

STRONG BERTINI THEOREMS

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ABSTRACT. We show that the singular locus of the general member of a linear system has dimension less than that predicted by Bertini's theorem, provided that the base locus is scheme-theoretically smooth. As corollaries, we obtain a result about complete intersection varieties containing a given subvariety and a result concerning liaison.

1. INTRODUCTION

The classical Bertini theorem in characteristic zero states that the general element of a linear system of divisors on a variety X is nonsingular away from the base locus of the linear system and the singularities of X . In characteristic p , if the linear system is assumed to be very ample, then the same conclusion is obtained. In this paper we show that if the base locus of the linear system is scheme-theoretically reduced and nonsingular, the singularities of the general element of the linear system have dimension less than that predicted by Bertini, and in fact (when such singularities actually occur) this dimension may be computed exactly. We present two applications of these strengthened Bertini theorems: In §3 we generalize a theorem of Nishi [N] about finding nonsingular complete intersection varieties that contain a given nonsingular variety. In §4 we obtain some results about when a given nonsingular subvariety of projective space can be directly linked in liaison to another nonsingular variety.

We fix our conventions for the paper: The field K will be algebraically closed and of arbitrary characteristic (although we will sometimes explicitly add the assumption that $\text{char } K = 0$ and will also sometimes consider nonalgebraically closed subfields of K). A *variety* X over K will be an integral, separated scheme of finite type over K . The *singular* points of X are the points whose local rings are not regular, and X is *nonsingular* if all its local rings are regular. The *dimension* of a scheme is the maximum of the dimensions of its irreducible components. When we say that the "general element" of a linear system has a certain property, we mean that there is a Zariski dense open subset U of the projective space parametrizing the linear system such that all divisors corresponding to points in U have the given property.

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2. STRONG BERTINI THEOREMS

In this section we show the main result of the paper (Theorem 2.1), concerning the singularities of the general member of a linear system whose base locus is reduced and nonsingular. In the characteristic 0 case, we show that the codimension of this singular locus in each component of the base locus is equal to the codimension of that component of the base locus in the ambient variety. In many cases this implies that the general member is nonsingular; see Corollary 2.4.

For this section we fix a linear system Λ of effective Cartier divisors on a complete variety X . Let B be the scheme-theoretic base locus of Λ , $\pi: X' \rightarrow X$ the blow-up of X along B , and E the exceptional divisor of this blow-up. Let X^0 be the set of nonsingular points of X . Let D be a general element of Λ , and D_s the set of singular points of D . Let B^0 be an irreducible component of $B \cap X^0$. It might be that $B^0 \cap D_s$ is empty; let us assume that this is not the case and let S be an irreducible component of $S^0 = B^0 \cap D_s$. Let a be the codimension of B^0 in X and let b be the codimension of S in B^0 .

(2.1) **Theorem.** *In the preceding situation, assume either that K has characteristic zero or that the linear system $\pi^*\Lambda - E$ is very ample on X' . Then*

- (1) $D_s \cap X^0 \subset B \cap X^0$.
- (2) *If we further assume that $B \cap X^0$ is reduced and nonsingular, then $a = b$.*

Proof. Because B is the scheme-theoretic base locus of Λ , the linear system $\pi^*\Lambda - E$ is base point free. Also, when the characteristic of K is not zero we have assumed that $\pi^*\Lambda - E$ is very ample. Thus, in any characteristic the classical Bertini theorem applies to tell us that a general element D' of the linear system $\pi^*\Lambda - E$ is nonsingular away from the singularities of X' . All singularities of X' and thus D' lie over B or singularities of X . We may then take our general divisor D of Λ to be $\pi(D')$ and from the previous few sentences conclude that D satisfies (1). To prove (2) note that the additional assumptions on B say that all singularities of X' and thus D' lie over singularities of X . We must determine when a nonsingular point of D' lying over a nonsingular point of X contained in B maps to a singular point of D .

(2.2) *Claim.* In case (2) of (2.1) let p be a point of $B \cap X^0$. Then p is a singular point of D if and only if D' contains the entire fiber $\pi^{-1}(p)$.

Proof of claim. This is an explicit local calculation and may be done in the complete local ring of X at p . Let z_1, \dots, z_n be a system of local coordinates on X at p . We may assume B is given by $z_{k+1} = \dots = z_n = 0$ and D by $f(z_1, \dots, z_n) = 0$. Because $B \subset D$, each monomial of f will be divisible by

some z_i with $i \geq k+1$. If all divisors in Λ were singular everywhere on B , then B could not be reduced; thus we may assume that D is nonsingular on an open dense subset of B . By taking partial derivatives of f we may see that this implies that at least one monomial of f has degree one in z_{k+1}, \dots, z_n . Note that D is singular at $p = (0, \dots, 0)$ if and only if all monomials of f have degree at least two.

Now blow up B . To determine the equation of D' on a typical affine patch U_j of the blow-up, we make the following substitutions into f :

$$\begin{aligned} z_i &\mapsto z_i; & i = 1, \dots, k; \\ z_i &\mapsto z_i z_j; & i = k+1, \dots, \hat{j}, \dots, n; \\ z_j &\mapsto z_j. \end{aligned}$$

The equation for E is $z_j = 0$. From what we have said about f , we see that after making the preceding substitution into f it will be divisible by z_j exactly once. Call the quotient \bar{f} ; this is the equation for D' . The equations for $\pi^{-1}(p)$ are $z_1 = \dots = z_k = z_j = 0$. Thus D' will fail to contain $\pi^{-1}(p)$ if and only if \bar{f} contains a constant term or a monomial containing only z_i 's for $i \geq k+1$, $i \neq j$. Reconsidering how monomials of \bar{f} arise from monomials of f , we see that this is the case if and only if f has a monomial of degree exactly one, that is, if and only if D is nonsingular at p . This completes the proof of the claim.

To see how often D' can contain a fiber of π over a point of B we consider the morphism $\phi: X' \rightarrow \mathbf{P}^N$ given by the linear system $\pi^*\Lambda - E$. Because of the local nature of what we are trying to show, we may assume that X is nonsingular and B has only one component. If $N = a-1$ then B is a complete intersection of a elements of Λ and all divisors of Λ must be nonsingular everywhere on B ; so we may assume $N > a-1$. For each point p of B , ϕ embeds $\pi^{-1}(p)$ in \mathbf{P}^N as an $a-1$ plane. This follows from standard knowledge about blow-ups: $\pi^{-1}(p)$ is the projectivization of the normal space to B in X at p , the divisors of Λ generate the ideal of B in X at p , and $\pi^*\Lambda - E$ restricts to the bundle $\mathcal{O}(1)$ on $\pi^{-1}(p) = \mathbf{P}^{a-1}$.

Let \mathbf{P}^{N^*} be the projective space of hyperplanes in \mathbf{P}^N . Consider the product $B \times \mathbf{P}^{N^*}$ with projections π_1 and π_2 onto the first and second factors. Inside $B \times \mathbf{P}^{N^*}$ define $I = \{(b, H): \pi^{-1}(b) \subset H\}$. For any point b of B , $\pi_1^{-1}(b) =$ hyperplanes containing a fixed $a-1$ plane, so dimension $\pi^{-1}(b) = N-a$. This says that $\dim I = \dim B + N - a$. So the dimension of a general fiber of π_2 is at most $\dim B - a$. But by the claim (2.2) $\pi_1(\pi_2^{-1}(H))$ is the singular locus of the general D corresponding to this general H . This shows that $b \geq a$.

Finally, to show that $b \leq a$, let \mathcal{I} be the ideal sheaf of B in X and $\mathcal{L} = \mathcal{O}(D)$, so that Λ is contained in the complete linear system corresponding

to \mathcal{L} . Consider the following exact sequence of sheaves on B :

$$(2.3) \quad 0 \rightarrow \mathcal{F}^2 \otimes \mathcal{L} \rightarrow \mathcal{F} \otimes \mathcal{L} \rightarrow \mathcal{F}/\mathcal{F}^2 \otimes \mathcal{L} \rightarrow 0.$$

The sheaf $\mathcal{F}/\mathcal{F}^2$ is the conormal bundle to B in X ; therefore, $\mathcal{F}/\mathcal{F}^2 \otimes \mathcal{L}$ is a vector bundle of rank a . A divisor D in Λ gives a global section of $\mathcal{F} \otimes \mathcal{L}$ and therefore a global section of $\mathcal{F}/\mathcal{F}^2 \otimes \mathcal{L}$ vanishing exactly on the singularities of D contained in B . The zero locus of a section of a vector bundle of rank a on a nonsingular variety is either empty or has codimension at most a . Q.E.D.

Remark. The proof of (2.1) also shows that for any divisor $D \in \Lambda$, if $D_s \cap B^0$ has $\text{codim} > a$ in B^0 then $D_s \cap B^0$ is empty.

(2.4) **Corollary.** *Under the hypotheses of Theorem 2.1(2), if the dimension of a component of $B \cap X^0$ is less than $\frac{1}{2} \dim X$, then a general element of the linear system is nonsingular along this component.*

Warning. Even if the dimension of a component of $B \cap X^0$ is greater than $\frac{1}{2} \dim X$ it may be that a general element of Λ is nonsingular along this component. (For example, consider a nonsingular complete intersection of nonsingular hypersurfaces in \mathbf{P}^n .) See (2.5) and (3.4) for more on this.

By looking more carefully at the proof of Theorem 2.1, we can say even more about the singularities of our general D —in particular that the singular locus is itself nonsingular. To do this we describe the singularities separately on each irreducible component of $B \cap X^0$. Let B^0 be a fixed irreducible component of $B \cap X^0$ of dimension d and codimension a in X and let $S^0 = B \cap D_s$. Denote by $A_i(B^0)$ the i th Chow group of i -cycles modulo rational equivalence, as defined for instance in [F]. Let F be the twisted conormal bundle $\mathcal{F}/\mathcal{F}^2 \otimes \mathcal{L}$ of (2.3) for B^0 in X , and let $c_i(F)$ be the i th Chern class of F . For our general divisor D let s_D be a section of F induced by D (so that s_D is determined up to a nonzero constant multiple). Let Z be the zero scheme of s_D and let $[Z]$ be the class of the cycle associated to Z in $A_{d-a}(B^0)$ as defined for instance in [F].

(2.5) **Corollary.** *With the preceding notation and under the assumptions of part (2) of (2.1) the following are true:*

- (a) *Set theoretically, $Z = S^0$.*
- (b) *$[Z] = c_a(F) \cap [B^0]$ in $A_{d-a}(B^0)$.*
- (c) *S^0 is empty if and only if $c_a(F) = 0$.*
- (d) *The singularities of D in X^0 are all double points.*
- (e) *Z is a nonsingular reduced scheme. Thus $[Z]$ is a sum of disjoint nonsingular varieties each counted with multiplicity one.*

Proof. Part (a) is clear from the discussion surrounding (2.3). Parts (b) and (c) follow from standard knowledge about Chern classes [F, Example 3.2.16,

Appendix A.7] and the fact (2.1(2)) that if nonempty, Z has the expected dimension. Parts (d) and (e) will follow from a closer look at the local calculation used in the proof of (2.2). Part (d) is straightforward: If $p = (0, \dots, 0)$ were a triple point or worse of D , then all the monomials of f would have degree at least 3. This would imply that all monomials of \bar{f} have degree at least 2, meaning that D' is singular, contrary to assumption. Part (e) requires more work:

First, let us write down the section s_D in terms of our local coordinates z_i and local equation f . Keep the notation of the proof of (2.2). Write f as follows: $g_{k+1}(z_1, \dots, z_k)z_{k+1} + \dots + g_n(z_1, \dots, z_k)z_n +$ terms of degree at least two in z_{k+1}, \dots, z_n . The terms of degree one in z_{k+1}, \dots, z_n give a local expression for s_D . The zero scheme Z is the subscheme of B^0 defined by the vanishing of the g_i 's. We may assume dimensions are such that D does have singular points. Thus we may assume $k \geq n - k$ and $p = (0, \dots, 0)$ is a singular point of D , so that all the g_i 's have constant term zero. To show that Z is reduced and nonsingular we must show that the matrix

$$(2.6) \quad J = \left[\frac{\partial g_i}{\partial z_l}(0, \dots, 0) \right], \quad k+1 \leq i \leq n, \quad 1 \leq l \leq k,$$

has rank $n - k$.

The linear system $\pi^* \Lambda - E$ on X' is either very ample or base point free (depending on our assumptions according to the characteristic of K). It will restrict to a linear system on E which is again either very ample or base point free. Applying classical Bertini to this restricted linear system, we conclude that, for general D' , $D' \cap E$ is reduced and nonsingular over B^0 . Now suppose the matrix J of (2.6) does not have rank $n - k$. This means that the system of linear equations

$$J \begin{bmatrix} z_{k+1} \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

has at least a one-dimensional family of solutions. In particular, for some j it has a solution of the form $z_j = 1$ and $z_i = x_i$, $i \geq k+1$, $i \neq j$. One then checks by looking at the appropriate matrix of partial derivatives that on the patch of the blow-up U_j where $E = \{z_j = 0\}$, the point $z_1 = \dots = z_k = z_j = 0$, $z_i = x_i$, $i \geq k+1$, $i \neq j$, is a singular point of $D' \cap E$, contrary to assumption. Q.E.D.

Remarks. (a) A divisor D in Λ satisfying the conclusions of (2.1) might not satisfy all the conclusions of (2.5). In the proof of (2.1) we only need that D' is nonsingular, whereas in the proof of (2.5) we also need that $D' \cap E$ is nonsingular over B^0 . Note also that the singularities of our general D over B^0 have resolutions with projective spaces as fibers, and thus they are rational double points.

(b) Concerning possible generalizations of (2.1); note that standard counterexamples show that in characteristic zero all divisors of Λ may have moving singularities contained in the base locus. Similarly, in characteristic $p > 0$ without a very ampleness assumption, all divisors may have moving singularities outside the possibly empty base locus. See [H1, exercises III.10.7, III.10.8]; [Z, p. 140]. Also, it is clear that all divisors of Λ will be singular where they meet singularities of X . But see Proposition 2.7 which follows.

Keep all of the assumptions of (2.1)(2) except allow B to have singularities outside the singularities of X . Allow these singularities to be either singularities of the underlying reduced scheme of B or points at which B is not reduced. The singularities of a general D contained in the locus where both X and B are nonsingular still obey (2.1) and (2.5). As for the singularities of a general D which lie in the singular locus of $B \cap X^0$, we have the following result.

(2.7) **Proposition.** *In the preceding situation, let Y be an irreducible component of the singular locus of $B \cap X^0$ and let Y_1 be an irreducible component of $D_s \cap Y$ for a general D . Assume that for some nonempty Zariski open set $U \subset Y$ and each closed point $y \in U$, the dimension of the tangent space to B at y is strictly less than the dimension of X . Then Y_1 has codimension at least one in Y .*

Proof. It is sufficient to show that for each $y \in U$ at least one D (possibly dependent on y) is nonsingular at y ; this of course implies that a general D is nonsingular at y . If every D were singular at y , then every D would have tangent space at y equal to the tangent space to X at y . This would imply that B , which is the intersection of all the D 's, would have tangent space at y equal to the tangent space to X at y . This is contrary to our assumption on the dimensions of the tangent spaces to B at points of U . Q.E.D.

To end this section we consider Bertini's theorem for nonalgebraically closed fields.

(2.8) **Proposition.** *Let L be any infinite subfield of the algebraically closed field K , and assume that the linear system Λ is defined over L . Then a general divisor D satisfying the conclusions of (2.1), (2.5), and (2.7) may be taken to be defined over L .*

Proof. By assumption, the linear system Λ over K is induced by a linear system Δ over L . Here Δ may be identified with a projective space \mathbf{P}_L^n , and the induced system Λ over K may compatibly be identified with $\mathbf{P}_K^n = \mathbf{P}_L^n \times_L K$. Note that after replacing K by the algebraic closure of L in K , we may assume that K is algebraic over L .

Now according to (2.1), (2.5), and (2.7), there is a nonempty Zariski open set U in \mathbf{P}_K^n , in which any K -valued point corresponds to a divisor D having the desired properties. After shrinking U , we may assume that U is a basic open set, whose complement is the locus of a homogeneous polynomial $F \in K[X_0, \dots, X_n]$. Since F has only finitely many coefficients,

$F \in M[X_0, \dots, X_n]$ for some subfield $M \subset K$ which is finite over L . After enlarging M , we may assume that the field extension $L \subset M$ is normal. Let U_M be the complement of the zero locus of F in \mathbf{P}_M^n . Then U_M is open, and $U_M \times_M K = U$.

We claim that there is a basic open set V in \mathbf{P}_L^n such that $V \times_L M \subset U_M$, and thus $V \times_L K \subset U$. Namely, let $F_1, \dots, F_m \in M[X_0, \dots, X_n]$ be the images of F under the elements of the Galois group $\text{Gal}(M/L)$, and let $E = \prod F_i$. Then $E \in L'[X_0, \dots, X_n]$, where L' is the fixed field in M of $\text{Gal}(M/L)$. If L has characteristic zero, then $L' = L$ and so we may take V to be the complement in \mathbf{P}_L^n of the zero locus of E . If L has characteristic $p \neq 0$, then L' is purely inseparable over L , say of degree $q = p^r$. Thus $E^q \in L[X_0, \dots, X_n]$ and we may take V to be the complement in \mathbf{P}_L^n of the zero locus of E^q .

Now V is nonempty, since it is a basic open set in \mathbf{P}_L^n . Since L is infinite, V contains L -valued points (by an easy induction on n). Choose such a point P , corresponding to some divisor in the linear system Δ over L . But this divisor induces a divisor D over K which is a member of the linear system Λ over K and which corresponds to the image of P in the open set $U \subset \mathbf{P}_K^n$. So indeed the divisor D , which is defined over L , has the property in question. Q.E.D.

3. COMPLETE INTERSECTIONS

For this section we fix a projective nonsingular variety X of dimension n and a very ample divisor D on X . By a *hypersurface* we mean a divisor from the linear system $|mD|$ for some $m \geq 1$. A purely dimension l closed subscheme Y of X will be called a *complete intersection* if there exist $n-l$ hypersurfaces H_1, \dots, H_{n-l} such that $Y = H_1 \cap \dots \cap H_{n-l}$ scheme theoretically; that is, on any affine patch of X the ideal of Y is the sum of the ideals of the H_i 's. The following theorem was proved by Nishi [N] in the special case that $X = \mathbf{P}^n$ and D is a hyperplane.

(3.1) **Theorem.** *Let Y be a nonsingular closed subscheme of X of dimension l . Assume that $2l < n$. Then there exist distinct nonsingular hypersurfaces H_1, \dots, H_{n-2l} containing Y such that for each i , $1 \leq i \leq n-2l$, $H_1 \cap \dots \cap H_i$ is a nonsingular complete intersection of dimension $n-i$.*

Proof. If we think of X as embedded in projective space \mathbf{P}^N , then we know that the homogeneous ideal of Y in \mathbf{P}^N will be finitely generated. This says that for $k \gg 0$ the subsystem of $|kD|$ of all divisors containing Y will have Y as its scheme theoretic base locus. If we let $\pi: X' \rightarrow X$ be the blow-up of X along Y with exceptional divisor E , then again, for $k \gg 0$, $\pi^*(kD) - E$ is very ample on X' [H1, exercise II.5.12, Proposition II.7.10]. Since $2l < n$, Theorem 2.1 then applies to tell us that a general divisor of $|kD|$ containing Y will be a nonsingular hypersurface in X containing Y . Pick one such hypersurface and call it H_1 .

We now proceed inductively. Having chosen H_1, \dots, H_j , one chooses H_{j+1}

as follows. By the previous paragraph there is clearly no problem in finding a nonsingular hypersurface H_{j+1} containing Y ; the problem is making sure that $H_1 \cap \cdots \cap H_{j+1}$ is also nonsingular. To do this, apply the previous argument used to obtain H_1 to the linear system given by the restriction of $k'D$ (for appropriate $k' \geq k$) to $H_1 \cap \cdots \cap H_j$. Q.E.D.

(3.2) **Corollary.** *If $\dim Y < \frac{1}{2} \dim X$, then Y is contained in a nonsingular complete intersection subvariety of dimension equal to $2 \dim Y$.*

For example, a nonsingular curve in \mathbf{P}^n lies on a nonsingular complete intersection surface. See also [N].

Remarks. (a) It is easy to see that a complete intersection of dimension at least one is connected. Namely, a general complete intersection—i.e., the intersection of general hypersurfaces—is connected because the intersections can be viewed successively as very ample divisors on nonsingular varieties. Any complete intersection is a flat specialization of a general complete intersection and hence still connected.

(b) Nishi [N] gives an example to show that the numerical restriction $2l < n$ is needed. We consider this example from a more abstract point of view, using a suggestion of Donu Arapura. According to the Grothendieck-Lefschetz theorem [H2, p. 179], if Y is a subscheme of dimension ≥ 3 that is a (strict) complete intersection in \mathbf{P}_K^n , then $\text{Pic } Y \cong \mathbb{Z}$ and is generated by the class $\mathcal{O}_Y(1)$. It follows that a subvariety of dimension at least 2 in projective space that is not a complete intersection cannot be a Cartier divisor on a complete intersection. For \mathbf{P}^4 this says that a nonsingular surface that is not a complete intersection cannot be contained in a nonsingular hypersurface. (There are nonsingular surfaces in \mathbf{P}^4 that are not complete intersections. For instance, embed the blow-up of \mathbf{P}^2 at a point into \mathbf{P}^4 by the system of conics through that point. The degree of the image is three, so it cannot be a complete intersection.)

(3.3) **Corollary.** *In (3.1) allow Y to have isolated singularities but require that all these singularities have tangent spaces of dimensions $\leq k$. Set $i = \min(n - 2l, n - k)$. Then there exist distinct nonsingular hypersurfaces H_1, \dots, H_i containing Y such that for each j , $1 \leq j \leq i$, $H_1 \cap \cdots \cap H_j$ is a nonsingular complete intersection of dimension $n - j$.*

Proof. Follow the proof of (3.1), using (2.7) where needed. Q.E.D.

In particular, a curve in \mathbf{P}^n with isolated planar singularities is contained in a nonsingular surface.

(3.4) **Theorem.** *Let Y be a nonsingular closed subscheme of X with Y of dimension l .*

(i) *If $2l < n$, then there are infinitely many positive integers d_i such that there is a nonsingular hypersurface $H_i \in |d_i D|$ containing Y .*

(ii) If $2l \geq n$, then there are at most $n-l$ positive integers d_i such that there is a nonsingular hypersurface $H_i \in |d_i D|$ containing Y . If $Y = H_1 \cap \cdots \cap H_{n-l}$ is a complete intersection, then any nonsingular hypersurface containing Y must be linearly equivalent to one of these H_i .

Proof. Statement (i) follows from the proof of (3.1). To prove statement (ii) let H be a nonsingular hypersurface containing Y . If C denotes the conormal bundle to Y in X , then H gives a nowhere vanishing section of $C \otimes \mathcal{O}(H)$ on Y (see (2.3) and surrounding discussion). This means that $(n-l)$ th Chern class of $C \otimes \mathcal{O}(H)$ must be zero. From standard Chern class calculations (see [F]) we see that

$$c_{n-l}(C \otimes \mathcal{O}(H)) = \sum_{i=0}^{n-l} c_i(C) H^{n-l-i}.$$

Say $H \in |mD|$ and the degree of $c_i(C) \cdot D^{l-i}$ is k_i . If $c_{n-l}(C \otimes \mathcal{O}(H)) = 0$ then

$$(3.5) \quad \sum_{i=0}^{n-l} k_i m^{n-l-i} = 0.$$

This is a polynomial of degree $n-l$ in m , so it has at most $n-l$ solutions.

Now suppose Y is a complete intersection $Y = H_1 \cap \cdots \cap H_{n-l}$ with $H_j \in |m_j D|$. For any containment $Z \subset W$ of nonsingular schemes we have the following exact sequence involving tangent and normal bundles.

$$0 \rightarrow T_Z \rightarrow T_{W|Z} \rightarrow N_Z W \rightarrow 0.$$

Some of the H_i may be singular, but they all must be nonsingular along Y . We also know that the normal bundle to a Cartier divisor E is $\mathcal{O}_E(E)$. Using these facts, one explicitly computes the Chern classes of C and the k_i in (3.5) (up to knowing D^l) and finds that the roots of (3.5) are exactly the m_j . Q.E.D.

(3.6) *Remark.* Consider equation (3.5). One would expect that a random polynomial would not have any positive integer zeros. Furthermore, the vanishing of the polynomial in (3.5) only says that $c_{n-l}(C \otimes \mathcal{O}(mD))$ has degree zero with respect to D , not that it actually is zero. If the Chern classes of C involve terms other than powers of D (as they might if Y is not a complete intersection), then it is even more unlikely that $c_{n-l}(C \otimes \mathcal{O}(mD))$ will ever be zero. Finally, we have only shown that the vanishing of $c_{n-l}(C \otimes \mathcal{O}(mD))$ is a necessary condition for Y to be contained in a nonsingular hypersurface. All this leads one to suspect that it is very difficult for a subvariety of X of dimension $\geq \frac{1}{2}n$ to be contained in a nonsingular hypersurface. But it is unclear how to make this observation into a precise assertion.

Remark. As at the end of §2, suppose that $L \subset K$ is an infinite subfield. Assume also that X and the linear system $|D|$ are defined over L . Then, as at the end of §2, one may show that the hypersurfaces occurring in (3.1), (3.3), and (3.4)(i) (for most d_i) can be chosen to be defined over L .

4. LIAISON

Continue with the assumptions of §3 but assume further that $X = \mathbf{P}^n$ and D is a hyperplane. Given two reduced closed subschemes Y_1 and Y_2 of \mathbf{P}^n , we say that Y_1 and Y_2 are *directly linked in liaison* if the scheme-theoretic union $Y_1 \cup Y_2$ (locally defined by the intersection of the ideals of Y_1 and Y_2) is a complete intersection, and no irreducible component of either of the subschemes is contained in the other. Thus Y_2 is minimal among subschemes Y of \mathbf{P}^n such that $Y_1 \cup Y = Y_1 \cup Y_2$. Consider the homogeneous coordinate ring S of \mathbf{P}^n :

$$S = \bigoplus_{i=0}^{\infty} H^0(\mathbf{P}^n, \mathcal{O}(i)) = K[X_0, \dots, X_n].$$

A subscheme Y of \mathbf{P}^n of pure dimension l can be represented as a complete intersection $Y = H_1 \cap \dots \cap H_{n-l}$ if and only if the H_i generate the homogeneous ideal of Y in S ; see [H1, exercise II.8.4(a)].

(4.1) **Theorem.** *Let $Y \subset \mathbf{P}^n$ be a nonsingular closed subscheme of pure dimension l that is a complete intersection. If $l \geq \frac{2}{3}n$, then Y cannot be directly linked to any nonsingular closed subscheme of \mathbf{P}^n that is not a complete intersection.*

Proof. Say Y is directly linked to a nonsingular Y' . Thus $Y = H_1 \cap \dots \cap H_{n-l}$ and $Y \cup Y' = H'_1 \cap \dots \cap H'_{n-l}$. We may assume that in the expressions for Y and $Y \cup Y'$, the H 's are listed so that their degrees are nondecreasing. We wish to show that Y' is a complete intersection.

(4.2) *Claim.* For some m , $1 \leq m \leq n-l$, we may choose H_i and H'_i above so that

- (i) $H'_i = H_i$ for $m < i \leq n-l$,
- (ii) $(H'_1, \dots, H'_m) \subset (H_1, \dots, H_m)$,
- (iii) $\deg H'_m > \deg H_m$.

Proof of (4.2). Note that (i) holds trivially with $m = n-l$; let $j \geq 0$ be the least value of m for which there is a choice of H_i, H'_i satisfying (i). Here $j \geq 1$, since $Y \neq Y \cup Y'$. Observe that for some such choice of H_i, H'_i , property (ii) also holds. Namely, first make any choice satisfying (i) as above; thus the inclusion $Y \subset Y \cup Y'$ implies that we may write

$$H'_i = \sum_{k=1}^{n-l} G_{ik} H_k = \sum_{k=1}^j G_{ik} H_k + \sum_{k=j+1}^{n-l} G_{ik} H'_k \quad \text{for } 1 \leq i \leq j,$$

where the G_{ik} are homogeneous polynomials, with $\deg G_{ik} + \deg H_k = \deg H'_i$ for all i and k . Replacing H'_i by $\sum_{k=1}^j G_{ik} H_k$ for $1 \leq i \leq j$, and leaving the other H'_i and all of the H_i as before, we obtain a choice satisfying (i) and (ii).

It remains to show that (possibly after again altering our choice) we may satisfy (iii) as well. With G_{ik} as above, note that $G_{hj} \neq 0$ for some $h \leq j$;

otherwise, we would have the containment $(H'_1, \dots, H'_j) \subset (H_1, \dots, H_{j-1})$, which is impossible because the zero set of the former ideal has smaller dimension than that of the latter. So with h as above, $\deg H'_j \geq \deg H'_h = \deg G_{hj} + \deg H_j \geq \deg H_j$. If the inequality is strict, we are done. Otherwise, we have that $\deg G_{hj} = 0$; i.e., G_{hj} is a nonzero constant. Also, since the H'_k are listed in order of nondecreasing degree, we have $\deg H'_j = \deg H'_h$. So interchanging H'_h and H'_j , we may assume that $G_{jj} \neq 0$. Multiplying H'_j by the constant G_{jj} , we have that $H'_j = H_j +$ terms involving lower H_i . Thus we may replace H_j by H'_j . So (i) holds with $m = j - 1$, contradicting the minimality of j . Therefore (iii) does in fact hold for $m = j$ and the above choice of H_i and H'_i , and the claim is shown.

(4.3) *Claim.* Let m be as in (4.2), let $1 < k \leq m$, and suppose that some choice of H_i, H'_i satisfies the condition

$$(C_k) \quad Y \cup Y' = H'_1 \cap \dots \cap H'_{k-1} \cap H_{k+1} \cap \dots \cap H_{n-l} \cap H'_m.$$

Then

- (1) $\deg H'_{k-1} \leq \deg H_k$ and
- (2) the singular locus of H'_m is nonempty and meets Y in codimension at most m on Y .

Proof of (4.3). Make some choice of H_i, H'_i as in (4.2). Using (C_k) and the inclusion $Y \subset Y \cup Y'$, and proceeding as in the first paragraph of the proof of (4.2), we may alter our choice of H'_1, \dots, H'_{k-1} (without changing their degrees, and while leaving alone the other H'_i and all of the H_i) so that the ideal they generate is contained in the ideal (H_1, \dots, H_k) .

Now suppose that $h, j \leq m$ and that we have the containment of ideals $(H'_1, \dots, H'_j) \subset (H_1, \dots, H_h)$. Then the locus of H'_j contains the complete intersection $H_1 \cap \dots \cap H_h$. Now $H_1 \cap \dots \cap H_h$ may be singular, but since it is a complete intersection of Cartier divisors it still has a conormal bundle C whose Chern classes may be computed as in the proof of (3.4). If, in addition, the degree of H'_j is greater than the degrees of H_1, \dots, H_h , then the Chern class $c_{n-h}(C \otimes \mathcal{O}(H'_j))$ is nonzero. Restricting this class to Y , we see that the singularities of H'_j on Y occur in codimension at most h in Y .

In particular, by (4.2) we may take $h = j = m$ and thus obtain (2). Meanwhile, if (1) fails, then by the first paragraph of this proof we may take $j = k - 1$ and $h = k$ and conclude that H'_{k-1} has singularities of codimension at most k in Y . Furthermore, the singular locus supports a class of codimension k in Y which is a multiple of a power of the restriction of $D \sim \mathcal{O}(1)$ to Y , and therefore meets everything on Y that it dimensionally should. Since $l \geq \frac{2}{3}n$, we have $m + k \leq 2m \leq 2(n - l) \leq l$, and so the singularities of H'_{k-1} meet the singularities of H'_m on Y . At any such point of intersection, the multiplicities of H'_{k-1} and of H'_m are each at least 2, and so the multiplicity of

$Y \cup Y' = H'_1 \cap \cdots \cap H'_{n-l}$ must be at least 4. But this cannot happen since both Y and Y' are nonsingular. This contradiction implies the claim.

(4.4) *Claim.* With m as in (4.2), there is a k ($1 \leq k \leq m$) such that some choice of H_i and H'_i will satisfy the condition

$$(D_k) \quad Y \cup Y' = H_1 \cap \cdots \cap H_{k-1} \cap H_{k+1} \cap \cdots \cap H_{n-l} \cap H'_m.$$

Proof of (4.4). By (4.2)(ii), the locus of that H'_m contains $H_1 \cap \cdots \cap H_m$, which is a complete intersection. For each k ($1 \leq k \leq m$), we may consider the condition (C_k) of (4.3). Thus $(C_1) = (D_1)$, and (C_m) is immediate from (4.2). It thus suffices to show that for $1 < k \leq m$, if there is a choice of H_i and H'_i satisfying (C_k) , then there also exists a choice which satisfies (D_k) or (C_{k-1}) .

So suppose (C_k) holds for some H_i, H'_i . As shown in the proof of (4.3), we may choose the H_i and H'_i such that $(H'_1, \dots, H'_{k-1}) \subset (H_1, \dots, H_k)$. There are two possibilities:

Case (a). $(H'_1, \dots, H'_{k-1}) \subset (H_1, \dots, H_{k-1})$.

It suffices to show that the inclusion must be an equality. For if this is shown, then we may replace H'_i by H_i for $1 \leq i \leq k-1$, and then (D_k) holds.

So suppose that inclusion is strict. Since both of these ideals must have reduced complete intersection zero sets of the same dimension, we conclude that the zero set of (H'_1, \dots, H'_{k-1}) is reducible. Thus $H'_1 \cap \cdots \cap H'_{k-1}$ has two components, and they meet in codimension one ([H3] and the remark following (3.2)). Now $H_1 \cap \cdots \cap H_{k-1}$ must be nonsingular along Y , since Y is nonsingular and $Y = (H_1 \cap \cdots \cap H_{k-1}) \cap (H_k \cap \cdots \cap H_{n-l})$. Also, since Y is a complete intersection, it must meet every cycle on $H_1 \cap \cdots \cap H_{k-1}$ that it dimensionally should meet. Thus $H'_1 \cap \cdots \cap H'_{k-1}$ is singular in codimension at most one on Y , namely along the locus in Y of the intersection of the two components. These singularities then meet the singularities of H'_m on Y (see (4.3)(2)). As in the proof of (4.3), this contradicts the nonsingularity of Y and Y' , thus proving the claim in this case.

Case (b). (H'_1, \dots, H'_{k-1}) is not contained in (H_1, \dots, H_{k-1}) .

By (4.3), we have $\deg H'_{k-1} \leq \deg H_k$. So, as in the proof of (4.2), we have (possibly after renumbering and multiplying by a nonzero constant) that $H'_{k-1} = H_k +$ terms involving lower H_i . Thus we may replace H_k by H'_{k-1} and still have the same ideal. Thus (C_{k-1}) holds.

(4.5) *Completion of the proof of (4.1).* Let m and k be as in (4.4). For the corresponding choice of H_i and H'_i , the condition (D_k) and the inclusion $Y \subset Y \cup Y'$ imply that we may write $H'_m = \sum_{j=1}^{n-l} G_j H_j$, where G_j is homogeneous and $\deg G_j + \deg H_j = \deg H'_m$. Thus

$$\begin{aligned} Y \cup Y' &= H_1 \cap \cdots \cap H_{k-1} \cap H_{k+1} \cap \cdots \cap H_{n-l} \cap H'_m \\ &= H_1 \cap \cdots \cap H_{k-1} \cap H_{k+1} \cap \cdots \cap H_{n-l} \cap G_k H_k, \end{aligned}$$

scheme-theoretically. Since $Y \cup Y'$ is reduced, G_k and H_k have no common factors, and so the locus of $G_k H_k$ is the scheme-theoretic union of that of G_k and that of H_k . So

$$Y \cup Y' = H_1 \cap \cdots \cap H_{k-1} \cap H_{k+1} \cap \cdots \cap H_{n-l} \cap (G_k \cup H_k) = Y \cup Y'',$$

where $Y'' = H_1 \cap \cdots \cap H_{k-1} \cap H_{k+1} \cap \cdots \cap H_{n-l} \cap G_k$, which is a complete intersection. Now by the definition of liaison, Y' is minimal among subschemes of \mathbf{P}^n whose union with Y is $Y \cup Y'$. So $Y' \subset Y''$. But

$$\begin{aligned} \deg Y + \deg Y' &= \deg Y \cup Y' = \left[\prod_{i \neq k} h_i \right] [g_k + h_k] \\ &= \prod_{i=1}^{n-l} h_i + \left[\prod_{i \neq k} h_i \right] g_k = \deg Y + \deg Y'', \end{aligned}$$

where $h_i = \deg H_i$ and $g_k = \deg G_k$. So in fact $Y' = Y''$. Thus Y' is a complete intersection. Q.E.D.

One might wonder whether the $\frac{2}{3}$ in (4.1) is in any way related to the $\frac{2}{3}$ in the following conjecture of Hartshorne [H4].

Conjecture (Hartshorne). If Y is a nonsingular subvariety of dimension r of \mathbf{P}^n and $r > \frac{2}{3}n$, then Y is a complete intersection.

Notice that if Hartshorne's conjecture is true, then Theorem (4.1) is automatic except possibly when $l = \frac{2}{3}n$. Hartshorne has shown by example that the strict inequality in this conjecture cannot be replaced by a \geq . Namely, the Grassmannian G of lines in \mathbf{P}^4 embeds via the Plucker embedding as a 6-fold in \mathbf{P}^9 that is not a complete intersection. Note that this provides a nontrivial example of Theorem (4.1), namely that G with this embedding cannot be directly linked to a nonsingular complete intersection. Also observe that in the absence of a proof of Hartshorne's conjecture, Theorem (4.1) does indicate one place *not* to look for a counterexample to Hartshorne's conjecture. Namely, it says that a counterexample (if one exists) could not be directly linked to a nonsingular complete intersection. See also [HU], where Theorem 4.1 is asserted in the strict inequality case, via other methods.

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