EXTENDING DISCRETE-VALUED FUNCTIONS

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ABSTRACT. In this paper, we show that for a separable metric space X, every continuous function from a subset S of X into a finite discrete space extends to a continuous function on X if and only if every continuous function from S into any discrete space extends to a continuous function on X. We also show that if there is no inner model having a measurable cardinal, then there is a metric space X with a subspace S such that every 2-valued continuous function from S extends to a continuous function on all of X, but not every discrete-valued continuous function on S extends to such a map on X. Furthermore, if Martin's Axiom is assumed, such a space can be constructed so that not even ω-valued functions on S need extend. This last result uses a version of the Isbell-Mrowka space Ψ having a C^*-embedded infinite discrete subset. On the other hand, assuming the Product Measure Extension Axiom, no such Ψ exists.

0. INTRODUCTION

Suppose that X is a metric space and S is a subset of X. Then the following statements are equivalent: (i) Every continuous function $f: S \to \mathbb{R}$ extends to a continuous function $F: X \to \mathbb{R}$, and (ii) Every continuous function $f: S \to \mathbb{R}$ having a compact image extends to a continuous function $F: X \to \mathbb{R}$. (Each statement is equivalent to S being a closed subset of X. To see this, notice that (i) trivially implies (ii), and every closed set satisfies (i) by Tietze's Extension Theorem. Finally, if S is not closed in X, there is a sequence σ in S converging to a point p not in X. Then any function $f: \sigma \to \{0, 1\}$ such that each fiber is infinite extends to a continuous $F: S \to [0, 1]$. If F is not onto, it can be composed with a continuous function from the range of F onto $\{0, 1\}$. In any case, we get a continuous function from S onto a compact set which does not extend continuously to p.) In this paper we deal with the question of what happens when $\mathbb{R}$ is replaced by a discrete space. In particular, we discuss the following problem: Suppose that X is a metric space and S is a subset such that every continuous function $f: S \to \{0, 1\}$ extends to a continuous function $F: X \to \{0, 1\}$. Does it follow that every continuous function from S to a discrete space extends to a continuous function defined on all of X? We show that if X is separable, then the answer is "yes".

However, if X fails to be separable, the situation becomes clouded by set-theoretic considerations. We show that if there is no inner model having a measurable cardinal, then there is a metric space and a subset such that every $\{0, 1\}$-valued continuous function extends, but not every function having a discrete image extends. Furthermore, assuming something weaker than Martin's Axiom, there is a metric space and a subset such that every $\{0, 1\}$-valued function extends, but not every ω-valued function extends. The construction of both of these examples involves finding a first countable space which has a certain clopen reflection, and then applying a technique which gives a metric space having the same clopen reflection as that space. It is shown that assuming the Product Measure Extension Axiom, these examples cannot be constructed by this technique.

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1. PRELIMINARIES.

We will use standard set-theoretic and topological terminology. See [E], [GJ], [K], and [W] for notions not defined here. Cardinals (that is, initial ordinals) will be assumed to have the *discrete topology*. If X is a topological space, S is a subset of X, and α is a cardinal number, then S is α-*embedded* in X if every continuous function $f: S \to \alpha$ extends to a continuous function $F: X \to \alpha$. Clearly, if $\alpha \leq \gamma$ and S is γ-embedded in X, then S is α-embedded in X. It follows from the Urysohn Extension Theorem that every 2-embedded discrete subset of a topological space is C^*-embedded.

Suppose that α is an infinite cardinal. Then the *hedgehog with α spines*, denoted $H(\alpha)$, is the subspace $\{ce_\lambda \in l_2(\alpha) : 0 \leq c \leq 1, \ e_\lambda \in \mathscr{B}\}$ of $l_2(\alpha)$, where $\mathscr{B}$ is the standard orthonormal basis for the Hilbert space $l_2(\alpha)$. The convex hull of $\{0, e_\lambda\}$ is called the λth *spine*, and e_λ is called the *end* of the λth spine.

An infinite collection $\mathfrak{E}$ of infinite subsets of ω is *almost disjoint* if every pair of distinct elements of $\mathfrak{E}$ has finite intersection. $\mathfrak{E}$ is a *maximal almost disjoint* (MAD) family if $\mathfrak{E}$ is almost disjoint and is not a proper subset of any almost disjoint family of subsets of ω. It is easy to see that $\mathscr{E}$ is an almost disjoint family if and only if the collection $\{(\mathrm{Cl}_{\beta\omega} E)\backslash E : E \in \mathscr{E}\}$ is a disjoint collection of clopen subsets of ω^*, and that $\mathscr{E}$ is a MAD family if and only if $\{(\mathrm{Cl}_{\beta\omega} E)\backslash E : E \in \mathscr{E}\}$ is a disjoint collection of clopen subsets of ω^* whose union is dense in ω^*. Given a MAD family $\mathscr{E}$, there is a pseudocompact space $\Psi_{\mathscr{E}}$ whose points are $\omega \cup \{p_E : E \in \mathscr{E}\}$ where each element of ω is isolated and a typical neighborhood of p_E is $\{p_E\} \cup (E\backslash F)$ where F is a finite subset of E. Another way of describing $\Psi_{\mathscr{E}}$ is that it is obtained from the subspace $\omega \cup \{(\mathrm{Cl}_{\beta\omega} E)\backslash E : E \in \mathscr{E}\}$ of $\beta\omega$ by collapsing each element of $\{(\mathrm{Cl}_{\beta\omega} E)\backslash E : E \in \mathscr{E}\}$ to a single point. We call any space of the form $\Psi_{\mathscr{E}}$ a *version* of Ψ.

If α is a cardinal number, an α-additive measure on a σ-algebra Σ of subsets of a set S is a nonnegative real-valued function μ defined on Σ with the property that given any collection $\mathscr{C}$ of fewer than α pairwise disjoint elements of Σ, at most countably many elements, say $C_1, C_2, \ldots$, satisfy $\mu(C_k) > 0$, and $\mu(\bigcup C_k) = \sum \mu(C_k)$. The Product Measure Extension Axiom, abbreviated PMEA, is the following statement: Given any set X, there is a C-additive measure μ on the power set $\mathscr{P}(X)$ of X with the property that if F is any finite subset of X, then $\mu\{A \subseteq X : F \subseteq A\} = 1/2^{|F|}$. It is known that the consistency of the existence of a strongly compact cardinal implies the consistency of PMEA.

The organization of the paper is as follows. In §2, we prove that if X is a separable metric space and S is a subset of X, then S is 2-embedded in X if and only if S is α-embedded in X for every cardinal α. A consequence is that if X is a separable metric space, S is a subset of X, and Y is a subset of $\mathbb{R}$, then every continuous function $f : S \to Y$ extends to a continuous function $F : X \to Y$ if and only if every bounded continuous function $f : S \to Y$ extends to a continuous function $F : X \to Y$. In §3, we discuss a process which we call hedgehogging a first countable space. The idea of hedgehogging a first countable space is to replace each point of the space with a hedgehog and topologizing the resulting set in such a way that the topology is metrizable and has the same zero-dimensional reflection as the original space. Hedgehogging certain familiar spaces gives consistent examples of metric spaces having 2-embedded subsets which are not α-embedded for certain cardinals α. We also construct a consistent example of a version of Ψ which has a 2-embedded infinite discrete subset. Hedgehogging this version of Ψ gives a consistent example of a metric space having a 2-embedded subset which is not ω-embedded. In §4, we give some consequences of PMEA. In particular, we show that assuming PMEA, in many cases the zero-dimensional reflection of a space cannot have a 2-embedded subset which is not α-embedded. It will follow that, assuming PMEA, in many cases a metric space, or more generally a first countable space, cannot have a 2-embedded subset which is not α-embedded. Furthermore, assuming PMEA, a pseudocompact space of small character cannot have an infinite discrete C^*-embedded subset, so, in particular, there cannot be a version of Ψ having an infinite 2-embedded discrete subset.

2. SEPARABLE METRIC SPACES

In this section, we show that if S is a subset of a separable metric space X, then if every 2-valued continuous function from S extends to such a function from X, then for each cardinal α, every α-valued continuous function from S extends to such a function from X. It will be helpful to use the following observation: If α is a cardinal number and X is any topological space, then a continuous function from X to α gives rise to a partition of X into clopen subsets, one for each nonempty fiber. Conversely, every partition of a space into clopen subsets induces a continuous function into a discrete space by mapping

the elements of the partition to different points of the discrete space. Thus, for example, to say that each 2-valued continuous function from S extends to a 2-valued continuous function on X says exactly that every disconnection of S is the trace on S of a disconnection of X.

2.1. **Proposition.** *Suppose that X is a separable metric space and S is a 2-embedded subset of X. Then S is ω-embedded in X.*

Proof. Suppose that S is a 2-embedded subset of the separable metric space X and $f: S \to \omega$ is a continuous function. For each $n \in \omega$, let $C_n = f^{\leftarrow}(n)$. Then $\{C_n : n \in \omega\}$ is a partition of S into clopen subsets of S. We must show that there is a partition $\{A_n : n \in \omega\}$ of X into clopen subsets of X such that for each $n \in \omega$, $A_n \cap S = C_n$. Since every 2-valued continuous function on S extends to a 2-valued continuous function on X, for each $n \in \omega$, there exists a clopen subset W_n of X such that $W_n \cap S = C_n$. Let $U_0 = W_0$ and for $n > 0$, let $U_n = W_n \setminus \bigcup \{U_k : k < n\}$. Then the sets U_n are pairwise disjoint clopen subsets of X such that $U_n \cap S = C_n$ for all n. Let L be the boundary in X of the set $\bigcup \{U_k : k \in \omega\}$, that is, $L = \{x \in X : \text{Every neighborhood of } x \text{ in } X \text{ intersects } U_n \text{ for infinitely many } n \in \omega\}$. For $x \in L$ and $n = 1, 2, \ldots$, let $B_n^x = \{k \in \omega : \text{Every clopen neighborhood of } C_k \text{ in } X \text{ intersects } S_{1/n}(x)\}$, where $S_\varepsilon(x)$ is the open ε-sphere of x in X. Notice that for $x \in L$ and n a positive integer, $B_n^x \supseteq B_{n+1}^x$. We now show that given an $x \in L$, there is an m such that B_m^x finite. For if not, we can find $k_1, k_2, \ldots$ such that $k_i \neq k_j$ for $i \neq j$ and $k_i \in B_i^x$. Define $g: S \to 2$ by $g(z) = 0$ if $z \in C_{k_{2i}}$ for some i, and $g(z) = 1$ otherwise. Let $G: X \to 2$ be a continuous extension of g. Then either $G(x) = 0$ or $G(x) = 1$. Since the argument is similar in both cases, assume that $G(x) = 1$. By continuity of G, there exists a positive integer n such that G is identically 0 on $S_{1/n}(x)$. By the definitions of B_m^x and the k_i's, there exists an integer i such that every clopen neighborhood of $C_{k_{2i+1}}$ intersects $S_{1/n}(x)$, a contradiction of the fact that G is identically 0 on $S_{1/n}(x)$ and identically 1 on $C_{k_{2i+1}}$. Therefore, there exists an integer m such that B_m^x is finite, say $B_m^x \subseteq \{0, 1, \ldots, m_0^x\}$. Let $V_x = S_{1/m}(x) \setminus \bigcup \{U_i : 0 \leq i \leq m_0^x\}$. Then V_x is a neighborhood of x in X such that there exist clopen neighborhoods W_n^x of C_n in X with $V_x \cap \bigcup_{n \in \omega} (W_n^x) = \varnothing$. By intersecting the sets W_n^x with U_n, we may assume that $W_n^x \subset U_n$ for all $x \in L$ and for all $n \in \omega$. The open cover $\{V_x : x \in L\}$ has a countable subcover, say $\{V_{x_0}, V_{x_1}, \ldots\}$. Let $R_k = \bigcap \{W_k^{x_i} : i = 1, \ldots, k\}$. Then $\bigcup_{k \in \omega} R_k$ is clopen, so $F: X \to \omega$ given by $F(x) = k$ if $x \in S_k$, $F(x) = 0$ otherwise is a continuous extension of f. $\square$

We remark that the above proof works if X is assumed to be any Lindelöf first countable space. At first glance, it might seem that hereditary Lindelöfness is required because a countable subcover is required for an open cover of the subset L of X. However, L is closed in X, so L is Lindelöf if X is.

We also remark that a separable metric space hereditarily has countable cellularity. Therefore, no subset of a separable metric space admits a continuous map onto an uncountable discrete space. Therefore, Proposition 2.1 actually gives the result that a 2-embedded subset of a separable metric space is α-embedded for every cardinal α. Furthermore, it is easy to see that if a subset of any space is γ-embedded for some cardinal γ, then it is α-embedded for every $\alpha < \gamma$.

Before giving the next result, we point out an example. Let X be the Euclidean plane, let S be the closure in X of the graph of the function $s: (0, \infty) \to \mathbb{R}$ given by $s(x) = \frac{\sin(1/x)}{x}$, and let $Y = S$. Then if $f: S \to Y$ is a bounded continuous function, the image of f will be a subset of either the graph of s or of the y-axis. It follows from Tietze's Extension Theorem that f has an extension to a continuous function $F: X \to Y$. On the other hand, the identity function $\iota: S \to Y$ does not have an extension to a continuous function from X to Y, because S is not arcwise connected, whereas X is. The point of the next result is that this sort of example cannot exist if Y is a subset of $\mathbb{R}$.

2.2. Proposition. *Suppose that X is a separable metric space and S is a subset of X. Let Y be a subset of $\mathbb{R}$ such that every bounded continuous function $f: S \to Y$ extends to a continuous function $F: X \to Y$. Then every continuous function $f: S \to Y$ extends to a continuous function $F: X \to Y$.*

Proof. We may obviously assume that Y is an unbounded proper subset of $\mathbb{R}$. Since for any given $r \in Y$, the space $\{y - r : y \in Y\}$ is homeomorphic to Y, we may assume that $0 \in Y$. Suppose that $f: S \to Y$ is continuous. We may assume that f is bounded below by 0, because if any such function extends to a continuous function on X, then we can extend $f^+ = 0 \vee f$ to $F^+: X \to Y$ and $f^- = 0 \vee (-f)$ to $F^-: X \to Y$. Then $F = F^+ - F^-$ will be a continuous extension of f. Under the assumption that f is bounded below by 0, we consider two cases.

Case (i). If Y contains an interval of the form (b, ∞) where $b > 0$, then S is closed in X, because in this case, the assumption that every bounded continuous function from S to Y extends to a continuous function on X implies that S is C^*-embedded in X. Let $h: Y \to [0, b + 2)$ be a homeomorphism such that $h(y) = y$ for each $y \in Y \cap [0, b + 1]$. Then $h^\to(Y) \subset Y$, so $h \circ f$ maps S into a bounded subset of Y. By assumption, $h \circ f$ extends to a continuous function $g: X \to Y$. Then $G: X \to [0, b + 2]$ given by $G = g \wedge (b + 2)$ is also a continuous extension of $h \circ f$. By the normality of X, we can find a continuous function $k: X \to [\frac{b+1}{b+2}, 1]$ such that $k(x) = 1$ for each x in $G^\leftarrow[0, b + 1]$ and $k(x) = \frac{b+1}{b+2}$ for each x in $G^\leftarrow(\{b + 2\})$. Then $k \cdot G$ maps X into $[0, b + 2)$ and the restriction of $k \cdot G$ to S is $h \circ f$. Now let $F = h^{-1} \circ (k \cdot G)$. Then F maps X to Y, F is continuous, and F extends f.

Case (ii). Suppose that Y contains no interval which is cofinal in $\mathbb{R}$. Then Y can be written as the union of a discrete family of nonempty clopen subsets,

C_n, $n \in \omega$, of Y, where every element of C_{n+1} is larger than every element of C_n for each n. Choose $y_n \in C_n$. Then $\Omega = \{y_n : n \in \omega\}$ is homeomorphic to ω. If $g: S \to \Omega$ is continuous and assumes only finitely many values, then g extends to a continuous function $G: X \to \Omega$. To see this, notice that such a g extends to a continuous $g': X \to Y$ by assumption, so if $h: Y \to \Omega$ is given by $h(y) = y_n$ for $y \in C_n$, then we can let $G = h \circ g'$. Therefore, by Proposition 2.1, every continuous function $g: S \to \Omega$ extends to a continuous function $G: X \to \Omega$. Translating this into terms of clopen partitions, it means that any partition of S into clopen sets arises as the trace on S of a partition of X into clopen sets. Now let $B_n = f^{\leftarrow}(C_n)$. Then $\{B_n : n \in \omega\}$ is a partition of S into clopen sets, so there is a partition $\{A_n : n \in \omega\}$ of X into clopen sets so that $A_n \cap X = B_n$ for all n. For $n \in \omega$, let $f_n: S \to Y$ be defined by $f_n(x) = f(x)$ if $x \in B_n$, $f_n(x) = 0$ otherwise. Then each f_n is bounded and continuous so by assumption, each f_n extends to a continuous function $F_n: X \to Y$. Now define $F: X \to Y$ by $F(x) = F_n(x)$ if $x \in A_n$. Then F is a continuous extension of f. $\square$

3. Hedgehogging first countable spaces

In this section, we show how to replace the points of a space with hedgehogs in such a way that the clopen structure of the new space is the same as the clopen structure of the original space, and the new space is metrizable. More precisely, we define a metrizable topology on $X \times H$, where H is a hedgehog, such that the zero-dimensional reflection of $X \times H$ is the same as the zero-dimensional reflection of X. (The zero-dimensional reflection of a space is obtained from a space by identifying each quasicomponent of the space to a point and giving the resulting set the topology obtained by declaring a set open if its inverse image under the identification function is clopen.)

Let X be a first countable space and let $\xi = |X|$. Let $H = H(\xi)$ be the hedgehog on ξ spines and label the spines of H as S_x, $x \in X$. Denote the end of the spine S_x as p_x. Define a topology on $X \times H$ as follows. A neighborhood of (x, y) where $y \neq p_x$ is $\{x\} \times V$ where V is a neighborhood of y in H. A neighborhood of (x, p_x) is $U \times V$ where U is a neighborhood of x and V is a neighborhood of p_x. We will denote the resulting space $\mathscr{H}(X)$, and call this space the *hedgehog of* X.

We will use Frink's Metrization Theorem (see [N]): A Hausdorff space Z is metrizable if and only if for each $p \in Z$, there is a countable nested open neighborhood base $\{U_p(n) : n \in \omega\}$ at p such that for each $n \in \omega$, there exists an $m \in \omega$ such that $U_p(m) \cap U_q(m) \neq \varnothing$ implies that $U_q(m) \subseteq U_p(n)$. We will be applying this theorem to $\mathscr{H}(X)$ for X a first countable space.

3.1. Proposition. *If X is first countable, then $\mathscr{H}(X)$ is metrizable.*

Proof. For each $p \in X$, let $\{V_p(n) : n \in \omega\}$ be a nested local base at p in X, and for each $q \in H$ and $n \in \omega$, let $S_q(n)$ denote $\{h \in H:$ the distance in H from h to q is less than $\frac{1}{n+1}\}$. Then for $(x, y) \in \mathscr{H}(X)$, define a nested local

base $\{U_{(x,y)}(n): n \in \omega\}$ as follows:

(i) If $y \neq p_x$, $U_{(x,y)}(n) = \{x\} \times S_y(n)$, and

(ii) If $y = p_x$, $U_{(x,y)} = V_x(n) \times S_y(n)$.

Observe that $\{x\} \times H$ is just a copy of the metric space H and for $y \neq p_x$, $U_{(x,y)}(n)$ is just the set of points in $\{x\} \times H$ at a distance of less than $\frac{1}{n+1}$ from (x, y).

We will show that the U's satisfy the conditions of Frink's Theorem. Suppose that $(x, y) \in \mathscr{H}(X)$ and $n \in \omega$ are given. First suppose $y \notin S_x$. Then let $\tilde{y}$ be the point of X such that $y \in S_{\tilde{y}}$. (If $y = 0$, choose $\tilde{y}$ arbitrarily.) Choose $r \geq n$ so that $V_{\tilde{y}}(r)$ does not contain x. Then if $U_{(x',y')}(m) \cap U_{(x,y)}(m) \neq \varnothing$, it follows that $x' = x$ and $y' = p_x$. Now choose m so that $\frac{3}{m+1} < \frac{1}{r+1}$. If $U_{(x,y')}(m) \cap U_{(x,y)}(m) \neq \varnothing$, then the distance in $\{x\} \times H$ from a point (a, b) in $U_{(x,y')}$ to (x, y) is less than $\frac{3}{m+1}$ which is less than $\frac{1}{r+1}$. Since $U_{(x,y)}(r)$ is the set of points whose $\{x\} \times H$ distance to (x, y) is less than $\frac{1}{r+1}$, it follows that $U_{(x,y')}(m) \subseteq U_{(x,y)}(r)$.

Now suppose $y \in S_x$ and $y \neq p_x$. This case is similar to the previous one, only this time choose r so that the H-distance from y to p_x is greater than $\frac{2}{r+1}$. (This guarantees that if $U_{(x',y')}(r) \cap U_{(x,y)}(r) \neq \varnothing$, then $y' \neq p_{x'}$.) Now proceed as in the previous case. Finally, if $y = p_x$, choose m so that $\frac{3}{m+1} < \frac{1}{r+1}$. If $U_{(x',y')}(m) \cap U_{(x,y)}(m) \neq \varnothing$ and $(x', y') \neq (x, y)$, then $y' \neq p_{x'}$ (since, in that case $U_{(x',y')} \subseteq V_{x'}(m) \times S_{x'}$, but $S_{x'} \cap S_x = \varnothing$). Thus, $U_{(x',y')}(m) = \{x'\} \times S_{y'}(m)$, $x' \in V_x(m)$, and $S_y(m) \cap S_{y'}(m) \neq \varnothing$, so $S_{y'}(m) \subseteq S_y(r)$; therefore, $U_{(x',y')} \subseteq V_x(m) \times S_y(r) \subseteq U_{(x,y)}(r)$. □

3.2. Lemma. *If D is a discrete subset of the space X and α is a cardinal number, then D is α-embedded in X if and only if $\{(x, p_x): x \in D\}$ is α-embedded in $\mathscr{H}(X)$.*

Proof. For each $f: D \to \alpha$ let $h_f: \{(x, p_x): x \in D\} \to \alpha$ be defined by $h_f((x, p_x)) = f(x)$. The lemma will be proved if we show that f extends to a continuous function $F: X \to \alpha$ if and only if h_f extends to $H_f: \mathscr{H}(x) \to \alpha$. If f extends to $F: X \to \alpha$, define $H_f: \mathscr{H}(X) \to \alpha$ by $H_f((x, y)) = F(x)$.

Now suppose $H_f: \mathscr{H}(x) \to \alpha$ extends h_f. Observe that H_f is constant on each $\{x\} \times H$ because α is discrete and $\{x\} \times H$ is connected. Thus $F(x) = H_f((x, y))$ where (x, y) is any element of $\{x\} \times H$ is well-defined. F is continuous because if $F(x) = \delta$, then $H_f^{\leftarrow}\{\delta\}$ contains a neighborhood $V_x(n) \times S_{p_x}(n)$ of (x, p_x), since H_f is continuous and thus contains $V_x(n) \times H$. Therefore, $F(V_x(n)) = \delta$. □

Combining 3.1 and 3.2 allows us to find examples of metric spaces with 2-embedded subsets which are not α-embedded by finding first countable spaces with these properties. In particular, assuming that no inner model has a measurable cardinal, Fleissner has constructed a normal Moore space which

has a family $\{C_\lambda : \lambda \in \alpha\}$ of closed subsets which exhibits noncollectionwise normality [F]. The C_λ's are totally disconnected. If we cone over each C_λ, that is, replace each C_λ by $C_\lambda \times [0, 1]$ with $C_\lambda \times \{1\}$ identified to a point for each λ, we get a 2-embedded subset which for some infinite cardinal α is not α-embedded, namely, the "points" of the cones. From this example, we get the following.

3.3. Example. If there is no inner model with a measurable cardinal, then there is a metric space X, a cardinal α, and a subset of X which is 2-embedded in X but not α-embedded in X.

Example 3.3 does not give a metric space which has a 2-embedded subset which is not ω-embedded. To get such an example from 3.1 and 3.2, we need a first countable space having a 2-embedded subset which is not ω-embedded. Recall that $\mathfrak{p}$ is the minimal cardinality of a centered family of clopen subsets of ω^* whose intersection has empty interior in ω^*. We will use a hypothesis which follows from the hypothesis that $\mathfrak{p} = \mathfrak{c}$, to construct a version of Ψ having this property. (Of course, the hypothesis $\mathfrak{p} = \mathfrak{c}$ follows from Martin's Axiom.) We note that since Ψ is pseudocompact, it suffices to find a version of Ψ having an infinite, discrete, 2-embedded subset. For the definitions of the cardinals $\mathfrak{b}$, and $\mathfrak{s}$, see [vD].

3.4. Proposition. *Assuming* $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$, *there is a MAD family* $\mathscr{E}$ *such that every countable set of nonisolated points of* $\Psi_\mathscr{E}$ *is 2-embedded in* $\Psi_\mathscr{E}$.

Proof. Let $\{A_\alpha : \alpha < \mathfrak{c}\}$ list all countable subsets of $\mathfrak{c}$ such that $A_n \subseteq \omega$ for all finite n, $A_\alpha \subseteq \alpha$ for all infinite α, and A_α is listed $\mathfrak{c}$ times for all α. Let $\{Z_\alpha : \alpha < \mathfrak{c}\}$ list the infinite subsets of ω. Let $\{S_n : n \in \omega\}$ be an almost disjoint collection of infinite subsets of ω. For each $\alpha \leq \omega$, let D_α be a subset of ω almost containing all S_n with $n \in A_\alpha$ and almost disjoint from all S_n with $n \notin A_\alpha$.

We will find collections $\{S_\alpha : \alpha < \mathfrak{c}\}$ and $\{D_\alpha : \alpha < \mathfrak{c}\}$ extending those above and satisfying, for all γ such that $\omega \leq \gamma < \mathfrak{c}$:

(i) $\{S_\alpha : \alpha < \gamma\}$ is an almost disjoint collection such that for $\omega \leq \alpha < \gamma$, $S_\alpha \subseteq Z_\alpha$, and S_α is nonempty if and only if S_α is infinite and $Z_\alpha \cap S_\beta$ is finite for each $\beta < \alpha$ and, for $\beta < \alpha$, S_α is almost contained in or is almost disjoint from D_β.

(ii) For each $\alpha \leq \mathfrak{c}$, $D_\alpha \subseteq \omega$ and D_α almost contains all S_β with $\beta \in A_\alpha$ and is almost disjoint from all S_β with $\beta \in \alpha \setminus A_\alpha$.

Proceeding inductively, suppose $\gamma < \mathfrak{c}$ and $\{S_\alpha : \alpha < \gamma\}$ and $\{D_\alpha : \alpha < \gamma\}$ are as above. If $Z_\gamma \cap S_\beta$ is infinite for some $\beta < \gamma$ then let $S_\gamma = \varnothing$. Otherwise, using $\mathfrak{s} = \mathfrak{c}$, there is an infinite subset S_γ of Z_γ such that for each $\alpha < \gamma$, S_γ is either almost contained in or almost disjoint from D_α. Using $\mathfrak{b} = \mathfrak{c}$, and Theorem 3.3 of [vD], we can choose D_γ to satisfy condition (ii).

Claim (1). $\mathscr{E} = \{S_\alpha : \alpha < \mathfrak{c}\}$ is a MAD family. For suppose that X is an infinite subset of ω. Then $X = Z_\alpha$ for some $\alpha < \mathfrak{c}$. If $S_\alpha = \varnothing$, by (i)

$\omega = |Z_\alpha \cap S_\beta| = |X \cap S_\beta|$ for some $\beta < \alpha$. By (i) S_α is empty or infinite; in either case, X has infinite intersection with an element of $\mathscr{E}$.

Claim (2). If N is a countable subset of the nonisolated points of $\Psi_\mathscr{E}$, then N is 2-embedded in $\Psi_\mathscr{E}$. To prove this, notice first that N can by written as $\{S_\alpha : \alpha \in H\}$ for some countable subset H of $\mathfrak{c}$. Let $\beta = \sup\{\alpha : \alpha \in H\}$. Let $f: N \to \{0, 1\}$ be a function. Then $N = N_0 \cup N_1$ where $N_i = f^\leftarrow\{i\}$ for $i = 0, 1$, and there are sets H_0 and H_1 such that $N_i = \{S_\alpha : \alpha \in H_i\}$ for $i = 0, 1$. There is $\gamma \geq \beta$ with $A_\gamma = H_0$; it follows that $H_1 \subseteq \gamma \backslash A_\gamma$. Thus D_γ almost contains each element of N_0 and is almost disjoint from each member of N_1. Since each S_α is almost contained in D_γ or almost disjoint from it, $\mathrm{Cl}\, D_\alpha$ is a clopen subset of $\Psi_\mathscr{E}$. Thus, $F: \Psi_\mathscr{E} \to \{0, 1\}$ defined by $F(x) = 0$ if $x \in \mathrm{Cl}\, D_\gamma$ and $F(x) = 1$ otherwise is a continuous extension of f. □

We remark that a consequence of Urysohn's Extension Theorem is that a discrete 2-embedded subset of a topological space is C^*-embedded. Hence, Proposition 3.4 establishes the consistency of the existence of a version of Ψ having an infinite, C^*-embedded discrete subset.

3.5. Corollary. *Assuming* $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$ *there is a metric space which has a 2-embedded subset which is not* ω-*embedded.*

4. A CONSEQUENCE OF PMEA

In §3, we saw that, given the right set theoretic assumptions, it is possible to find a metric space having a subset which is 2-embedded but not ω-embedded. The key to the construction is the construction of a pseudocompact first countable space which has an infinite 2-embedded subset. In this section, we show that assuming PMEA, no such pseudocompact space can exist. In fact, a somewhat stronger result holds.

4.1. Theorem. *Assuming PMEA, a space* X *of character less than* $\mathfrak{c}$ *is pseudocompact if and only if* X *has no infinite* C^*-*embedded discrete subset.*

Proof. X is pseudocompact if and only if X has no C-embedded copy of ω. (See [GJ].) Hence, a space which has no C^*-embedded copy of ω is pseudocompact.

Now suppose $P = \{p_i : i \in \omega\}$ is a C^*-embedded discrete subset of X. We will find a discrete family $\{\mathscr{O}_i : i \in \omega\}$ of open sets with $p_i \in \mathscr{O}_i$ for all $i \in \omega$. For each $S \in \mathscr{P}(P)$, let $f_S : X \to [0, 1]$ be such that $f_S|S = 0$ and $f_S|(P \backslash S) = 1$, and for each $x \in X$ choose a neighborhood $N_S(x)$ of x such that the diameter of $f_S(N_S(x)) < 1/4$. Let μ denote the measure guaranteed by PMEA. Using PMEA and the fact that the character of X is less than $\mathfrak{c}$, we can find a neighborhood $N(x)$ of x such that $\mu(\{S : N(x) \subseteq N_S(x)\}) > 15/16$. We claim that $\{N(p_i) : i \in \omega\}$ is a discrete family. Suppose $N(x) \cap N(p_i) \neq \varnothing$. Then $\mu(\{S : f_S^\rightarrow(N(x))$ is within $1/2$ of $f_S(p_i)\}) \geq 7/8$. For distinct p_i, p_j, $\mu(\{S : f_S(p_i) \neq f_S(p_j)\}) = 1/2$. Thus $\mu(\{S : f_S(N(x))$ misses the $1/2$-neighborhood of $f_S(p_j)\}) \geq 3/8$. So $N(x) \cap N(p_j) = \varnothing$. Thus $\{N(p_i) : i \in \omega\}$ is discrete. □

4.2.   Corollary. *Assuming PMEA, no first countable pseudocompact space has an infinite 2-embedded discrete subset.*

Of course, it does not follow from 4.2 that assuming PMEA every 2-embedded subset of a metric space is ω-embedded, but only that an example cannot be constructed using the method of §3. We do not know if there is a ZFC example of a metric space having a 2-embedded subset which is not α-embedded for some α.

We point out that the full strength of PMEA was not used to prove 4.1, but rather the statement that the product measure on 2^{ω} extends to a c-additive measure defined on the collection of all subsets of 2^{ω}. This statement is equivalent to the statement that Lebesgue measure on the real line extends to a c-additive measure μ with the property that every subset of $\mathbb{R}$ is μ-measurable.

We close with a question. We have seen that if there is no inner model having a measurable cardinal, then there is a metric space having a 2-embedded subset which is not α-embedded. Is there a ZFC example of a metric space having a 2-embedded subset which is not α-embedded? Specifically, is there a ZFC example of a metric space having a 2-embedded subset which is not ω-embedded?

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