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ON THE DISTANCE OF SUBSPACES OF l_p^n TO l_p^k

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ABSTRACT. It is proved that if l_p^n is well-isomorphic to $X \oplus Y$ and X either has small dimension or is a Euclidean space, then Y is well-isomorphic to l_p^k, $k = \dim Y$. The proofs use new forms of the finite dimensional decomposition method. It is shown that the constant of equivalence between a normalized K-unconditional basic sequence in l_p^n and a subsequence of the unit vector basis of l_p^n is greatest, up to a constant depending on K, when the sequence spans a 2-Euclidean space.

1. INTRODUCTION

The structure of infinite dimensional subspaces of L_p has been a central topic in Banach space theory for a long time. Now that attention has shifted to finite dimensional Banach spaces, it seems appropriate to consider finite dimensional versions of classical questions about the structure of subspaces of L_p. Perhaps the main problem in the infinite dimensional theory is to classify the isomorphic types of complemented subspaces of L_p, and the finite dimensional version of this problem seems to us of comparable interest. Just as in the infinite dimensional setting, the cases when $1 < p \neq 2 < \infty$ must be treated separately from the cases when $p = 1$ and $p = \infty$.

For $p = 1$, the famous infinite dimensional conjecture is that every infinite dimensional complemented subspace of $L_1[0, 1]$ is isomorphic to l_1 or $L_1[0, 1]$; the finite dimensional analogue is equivalent (by duality) to the equally famous finite injective problem of whether any "well"-complemented n-dimensional subspace of L_∞ is "well"-isomorphic to l_∞^n (see [Z] for a solution to the "almost isometric" version and a discussion of the finite injective problem).

When $1 < p \neq 2 < \infty$, there are known to exist uncountably many isomorphic types of complemented subspaces of L_p [BRS] and it is also known that

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for fixed n, the n-dimensional "well"-complemented subspaces of L_p form a rich class. However, there is the following analogue for L_p to the finite injective problem:

Conjecture 1. If X is a complemented subspace of l_p^n whose dimension (say, k) is proportional to n, then X is isomorphic to l_p^k.

Of course, as stated the conjecture lacks content since all k-dimensional spaces are isomorphic. Here and throughout this paper we follow the usual convention in local Banach space theory that qualitative statements about finite dimensional spaces should be quantified so as not to depend on dimension (but there may be dependence on other parameters). Thus the formal version of Conjecture 1 reads:

There exists a function $f(K, \delta, p)$ such that if X is a subspace of l_p^n which is complemented by means of a projection of norm at most K and $\dim X = k \geq \delta n$, then the Banach-Mazur distance of X from l_p^k is at most $f(K, \delta, p)$.

§2 below is devoted to two partial solutions to Conjecture 1: In Corollary 3 the conclusion of Conjecture 1 is shown to hold under the stronger hypothesis that $k \geq n - n^{2/p^*}[\operatorname{Log} n]^{-\beta(p)}$, where $p^* \equiv p \vee p'$ and p' is the conjugate index to p. This improves [BLM, Proposition 8.2], where a slightly weaker result is proved under the additional assumption that the complement of X has a good basis. In Corollary 1 Conjecture 1 is verified for the case where the complement of X is isomorphic to l_2^{n-k} (here there is no need to assume that the dimension of X is large, but of course this follows from the assumption); i.e. if l_p^n is isomorphic to $X \oplus l_2^{n-k}$ then X is isomorphic to l_p^k. Here our "convention" allows the Banach-Mazur distance of X from l_p^k to depend on the Banach-Mazur distance of $X \oplus l_2^{n-k}$ from l_p^n. This solves a problem of Bourgain and Tzafriri [BT1], who proved the same result under some restriction on k. Major parts of the proofs of Corollaries 3 and 1 are due to Bourgain and Tzafriri [BT1], [BT2], and results in [BLM] are used in the proof of Corollary 3. When $l_p^n = X \oplus Y$ with $\dim Y = n - k$, Bourgain and Tzafriri [BT1] break the proof that X is isomorphic to l_p^k into two steps: First, show that X is uniquely determined in the sense that if l_p^n is isomorphic to $X' \oplus Y$ then X and X' are isomorphic; [BT1, Theorem 1], gives a simple condition which guarantees this. We give in §4 a modification of the Bourgain-Tzafriri proof of step one which avoids some technical details in [BT1]. Secondly, check directly that $l_p^k \oplus Y$ is isomorphic to l_p^n. Our main contribution is to do the second step in some new cases. The ingredients we add are a new finite dimensional decomposition method, contained in Proposition 1, and various ways of applying the method. This new decomposition method allows two improvements on existing techniques: it can be used for subspaces which are not assumed to have a good basis and there can be an unbounded number of terms in the decomposition without increasing the dimension of the entire space. As is usual for decomposition methods, the proof of Proposition 1 is very simple and requires no background.

The results mentioned above give nothing when $p = 1$ or $p = \infty$ and, indeed, we have no new information about these cases even though Proposition 1 holds for these indices.

Here we mention also two weakenings of Conjecture 1. The first is the finite dimensional version of the statement that L_p is *primary*; that is, if $L_p = X \oplus Y$, then either X or Y is isomorphic to L_p.

Conjecture 1.a. If $l_p^n = X \oplus Y$, then either X is isomorphic to l_p^k $(k = \dim X)$ or Y is isomorphic to l_p^{n-k}.

Here our "convention" allows the isomorphism from X or Y onto l_p^k or l_p^{n-k} to depend on the norm of the projection from l_p^n onto X with kernel Y (and also on p, although we expect constants to be independent of p). We do not know the answer to Conjecture 1.a even when X is assumed to be isomorphic to Y:

Conjecture 1.b. If l_p^{2n} is isomorphic to $X \oplus X$, then X is isomorphic to l_p^n.

The various forms of Conjecture 1 are interesting when specialized to the case when X is a translation invariant subspace of

$$Z = \operatorname{span}\{e^{ikt} : 0 \le k \le n\}$$

in $L_p[0, 2\pi]$ and $1 < p < \infty$ (this restriction on p guarantees that Z is isomorphic to l_p^n). Note that Corollary 1 solves the translation invariant case when the complementary set of characters in $\{e^{ikt} : 0 \le k \le n\}$ to the characters which span X forms a Λ_{p^*}-set, which is the "natural" hypothesis for guaranteeing that X is complemented in $L_p[0, 2\pi]$.

§3 is motivated by the well-known

Conjecture 2. The maximal distance of an n-dimensional subspace X of L_p to l_p^n is attained when $X = l_2^n$; that is, $d(X, l_p^n) \le C n^{|1/p - 1/2|}$.

Conjecture 2 was recently disproved for $p = \infty$ by Szarek [S], who constructed a sequence $\{X_n\}_{n=1}^{\infty}$ with $\dim X_n = n$ and $d(X_n, l_\infty^n) \div \sqrt{n} \to \infty$. Consequently, if Conjecture 2 is true for $p < \infty$, the constant C must tend to infinity as $p \to \infty$. The finite dimensional analogue of Conjecture 2 is false for $p > 2$; the maximal distance of a subspace X of l_p^n from $l_p^{\dim X}$ is not attained for a Hilbert space; indeed, it follows from [FKP] (or see [FJ]) that there are subspaces X of l_p^n of distance of order $n^{|1/p-1/2|}$ from $l_p^{\dim X}$, while the highest dimension a Euclidean subspace of l_p^n can have is of order $n^{2/p}$ [BDGJN]. The spaces constructed in [FKP], [FJ] even have GL_2 and hence unconditional constant of order $n^{|1/p-1/2|}$. Now it is easy to see that Conjecture 2 is true for subspaces of L_p, $p < \infty$, which have a good unconditional basis. In §3 below we prove that if we restrict attention to subspaces of l_p^n with good unconditional bases, then the finite dimensional analogue of Conjecture 2 holds for $2 < p < \infty$; i.e. the maximal distance of a subspace X of l_p^n with an unconditional basis to $l_p^{\dim X}$ is of order at most $n^{(2/p)(1/2-1/p)}$. For general subspaces of l_p^n, we mention a weakened version of Conjecture 2:

Conjecture 2.a. If X is a subspace of l_p^n, $1 \le p < \infty$, then $d(X, l_p^{\dim X}) \le Cn^{|1/p-1/2|}$.

Conjecture 2.a is open even for complemented subspaces of l_p^n.

The proofs below are written for real scalars; with the obvious changes they work also in the complex case. This is important in Remark 1 below.

II. Complemented subspaces of l_p^n

We begin with the new decomposition method mentioned in the Introduction. For ease of reference, we ignore the convention about constants mentioned in the Introduction and write the constants explicitly.

Proposition 1. *Suppose that for each* $0 \le i \le m+1$. Y_i *is a* K*-complemented subspace of* $l_p^{s_i}$, $1 \le p \le \infty$, *and* $\mathrm{d}(Y_i, Y_{i+1}) < M$ *for* $0 \le i \le m$. *Set* $s = \sum_{i=1}^m s_i$. *Then* $Y_0 \oplus_p l_p^s$ *is* CKM*-isomorphic to* $Y_{m+1} \oplus_p l_p^s$ *for some absolute constant* C.

Proof. Let X_i be the complement to Y_i in $l_p^{s_i}$. Then

$$\begin{aligned} Y_0 \oplus_p l_p^s &= Y_0 \oplus_p l_p^{s_1} \oplus_p l_p^{s_2} \oplus_p \cdots \oplus_p l_p^{s_m} \\ &= Y_0 \oplus_p (Y_1 + X_1) \oplus_p \cdots \oplus_p (Y_m + X_m) \\ &\approx Y_0 \oplus_p (Y_1 \oplus_p X_1) \oplus_p \cdots \oplus_p (Y_m \oplus_p X_m) \\ &\equiv (Y_0 \oplus_p X_1) \oplus_p (Y_1 \oplus_p X_2) \oplus_p \cdots \oplus_p (Y_{m-1} \oplus_p X_m) \oplus_p Y_m \\ &\approx (Y_1 \oplus_p X_1) \oplus_p \cdots \oplus_p (Y_m \oplus_p X_m) \oplus_p Y_{m+1} \approx Y_{m+1} \oplus_p l_p^s. \quad \square \end{aligned}$$

Corollary 1 solves a problem raised by Bourgain and Tzafriri [BT1]; they proved the result under the restriction that $k \le n^{(2-\alpha)/p}$ for some $\alpha > 0$.

Corollary 1. *Suppose that* $X + Y = l_p^n$, $1 < p < \infty$, *with* $\mathrm{d}(Y, l_2^k) \le K_1$ *and where the projection onto* Y *has norm at most* K_2. *Then* $d(X, l_p^{n-k}) \le C$ *for some constant* $C = C(K_1, K_2, p)$.

Proof. Without loss of generality we can assume that $2 < p < \infty$ and $n = \delta k^{p/2}$ with δ bounded away from 0 (see [BDGJN]). Also, for notational simplicity, assume that $k = 2^m$. For $0 \le i \le m$, let Y_i be the $l_p^{2^i}$-sum of $l_2^{2^{m-i}}$ and set $Y_{m+1} = l_p^k$. Thus $d(Y_i, Y_{i+1}) \le 2$ for each i. It is enough by [BT1] to check that $l_2^k \oplus l_p^{n-k} \approx l_p^n$. Notice that we can write l_p^{n-k} as the l_p^m-sum of $l_p^{s_i}$, $1 \le i \le m$, where $s_i = \delta_i 2^{(m-i)p/2} 2^i$, with δ_i bounded away from 0.

Now by [Mi], [BGN], [BDGJN] (or see [FLM]), Y_i embeds into $l_p^{s_i}$ as a well-complemented subspace, so the desired conclusion follows from Proposition 1. □

Remark 1. It is well known [Zy, Theorem 7.10] that for $1 < p < \infty$, $Z_n = \operatorname{span}\{\exp(2\pi ik\theta) : 0 \le k \le n\}$ is complemented in $L_p[0, 1]$ and is isomorphic to l_p^{n+1} (with constants dependent on p but of course independent of n). The natural way to guarantee that a translation invariant subspace $Z_\Gamma = \operatorname{span}\{\exp(2\pi ik\theta) : k \in \Gamma\}$ with $\Gamma \subset \{0, 1, \dots, n\}$ is complemented in Z_n is to insist that $\{0, 1, \dots, n\}\backslash\Gamma$ is a Λ_{p^*}-set in Rudin's sense [Ru], which means that it spans a Hilbert space in the L_{p^*}-norm. Corollary 1 gives that in this case Z_Γ is isomorphic to l_p^m, $m = |\Gamma|$. In fact, in view of Bourgain's [B] recent solution to Rudin's Λ_{p^*}-set problem and the results of [BT1], this special case of Corollary 1 for translation invariant subspaces of $L_p[0, 1]$ already implies Corollary 1! On the other hand, we do not know how to prove, using only the tools of harmonic analysis, that Z_Γ is isomorphic to l_p^m, $m = |\Gamma|$, even when $\{0, 1, \dots, n\}\backslash\Gamma$ satisfies a lacunarity condition.

In order to implement the decomposition method for a general complemented subspace X of l_p^n, we need to build a short path from Y, the complement of X in l_p^n, to $l_p^{\dim Y}$ through complemented subspaces of L_p. The idea that this can be useful was suggested by one of Zippin's approaches to the finite injective problem. However, we do not know how to build such paths in L_1 or L_∞; for the other values of p we build the paths through l_2^n and this cannot work in L_1 or L_∞.

Proposition 2. *Suppose that Y_0 is a K-complemented n-dimensional subspace of $L_p[0, 1]$, $2 < p < \infty$. Then there exist $C = C(K, p)$ and subspaces Y_1, $Y_2, \dots, Y_k$ of L_p so that for each $0 \le i \le k$, $d(Y_i, Y_{i+1}) < 4$, and Y_i is C-isomorphic to a C-complemented subspace of L_p, where $k \equiv [\operatorname{Log}_2 n]$ and $Y_{k+1} \equiv l_2^n$.*

Proof. Assume, by making a Lewis change of density [L1] followed by the change of density in [JJ], that $\|y\|_p \le 2n^{1/2-1/p}\|y\|_2$ for each $y \in Y_0$ and that the projection P onto Y_0 is also bounded in the L_2-norm (by $\sqrt{p}K$; see [JJ]). Define Y_i as a subspace of $L_p \oplus_p L_2 : Y_i = \{(y, 2^i y) : y \in Y\}$. It is evident that for each $0 \le i \le k$, $d(Y_i, Y_{i+1}) < 4$. For each $1 \le i \le k$, let Z_i be the subspace $\{(z, 2^i z) : z \in L_p\}$ of $L_p \oplus_p L_2$ and note that Z_i is $K(p)$-isomorphic to a $K(p)$-complemented subspace of L_p by Rosenthal's theorem [R]. Finally, define for each $0 \le i \le k$ a projection P_i from Z_i onto Y_i by $P_i(z, 2^i z) = (Pz, 2^i Pz)$. Then $\|P_i\| \le \sqrt{p}K$ and thus each Y_i is $K(p)$-isomorphic to a $\sqrt{p}KK(p)$-complemented subspace of L_p. □

As an immediate consequence of Propositions 1 and 2, Corollary 1, and [BLM, Theorem 8.1], we have the following corollary.

Corollary 2. *If Y is a K-complemented k-dimensional subspace of L_p, $1 < p < \infty$, then $Y \oplus l_p^{n-k} \approx l_p^n$ as long as $n > k^{p^*/2}(\operatorname{Log} K)^{\alpha(p)}$ (recall that $p^* \equiv p \vee p'$, where p' is the conjugate index to p).*

Corollary 2 combines with results of Bourgain-Tzafriri [BT1], [BT2] and the complemented embedding theorem in [BLM] to yield an improvement of [BLM, Proposition 8.2].

Corollary 3. *Suppose $X + Y = l_p^n$, $1 < p < \infty$, with $\dim Y = k$ and $n > k^{p^*/2}(\operatorname{Log} k)^{\beta(p)}$ and where the projection onto Y has norm at most K. Then $\mathrm{d}(X, l_p^{n-k}) \le C$ for some constant $C = C(K, p)$.*

Proof. Replace step 2 in the proof of [BLM, Proposition 8.2] with Corollary 2. □

In this paper, we are concerned only with the "isomorphic" theory of l_p^n. In the isometric theory there are no problems: Ando [A] proved that every contractively complemented subspace of an L_p space is isometric to another L_p space. Zippin [Z] proved an "almost isometric" version of Ando's theorem for $p = 1$ and there may be such a result for other values of p; this would give a much stronger result than Conjecture 1 in the almost isometric setting. However, in the isomorphic theory, for any fixed value of $1 < p \neq 2 < \infty$, no stronger version of Conjecture 1 is true:

Remark 2. If $f(n) = o(n)$, then for all $1 < p \neq 2 < \infty$ there are $\sqrt{p^*}$-complemented subspaces X_n of l_p^n with $\dim X_n = k(n) \ge f(n)$ and $d(X_n, l_p^{k(n)}) \to \infty$ as $n \to \infty$. Indeed, each X_n can be chosen to be an l_p^m-sum of l_2^s for appropriate m and s, because such a space embeds into l_p^n, $n \approx ms^{p^*/2}$, as a $\sqrt{p^*}$-complemented subspace while clearly

$$d\left(\left[\sum_{i=1}^{m} l_2^s\right]_p, l_2^{k(n)}\right) \le m^{|1/p-1/2|} \quad \text{and} \quad d(l_p^{k(n)}, l_2^{k(n)}) = k(n)^{|1/p-1/2|}.$$

III. Subspaces of l_p^n with unconditional bases

We begin with two lemmas.

Lemma 1. *Let $\{x_i\}_{i=1}^k$ be a normalized sequence in l_p^n, $2 < p < \infty$, and let $H = H_2(\{x_i\}_{i=1}^k)$ be the 2-Hilbertian constant of $\{x_i\}_{i=1}^k$; that is, $\|\sum_{i=1}^k a_i x_i\| \le H(\sum_{i=1}^k a_i^2)^{1/2}$ for all scalars $\{a_i\}_{i=1}^k$. Then*

$$\left\| \left(\sum_{i=1}^{k} x_i^2 \right)^{1/2} \right\| \le H n^{1/p}.$$

Proof. We follow the proof on p. 182 in [BDGJN]. For any $1 \le j \le n$,

$$\left(\sum_{i=1}^k x_i^2(j)\right)^p \le \sum_{m=1}^n \left|\sum_{i=1}^k x_i(j)x_i(m)\right|^p = \left\|\sum_{i=1}^k x_i(j)x_i\right\|^p$$
$$\le H^p \left(\sum_{i=1}^k x_i^2(j)\right)^{p/2}.$$

Thus,

$$\left(\sum_{i=1}^k x_i^2(j)\right)^{p/2} \le H^p,$$

and

$$\left\|\left(\sum_{i=1}^k x_i^2\right)^{1/2}\right\| = \left(\sum_{j=1}^n \left(\sum_{i=1}^k x_i^2(j)\right)^{p/2}\right)^{1/p} \le Hn^{1/p}. \quad \square$$

Lemma 2. *Let $\{x_i\}_{i=1}^k$ be a normalized sequence in l_p^n, $2 < p < \infty$, and let H be the 2-Hilbertian constant of $\{x_i\}_{i=1}^k$. Then*

$$\left\|\left(\sum_{i=1}^k a_i^2 x_i^2\right)^{1/2}\right\| \le 12Hn^{(1/2-1/p)2/p}\left(\sum_{i=1}^k |a_i|^p\right)^{1/p}.$$

Proof. Suppose that $(\sum_{i=1}^k |a_i|^p)^{1/p} = 1$ and define

$$A_j = \{i\colon 2^{-j} < |a_i| \le 2^{-j+1}\}$$

so that also

$$\sum_{j=1}^\infty 2^{-pj}|A_j| \le 1.$$

For a value of B to be specified later, set

$$E = \{j\,;\, |A_j| \le n^{2/p} \text{ and } 2^j \ge B\},$$
$$F = \{j\,;\, |A_j| \le n^{2/p} \text{ and } 2^j < B\},$$
$$G = \{j\,;\, |A_j| > n^{2/p}\}.$$

Then

$$\left\|\left(\sum_{j\in E}\sum_{i\in A_j} a_i^2 x_i^2\right)^{1/2}\right\| \le 2\left(\sum_{j\in E} 2^{-2j}|A_j|\right)^{1/2} \le 4B^{-1}n^{1/p},$$

and

$$\left\| \left(\sum_{j\in F} \sum_{i\in A_j} a_i^2 x_i^2 \right)^{1/2} \right\| \le 2 \left(\sum_{j\in F} 2^{-2j} |A_j| \right)^{1/2}$$

$$= 2 \left(\sum_{j\in F} 2^{-pj} |A_j| 2^{(p-2)j} \right)^{1/2} \le 2B^{(p-2)/2}.$$

Let $J = \min G$. Then

$$\left\| \left(\sum_{j\in G} \sum_{i\in A_j} a_i^2 x_i^2 \right)^{1/2} \right\| \le 2 \sum_{j\in G} 2^{-j} \left\| \left(\sum_{i\in A_j} x_i^2 \right)^{1/2} \right\|$$

$$\le 2Hn^{1/p} \sum_{j\in G} 2^{-j} \qquad \text{(by Lemma 1)}$$

$$\le 4Hn^{1/p} 2^{-J} \le 4Hn^{1/p} 2^{-J} |A_J|^{1/p} n^{-2/p^2} \le 4Hn^{(1/2-1/p)2/p}.$$

Setting $B = n^{2/p^2}$, we get the desired conclusion. □

We are now ready for the main result of this section. Recall that the dimension of the largest 2-Euclidean subspace of l_p^n is of order $n^{2/p}$ (actually, of order $B_p^2 n^{2/p}$ up to constants independent of p; see [MS, p. 145, Remark 5.7]). Thus Proposition 3 implies that the maximal distance of a subspace X of l_p^n with an unconditional basis to $l_p^{\dim X}$ is achieved, up to a constant depending only on the unconditionality constant, for X a Euclidean subspace of l_p^n.

Proposition 3. *Let $\{x_i\}_{i=1}^k$ be a normalized K-unconditional basic sequence in l_p^n, $2 < p < \infty$, and let H be the 2-Hilbertian constant of $\{x_i\}_{i=1}^k$. Then*

$$K^{-1} \left(\sum_{i=1}^k |a_i|^p \right)^{1/p} \le \left\| \sum_{i=1}^k a_i x_i \right\| \le 12HKB_p n^{(1/2-1/p)2/p} \left(\sum_{i=1}^k |a_i|^p \right)^{1/p}$$

where $B_p \approx \sqrt{p}$ is the Khintchine constant. In particular, the constant of equivalence of $\{x_i\}_{i=1}^k$ to the unit vector basis of l_p^k is at most $12K^3 B_p^2 n^{(1/2-1/p)2/p}$.

Proof. The left inequality is easy and well known; see, for example, [LT, p. 73]. The right side follows from Lemma 2 and Khintchine's inequality. □

Remark 2. When $p = \log n$, the unit vector basis of l_p^n is 3-equivalent to the unit vector basis of l_∞^n. Thus Proposition 3 yields the following corollary, which is a slight variation of a lemma due to D. R. Lewis [L2].

Corollary 4. *Let $\{x_i\}_{i=1}^k$ be a normalized K-unconditional basic sequence in l_∞^n and let H be the 2-Hilbertian constant of $\{x_i\}_{i=1}^k$. Then for some absolute* *constant* C,

$$\left\|\sum_{i=1}^{k} a_i x_i\right\| \leq CHK[\log n]^{1/2} \max\{|a_i|\colon 1 \leq i \leq k\}.$$

In particular, the constant of equivalence of $\{x_i\}_{i=1}^k$ *to the unit vector basis of* l_∞^k *is at most* $CHK^2[\log n]^{1/2} \leq CK^3 \log n$.

IV. Concluding remarks

We present a proof of the Bourgain-Tzafriri [BT1] result that complemented Euclidean subspaces of l_p^n have unique complements. The "unique complement" criterion in Proposition 4 is proved in [BT1, Theorem 1], although the statement there is for the special case when Z, H, and G are L_p-spaces. (The isomorphism constants for the "$\approx$'s" in the conclusions of Propositions 4 and 5 of course depend only on the isomorphism constants for the "$\approx$'s" and the norms of the projections represented by sums in the hypotheses.)

Proposition 4. *Suppose for* $i = 1, 2$ *that* $Z = Y_i + X_i = H_i + G_i$ *with* $Y \approx Y_i$ *and* $G \approx G_i$ *and* $H \approx H_i \subset X_i$. *Suppose that* $H \approx Y \oplus W$. *Then* $X_i \approx G \oplus W$. *In particular,* $X_1 \approx X_2$.

Proof. Set $F_i = X_i \cap G_i$. Then $Z = Y_i + X_i = Y_i + F_i + H_i$ so that $G \approx Y \oplus F_i$. Thus $X_i = F_i + H_i \approx F_i \oplus Y \oplus W \approx G \oplus W$. □

In the proof of Proposition 5, we use the known fact that for $2 < p < \infty$, if C is any constant then for $n \geq n(C)$ every n-dimensional subspace Z of L_p contains a subspace $H \approx l_2^k$ with $k \geq Cn^{2/p}$, and by [M, Théorème 76], H is complemented in L_p by a projection of norm of order $\sqrt{p}$ times the isomorphism constant $H \approx l_2^k$. The existence of such an H (say, H_1) for *some* $C_1 > 0$ was proved in [FLM]; the complement of H_1 in Z contains another H (say, H_2) of dimension almost $C_1 n^{2/p}$ and thus $H_1 + H_2 \approx l_2^m$ with $m \approx 2C_1 n^{2/p}$. An iteration argument now produces the "H" of the desired dimension. (V. Milman pointed out that the more sophisticated iteration argument in [MP] yields a similar result for every K-convex cotype p normed space.)

Proposition 5. *Suppose for* $i = 1, 2$ *that* $l_p^n = X_i + Y_i$, $1 < p < \infty$, *with* $l_2^k \approx Y_i$. *Then* $X_1 \approx X_2$.

Proof. Without loss of generality we can assume that $2 < p < \infty$ and $n = \delta k^{p/2}$ with δ bounded away from 0 (see [BDGJN]). Consequently, $\dim(X_1 \cap X_2)$ is proportional to n and thus $X_1 \cap X_2$ contains a subspace $H \approx l_2^k$. Now apply Proposition 4. □

We conclude with a third partial solution to Conjecture 1; namely, when the complement to X in l_p^n is isomorphic to l_p^{n-k} with $n-k$ smaller than a small proportion of n. The proof uses [BT1, Proposition 4] and an improvement of [BT1, Proposition 6] which follows easily from the results in [BT2].

Proposition 6. *Let $1 \le p \le \infty$. For every $C < \infty$ there exists $\delta = \delta(p, C) > 0$ so that if $l_p^n = X + Y$ and the projection P_X onto X perpendicular to Y has norm at most C and $\dim Y \le \delta n$, then $l_p^n = H + G$, where $H \subset X$, $H \approx l_p^m$ with $m \ge \delta n$, and $G = l_p^S$ for some subset S of $\{1, \dots, n\}$.*

Proof. By [BT2], there exists $\tau = \tau(p, C) > 0$ so that there is a subset S_1 of $\{1, \dots, n\}$ with $|S_1| > 2\tau n$ and $\|R_{S_1} P_X R_{S_1} - \operatorname{diag}(R_{S_1} P_X R_{S_1})\| < 4^{-1}$, where " R_T " denotes "restriction to T ". Letting $\{e_i\}_{i=1}^n$ (respectively, $\{e_i^*\}_{i=1}^n$) denote the unit vector basis for l_p^n (respectively, $l_{p'}^n$), we set

$$S_2 = \{i \in S_1 : |e_i^*(1 - P_X)e_i| > 4^{-1}\}.$$

Then $4^{-1}|S_2| < \dim Y \le \delta n$ so that if $4\delta < \tau$, we can set $S_3 = S_1 \backslash S_2$ and conclude that $\|R_{S_3} P_X R_{S_3} - R_{S_3}\| < 2^{-1}$ with $|S_3| > \tau n$. To finish the proof, just set $S = \{1, \dots, n\} \backslash S_3$ and $H = P_X l_p^{S_3}$. □

Corollary 5. *Let $1 \le p \le \infty$. For every $C < \infty$ there exists $\delta = \delta(p, C) > 0$ so that if $l_p^n = X + Y$ and the projection P_X onto X perpendicular to Y has norm at most C and $Y \approx l_p^m$ with $m \le \delta n$, then $X \approx l_p^{n-m}$.*

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