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BOX-SPACES AND RANDOM PARTIAL ORDERS

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ABSTRACT. Winkler [2] studied random partially ordered sets, defined by taking n points at random in $[0, 1]^d$, with the order on these points given by the restriction of the order on $[0, 1]^d$. Bollobás and Winkler [1] gave several results on the height of such a random partial order. In this paper, we extend these results to a more general setting. We define a box-space to be, roughly speaking, a partially ordered measure space such that every two intervals of nonzero measure are isomorphic up to a scale factor. We give some examples of box-spaces, including (i) $[0, 1]^d$ with the usual measure and order, and (ii) Lorentzian space-time with the order given by causality. We show that, for every box-space, there is a constant d which behaves like the dimension of the space. In the second half of the paper, we study random partial orders defined by taking a Poisson distribution on a box-space. (This is of course essentially the same as taking n random points in a box-space.) We extend the results of Bollobás and Winkler to these random posets. In particular we show that, for a box-space X of dimension d, there is a constant m_X such that the length of a longest chain tends to $m_X n^{1/d}$ in probability.

1. BOX-SPACES

The objects we shall be considering are partially ordered measure spaces with a very regular and homogeneous structure: we demand that all intervals $\langle x, y\rangle$ of nonzero measure are isomorphic up to a scale-factor. The prime examples we have in mind are (i) the d-dimensional cube, with componentwise order, and (ii) the space $\mathbf{R}^{d-1} \times \mathbf{R}$ with $(x, t) \leq (y, u)$ if $|x-y| \leq u-t$.

First we make some definitions. We define a *partially ordered measure space* to be a quadruple $(X, \mathcal{F}, \mu, <)$ such that $(X, \mathcal{F}, \mu)$ is a measure space, $(X, <)$ is a partially ordered set, and $\langle x, y\rangle \equiv \{z \in X : x \leq z \leq y\} \in \mathcal{F}$ for every x, y with $x < y$. We shall usually abbreviate $(X, \mathcal{F}, \mu, <)$ to X. An *interval* in a partially ordered measure space is a set $\langle x, y\rangle$ with $x < y$ and $\mu\langle x, y\rangle > 0$.

We define two partially ordered measure spaces $(X, \mathcal{F}, \mu, <)$ and $(X', \mathcal{F}', \mu', <')$ to be *order-isomorphic* if there is a bijection λ from X to X' such that (i) $x < y$ iff $\lambda x <' \lambda y$, and (ii) $\lambda\mathcal{F} = \mathcal{F}'$. If, in addition, we have (iii) there is a constant α, called the *scale-factor*, such that $\mu(\lambda A) = \alpha\mu(A)$ for every $A \in \mathcal{F}$, then we say that the two spaces are *scale-isomorphic*.

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A partially ordered measure space X is called *homogeneous* if every two intervals in X are scale-isomorphic.

There is a trivial possibility we wish to exclude. Any measure space with the trivial partial order (no relations) is homogeneous. Indeed, the homogeneity condition is vacuously satisfied whenever there are no intervals. In this case we call our space *trivial*. We call a nontrivial homogeneous partially ordered measure space an HPO-*space*.

For convenience, we would like to have the whole space isomorphic to an interval inside it. Thus we define a *box-space* to be an HPO-space X of measure 1, with a unique minimal element, denoted by 0, and a unique maximal element 1. Thus $\mu(X) = \mu\langle 0, 1\rangle = 1$. We shall work with box-spaces throughout, although most of our results carry over immediately to HPO-spaces. Clearly any interval in an HPO-space is a box-space, at least after normalization. Furthermore, any order-convex subset of an HPO-space X is itself homogeneous: we would not usually want to consider an HPO-space obtained in this way as being essentially different from X.

Note that if X is a box-space, equipped with a collection of scale-isomorphisms, one for every pair of intervals, then we can insist that the scale-isomorphisms are compatible, i.e., that if λ_1 is the scale-isomorphism from A to B and λ_2 is the one from B to C, then $\lambda_2\lambda_1$ is the scale-isomorphism from A to C. Indeed, given an arbitrary set of scale-isomorphisms from X, one to every interval in X, we define, for every pair (A, B) of intervals, the scale-isomorphism from A to B to be that given by the inverse of the map from X to A, composed with the map from X to B.

The simplest examples of box-spaces are the spaces $[0, 1]^d$ for $d \in \mathbf{N}$, with $(x_1, \dots, x_d) \le (y_1, \dots, y_d)$ iff $x_i \le y_i$ for each i. For instance, a scale isomorphism from $\langle(0, \dots, 0), (1, \dots, 1)\rangle$ to $\langle(0, \dots, 0), (a_1, \dots, a_d)\rangle$ is given by $\lambda(x_1, \dots, x_d) = (a_1x_1, \dots, a_dx_d)$, the scale-factor being the product of the a_i. We call this space the *d-dimensional cube space,* Cu_d. An example of an HPO-space which is not a box-space is $\mathbf{R}^d$, with the same definition of the order as above. Of course, the intervals of $\mathbf{R}^d$ are isomorphic to Cu_d.

A further example of an HPO-space is that of $\mathbf{R}^{d-1} \times \mathbf{R}$, with $(x, t) \le (y, t')$ if $|x - y| \le t' - t$, where $|\cdot|$ denotes the Euclidean metric. Here the intervals are cones, and the associated box-space is called the *d-dimensional cone space* Co_d. This space can be viewed as a Lorentzian space-time, with t being time and x position, and the scale-isomorphisms are Lorentz transforms combined with expansions. In two dimensions, the map from Cu_2 to Co_2 given by $(x, y) \mapsto (\frac{x-y}{2}, \frac{x+y}{2})$ is an isomorphism, but for $d > 2$, Cu_d and Co_d are not isomorphic.

Before introducing some more examples of box-spaces, let us prove what appears to be a fairly fundamental result, showing that every box-space has a 'dimension.' The spaces Cu_d and Co_d both do have dimension d: we shall give examples of spaces with noninteger dimension shortly.

Given a box-space X, define V_n to be the supremum over all sequences of points $x_1, \ldots, x_{n-1}$ in X of $\min_{i=0,\ldots,n-1} \mu\langle x_i, x_{i+1}\rangle$, where (as always when we take such a splitting) $x_0 = 0$ and $x_n = 1$. Thus for Cu_d, the best choice of (x_i) is $((i/n, i/n, \ldots, i/n))_{i=1,\ldots,n-1}$, and $V_n = n^{-d}$. The next theorem asserts essentially that this is the case for any box-space X.

Theorem 1. *For every box-space X there is a constant $d \in [1, \infty]$ such that $V_n = n^{-d}$ for every $n \in \mathbf{N}$.*

Proof. Let X be a box-space. First we claim that $V_{n^s} = (V_n)^s$ for every $n, s \in \mathbf{N}$. We use induction on s. The statement is trivially true for $s = 1$, so we turn to the induction step. Suppose the result holds for $s = r - 1 \geq 1$. Fix $\varepsilon > 0$, and let $x_1, \ldots, x_{n^r-1}$ be a sequence of points in $\langle 0, 1\rangle$ such that $\mu\langle x_i, x_{i+1}\rangle > V_{n^r} - \varepsilon$ for every i. We are guaranteed such a sequence by the definition of V_{n^r}. Now consider the points $x_n, x_{2n}, \ldots, x_{n^r-n}$. What is $\mu\langle x_{in}, x_{(i+1)n}\rangle$? It is certainly at least $(V_{n^r} - \varepsilon)/V_n$ for each i, since otherwise choosing the points $x_{in+1}, \ldots, x_{in+n-1}$ shows that $V_n > (V_{n^r} - \varepsilon)/((V_{n^r} - \varepsilon)/V_n) = V_n$ in the interval $\langle x_{in}, x_{(i+1)n}\rangle$. On the other hand, for at least one i we must have $\mu\langle x_{in}, x_{(i+1)n}\rangle \leq V_{n^{r-1}}$, and so $V_n V_{n^{r-1}} \geq V_{n^r} - \varepsilon$. The choice of ε was arbitrary, so $V_{n^r} \leq V_n V_{n^{r-1}} = (V_n)^r$ by the induction hypothesis.

Conversely, take any $\varepsilon > 0$ and choose a sequence $x_n, x_{2n}, \ldots, x_{n^r-n}$ in $\langle 0, 1\rangle$ such that $\mu\langle x_{in}, x_{(i+1)n}\rangle \geq V_{n^{r-1}}(1-\varepsilon)^{1/2}$ for every i. For each i, take points $x_{in+1}, \ldots, x_{in+n-1}$ in the interval $\langle x_{in}, x_{(i+1)n}\rangle$ such that, for every j, $\mu\langle x_j, x_{j+1}\rangle \geq V_{n^{r-1}}(1-\varepsilon)^{1/2} V_n (1-\varepsilon)^{1/2}$. Then $\mu\langle x_j, x_{j+1}\rangle \geq (V_n)^r(1-\varepsilon)$ by the induction hypothesis. This holds for arbitrary positive ε, so $V_{n^r} \geq (V_n)^r$, establishing the claim.

Now let $d_n = -\log V_n / \log n$, for $n \in \mathbf{N}$. Suppose there are natural numbers n and m such that $d_m = d_n(1+\varepsilon)$ for some $\varepsilon > 0$. We choose integers r and s such that

$$\frac{\log m}{\log n} < \frac{r}{s} < \frac{\log m}{\log n}\left(1 + \frac{\varepsilon}{2}\right).$$

Thus $m^s < n^r$, and so $V_{m^s} \geq V_{n^r}$ by definition. Therefore

$$(V_m)^s \geq (V_n)^r, \quad s\log V_m \geq r \log V_n, \quad -sd_m \log m \geq -rd_n \log n.$$

Thus

$$\frac{r}{s} \geq \frac{\log m}{\log n}\frac{d_m}{d_n} = \frac{\log m}{\log n}(1+\varepsilon),$$

which is a contradiction. Thus $d_n = d_m$ for every n and m, as required.

Finally we note that $V_n \leq 1/n$, and so $d_n \geq 1$. □

The number d given by Theorem 1 is called the *dimension* of the box-space X.

There are box-spaces with infinite dimension; for instance, the space $[0, 1]^\infty$ with $\underline{x} \leq \underline{y}$ if $x_i \leq y_i$ for every i, and the usual measure. In an infinite-dimensional box-space, there is no point z in $\langle 0, 1\rangle$ such that both $\langle 0, z\rangle$ and

$\langle z, 1\rangle$ have positive measure. Another way of looking at this is that, if we take say three random points in $\langle 0, 1\rangle$, the probability that they are related in $<$ is 0. For our purposes, this makes infinite-dimensional box-spaces uninteresting, and from now on we shall consider only finite-dimensional box-spaces.

Let $X = \langle 0, 1\rangle$ be a box-space of dimension d. As one of our aims is to study long chains, and we wish to build these up by concatenating chains in small intervals, we are interested in good 'splittings' of X into intervals one above the other. Thus we want to know, for a fixed U with $0 \le U \le 1$, how large $\mu\langle x, 1\rangle$ can be, given that $\mu\langle 0, x\rangle \ge U$. For $U = 2^{-d}$, Theorem 1 with $n = 2$ gives us the answer. As we now show, we can in fact use Theorem 1 to answer this question fully.

Theorem 2. *Let X be a box-space of finite dimension d. For $0 < U < 1$, we define $R(U) = \sup_{x\in X}\{\mu\langle x, 1\rangle : \mu\langle 0, x\rangle \ge U\}$. Then $R(U) = (1 - U^{1/d})^d$.*

Proof. Suppose, for some $\varepsilon > 0$, there exists $x \in X$ with $\mu\langle 0, x\rangle \ge U$ and $\mu\langle x, 1\rangle \ge (1 - U^{1/d})^d(1+\varepsilon)^d$. Take integers r and s such that

$$U^{1/d} - \tfrac{\varepsilon}{4}(1 - U^{1/d}) > \tfrac{r}{s} > U^{1/d} - \tfrac{\varepsilon}{2}(1 - U^{1/d}).$$

Now find points $x_1, \dots, x_{r-1}$ in $\langle 0, x\rangle$ such that

$$\mu\langle x_i, x_{i+1}\rangle \ge Ur^{-d}\{1 - \tfrac{\varepsilon}{4}(1 - U^{1/d})/U^{1/d}\}$$

for each $i = 0, \dots, r-1$ (taking $x_0 = 0$ and $x_r = x$); and points $x_{r+1}, \dots, x_{s-1}$ in $\langle x, 1\rangle$ with

$$\mu\langle x_j, x_{j+1}\rangle \ge (1 - U^{1/d})^d(1+\varepsilon)^d(s-r)^{-d}((1+\varepsilon/2)/(1+\varepsilon))^d,$$

for every j. This gives us s intervals in X, one of which must have measure at most s^{-d}. So either

(i) $Ur^{-d}\{1 - \tfrac{\varepsilon}{4}((1 - U^{1/d})/U^{1/d})\}^d \le s^{-d}$, which implies that $\tfrac{r}{s} \ge U^{1/d} - \tfrac{\varepsilon}{4}(1 - U^{1/d})$; or

(ii) $(s-r)/s \ge (1-U^{1/d})(1+\varepsilon/2)$, which implies that $\tfrac{r}{s} \le U^{1/d} - \tfrac{\varepsilon}{2}(1-U^{1/d})$, in both cases contradicting the definitions of r and s. Thus $R(U) \le \mathrm{wd}(B)(1 - U^{1/d})^d$.

For the converse, we take r/s just greater than $U^{1/d}$, split X into s intervals $\langle x_i, x_{i+1}\rangle$ each of measure at least s^{-d}, and set $x = x_r$. Clearly this point x has the desired properties. □

Corollary 3. *If $x < y < z$ in a box-space X of dimension d, then $\mu\langle x, z\rangle^{1/d} \ge \mu\langle x, y\rangle^{1/d} + \mu\langle y, z\rangle^{1/d}$.* □

For $x < y$ in Co_d, $\rho(x, y) = \mu\langle x, y\rangle^{1/d}$ is essentially the usual metric on a Lorentzian space-time, and Corollary 3 is the appropriate form of the triangle inequality for a metric of negative signature.

Let us now give the promised examples of box-spaces with noninteger dimension. These spaces are certainly a little artificial, and it would be of considerable interest to find a class constructed in a more natural way.

Roughly, the idea is to take a subset of $[0, 1]^2$ with a strange order and a nonstandard measure. We take $X = \{(x, y) \in [0, 1]^2 : x < y\}$, with $(x, y) < (u, v)$ if $y \leq u$. Thus the interval $\langle (x, y), (u, v) \rangle$ is just $\{(w, z) \in X : y \leq w < z \leq u\}$, which is certainly order-isomorphic to X. This interval depends only on y and u, so we may call it $\langle \widetilde{y, u} \rangle$ with no danger of confusion. Thus $X = \langle \widetilde{0, 1} \rangle$.

We now define a family $(\rho_\alpha)_{\alpha > -1}$ of density functions on X by $\rho_\alpha(x, y) = (y - x)^\alpha$, for each real $\alpha > -1$, thus giving a family of measures on the Lebesgue-measurable subsets $\widetilde{A}$ of X:

$$\mu_\alpha(\widetilde{A}) = \iint_{\widetilde{A}} \rho_\alpha(x, y)\, dx\, dy.$$

Hence the measure of the interval $\langle \widetilde{u, v} \rangle$ is $(1/(\alpha + 1)(\alpha + 2))(v - u)^{\alpha+2}$. It is easy to see that the partially ordered measure spaces $(X, \mathscr{F}, \mu_\alpha, <)$ we have defined are HPO-spaces, with scale-isomorphisms $\lambda : X \to \langle \widetilde{u, v} \rangle$ given by $\lambda(x, y) = ((v - u)x + u, (v - u)y + u)$ and scale-factor $(v - u)^{\alpha+2}$. Furthermore, the dimension of the box-space with parameter α is $\alpha + 2$, since $V_2 = \lim_{\varepsilon \to 0} \mu_\alpha \langle 0, \frac{1}{2} - \varepsilon \rangle \cdot (\alpha + 1)(\alpha + 2) = (1/2)^{\alpha+2}$. These spaces are almost box-spaces: we have to normalize the measure and adjoin a 0 and a 1 to satisfy the requirements. We call the space thus defined, with parameter α and therefore dimension $\alpha + 2$, the *minimal space* $M_{\alpha+2}$. Roughly speaking, the minimal space has the smallest set of relations possible for a box-space of that dimension.

There is a more general construction of box-spaces with noninteger dimension. The minimal space above can be viewed as the set of intervals in the cube space of dimension 1, namely, the interval $[0, 1]$. We can do the same with any box-space in place of $[0, 1]$.

Given a box-space X of dimension d, form $\widetilde{X} = \{(x, y) \in X^2 : x < y\} \cup \{(0, 0), (1, 1)\}$, with $(x, y) < (u, v)$ iff $y \leq u$ and the completed product measure $dx\, dy$. The interval $\langle (x, y), (u, v) \rangle$ is thus given by $\{(w, z) : y \leq w < z \leq u\}$, which we denote by $\langle \widetilde{y, u} \rangle$. Thus $\widetilde{X}$ is homogeneous with this measure.

We now define $\widetilde{\rho}_\beta(w, z) = (\mu\langle w, z \rangle)^\beta$ and

$$\widetilde{\mu}_\beta(\widetilde{A}) = \iint_{\widetilde{A}} \widetilde{\rho}_\beta(x, y)\, dy\, dx,$$

for every β for which the integral is always defined, and every $\widetilde{A}$ which is product-measurable. The partially ordered measure space given by $\widetilde{\mu}_\beta$ is called the $(\beta + 2)$*-expansion* $\widetilde{X}_{\beta+2}$ of X.

Theorem 4. *For a box-space X of dimension d, and any appropriate β, the expansion $\widetilde{X}_{\beta+2}$ of X is (after normalization) a box-space of dimension $d(\beta + 2)$.*

Proof. We define scale-isomorphisms $\tilde{\lambda}: \widetilde{\langle 0, 1\rangle} \to \widetilde{\langle u, v\rangle}$ as follows. Let λ be a scale-isomorphism from $\langle 0, 1\rangle$ to $\langle u, v\rangle$, with scale-factor α say. Now set $\tilde{\lambda}(x, y) = (\lambda x, \lambda y)$. We have to check that $\tilde{\mu}_\beta(\tilde{\lambda}\tilde{A}) = \alpha^{\beta+2}\tilde{\mu}_\beta(\tilde{A})$, for every measurable set $\tilde{A}$. We have

$$\begin{aligned}\tilde{\mu}_\beta(\tilde{\lambda}\tilde{A}) &= \iint_{\tilde{\lambda}\tilde{A}} \tilde{\rho}_\beta(x, y)\, dx\, dy \\ &= \iint_{\tilde{A}} \tilde{\rho}_\beta(\lambda u, \lambda v)\alpha\, du\, \alpha\, dv \\ &= \alpha^2 \iint_{\tilde{A}} \mu\langle\lambda u, \lambda v\rangle^\beta\, dv\, du \\ &= \alpha^{\beta+2} \iint_{\tilde{A}} \mu\langle u, v\rangle^\beta\, dv\, du \\ &= \alpha^{\beta+2}\tilde{\mu}_\beta(\tilde{A}).\end{aligned}$$

One can readily check that the dimension is $d(\beta+2)$. □

As a consequence of Theorem 4, we certainly have that the minimal space M_α is a box-space. The box-space $X_{\beta+2}$ is always well defined for $\beta \geq 0$: the negative values of β for which it is defined depend on the space X. For the cube space $\beta > -1$ is necessary and sufficient, whereas for the minimal space of dimension d, we require instead $\beta > -1/d$.

The above process of expansion, forming X_β from X, is one way of generating new box-spaces from old. Another way is to take the *Cartesian product*: if $X_1, \dots, X_r$ are box-spaces, set X equal to the Cartesian product of the X_i, with the product measure and with $(x_1, \dots, x_r) \leq (y_1, \dots, y_r)$ if $x_i \leq y_i$ in X_i for every i. This makes X a box-space with dimension equal to the sum of the dimensions of the X_i. Thus the cube space $[0, 1]^d$ is the Cartesian product of d copies of $[0, 1]$.

Another procedure for obtaining new box-spaces is perhaps a little too simple. Given a box-space X, the box-space $2X$ is defined by taking two incomparable copies x_1 and x_2 of each point $x \in X$ with $x_i < y_j$ iff $x < y$. We can regard the set $2X$ as a product of X and $\{0, 1\}$, and give it the product measure, for any probability measure on $\{0, 1\}$. This defines a box-space, but not one that is fundamentally different from X. We can replace $\{0, 1\}$ by any measure-space with the trivial order.

We say that a box-space X is a *reduction* of a box-space Y if there is a measurable function $\lambda: X \to Y$ such that (i) $x < y$ in X iff $\lambda x < \lambda y$ in Y (this implies that $\lambda^{-1}(y)$ is an antichain for every $y \in Y$) and (ii) $\mu(\lambda^{-1}A) = \mu(A)$ for every measurable $A \subseteq Y$. We say that a box-space X is *reduced* if all reductions of X are isomorphic to X.

As another example, it is easy to see that the space $(X_\beta)_\gamma$ formed by applying a β-expansion followed by a γ-expansion to X is not reduced, since $X_{\beta\gamma}$ is a reduction.

Ideally, one would like to classify all finite-dimensional reduced box-spaces. This may be a little ambitious, and it would be pleasant just to find a fairly natural family of these spaces that includes some or all of our examples. Despite some effort, we have failed to find any such family, and the only examples we know of are: the real interval $[0, 1]$, the cone spaces Co_d for $d = 2, 3, \ldots$, and other spaces that can be obtained from these by repeatedly taking Cartesian products and performing β-expansions. It is difficult to believe that these are the only ones, but it has proved difficult to test conjectures about box-spaces with this rather small supply of examples.

2. Random partial orders

We now move on to study random partially ordered sets. Winkler [2] studied random orders generated by taking n points at random in Cu_d (see also [3, 1]). The basic idea is to replace Cu_d by an arbitrary box-space.

In fact, for our purposes it is a little more convenient to consider the following random structure. Let X be a box-space. We take as elements of the ground-set the points (y_i) in a Poisson distribution of density n on X, so the number of y_i in a set of measure α is a Poisson random variable with parameter $n\alpha$. The partial order on the y_i is given by the restriction of the order on X. We denote the random poset thus obtained by $P_{X,n}$. The relationship between $P_{X,n}$ and the random poset given by taking n random points in X is similar to that between the models G_p and G_M of random graphs, and it is easy to translate results about one model into results about the other.

The question in which we are most interested is that of finding the longest chain in a random partial order. Of course, we want the expected length, together with some estimate of the error. Bollobás and Winkler [1] studied this problem for Cu_d: our main motivation was to do the same for Co_d. In fact we lose nothing by considering a general box-space X.

Let us first prove some results concerning the probability G_s that s random points form a chain. More precisely, for a fixed box-space X, we let A be an X-valued random variable with the uniform distribution on X. Now let A_i, for $i \in \mathbf{N}$, be independent copies of A, so that A_i are 'random points' in X. Denote by G_s the probability that s random points form a chain in the given order: $G_s = \mathbf{P}(A_1 \leq A_2 \leq \cdots \leq A_s)$. As we shall see shortly, $sG_s^{1/ds}$ tends to a constant as $s \to \infty$. Let us begin by giving an upper bound on G_s whenever $s + 1$ is a power of 2.

Lemma 5. *Let X be a box-space of dimension d. Let m be a natural number and set $s = 2^m - 1$. Then $G_s \leq F_m \equiv 2^{-d2^m(m-2)-2d}$.*

Proof. Let us apply induction on m. For $m = 1$, we have $F_m = 1$, so the assertion holds.

Suppose the assertion is true for $m-1$ $(m \geq 2)$, and let $A_1, \dots, A_{2^m-1}$ be random points as above. Letting x represent the value of $A_{2^{m-1}}$, we have

$$\mathbf{P}(A_1 \leq \cdots \leq A_{2^m-1}) = \int_{x \in X} \mathbf{P}(A_i \leq x)^{2^{m-1}-1} \mathbf{P}(A_i \geq x)^{2^{m-1}-1} F_{m-1}^2 \, d\mu(x),$$

since, conditional on $A_1, \dots, A_{2^{m-1}-1}$ all being in $\langle 0, x\rangle$, the probability that they form a chain in the appropriate order is just F_{m-1}, and similarly for the upper portion. Thus

$$\begin{aligned} G_s &\leq \left[\sup_{x \in X}(\mathbf{P}(A_i \leq x)\mathbf{P}(A_i \geq x))\right]^{2^{m-1}-1} F_{m-1}^2 \\ &= \sup_x(\mu\langle 0, x\rangle \mu\langle x, 1\rangle)^{2^{m-1}-1} F_{m-1}^2 \\ &= \sup_U(U(1-U^{1/d})^d)^{2^{m-1}-1} F_{m-1}^2 \\ &\leq 2^{-2d2^{m-1}-1} 2^{-(d2^{m-1}(m-3)+2d)\cdot 2} \\ &= 2^{-d(2^m(m-2)+2)}, \end{aligned}$$

as desired. □

Setting again $s = 2^m - 1$, we note that $F_m = (\frac{s+1}{4})^{-ds}(s+1)^{-d}$.

Our immediate aim is to prove that G_s is of order $(\frac{c}{s})^{ds}$, for some constant c depending only on the space. To this end, we prove the following lemma.

Lemma 6. *Let X be a box-space of dimension d. If $G_s > (\frac{c}{s})^{ds}$ for some constant c and some integer s, and ε is any positive constant, then there exists t_0 such that $G_t > (\frac{c-\varepsilon}{t})^{dt}$, for every $t \geq t_0$.*

Proof. Let c, s, ε be as above, and take any $t \geq s$. Let k and l be the nonnegative integers such that $t = ks - l$ and $l < s$. Also let $x_1, \dots, x_{k-1}$ be elements of X such that $\mu\langle x_i, x_{i+1}\rangle \geq k^{-d}(1-\varepsilon/c)^{d/2}$.

We see that

$$\begin{aligned} G_t &\geq \mathbf{P}(A_1, \dots, A_s \in \langle 0, x_1\rangle; A_{s+1}, \dots, A_{2s} \in \langle x_1, x_2\rangle; \cdots; \\ &\qquad A_{(k-1)s+1}, \dots, A_t \in \langle x_{k-1}, 1\rangle) \times G_s^k \\ &> \left[k^{-d}\left(1-\frac{\varepsilon}{c}\right)^{d/2}\right]^t \left(\frac{c}{s}\right)^{dks} \\ &= \left(\frac{c-\varepsilon}{t}\right)^{dt} \left[\left(\frac{c}{c-\varepsilon}\right)^t \frac{c^l}{s^l}\left(1-\frac{\varepsilon}{c}\right)^{t/2}\left(\frac{t}{ks}\right)^t\right]^d. \end{aligned}$$

So we are done provided

$$\left(1-\frac{\varepsilon}{c}\right)^{-t/2}\left(\frac{c}{s}\right)^l\left(1+\frac{l}{t}\right)^{-t} \geq 1.$$

Recalling that $l < s$, we see that this is true for sufficiently large t, as desired. □

Theorem 7. *For every box-space X of dimension d, there is a constant c_X between* 1 *and* 4 *such that $sG_s^{1/ds} \to c_X$.*

Proof. Let $c_s = sG_s^{1/ds}$. Lemma 5 tells us that $c_{2^m-1} < 4$ for every m, and by Lemma 6 this implies that $c_s \leq 4$ for all s. Let $c_X = \limsup c_s \leq 4$, and fix $\varepsilon > 0$. Now take s_0 such that $c_{s_0} \geq c_X - \varepsilon/2$. Lemma 6 tells us that, for some t_0, $c_t \geq c_X - \varepsilon$ whenever $t \geq t_0$. But ε was arbitrary, and so $c_s \to c_X$.

Finally we have to check that $c_X \geq 1$. In fact we even have $c_s \geq 1$ for every s, since, for every $\varepsilon > 0$ we can split X into s intervals each of size at least $s^{-d} - \varepsilon$, so that the probability that A_i is in the ith interval for every i is at least $(s^{-d} - \varepsilon)^s$. □

The constant c_X of Theorem 7 is called the *chain constant* of the space X. Our next aim is to calculate the chain constants of particular box-spaces X. For Cu_d this is straightforward: $G_s = (s!)^{-d}$, since for s points to be in a particular order, their ith coordinates have to be in that order for every i. Thus $c_X = \mathrm{e}$ for the cube. For other spaces, the following result is useful.

For a box-space X of dimension d, and $0 < \delta < 1$, define

$$g(\delta) = \delta^{-d}\mu\{x \in X : \mu\langle x, 1\rangle \geq (1-\delta)^d\}.$$

By Theorem 2, $g(\delta) \geq 1$ for every δ.

Theorem 8. *Let X be a box-space with dimension d, and chain constant c_X. Then $g(\delta)$ is a decreasing function of δ, and*

$$\lim_{\delta\to 0} g(\delta) = (c_X d/e)^d (\Gamma(d+1))^{-1}.$$

Proof. We first prove that $g(\delta)$ is a decreasing function of δ.

Fix $\varepsilon > 0$, and take any $0 < \delta_1 < \delta_2 < 1$. Set $\eta = \eta(\varepsilon) = (\delta_1 - \varepsilon)/(\delta_2 - \varepsilon) < 1$. Now we let x be a point in X such that $\mu\langle 0, x\rangle \geq \eta^d$ and $\mu\langle x, 1\rangle \geq (1-\eta)^d(1-\varepsilon)^d$. Let λ be a scale-isomorphism from $\langle 0, 1\rangle$ to $\langle 0, x\rangle$ with scale-factor $\alpha \geq \eta^d$.

Let y be any point such that $\mu\langle y, 1\rangle \geq (1-\delta_2)^d$. Then $\mu\langle \lambda y, x\rangle \geq \eta^d(1-\delta_2)^d$, so by Corollary 3,

$$\begin{aligned}\mu\langle \lambda y, 1\rangle^{1/d} &\geq \mu\langle \lambda y, x\rangle^{1/d} + \mu\langle x, 1\rangle^{1/d}\\ &\geq \eta(1-\delta_2) + (1-\eta)(1-\varepsilon)\\ &= (1-\delta_1).\end{aligned}$$

So λ maps $\{y : \mu\langle y, 1\rangle \geq (1-\delta_2)^d\}$ into $\{z : \mu\langle z, 1\rangle \geq (1-\delta_1)^d\}$. Thus

$$\frac{\mu\{y : \mu\langle y, 1\rangle \geq (1-\delta_2)^d\}}{\mu\{z : \mu\langle z, 1\rangle \geq (1-\delta_1)^d\}} \leq \eta^{-d} = \frac{(\delta_2 - \varepsilon)^d}{(\delta_1 - \varepsilon)^d}.$$

But ε was arbitrary, so $g(\delta_2) \leq g(\delta_1)$, as desired.

Now we turn to the evaluation of

$$L \equiv \lim_{\delta\to 0} \delta^{-d}\mu\{x \in X : \mu\langle x, 1\rangle \geq (1-\delta)^d\}.$$

Suppose for the moment that $L < \infty$. Fix $\varepsilon > 0$, and choose $\delta_0 > 0$ such that $(L-\varepsilon)\delta^d \leq \mu\{x : \mu\langle x, 1\rangle \geq (1-\delta)^d\}$ whenever $\delta < \delta_0$.

Now

$$\int_{\delta=0}^{\delta_0} \frac{d}{d\delta}((L-\varepsilon)\delta^d)(1-\delta)^{d(s-1)} G_{s-1}\,d\delta$$
$$\leq G_s \leq \int_{\delta=0}^{\delta_0} \frac{d}{d\delta}(L\delta^d)(1-\delta)^{d(s-1)} G_{s-1}\,d\delta + \int_{\delta_0}^{1} (1-\delta)^{d(s-1)} G_{s-1}\,d\delta.$$

For large s, the final integral is negligible, and so we certainly have G_s bounded between $G_{s-1}(L\pm\varepsilon)d(\Gamma(d)\Gamma(ds-d+1)/\Gamma(ds+1))(1+o(1))$, these bounds being equal to $G_{s-1}(L\pm\varepsilon)\Gamma(d+1)(ds)^{-d}(1+o(1))$. Thus $(G_s/G_{s-1})^{1/d}s$ converges to $\frac{1}{d}(L\Gamma(d+1))^{1/d}$. It is an elementary exercise to check that, if this quantity converges, it converges to c_X/e, which is the required result.

If L is infinite, i.e., $g(\delta)$ increases without limit as $\delta \to 0$, then the same proof shows that, for every M, $(G_s/G_{s-1})^{1/d}s > \frac{1}{d}(M\Gamma(d+1))^{1/d}$ for sufficiently large s. This, combined with Theorem 7, gives a contradiction. □

Corollary 9. *The box-spaces* Cu_d, Co_d, *and* M_d *have the following chain constants*:

(i) $c_{\mathrm{Cu}_d} = e$,

(ii) $c_{\mathrm{Co}_d} = 2^{1-1/d}e(\Gamma(d+1))^{1/d}d^{-1}$,

(iii) $c_{M_d} = e(\Gamma(d+1))^{1/d}d^{-1}$.

For every box-space X *of dimension* d, $c_X \geq e(\Gamma(d+1))^{1/d}d^{-1}$.

Proof. We have already seen (i). For (ii), let $\langle(\underline{0}, 0), (\underline{0}, 1)\rangle$ be an interval in Co_d, with volume normalized to 1. The measure of the set of points x, such that $\mu\langle x, 1\rangle$ is at least $(1-\delta)^d$, is asymptotically the volume of $\{(\underline{y}, t) \geq (\underline{0}, 0) : t \leq 2\delta\}$. This volume is $2^{d-1}\delta^d$, and Lemma 8 now gives us the result.

For the minimal space M_d of dimension d, it is clear that the measure of the set of points x with $\mu\langle x, 1\rangle \geq (1-\delta)^d$ is precisely δ^d, which means that $g(\delta) \equiv 1$ is attained. Thus $c_{M_d} = e(\Gamma(d+1))^{1/d}d^{-1}$, as desired for (iii). The final assertion is immediate from $g(\delta) \geq 1$. □

Corollary 9 improves the lower bound of 1 (from Theorem 7) on the chain constant, although as $d \to \infty$, our new bound tends to 1. For large d, c_{Co_d} tends to 2: for $d = 2$ it equals e, which agrees with the observation that the cube- and cone-spaces of dimension 2 are isomorphic.

Let us at this point conjecture that the upper bound of 4 in Theorem 7 can also be improved.

Conjecture. The chain constant of every finite-dimensional box-space X is at most e.

The only evidence we offer for this conjecture is that the cube-space attains this bound and no other known space beats it; however, it seems likely to be true.

Throughout the remainder of this paper, we use the term '*almost surely*' as in the theory of random graphs: a random partial order $P_{X,n}$ has a property Q *almost surely* if $\mathbf{P}(P_{X,n}$ has $Q) \to 1$ as $n \to \infty$.

We now apply our results to the problem of finding the length of a longest chain in $P_{X,n}$. Firstly, we have the following simple deduction from Theorem 7.

Theorem 10. *Let X be a box-space of dimension d. For every $\varepsilon > 0$, there is almost surely no chain of length $(c_X + \varepsilon)n^{1/d}$ in $P_{X,n}$.*

Proof. Fix $\varepsilon > 0$, and take s_0 such that $G_s \le ((c_X + \varepsilon/2)/s)^{ds}$ whenever $s \ge s_0$.

Now take any $n > (s_0/(c_X + \varepsilon))^d$, and set $s = s(n) = \lceil (c_X + \varepsilon)n^{1/d} \rceil \ge s_0$. The expected number of chains of length s in $P_{X,n}$ is

$$\binom{n}{s} s! G_s \le n^s \left(\frac{c_X + \varepsilon/2}{s}\right)^{ds} \le \left(\frac{c_X + \varepsilon/2}{c_X + \varepsilon}\right)^{ds} = o(1),$$

as $n \to \infty$, as required. □

From the other direction, we can estimate the length of a 'greedy' chain.

Theorem 11. *Let X be a box-space of dimension d. For every $\varepsilon > 0$, there is almost surely a chain in $P_{X,n}$ of length*

$$\frac{c_X d}{e(\Gamma(d+1))^{1/d}\Gamma(1+1/d)} n^{1/d}(1-\varepsilon).$$

Proof. The idea is to build a 'greedy' chain (x_i) in $P_{X,n}$ by trying to keep $\mu\langle x_i, 1\rangle$ as large as possible at all stages. The technique is basically just to use Lemma 8, but the need to avoid 'edge-effects' makes the proof a little fiddly.

Fix any $\varepsilon > 0$, also with $\varepsilon < 1/2$ for convenience. Take $\delta_0 < 1$ such that whenever $\delta < \delta_0$ we have $\mu\{x \in X : \mu\langle x, 1\rangle \ge (1-\delta)^d\} \ge \delta^d L(1-2\varepsilon)^d$, where $L = (c_X d/e)^d (\Gamma(d+1))^{-1}$. Finally let $n_0 > (\delta_0\varepsilon)^{-2d}$: and in future we shall always assume without loss of generality that $n > n_0$.

We now define a random walk (x_i) in X as follows. We set $x_0 = 0$. Now, given x_i, we set x_{i+1} equal to that y_j in $\langle x_i, 1\rangle$ maximizing $\mu\langle y_j, 1\rangle$, *provided* that this y_j satisfies $\mu\langle y_j, 1\rangle^{1/d} \ge \mu\langle x_i, 1\rangle^{1/d} - n^{-1/2d}$. If no such y_j exists, we set $x_{i+1} = x_i$.

Clearly if this last clause is ever invoked, the process then remains stationary. If it is not invoked, the sequence $(x_i)_1^r$ is a chain in $P_{X,n}$. We shall later prove that, for the appropriate value of r, the process almost surely never becomes stationary.

The random variable X_i is defined to be equal to $\max\{\mu\langle x_i, 1\rangle^{1/d}, \varepsilon\}$. Roughly speaking, X_i is the 'distance left to work in' after the ith jump.

The key step in the proof is to establish that, for every $a \geq \varepsilon$ and every $\delta \leq a$,

$$\mathbf{P}(X_i \leq a - \delta | X_{i-1} = a) \leq \exp[-n\delta^d L(1-2\varepsilon)^d].$$

This is certainly true if $\delta > n^{-1/2d}$, so in proving the above inequality we may assume that $\delta/a \leq n^{-1/2d}/\varepsilon < \delta_0$. Under this assumption, we have

$$\begin{aligned}
&\mathbf{P}(X_i \leq a - \delta | X_{i-1} = a)\\
&\quad \leq \mathbf{P}\text{ (there is no } y_j \text{ in } \langle x_{i-1}, 1\rangle \text{ with } \mu\langle y_j, 1\rangle^{1/d} \geq a - \delta)\\
&\quad = \exp[-n\mu\{x \in \langle x_{i-1}, 1\rangle : \mu\langle x, 1\rangle \geq (a-\delta)^d\}]\\
&\quad \leq \exp[-na^d \left(\tfrac{\delta}{a}\right) L(1-2\varepsilon)^d]\\
&\quad = \exp[-n\delta^d L(1-2\varepsilon)^d],
\end{aligned}$$

as required.

What this means for us is that X_i dominates the random variable Z_i defined by $Z_0 = 1$ and $\mathbf{P}(Z_i \leq a - \delta \mid Z_{i-1} = a) = \exp[-n\delta^d L(1-2\varepsilon)^d]$ for all a and δ. The random variable $1 - Z_i$ is very easy to deal with, since it is the sum of i independent identically distributed random variables W_k, each with distribution given by $\mathbf{P}(W_k \geq \delta) = \exp[-n\delta^d L(1-2\varepsilon)^d]$. The mean of W_k is $\Gamma(1/d)/(dn^{1/d}L^{1/d}(1-2\varepsilon))$.

Set $r = dn^{1/d}L^{1/d}(1-2\varepsilon)^2/\Gamma(1/d)$, and consider Z_r. By the Weak Law of Large Numbers, say, we have that $\mathbf{P}(|1 - Z_r - r\mathbf{E}W_1| > \varepsilon) \to 0$ as $n \to \infty$.

So, almost surely, $1 - Z_r \leq (1-2\varepsilon) + \varepsilon = 1 - \varepsilon$. Thus, almost surely, $1 - X_r \leq 1 - \varepsilon$: in other words $\mu\langle x_r, 1\rangle^{1/d} \geq \varepsilon$.

Now, conditional on $\mu\langle x_r, 1\rangle^{1/d} \geq \varepsilon$, the probability that at any stage we were forced to set $x_i = x_{i-1}$ is at most

$$r \sup_{\{x:\mu\langle x, 1\rangle^{1/d} \geq \varepsilon\}} \mathbf{P}(\text{there is no } y_j \text{ above } x \text{ such that}$$
$$\mu\langle y_j, 1\rangle^{1/d} \geq \mu\langle x, 1\rangle^{1/d} - n^{-1/2d}),$$

which as we have already seen is at most $r\exp[-n \cdot n^{-1/2}L(1-2\varepsilon)^d] - o(1)$.

We have proved that the sequence $(x_i)_1^r$ is almost surely a chain in $P_{X,n}$. The length of this chain is

$$r = r(\varepsilon) = \frac{dn^{1/d}L^{1/d}(1-2\varepsilon)^2}{\Gamma(1/d)} = \frac{c_X d}{e(\Gamma(d+1))^{1/d}\Gamma(1+1/d)} n^{1/d}(1-2\varepsilon)^2.$$

Since ε was chosen arbitrarily, this implies the desired result. □

Theorems 10 and 11 combine to tell us that the height $H_{X,n}$ of $P_{X,n}$ is of order $n^{1/d}$ for every box-space X of dimension d. Bollobás and Winkler [1] proved that, for the cube, in fact $n^{-1/d}H_{X,n}$ converges to some constant m_X. Their proof translates directly to our setting.

Theorem 12. *Let X be a box-space of dimension d. There is a constant m_X called the maximal chain constant satisfying*

$$\frac{c_X d}{e(\Gamma(d+1))^{1/d}\Gamma(1+1/d)} \leq m_X \leq c_X,$$

such that $n^{-1/d}H_{X,n} \to m_X$ in probability.

Proof. Define $m_X = \sup_{n\in\mathbf{R}^+} \mathbf{E}H_{X,n}n^{-1/d}$. Theorems 10 and 11 tell us that m_X lies between the bounds given above: we have to prove that $n^{-1/d}H_{X,n} \to m_X$ in probability.

Take any $\varepsilon > 0$ and choose s such that $\mathbf{E}(H_{X,n}) > s^{1/d}m_X(1-\varepsilon)^{1/4}$. Next find t_0 sufficiently large that, whenever $t \geq t_0$, $\mathbf{P}($ the sum of t copies of $H_{X,s}$ is greater than $ts^{1/d}m_X(1-\varepsilon)^{1/2}) \geq 1-\varepsilon$. Now take n_0 sufficiently large that (i) $n_0 \geq t_0^d s(1-\varepsilon)^{-d/4}$ and (ii) $((n_0/s)(1-\varepsilon)^{d/2})^{1/d} \leq \lfloor((n_0/s)(1-\varepsilon)^{d/4})^{1/d}\rfloor$.

Take any $n \geq n_0$, and set $t = \lfloor((n/s)(1-\varepsilon)^{d/4})^{1/d}\rfloor$. Thus $t \geq t_0$. Now choose $x_1, \dots, x_t$ in X such that $\mu\langle x_i, x_{i+1}\rangle \geq t^{-d}(1-\varepsilon)^{d/4}$ for every i.

Construct a chain in $P_{X,n}$ by taking the longest chain in each $\langle x_i, x_{i+1}\rangle$, and joining these together. The length of the longest chain in one of these small intervals is distributed as $H_{X,s'}$, where $s' \geq nt^{-d}(1-\varepsilon)^{d/4} \geq s$. So with probability at least $1-\varepsilon$, the chain constructed has length at least

$$ts^{1/d}m_X(1-\varepsilon)^{1/2} \geq n^{1/d}m_X(1-\varepsilon).$$

From this we deduce that $H_{X,n}n^{-1/d} \to m_X$ in probability, as desired. □

In conclusion, let us make a few remarks about possible values for the maximal chain constant m_X. In the minimal space, the greedy chain is a longest chain, and thus it is easy to prove that m_{M_d} attains the lower bound of Theorem 12. Since the minimal space also has the lowest possible value of its chain constant, the bound $m_X \geq 1/\Gamma(1+1/d)$ is attained. For $d = 2$, this gives $H_{M_2,n}/\sqrt{n} \to 2\sqrt{\pi}$ in probability. This result for M_2 has been obtained independently by Justicz, Scheinerman, and Winkler [4], who studied "Random interval orders," which can be thought of as random orders in M_2.

Bollobás and Winkler [1] proved that in fact $m_X < c_X = e$ for X a cube space of any dimension. One can repeat their proof to show that the maximal chain constant is strictly less than the chain constant for every box-space. Apart from the minimal space, the only space for which the value of m_X has been calculated is the 2-dimensional cube: the result is that $m_X = 2$ (see [1] and the references therein for details).

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