k,\alpha,j}\xi^j$, with $P_{k,\alpha,k} = E_\alpha$, such that*

$$\partial_{k,\alpha}(\Psi_W[\lambda]) = P_{k,\alpha}(\Psi_W[\lambda]).$$

If we substitute $\Psi_W[\lambda] = \hat{\Psi}_W[\lambda]\gamma[\lambda]$ in this relation we then get

$$\{\partial_{k,\alpha}(\hat{\Psi}_W[\lambda]) + \hat{\Psi}_W[\lambda]\lambda^k E_\alpha\}\gamma[\lambda] = P_{k,\alpha}(\Psi_W[\lambda]).$$

By translating this back to $B_W[\xi, \xi^{-1}]$, we obtain the relation

$$\partial_{k,\alpha}K_W + K_W \xi^k E_\alpha = P_{k,\alpha}K_W,$$

which implies $P_{k,\alpha} = (K_W \xi^k E_\alpha K_W^{-1})_+ = (L_W^k U_\alpha)_+$.

Proof. First we have

$$\partial_{k,\alpha}(\Psi_W[\lambda]) = \hat{\Psi}_W[\lambda]E_\alpha\lambda^k\gamma[\lambda] + \partial_{k,\alpha}(\hat{\Psi}_W[\lambda])\gamma[\lambda]$$

$$= \hat{\Psi}_W[\lambda]E_\alpha\lambda^k\gamma[\lambda] + \left(\sum_{i<0}c_i\lambda^i\right)\gamma[\lambda]$$

$$= \left\{E_\alpha\lambda^k + \sum_{j<k}b_j\lambda^j\right\}\gamma[\lambda].$$

Next we consider the action of ξ^r on $M(\gamma[\lambda])$. Here we look at the leading term modulo $\gamma[\lambda]$:

$$\partial^r(\Psi_W[\lambda]) = \sum_{j=0}^{r}\binom{r}{j}\partial^j(\hat{\Psi}_W[\lambda]^{r-j})\gamma[\lambda]$$

$$= \left(\mathrm{Id}\,\lambda^r + \sum_{j<r}d_j\lambda^j\right)\gamma[\lambda].$$

Hence there is an $s < k$ such that

$$\partial_{k,\alpha}(\Psi_W[\lambda]) - E_\alpha\partial^k(\Psi_W[\lambda]) = \left(A_s\lambda^s + \sum_{j<s}\rho_j\lambda^j\right)\gamma[\lambda].$$

Continuing in this way, after at most k steps we obtain a polynomial $\sum_{i=0}^{k}A_i\partial^i$, with all the A_i in B_W and $A_k = E_\alpha$, satisfying

$$\left(\partial_{k,\alpha} - \sum_{i=0}^{k}A_i\partial^i\right)\Psi_W[\lambda] = \left(\sum_{j<0}\mu_j\lambda^j\right)\gamma[\lambda].$$

Since multiplying a matrix from the left by another one involves only linear operations with the rows, all the rows of the $A_i\partial^i(\Psi_W(\gamma))$ belong again to W. On the other hand, the rows of $(\partial_{k,\alpha} - \sum_{i=0}A_i\partial^i)\Psi_W(\gamma)$ also belong to $(\mathbf{K}_{<0})\gamma$. Hence by the definition of Γ^W, they must be zero and we have the desired equality

$$\left(\partial_{k,\alpha} - \sum_{i=0}^{k}A_i\xi^i\right)\Psi_W[\lambda] = 0.$$

3.4. In this subsection we explain the connection between the scaling in (1.5) and that in (2.5). Take c in $\mathbb{C}^*$ as in (2.5). If

$$\lambda = \exp\left(\sum_{j=1}^{\infty}\mathrm{diag}(t_{j1}, \ldots, t_{jm})\Delta^j\right) \in \Gamma,$$

then we write

$$\tilde{\gamma} = \exp\left(\sum_{j \geq 1} \text{diag}(c^j t_{j1}, \ldots, c^j t_{jm})\underline{\Delta}^j\right)$$

$$= \exp\left(\sum_{j \geq 1} \text{diag}(t_{j1}, \ldots, t_{jm})(c\underline{\Delta})^j\right)$$

$$= \sum_{k \geq 0} c^k a_k \underline{\Delta}^k,$$

with $a_k \in \text{End}(\mathbb{C}^m)$. The relation needed is $R_c \tilde{\gamma} W = \gamma R_c W$ for all $\gamma \in \Gamma$ and all $W \in \text{Gr}_0(\mathbf{K})$. Since each $\gamma = \sum_{i \geq 0} a_i \underline{\Delta}^i = \lim_{n \to \infty} \sum_{i=0}^n a_i \underline{\Delta}^i$, it suffices to prove this relation for $\gamma \in \Gamma$ of the form $\gamma = \sum_{i=1}^k a_i \Lambda^i$; then also $\tilde{\gamma} = \sum_{i=1}^n c^i a_i \underline{\Delta}^i$. Now for $\underline{w} = \sum_{j \leq N} w_j \lambda^j \in W$, we have

$$R_c \tilde{\gamma}\left(\sum_{j \leq N} W_j \lambda^j\right) = R_c\left(\sum_{i=1}^n \sum_{j \leq N} c^i a_i (w_j) \lambda^{i+j}\right)$$

$$= \sum_{i=1}^n \sum_{j \leq N} c^{-j} a_i (w_j) \lambda^{i+j},$$

$$\gamma R_c\left(\sum_{j \leq N} w_j \lambda^j\right) = \sum_{i=1}^n \sum_{j \leq N} a_i (c^{-j} w_j) \lambda^{i+j} = R_c \tilde{\gamma} \underline{w}.$$

The elements of W of this form are dense in W, hence we get the desired equality. This implies

3.4.1. Proposition. *The wave functions of W and $R_c W$ are related through*

$$\hat{\Psi}_{R_c W}(\gamma)[\lambda]\gamma[\lambda] = \hat{\Psi}_W(\tilde{\gamma})[\lambda/c]\tilde{\gamma}[\lambda/c].$$

Proof. Since $\gamma[\lambda] = \tilde{\gamma}[\lambda/c]$ one merely has to show the equality for $\hat{\Psi}_W$ and $\hat{\Psi}_{R_c W}$. First we have $R_c(\hat{\Psi}_W(\tilde{\gamma})[\lambda]) = \hat{\Psi}_W(\tilde{\gamma})[\lambda/c]$. Thus the ith row of $\hat{\Psi}_W(\tilde{\gamma})[\lambda/c]$ is the scaling by c of the unique element of $\tilde{\gamma}^{-1} W$ of the form $f_i + \sum_{i<0}^c w_j \lambda^j$. In other words it is the unique element in $R_c \tilde{\gamma}^{-1} W = \gamma^{-1} R_c W$ of the form $f_i +$ lower order in λ and that is exactly the ith row of $\hat{\Psi}_{R_c W}(\gamma)[\lambda]$. This proves the proposition.

In particular, we see that $K_{R_c W}$ in $B[\xi, \xi^{-1}]$ is given by

$$K_{R_c W} = \text{Id} + \sum_{i>0} c^i \tilde{k}_i \xi^{-i},$$

with $K_W = \text{Id} + \sum_{i>0} k_i \xi^{-i}$. By making use of a grading argument as in (1.5), one shows that if $K_W^{-1} = \text{Id} + \sum_{i>0} \underline{k}_i \xi^{-i}$, then

$$K_{R_c W}^{-1} = \text{Id} + \sum_{i>0} c^i \underline{\tilde{K}}_i \xi^{-i}.$$

Hence we get

3.4.2. Proposition. $L_{R_cW} = K_{R_cW} \xi K_{R_cW}^{-1} = \tilde{L}_W$ *and* $\tilde{U}_\alpha = K_{R_cW} E_\alpha K_{R_cW}^{-1}$.

Remark. Assuming that condition (2.5.1) holds for all sufficiently small c, we consider a w in **P** with $\pi(w) = W_k$ such that $w(f_\alpha \lambda^i) = \sum_{k \leq i} \sum_{\beta=1}^m v_{k\beta} f_\beta \lambda^k$ for all $i \geq 0$ and $1 \leq \alpha \leq m$. Then we have that an embedding $R_c w$ corresponding to $R_c W$ is given by $R_c w(f_\alpha \lambda^i) = \sum_{k \leq i} \sum_{\beta=1}^m C^{i-k} v_{k\beta} f_\beta \lambda^k$. The coefficients $v_{k\beta}$ for $k \leq 0$ can be estimated by

$$|v_{k\beta}| \|f_\beta\| \leq \|p_\beta\| \|\Delta\|^{-k} \|w\| \|\Delta\|^i,$$

where p_β denotes the natural projection $K \to \langle f_\beta \rangle$. Hence given $p \geq 1$ one can apply a suitable small scaling by c on w such that $\mathbb{N}$-matrix corresponding to $R_c w$ has all columns in $l^p(\mathbb{Z})^m$. In this sense, the convergent power series solution of (1.4.1)–(1.4.3) to K can be scaled into a solution corresponding to $l^p(\mathbb{Z})^m$.

4. RELATION WITH τ-FUNCTIONS

4.1. Supposing that $W \in \text{Gr}$, we will introduce some other elements of Gr closely related to W. In 4.3 we will need these elements to find a relation between the wave function $\Psi_W[\lambda]$ and some τ-functions in the case $W = \mathbf{O}$.

We will need $\Lambda: H \to H$ to be invertible, so we require that $\Lambda: H \to H$ be surjective. By the bounded inverse theorem, Λ^{-1} is also bounded.

Let $\Lambda_{k_1, \ldots, k_m}$ be the automorphism of $\mathbf{K} = \bigoplus_{i=1}^m H^{(i)}$ of the form $\text{diag}(\Lambda^{k_1}, \ldots, \lambda^{k_m})$, and $W_{k_1, \ldots, k_m} = W \cdot \Lambda_{k_1, \ldots, k_m}$. Now $W_{k_1, \ldots, k_m}$ is again an element of Gr if $\Lambda_{k_1, \ldots, k_m}$ is an element of G_n, and this is the case when $k_1 + \cdots + k_m = 0$.

As is easily verified, $(\sum a_{1i} v_{i+k_1}, \ldots, \sum a_{mi} v_{i+k_m}) \in W_{k_1, \ldots, k_m}$ if and only if $(\sum a_{1i} v_i, \ldots, \sum a_{mi} v_i) \in W = W_{0, \ldots, 0}$.

In the sequel we are interested only in the case where most of the k_i are zero. Therefore we introduce $\Lambda_{i/j} = \Lambda_{k_1, \ldots, k_m}$ for $i \neq j$, with $k_i = 1$, $k_j = -1$, $k_l = 0$ $(l \neq i, j)$. We denote $W_{i/j} = W \Lambda_{i/j}$ correspondingly.

Now we want to extend the elements $\Lambda_{i/j} \in G_n$ to elements $(\Lambda_{i/j}, q_{i/j}) \in G$. To perform this extension, we consider the splitting

$$\Lambda_{i/j} = \begin{pmatrix} a_{i/j} & b_i \\ c_j & d_{i/j} \end{pmatrix}$$

with respect to the direct sum $\mathbf{K} = \mathbf{K}_{\geq 0} \oplus \mathbf{K}_{<0}$. As mentioned before, $a_{i/j}: \mathbf{K}_{\geq 0} \to \mathbf{K}_{\geq 0}$ is a Fredholm operator of index zero. In fact $\ker(a_{i/j}) = \mathbb{C} v_0^{(j)} = \mathbb{C} e_{j-1}$ and $\text{coker}(a_{i/j}) \cong \mathbb{C} v_0^{(i)} = \mathbb{C} e_{i-1}$. We introduce the operator $E_{k,l}$ mapping $\sum \alpha_i e_i$ to $\alpha_l e_k$. Then $q_{i/j} = a_{i/j} + E_{i-1, j-1}$ is invertible, moreover $q_{i/j}^{-1} a_{i/j} = \text{id} - E_{j-1, j-1}$, so it is surely of the form $\text{id} + \text{nuclear}$. Finally notice that $b_i = E_{i-1, i-1-m}$ is an operator $\mathbf{K}_{<0} \to \mathbf{K}_{\geq 0}$.

4.2. To introduce τ-functions, we have to fix an embedding $w \in \mathbf{P}$ such that $\text{im}(w) = W$. $\tau_w \colon G \to \mathbb{C}$ is defined in 2.4. Now suppose $\tau_w(g_1) \neq 0$. Then using the multiplicativity of the determinant one can prove

$$(4.2.1) \qquad \frac{\tau_w(g_1 g_2)}{\tau_w(g_1)} = \tau_{(g_1^{-1}w) \cdot (g_1^{-1}w)_+^{-1}}(g_2),$$

Notice that $w_1 = (g_1^{-1}w) \circ (g_1^{-1}w)_+^{-1}$ is of the form $\binom{\text{id}}{(w_1)_-}$. We apply this formula with $g_1 \in \Gamma^W$, and

$$g_2 = \text{diag}(\text{id}, \dots, \text{id}, q_\zeta, \text{id}, \dots, \text{id}) = q_\zeta^{(i)},$$

where $q_\zeta = \text{id} - \Lambda \zeta^{-1}$, is placed in the (i, i)-entry of g_2. We take $|\zeta| > \|\Lambda\|$ to ensure that $q_\zeta \in \Gamma(\Lambda)$ and $q_\zeta^{(i)} \in \Gamma$.

In the 1-component case it is proved that upon splitting $q_\zeta^{-1} = \binom{a\ b}{0\ d}$ with respect to $H_{\geq 0} \oplus H_{<0}$, one has that $a^{-1}b \colon H_{<0} \to H_{\geq 0}$ is the 1-dimensional projection of $H_{<0}$ onto H_0 given by

$$a^{-1}b \left(\sum_{j<0} a_j v_j \right) = \left(\sum_{j<0} a_j \zeta^j \right) v_0 \quad \text{(see [Se or He])}.$$

From this one easily derives that if

$$q_\zeta^{(i)^{-1}} = \begin{pmatrix} a^{(i)} & b^{(i)} \\ 0 & d^{(i)} \end{pmatrix}$$

with respect to $\mathbf{K}_{\geq 0} \oplus \mathbf{K}_{<0}$, then $a^{(i)^{-1}}b^{(i)} \colon \mathbf{K}_{<0} \to \mathbf{K}_{\geq 0}$ is the 1-dimensional projection on $\mathbf{K}_{i-1} = \mathbb{C}e_{i-1} = \mathbb{C}v_0^{(i)}$ given by

$$a^{(i)^{-1}}b^{(i)} \left(\sum_{j<0} a_{1j}v_j, \dots, \sum_{j<0} a_{mj}v_j \right) = \left(\sum_{j<0} a_{ij}\zeta^j \right) v_0^{(i)}.$$

Now it is easy to derive the following

Lemma. *Suppose* $\gamma \in \Gamma^W$ *and* $\text{Im}(w) = W$, $w \in \mathbf{P}$. *Then*

$$\Psi_{W,}[\zeta]_{ii} = 1 + \sum_{k<0} \alpha_{(k)ii} \zeta^k = \frac{\tau_w(\gamma q_\zeta^{(i)})}{\tau_w(\gamma)}.$$

Proof. We have to calculate

$$\begin{aligned} \tau_{w_1}(q_\zeta^{(i)}) &= \det((a^{(i)} + b^{(i)}(w_1)_-)a^{(i)^{-1}}) \\ &= \det(\text{id} + a^{(i)^{-1}}b^{(i)}(w_1)_-) \\ &= 1 + \text{trace}(a^{(i)^{-1}}b^{(i)}(w_1)_-). \end{aligned}$$

Since

$$(w_1)_-(e_{i-1}) = (w_1)_-(v_0^{(i)}) = \left(\sum_{k<0} \alpha_{(k)i1} v_k, \dots, \sum_{k<0} \alpha_{(k)im} v_k \right),$$

we have the desired result by using the explicit form of $a^{(i)^{-1}} b^{(i)}$.

4.3. The results of the last lemma enable us to express the diagonal elements of $\hat{\Psi}_W[\lambda]$ in terms of the function τ_w. For the off-diagonal element $\hat{\Psi}_W[\lambda]_{ji}$, $j \neq i$, we need the space $W_{i/j}$. We fix $w \in \mathbf{P}$ such that $\pi(w) = W$. Further we introduce $w_{i/j} = \Lambda_{i/j} w q_{i/j}^{-1}$. We will express $\hat{\Psi}_W[\lambda]_{ji}$ in terms of $\tau_{w_{i/j}}$ and τ_w. First we will prove this expression in the case $a_{(-1)ij} \neq 0$. This is an extra condition on $\gamma \in \Gamma^W$. This derivation shows how one comes to this expression. Finally we give a derivation which also holds when $a_{(-1)ji} = 0$.

To find $\Psi_W[\lambda]_{ji}$ in terms of $\tau_{w_{i/j}}$ we consider the complete row $\psi_w(\gamma)_j$:

$$\left(\sum_{k<0} \alpha_{(k)j1} v_k, \dots, v_0 + \sum_{k<0} \alpha_{(k)jj} v_k, \dots, \sum_{k<0} \alpha_{(k)jm} v_k \right) \in W\gamma^{-1}.$$

Multiplying by $\Lambda_{i/j}$ we find

$$\alpha_{(-1)ji} \left(\sum_{k<0} \frac{\alpha_{(k)j1}}{\alpha_{(-1)ji}} v_k, \dots, v_0 + \sum_{k<-1} \frac{\alpha_{(k)ji}}{\alpha_{(-1)ji}} v_{k+1}, \dots, \sum_{k<0} \frac{\alpha_{(k)jm}}{\alpha_{(-1)ji}} v_k \right)$$

$$\in W\gamma^{-1}\Lambda_{i/j} = W_{i/j}\gamma^{-1}.$$

Because we required $\alpha_{(-1)ji} \neq 0$, the natural projection $W_{i/j}\gamma^{-1} \to \mathbf{K}_{\geq 0}$ is a bijection. Using the results of 4.2 we find that

$$1 + \sum_{k<-1} \frac{\alpha_{(k)ji}}{\alpha_{(-1)ji}} \zeta^{k+1} = \frac{\tau_{w_{i/j}}(\gamma q_\zeta^{(i)})}{\tau_{w_{i/j}}(\gamma)}.$$

So we obtain the following result:

$$\hat{\Psi}_W[\zeta]_{ji} = \zeta^{-1} \alpha_{(-1)ji} \frac{\tau_{w_{i/j}}(\gamma q_\zeta^{(i)})}{\tau_{w_{i/j}}(\gamma)}.$$

Now it is easy to prove the following theorem in the case $\alpha_{(-1)ji} \neq 0$.

Theorem. *Suppose $\gamma \in \Gamma^W$ and define $\hat{\Psi}_W(\gamma)$ as before. Then*

$$\hat{\Psi}_W[\zeta]_{ji} = \zeta^{-1} \frac{\tau_{w_{i/j}}(\gamma q_\zeta^{(i)})}{\tau_w(\gamma)}, \qquad j \neq i.$$

Proof. For convenience we write $A = (\gamma^{-1} w a^{-1})_+^{-1} \in \mathbf{T}$, where $\gamma^{-1} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. Then $\tau_w(\gamma) = \det((\gamma^{-1} w a^{-1})_+) = \det A^{-1}$.

On the other hand, since $\det(aq_{i/j}a^{-1}q_{i/j}^{-1}) = 1$,

$$\tau_{w_{i/j}}(\gamma) = \det((\gamma^{-1}\Lambda_{i/j}w)_+ q_{i/j}^{-1}a^{-1})$$
$$= \det((\Lambda_{i/j}\gamma^{-1}wa^{-1})_+ q_{i/j}^{-1})$$
$$= \det(A^{-1}q_{i/j}^{-1}(\Lambda_{i/j}\gamma^{-1}wa^{-1}A)_+)$$
$$= \det(A^{-1}) \cdot \det(q_{i/j}^{-1}a_{i/j} + q_{i/j}^{-1}b_i(w_1)_-).$$

So $\tau_{w_{i/j}}(\gamma) = \det(q_{i/j}^{-1}a_{i/j} + q_{i/j}^{-1}b_i w_-) \cdot \tau_w(\gamma)$.

Now $q_{i/j}^{-1}a_{i/j} = \mathrm{id} - E_{j-1,j-1}$ and $q_{i/j}^{-1}b_i$ is the projection $E_{j-1,i-1-m}$. So $q_{i/j}^{-1}a_{i/j} + q_{i/j}^{-1}b_i(w_1)_-$ is of the form $\mathrm{id} + 1$-dimensional, so we are left with the calculation of

$$1 + \mathrm{trace}(-E_{j-1,j-1} + E_{j-1,i-1-m}(w_1)_-).$$

Since $(-E_{j-1,j-1} + E_{j-1,i-1-m}(w_1)_-)(e_{j-1}) = (-1 + \alpha_{(-1)ji})(e_{j-1})$ we find $\tau_{w_{i/j}}(\gamma) = \alpha_{(-1)ji} \cdot \tau_w(\gamma)$, proving the result.

We now turn to a proof for the last theorem without assuming $\alpha_{(-1)ji} \neq 0$. We define two elements of **P**, namely w_1 (as above) and $v_{i/j}$:

$$w_1 = (\gamma^{-1}wa^{-1})(\gamma^{-1}wa^{-1})_+^{-1} = \begin{pmatrix} \mathrm{id} \\ (w_1)_- \end{pmatrix},$$

$$v_{i/j} = \Lambda_{i/j}w_1 q_{i/j}^{-1}.$$

Now we have

Lemma. $\tau_{w_{i/j}}(\gamma q_\zeta^{(i)}) = \tau_w(\gamma)\tau_{v_{i/j}}(q_\zeta).$

Proof.

$$\tau_{w_{i/j}}(\gamma q_\zeta^{(i)}) = \det((q_\zeta^{(i)^{-1}}\gamma^{-1}\Lambda_{i/j}wq_{i/j}^{-1}a^{-1}a^{(i)^{-1}})_+)$$
$$= \det((q_\zeta^{(i)^{-1}}\Lambda_{i/j}\gamma^{-1}wa^{-1}q_{i/j}^{-1}a^{(i)^{-1}})_+)$$
$$= \tau_w(\gamma(\tilde{\Lambda}_{i/j}^{-1}q_\zeta^{(i)})).$$

Here we used that $\Lambda_{i/j}$ and γ^{-1} commute, and that $\det(aq_{i/j}a^{-1}q_{i/j}^{-1}) = 1$. Further $\tilde{\Lambda}_{i/j}^{-1} = (\Lambda_{i/j}, q_{i/j})^{-1}$. Using the multiplicity formula (4.2.1) we find

$$\tau_w(\gamma(\tilde{\Lambda}_{i/j}^{-1}q_\zeta^{(i)})) = \tau_w(\gamma) \cdot \tau_{w_1}(\tilde{\Lambda}_{i/j}^{-1}q_\zeta^{(i)}) = \tau_w(\gamma) \cdot \tau_{v_{i/j}}(q_\zeta^{(i)}).$$

It is left to prove that $\tau_{v_{i/j}}(q_\zeta^{(i)}) = \zeta\tilde{\Psi}_W[\zeta]_{ji}$. So let us investigate $v_{i/j} = \begin{pmatrix} v_+ \\ v_- \end{pmatrix}$, with

$$v_+ = a_{i/j}q_{i/j}^{-1} + b_i(w_1)_- q_{i/j}^{-1}$$
$$= (\mathrm{id} - E_{i-1,i-1}) + b_i(w_1)_- q_{i/j}^{-1}$$
$$= \mathrm{id} + \mathrm{proj} \quad \text{on } \mathbb{C}e_{i-1}.$$

Moreover $\tau_{v_{i/j}}(q_\zeta^{(i)}) = \det(v_+ + a^{(i)^{-1}} b^{(i)} v_-)$. Now $a^{(i)^{-1}} b^{(i)}$ is also a projection on $\mathbb{C}e_{i-1}$. So

$$
\begin{aligned}
\det(v_+ &+ a^{(i)^{-1}} b^{(i)} v_-) \\
&= 1 + \mathrm{trace}(-E_{i-1,i-1} + b_i(w_1)_- q_{i/j}^{-1} + a^{(i)^{-1}} b^{(i)} v_-) \\
&= 1 + (-1 + s(\zeta)) = s(\zeta),
\end{aligned}
$$

where $s(\zeta)$ is given by

$$
(b_i(w_1)_- q_{i/j}^{-1} + a^{(i)^{-1}} b^{(i)} v_-)(e_{i-1}) = s(\zeta)e_{i-1}.
$$

Now $b_i(w_1) - q_{i/j}^{-1}(e_{i-1}) = \alpha_{(-1)ji} e_{i-1}$ and

$$
(a^{(i)^{-1}} b^{(i)} v_-)(e_{i-1}) = \left(\sum_{k<-1} \alpha_{(k)ji} \zeta^{k+1} \right) e_{i-1},
$$

which proves the theorem in the general case.

4.4. Let us discuss these results. First notice that the spaces $W_{k_1,\dots,k_m}$ in [U] are what we called $W_{-k_1,\dots,-k_m}$, which can be seen by comparing the expression of $\hat{\Psi}_W$ in τ_w and $\tau_{w_{k_1,\dots,k_m}}$ in both cases.

Second we want to consider τ_w as a function of variables $(t_{j,\alpha})$, $j = 1, 2, \dots$, $\alpha = 1, \dots, m$. Therefore we write

$$
\gamma^{-1} = \mathrm{diag}\left(\exp\left(\sum_{j\geq 1} t_{j,1} \Lambda^j \right), \dots, \exp\left(\sum_{j\geq 1} t_{j,m} \Lambda^j \right) \right),
$$

and denote the set of $(t_{j,\alpha})$ for which $\gamma^{-1} \in \Gamma$ by T, and $\tau_w(\gamma)$ by $\tau_w((t_{j,\alpha}))$. In the case that $(t_{j,\alpha}) \in T$, we have the following assertions:

    $1°$ $\tau_w((t_{j,\alpha}))$ is well defined.
    $2°$ $\tau_w((t_{j,\alpha}))$ is a globally defined analytic function (of any finite number of variables).

This last statement can be proved using the expression of the determinant in traces of exterior powers of $(\gamma^{-1} w a^{-1})_+ - \mathrm{id}$ (see [He]). To prove that $T \neq \varnothing$, note that $(t_{j,\alpha})$ with only finitely many $t_{j,\alpha}$ nonzero are elements of T.

All this has consequences for the wave function $\hat{\Psi}_W[\zeta]$. Since

$$
(\gamma q_\zeta^{(i)})^{-1} = \mathrm{diag}\left(\exp\left(\sum t_{j,1} \Lambda^j \right), \dots, \right.
$$
$$
\left. \exp\left(\sum \left(t_{j,i} - \frac{1}{j\zeta} \right) \Lambda^j \right), \dots, \exp\left(\sum t_{j,m} \Lambda^j \right) \right)
$$

we see that for $\tau_w(\gamma) \neq 0$, $\hat{\Psi}_W[\zeta]_{ji}$ is an analytic function of (any finite number of) $(t_{j,\alpha})$ and ζ for $|\zeta| > \|\Lambda\|$.

5. REDUCTIONS

In this section we consider some reductions of the multicomponent KP-hierarchy. First in 5.1 we show that very general Lax equations, as considered in [Wi], are obtained as reductions. Moreover in subsection 5.2 we show that an easy case of the reductions leads to the AKNS-hierarchy, as defined in [FNR].

Throughout this section,

$$\underline{\Lambda}^l = \Lambda_{l,\ldots,l} = \mathrm{diag}(\Lambda^l, \ldots, \Lambda^l).$$

5.1. Suppose that $W \in \mathrm{Gr}$ satisfies

$$(5.1.1) \qquad w \in W \Rightarrow \sum_{\beta=1}^{m} c_\beta w E_\beta \underline{\Lambda}^l \in W \qquad (c_\beta \in \mathbb{C}).$$

A special case of this reduction is $c_\beta = 1$ leading to

$$w \in W \Rightarrow \sum_{\beta=1}^{m} w E_\beta \underline{\Lambda}^l = w \underline{\Lambda}^l \in W.$$

We denote $\mathscr{L}_w = \sum_{\beta=1}^{m} c_\beta L_w^l U_\beta$. We prove the following.

Assertion. $W \in \mathrm{Gr}$ *satisfies* $(5.1.1) \Rightarrow \mathscr{L}_w = (\mathscr{L}_w)_+$.

Proof (we write $\mathscr{L}$ instead of $\mathscr{L}_w$).

$$\sum_\beta c_\beta P_{l,\beta}(\psi_w[\lambda])$$

$$= \sum_\beta c_\beta \partial_{l,\beta}\left(\left(\sum_{k\leq 0}\alpha_{(k)}\lambda^k\right)\gamma[\lambda]\right)$$

$$= \left(\sum_{\substack{k\leq -1 \\ \beta}} c_\beta \partial_{l,\beta}(\alpha_{(k)})\lambda^k + \sum_{\substack{k\leq 0 \\ \beta}} c_\beta \alpha_{(k)}\lambda^k \lambda^l E_\alpha\right)\gamma[\lambda],$$

so

$$\sum_\beta c_\beta P_{l,\beta}\psi_w[\lambda] - \lambda^l \sum_\beta c_\beta \psi_w[\lambda]E_\beta$$

$$(5.1.2) \qquad = \left(\sum_{\substack{k\leq -1 \\ \beta}} c_\beta \partial_{l,\beta}(\alpha_{(k)})\lambda^k\right)\gamma[\lambda].$$

Because of (5.1.1), $\lambda^l \sum_\beta c_\beta \psi_w[\lambda] E_\beta \in W$ and with it the left-hand side of (5.1.2), so the right-hand side, is a member of $W \cap \mathbf{K}_{\leq 0} \cdot \gamma = \{0\}$. Therefore

$$\sum_\beta c_\beta P_{l,\beta} \psi_w[\lambda] = \lambda^l \sum_\beta c_\beta \psi_w[\lambda] E_\beta ,$$

$$\sum_\beta c_\beta \partial_{l,\beta} \alpha_{(k)} E_\beta = 0 \quad (\forall k \leq 0) .$$

Translating these equations back to $B[\xi, \xi^{-1}]$ and writing $\widehat{W} = K_W$, the first one yields

(5.1.3) $$\mathscr{L}_+ \widehat{W} = \sum_\beta c_\beta \widehat{W} \xi^l E_\beta ,$$

and the second one becomes

(5.1.4) $$\sum_\beta c_\beta \partial_{l,\beta}(\widehat{W}) E_\beta = 0 .$$

On the other hand we have $L^l \widehat{W} = \widehat{W} \xi^l$, from which we derive

$$\sum_\beta c_\beta \widehat{W} \xi^l E_\beta = \sum_\beta c_\beta L^l \widehat{W} E_\beta$$

$$= \sum_\beta c_\beta L^l U_\beta \widehat{W} = \mathscr{L} \widehat{W} .$$

Comparing with (5.1.3) yields $\mathscr{L}_+ = \mathscr{L}$, since $\widehat{W}$ is invertible. This proves the assertion.

$\mathscr{L}$ is a differential polynomial $\sum_{i=0}^l q_i \xi^i$ with leading terms

$$\mathscr{L} = \sum_{\beta=1}^m c_\beta E_\beta \xi^l + \sum_{\beta=1}^m c_\beta [\alpha_{-1}, E_\beta] \xi^{l-1} + \cdots .$$

Therefore $q_l = \mathrm{diag}(c_1, c_2, \ldots, c_m)$ and $(q_{l-1})_{ij} = (\alpha_{-1})_{ij}(c_i - c_j)$. In particular, $(q_{l-1})_{ij} = 0$ if $c_i = c_j$, so $\mathscr{L}$ is of the form studied by Wilson in [Wi]. From the equations for L one easily derives

$$\partial_{i,\beta}(\mathscr{L}) = [(L^i U_\beta)_+, \mathscr{L}] .$$

This yields partial differential equations for the matrix elements $(\alpha_k)_{ij}$ of $\widehat{W}$. In [Wi] conditions are discussed such that one finds equations expressed in the elements $(q_s)_{ij}$ of $\mathscr{L}$ only. This amounts to restricting oneself to certain derivations of the form $\sum_{\beta=1}^m d_\beta \partial_{i,\beta}$ such that the leading term of $\sum_{\beta=1}^m d_\beta L^i U_\beta$ ($\in Z(\mathscr{L})$) is admissible (see Definition 2.17 of [Wi]), that is $d_\alpha = d_\beta$ if $c_\alpha = d_\beta$.

5.2. We consider a special case of the reduction of 5.1, namely we take $c_\beta = 1$ $(\beta = 1, \ldots, m)$. In this case $\mathscr{L} = L^l$. Moreover from (5.1.4) one finds $\sum_\beta \partial_{l,\beta}(\widehat{W}) = 0$. In particular, for $l = 1$ we have $\partial(\widehat{W}) = 0$. Since $L = \widehat{W}\xi\widehat{W}^{-1}$ and $U_\alpha = \widehat{W}E_\alpha\widehat{W}^{-1}$ we also have

$$(5.2.1) \qquad \sum_\beta \partial_{l,\beta}(L) = \sum_\beta \partial_{l,\beta}(U_\alpha) = 0.$$

Next we consider the easiest case of the situation above, namely we take $m = 2$ and $l = 1$, which leads to

$$(5.2.2) \qquad \mathscr{L} = L = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\xi.$$

We have two series of equations, (1.3.1) and (1.3.2). As noted, $\partial(L^i U_\alpha)_+ = 0$ so (1.3.1) is trivially satisfied. We rewrite (1.3.2) in the following form:

$$(5.2.3) \qquad (\partial_{i,1} + \partial_{i,2})U_\beta = [(L^i(U_1 + U_2))_+, U_\beta],$$

$$(5.2.4) \qquad (\partial_{i,1} - \partial_{i,2})U_\beta = [(L^i(U_1 - U_2))_+, U_\beta].$$

Now $[(L^i(U_1+U_2))_+, U_\beta] = [\mathrm{Id}\cdot\xi, U_\beta] = 0$, so $(\partial_{i,1}+\partial_{i,2})U_\beta = 0$. This means that U_β depends only on $x_i = \frac{1}{2}(t_{i,1} - t_{i,2})$, and not on $y_i = \frac{1}{2}(t_{i,1} + t_{i,2})$. Therefore denote $\partial_i = \partial/\partial x_i = \partial_{i,1} - \partial_{i,2}$, so we have

$$(5.2.4)' \qquad \partial_i(U_\beta) = [(L^i(U_1 - U_2))_+, U_\beta],$$

which is equivalent to

$$(5.2.5) \qquad \partial_i(U_1 + U_2) = [(\xi^i Q)_+, U_1 + U_2],$$

$$(5.2.6) \qquad \partial_i(U_1 - U_2) = [(\xi^i Q)_+, U_1 - U_2].$$

We denoted $Q = U_1 - U_2$. Since $U_1 + U_2 = \mathrm{Id}$, (5.2.5) is trivial. We are left with

$$(5.2.7) \qquad \partial_i Q = [(\xi^i Q)_+, Q].$$

These are the equations of the AKNS-hierarchy, as defined by Flaschka, Newell, and Ratiu in [FNR]. Note that ξ takes the role of spectral parameter (it commutes with Q). One can show that

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}\xi^{-1} + \cdots$$

and all equations (5.2.7) can be expressed in q and r. Moreover $Q^2 = (U_1 - U_2)(U_1 - U_2) = \mathrm{Id}$.

5.3. To give an idea of what is going on, we associate groups to these reductions. We denote

$$\Omega = \mathrm{diag}(c_1\Lambda^l, c_2\Lambda^l, \ldots, c_m\Lambda^l)$$

$$= \sum_{k\in\mathbb{Z}} c_{k+1}E_{k, k+lm} \qquad (c_{\alpha+m} = c_\alpha).$$

Expressed in Ω, (5.1.1) becomes $\omega \in W \Rightarrow \omega\Omega \in W$. Consider now the subgroup G_Ω of G_n consisting of the elements of G_n that commute with Ω:

$$(5.3.1) \qquad G_\Omega = \{g \in G_n | g\Omega = \Omega g\}.$$

We consider the orbit $G_\Omega \cdot \mathbf{K}_{\geq 0}$ in Gr. One directly verifies that elements from this orbit satisfy (5.1.1). We describe G_Ω explicitly. For this we consider the matrix (g_{ij}) of g defined by $g \cdot e_j = \sum_i g_{ij} e_i$. Since e_j is a topological basis of $\mathbf{K}$, g is completely determined by its matrix. Condition (5.3.1) now yields

$$(5.3.2) \qquad g \in G_\Omega \Leftrightarrow g_{i,j-lm} c_{j+1} = g_{i+lm,j} c_{i+1} \quad (\forall i, j \in \mathbb{Z}).$$

A case of special interest is the case $c_\beta = 1$ $(\beta = 1, \ldots, m)$. Then (5.3.2) gives

$$g \in G_\Omega \Leftrightarrow g_{i,j} = g_{i+lm,j+lm} \quad (\forall i, j \in \mathbb{Z}).$$

Let us take $\mathbf{K} = l^2(\mathbb{Z}) \simeq L^2(S^1)$ and denote by $\mathrm{GL}_p(C)$ the group of twice differential maps $S^1 \to \mathrm{GL}_p(\mathbb{C})$. Then using (5.3.3) and Proposition 2.3 of [Se] we see that there is an embedding of $\mathrm{GL}_{lm}(\mathbb{C})$ into $G_\Omega \subset G_n$.

Conversely, one can define $\mathrm{Gr}^{(lm)}$ in $l^2(\mathbb{Z})$ to be the $\mathrm{GL}_{lm}(\mathbb{C})$-orbit of $\mathbf{K}_{\geq 0}$ in the m-component KP-hierarchy. In this way we associate to $\mathrm{GL}_p(\mathbb{C})$ as many hierarchies as p has positive divisors. In the case $p = 2$ one finds two hierarchies: the KdV-hierarchy $m = 1$, $l = 2$ (see [Se]), and the AKNS-hierarchy $m = 2$, $l = 1$ (see 5.2).

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