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OPTIMAL HÖLDER AND L^p ESTIMATES FOR $\bar{\partial}_b$ ON THE BOUNDARIES OF REAL ELLIPSOIDS IN $\mathbf{C}^n$

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ABSTRACT. Let D be a real ellipsoid in $\mathbf{C}^n$, $n \geq 3$, with defining function $\rho(z) = \sum_{k=1}^n (x_k^{2n_k} + y_k^{2m_k}) - 1$, $z_k = x_k + iy_k$, where $n_k, m_k \in N$. In this paper we study the sharp Hölder and L^p estimates for the solutions of the tangential Cauchy-Riemann equations $\bar{\partial}_b$ on the boundary ∂D of D using the integral kernel method. In particular, we proved that if $\alpha \in L^\infty_{(0,1)}(\partial D)$ such that $\bar{\partial}_b \alpha = 0$ on ∂D in the distribution sense, then there exists a $u \in \Lambda_{1/2m}(\partial D)$ satisfying $\bar{\partial}_b u = \alpha$ and $\|u\|_{\Lambda_{1/2m}(\partial D)} \leq c\|\alpha\|_{L^\infty(\partial D)}$ for some constant $c > 0$ independent of α, where $\Lambda_{1/2m}(\partial D)$ is the Lipschitz space with exponent $\frac{1}{2m}$ and $2m = \max_{1\leq k\leq n} \min(2n_k, 2m_k)$ is the type of the domain D.

0. INTRODUCTION

The tangential Cauchy-Riemann operators, or $\bar{\partial}_b$ complex, on the boundary of a domain D in $\mathbf{C}^n$, $n \geq 2$, have played an important role in the study of boundary values of holomorphic functions and holomorphic extension problems (see Kohn-Rossi [14] and Folland-Kohn [6]). When D is strongly pseudo-convex, the $\bar{\partial}_b$ complex has been studied extensively by many authors. The Sobolev estimates for $\bar{\partial}_b$ were obtained by Kohn [12] using subelliptic esimates for $\square_b$. The Hölder and L^p estimates were obtained first by Folland-Stein [7] using analysis on the Heisenberg group and by Skoda [24], Henkin [10], and Romanov [17] using kernel methods.

On weakly pseudo-convex boundaries, the C^∞ solvability for $\bar{\partial}_b$ was proved by Rosay [18] (see also Shaw [20]). The L^2 existence and estimates for $\bar{\partial}_b$ on weakly pseudo-convex boundaries have been proved by Shaw [21], Boas-Shaw [1], and by Kohn [13] independently. Much less is known about the regularity for $\bar{\partial}_b$ on weakly pseudo-convex boundaries in the Hölder and L^p classes for p other than 2. When $n = 2$ and the boundaries are of uniform strict type, the Hölder and L^p estimates for $\bar{\partial}_b$ were obtained by the author in [22]. Recently Fefferman and Kohn [5] have proved the Hölder estimates for $\bar{\partial}_b$ on

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weakly pseudo-convex boundaries of finite type in $\mathbf{C}^2$ using the L^2 results and microlocal analysis.

In this paper we study the optimal regularity of the solution for the equation

$$\bar{\partial}_b u = f, \tag{1}$$

where f is a $(0, 1)$ form on the boundary ∂D in Hölder and L^p classes on the boundaries of real ellipsoids in $\mathbf{C}^n$, $n \geq 2$. These domains are a special class of weakly pseudo-convex domain of finite type. Our results are the first concerning sharp Hölder and L^p estimates for $\bar{\partial}_b$ for $n > 2$ and should serve as prototypes of estimates for $\bar{\partial}_b$ on general weakly pseudo-convex boundaries of finite type. The Hölder and L^p estimates are particularly important since they are crucial in many applications. Using the L^p estimates, we are able to prove the existence of holomorphic functions in the Nevanlinna class with prescribed zeros.

The plan of the paper is as follows: In §1 we introduce the notation and the main results. In §2 we derive the integral formula which solves $\bar{\partial}_b$ explicitly. This follows from the previous work of Romanov [17], Henkin [10], and Harvey-Polking [9] based on the existence of holomorphic support functions. The holomorphic support function we use here was constructed by Diederich-Fornaess-Wiegerinck [4] to prove the Hölder estimates for $\bar{\partial}$ on real ellipsoids. We note that the obvious choice of the holomorphic support function, the defining function for the complex tangent plane of ∂D, does not yield the optimal estimates, even though the complex tangent plane has the maximal order of contact with the boundary. We estimate the kernels in Hölder and L^p spaces in §3. These estimates are much more complicated than the cases in $\mathbf{C}^2$ and they show the complexity of the problem in $\mathbf{C}^n$ when $n > 2$. In §4 we apply the L^p estimates to the Poincaré-Lelong equation. The results allow one to construct holomorphic functions in the Nevalinna class with prescribed zeros. In §5 we give some examples to show that the Hölder estimates we obtained are sharp.

1. Notation and the main results

Let D be a real ellipsoid in $\mathbf{C}^n$ with defining function

$$\rho(z) = \sum_{k=1}^{n} (x_k^{2n_k} + y_k^{2m_k}) - 1, \qquad z_k = x_k + i y_k, \tag{1.0}$$

where n_k, $m_k \in N$ and $n \geq 2$, i.e., $D = \{z \in \mathbf{C}^n | \rho(z) < 0\}$. It is easy to see that the order of contact of the boundary point $p_k = (0, \ldots, z_k = 1, 0, \ldots, 0)$ with any complex analytic curve Γ which passes through p_k is at most $\max_{j \neq k} \min(2n_j, 2m_j)$ and p_k is a point of finite type

$$\max_{j \neq k} \min(2n_j, 2m_j).$$

Thus the domain D is of finite type $2m = \max_{1 \leq k \leq n} \min(2n_k, 2m_k)$. By a linear holomorphic change of coordinates, we can always assume $m_k \leq n_k$. For more discussion of pseudo-convex domains of finite type and order of contact, see D'Angelo [3].

We define $D^\delta = \{z \in \mathbf{C}^n | \rho(z) < \delta\}$ and $D_\delta = \{z \in \mathbf{C}^n | \rho(z) < -\delta\}$ for a small number $\delta > 0$. The Lebesgue measure on ∂D is denoted by σ and $\nu(z)$ denotes the outward unit normal at $z \in \partial D$. The letter c will always denote a positive constant which might vary from line to line.

Let $A_{p,q}(\partial D)$ be the restriction of (p, q) forms in $\mathbb{C}^n$ to ∂D. The (p, q) forms on ∂D, denoted by $\beta_{p,q}(\partial D)$, where $0 \le p \le n$ and $0 \le q < n$, are forms which are in $A_{p,q}(\partial D)$ and are orthogonal to the ideal generated by $\bar\partial\rho$. Let τ be the projection operator from $A_{p,q}(\partial D)$ to $\beta_{p,q}(\partial D)$. We use $L^s_{p,q}(\partial D)$ to denote the completion of $\beta_{p,q}(\partial D)$ with $L^s(\partial D)$ norm, $1 \le s \le \infty$. As usual, $C^k(\partial D)$ denotes the space of functions whose kth derivatives are continuous. $C^k_{p,q}(\partial D)$ are (p, q) forms on ∂D with $C^k(\partial D)$ coefficients. The Hölder space of exponent α on ∂D, $0 < \alpha < 1$, is denoted by $\Lambda_\alpha(\partial D)$ and is defined by the following norm (see Stein [25]):

$$\|u\|_{\Lambda_\alpha(\partial D)} = \sup_{\partial D} |u| + \sup_{\substack{g \in \mathscr{C} \\ 0 \le t \le 1}} \frac{|u(g(t) - u(g(0))|}{|t|^\alpha} < \infty,$$

where $\mathscr{C} = \{g(t) : t \in [0, 1] \to g(t) \in \partial D,\ g \text{ is } C^1 \text{ and } |g'(t)| \le 1\}$.

The tangential Cauchy-Riemann complex $\bar\partial_b : \beta_{p,q}(\partial D) \to \beta_{p,q+1}(\partial D)$ is defined as follows. Given $f \in \beta_{p,q}(\partial D)$, choose $f \in \Lambda_{p,q}(D)$ such that $\tau f = f$. Then

$$\bar\partial_b f = \tau(\bar\partial f). \tag{1.1}$$

Note that $\bar\partial_b f$ does not depend on the extension f. We shall say $\bar\partial_b u = f$ in the weak sense for $u \in L^s_{p,q}(\partial D)$ and $f \in L^s_{p,q+1}(\partial D)$ if for any smooth $(n-p, n-q-1)$ form ψ on ∂D,

$$(-1)^{q-1} \int_{\partial D} u \wedge \bar\partial_b \psi = \int_{\partial D} f \wedge \psi. \tag{1.2}$$

It is easy to see that when u and f are smooth, (1.2) agrees with (1.1).

Our main results of the paper are the following three theorems.

Theorem 1. *Let D be a real ellipsoid in $\mathbf{C}^n$, $n \ge 2$, with defining function $\rho(z)$ in (1.0) and $2m = \max_{1\le j\le n} \min(2n_j, 2m_j)$. Let $f \in L^p_{0,1}(\partial D)$, $1 \le p \le \infty$, where f satisfies the compatibility conditions:*

(i) *If $n = 2$, $\int_{\partial D} f \wedge \phi = 0$ for every $\bar\partial$-closed $(2, 0)$ form ϕ on D and ϕ is continuous on $\bar D$.*

(ii) *If $n > 2$, $\bar\partial_b f = 0$ in the distribution sense.*

Then there exists a function $u \in L^p(\partial D)$ such that $\bar\partial_b u = f$ on ∂D in the weak sense.

Furthermore, u satisfies the following estimates:

(i) *If $p = 1$, then $\|u\|_{L^{\gamma-\varepsilon}(\partial D)} \le c\|f\|_{L^1(\partial D)}$, where*

$$\gamma = \frac{2(n-1)(m-1)+2n}{2(n-1)(m-1)+2n-1}$$

and $\varepsilon > 0$ is any small number.

(ii) *If $1 < p < 2(n-1)(m-1)+2n = p_0$, then $\|u\|_{L^q(\partial D)} \le c\|f\|_{L^p(\partial D)}$, where $q < q_0$ and q_0 satisfies $1/q_0 = 1/p - 1/p_0$.*

(iii) *If $p = 2(n-1)(m-1)+2n$, then $\|u\|_{L^{p'}(\partial D)} \le c\|f\|_{L^p(\partial D)}$ for all $p' < \infty$.*

(iii) *If $p > 2(n-1)(m-1)+2n$, then $\|u\|\Lambda_{1/2m-p_0/2mp}(\partial D) \le c\|f\|_{L^p(\partial D)}$.*

In particular, $\|u\|\Lambda_{1/2m}(\partial D) \le c\|f\|_{L^\infty(\partial D)}$, where the constants c in (i)–(iv) *depend only on p, m, and D.*

We introduce the Besov spaces $\Lambda^p_\alpha(\partial D)$, $0 < \alpha < 1$, $1 \le p \le \infty$. A function $u \in \Lambda^p_\alpha(\partial D)$ if and only if $u \in L^p(\partial D)$ and

$$\|u\|_{L^p(\partial D)} + \sup_{g\in\mathscr{C}} \frac{\|u(g(t)) - u(g(0))\|_{L^p(\partial D)}}{|t|^\alpha} = \|u\|_{\Lambda^p_\alpha(\partial D)} < \infty,$$

where $\mathscr{C}$ and $g(t)$ are defined as before. When $p = \infty$, we have $\Lambda^\infty_\alpha(\partial D) = \Lambda_\alpha(\partial D)$. Thus the Besov spaces are natural generalizations of the Hölder spaces. (For basic properties of Besov spaces, see [26] where $\Lambda^p_\alpha(\partial D)$ is denoted by $\Lambda^{p,\infty}_\alpha(\partial D)$.) We obtain below estimates for solutions of $\overline{\partial}_b$ in these spaces.

Theorem 2. *Under the same assumption as in Theorem* 1, *there exists a $u \in \Lambda^p_{1/2m}(\partial D)$ satisfying $\overline{\partial}_b u = f$ in the weak sense and the estimates*

$$\|u\|\Lambda^p_{1/2m}(\partial D) \le c\|f\|_{L^p(\partial D)},$$

where c depends on m, p, and D only.

We note that when $p = \infty$, Theorem 2 agrees with (iv) in Theorem 1.

We shall apply the L^p estimates obtained in Theorem 1 to the zeros of holomorphic functions in the Nevanlinna class. A holomorphic function h on D is said to be in the Nevanlinna class, denoted by $N(D)$, if

$$\sup_\delta \int_{\partial D_\delta} \log^+ |h(z)|\, d\sigma_\delta < \infty,$$

where σ_δ is the Lebesgue measure of ∂D_δ. An analytic variety $M = \sum_j \nu_j M_j$ in D, where $\nu_j \in N$ and the M_j's are irreducible varieties in D, is said to be of finite area if

$$\int_M dS_M = \sum \nu_j \int_{M_j} ds < \infty,$$

where ds is the induced $(2n-2)$ measure on M_j. Using the L^1 estimates for $\overline{\partial}_b$, we were able to prove the following:

Theorem 3. *Let D be a real ellipsoid in $\mathbf{C}^n$ with defining function $\rho(z)$ in (1.0). Given any analytic variety of complex dimension $(n-1)$ such that M is the zero sets of an analytic function on D of finite area, there exists a function $h \in N(D)$ such that M is the zero sets of h.*

The proof of Theorem 3 depends on solving the Poincaré-Lelong equation $\partial\bar\partial u = \alpha$ with L^1 boundary values. Its relation with the tangential Cauchy-Riemann equations was first investigated by Henkin [10] and Skoda [24] on strongly pseudo-convex domains. On the complex ellipsoids it was also studied by Bonami-Charpentier [2].

2. The kernel formula

In this section we shall use the Bochner-Martinelli-Koppelman formula and the Leray formula to derive the kernels for the solution of $\bar\partial_b$ on D explicitly. It is well known that such kernels exist for convex domains and the kernels can be constructed using the complex tangent plane as the holomorphic support function (see Range [16] and Harvey-Polking [9]). Harvey-Polking also showed that such kernels make sense in the Cauchy principal-value sense. However, we need precise estimates of the holomorphic support function in order to obtain the optimal Hölder and L^p estimates. The key in the present construction is to use the special holomorphic support function employed by Diederich-Fornaess-Wiegerinck [4]. This holomorphic support function is much more complicated than the defining function of the complex tangent plane. Since the derivation of the kernel formula follows closely the papers by Harvey-Polking [9], Henkin [10], and Romanov [17], we shall refer the readers to those papers for details.

(2.1) **Lemma.** *There exists a holomorphic support function $\Phi(\zeta, z)$ on $D^\delta \times D^\delta$ such that*

(i) *$\Phi(\zeta, z)$ is holomorphic in z.*
(ii) *$\Phi(\zeta, z) = 0 \Leftrightarrow \zeta = z$ and $d_\zeta\Phi(\zeta, z)|_{\zeta=z} = -\partial\rho(\zeta) \neq 0$.*
(iii) *$\Phi(\zeta, z) = \sum_{i=1}^n p_i(\zeta, z)(\zeta_i - z_i)$, where the p_i's are C^1 functions holomorphic in z.*
(iv)

$$|\operatorname{Re}\Phi(\zeta, z)| \geq c\left(\rho(\zeta) - \rho(z) + \sum_{i=1}^n \left[\frac{\partial^2\rho}{\partial\zeta_i\partial\bar\zeta_i}(\zeta)|\zeta_i - z_i|^2 + |\zeta_i - z_i|^{2m_i}\right]\right)$$

for all $\zeta \in D^\delta \backslash D$ and $z \in \bar D$, where c is a positive constant.

Proof. This follows essentially from Diederich-Fornaess-Wiegerinck. We let $\rho_j = \partial\rho/\partial\zeta_j$ and $\rho_{\bar j} = \partial\rho/\partial\bar\zeta_j$. Define

$$\Phi(\zeta, z) = \sum_{j=1}^n \rho_j(z_j - \zeta_j) - \alpha\sum_{k=1}^n[(\eta_k^{2m_k-2} - \xi_k^{2n_k-2})(z_k - \zeta_k)^2 + (z_k - \zeta_k)^{2m_k}],$$

where $\zeta_k = \xi_k + i\eta_k$. If α is chosen to be small enough, then there exists $\varepsilon > 0$ such that

$$(2.2)\quad 2\operatorname{Re}\Phi(\zeta, z) \le \rho(z)-\rho(\zeta)-\varepsilon\sum_{k=1}^{n}[(\xi_k^{2n_k-2}+\eta_k^{2m_k-2})|z_k-\zeta_k|^2+|z_k-\zeta_k|^{2m_k}].$$

To prove (2.1), we note that there exists a constant $\delta > 0$ such that for all $t, \tau \in \mathbf{R}$,

$$(2.3)\qquad t^{2p} - \tau^{2p} - 2pt^{2p-1}(t-\tau) \ge \delta\{t^{2p-2}(\tau - t)^2 + (1-\tau)^{2p}\}.$$

(2.2) is proved easily by the convexity of t^{2p} and comparing the highest term on both sides. Thus we have from (2.3)

(2.4)

$$\rho(z) - \rho(\zeta) - 2\operatorname{Re}\rho_j(z_j - \zeta_j)$$
$$\ge \delta\sum_{k=1}^{n}[\xi_k^{2n_k-2}(x_k-\xi_k)^2 + \eta_k^{2m_k-2}(y_k-\eta_k)^2 + (x_k-\xi_k)^{2n_k} + (y_k-\eta_k)^{2m_k}].$$

Since

$$-\alpha\operatorname{Re}\sum(\eta_k^{2m_k-2} - \xi_k^{2n_k-2})(z_k-\zeta_k)^2$$
$$= -\alpha\sum_{k=1}^{n}[\eta_k^{2m_k-2}(x_k-\xi_k)^2 + \xi_k^{2n_k-2}(y_k-\eta_k)^2$$
$$- \xi_k^{2n_k-2}(x_k-\xi_k)^2 - \eta_k^{2m_k-2}(y_k-\eta_k)^2]$$

if we choose $\alpha < \delta$, then the term

$$\alpha\sum_{k=1}^{n}[\xi_k^{2n_k-2}(x_k-\xi_k)^2 + \eta_k^{2m_k-2}(y_k-\eta_k)^2]$$

can be absorbed by the right-hand side of equation (2.4). To control the term $-\alpha\operatorname{Re}\sum_{k=1}^{n}(z_k-\zeta_k)^{2m_k}$ we only need to estimate the term $\operatorname{Re}(z_k-\zeta_k)^{2m_k} < 0$. This can be estimated in two parts:

(i) If $|y_k - \eta_k| \le \varepsilon_0|x_k - \xi_k|$, where $\varepsilon_0 > 0$ is chosen small enough, then $\operatorname{Re}(z_k - \zeta_k)^{2m_k} \ge \frac{1}{2}|x_k - \xi_k|^{2m_k} \ge 0$. There will be no extra condition on α.

(ii) If $|y_k - \eta_k| > \varepsilon_0|x_k - \xi_k|$, then $|z_k - \zeta_k|^{2m_k} \le (1 + 1/\varepsilon_0)^{2m_k}|y_k - \eta_k|^{2m_k}$. Thus if we choose α such that $\alpha(1 + 1/\varepsilon_0)^{2m_k} < \delta/2$, then this term can be absorbed by the last term in (2.1). Thus if α is chosen small enough, there exists an $\varepsilon > 0$ such that (2.2) holds.

Letting $p_i(\zeta, z) = \rho_i - \alpha[(\eta_i^{2m_i-2} - \xi_i^{2n_i-2})(z_i - \zeta_i) + (z_i - \zeta_i)^{2m_i-1}]$, (iii) is proved.

Next we use $\Phi(\zeta, z)$ to construct the kernel formula for $\overline{\partial}_b$. Let B denote the Bochner-Martinelli-Koppelman kernel for the $(0, 1)$ form, i.e.,

$$B(\zeta, z) = \frac{(-1)^{n(n-1)/2}}{(2\pi i)^n}\frac{\det(\overline{\zeta} - \overline{z}, d(\overline{\zeta} - \overline{z}), \dots, d(\overline{\zeta} - \overline{z}))_{n\times n}}{|\zeta - z|^{2n}},$$

where

$$(\bar\zeta - \bar z) = (\bar\zeta_1 - \bar z_1, \dots, \bar\zeta_n - \bar z_n)^t$$

and

$$d(\bar\zeta - \bar z) = (d\bar\zeta_1 - d\bar z_1, \dots, d\bar\zeta_n - d\bar z_n)^t,$$

t is the transpose. We define

$$\int_{\partial D} f(\zeta) \wedge B(\zeta, z) \wedge d\zeta_1 \wedge \dots \wedge d\zeta_n = \begin{bmatrix} B^+_{\partial D} f(z) & \text{when } z \in D, \\ B^-_{\partial D} f(z) & \text{when } z \in \mathbf{C}^n \backslash \bar D \end{bmatrix}$$

for any $f \in L^p_{0,1}(\partial D)$. We have the following jump formula.

(2.5) **Lemma.** *Let $D \subset \mathbf{C}^n$ with smooth boundary ∂D. Let $f \in L^p_{0,1}(\partial D)$ and $\bar\partial_b f = 0$ when $n > 2$. Then $B^+_{\partial D} f$ is $\bar\partial$-closed on D and $B^-_{\partial D} f$ is $\bar\partial$-closed on $\mathbf{C}^n \backslash D$. Furthermore, for any continuous $(n, n-2)$ form ψ on ∂D, we have*

$$\lim_{\varepsilon \to 0^+} \int_{z \in \partial D} [B^+_{\partial D} f(z - \varepsilon v(z)) - B^-_{\partial D} f(z + \varepsilon v(z))] \wedge \psi(z) = \int_{\partial D} f(z) \wedge \psi(z).$$

Proof. When f is smooth, see Harvey-Polking [9]. For any general f with L^p coefficients, an approximation argument can be applied.

Let L denote the Leray kernel generated by $\Phi(\zeta, z)$, i.e.,

$$L(\zeta, z) = \frac{(-1)^{n(n-1)/2}}{(2\pi i)^n} \frac{\det(P(\zeta, z), \bar\partial_{\zeta,z} P(\xi, z), \dots, \bar\partial_{\zeta,z} P(\zeta, z))_{n\times n}}{\Phi(\zeta, z)^n},$$

where

$$P(\zeta, z) = (p_1(\zeta, z), \dots, p_n(\zeta, z))^t$$

and

$$\bar\partial_{\zeta,z} P(\zeta, z) = (\bar\partial_{\zeta,z} p_1(\zeta, z), \dots, \bar\partial_{\zeta,z} p_n(\zeta, z))^t.$$

We set $L^*(\zeta, z) = (-1)^n L(z, \zeta)$ and define

$$L^+_{\partial D} f(z) = \int_{\zeta \in \partial D} f(\zeta) \wedge L(\zeta, z) \wedge d\zeta_1 \wedge \dots \wedge d\zeta_n \quad \text{when } z \in D,$$

$$L^-_{\partial D} f(z) = \int_{\zeta \in \partial D} f(\zeta) \wedge L^*(\zeta, z) \wedge d\zeta_1 \wedge \dots \wedge d\zeta_n \quad \text{when } z \in \mathbb{C}^n \backslash \bar D.$$

Let

$$\eta_j(\zeta, z, \lambda) = \lambda \frac{\bar\zeta_j - \bar z_j}{|\zeta - z|^2} + (1 - \lambda) \frac{p_j(\zeta, z)}{\Phi(\zeta, z)}$$

and

$$R(\zeta, z, \lambda) = \frac{(-1)^{n(n-1)/2}}{(2\pi i)^n} \det(\eta, (\bar\partial_{\zeta,z} + d_\lambda)\eta, \dots, (\bar\partial_{\zeta,z} + d_\lambda)\eta)_{n\times n},$$

where again $\eta = (\eta_1(\zeta, z, \lambda), \dots, \eta_n(\zeta, z, \lambda))^t$. We also define $R^*(\zeta, z, \lambda) = (-1)^n R(z, \zeta, \lambda)$.

We define $R^+_{\partial D}f(z)$ and $R^-_{\partial D}f(z)$ by

$$(2.6)\quad R^+_{\partial D}f(z) = \int_{\substack{\zeta\in\partial D\\ 0\le\lambda\le 1}} f(\zeta)\wedge R(\zeta, z, \lambda)\wedge d\zeta_1\wedge\cdots\wedge d\zeta_n \quad \text{for } z\in D,$$

$$(2.6')\quad R^-_{\partial D}f(z) = \int_{\substack{\zeta\in\partial D\\ 0\le\lambda\le 1}} f(\zeta)\wedge R^*(\zeta, z, \lambda)\wedge d\zeta_1\wedge\cdots\wedge d\zeta_n \quad \text{for } z\in \mathbf{C}^n\setminus\overline{D}.$$

Then we have the following lemma.

(2.7) **Lemma.** *Let $f\in L^p_{0,1}(\partial D)$ and $\overline{\partial}_b f = 0$ if $n > 2$. Then we have*

(2.8) (i) $\overline{\partial}_z R^+_{\partial D}f(z) = B^+_{\partial D}f(z)$ *for* $z\in D$,

(2.9) (ii) *if* $n = 2$, $\overline{\partial}_z R^-_{\partial D}f(z) = B^-_{\partial D}f(z) - L^-_{\partial D}f(z)$ *for* $z\in\mathbf{C}^2\setminus\overline{D}$,

(2.9′) *if* $n > 2$, $\overline{\partial}_z R^-_{\partial D}f(z) = B^-_{\partial D}f(z)$ *for* $z\in\mathbf{C}^n\setminus\overline{D}$.

Details of the proof for this lemma can be found in [22]. The key point in the proof is the fact that $(\overline{\partial}_{\zeta, z}+d_\lambda)R(\zeta, z, \lambda) = 0$ for all $(\zeta, z)\in(D^\delta\setminus\overline{D})\times D$, where $R(\zeta, z, \lambda)$ is smooth.

(2.10) **Lemma.** *Let $f\in L^p_{0,1}(\partial D)$, $1\le p\le\infty$. Then the integrals $R^+_{\partial D}f$ and $R^-_{\partial D}f$ have boundary values for almost $z\in\partial D$, i.e., if we define*

$$(2.11)\qquad Tf(z) = \lim_{\varepsilon\to 0^+} R^+_{\partial D}f(z-\varepsilon v(z)),$$

$$(2.12)\qquad Sf(z) = \lim_{\varepsilon\to 0^+} R^-_{\partial D}f(z+\varepsilon v(z)),$$

the limits on the right exist a.e. for $z\in\partial D$. Furthermore, T and S are compact operators from $L^p_{0,1}(\partial D)$ to $L^p(\partial D)$.

We shall postpone the proof of this lemma since it follows easily from the proof of Theorem 2. In Theorem 2 we prove that T and S are bounded operators from $L^p_{0,1}(\partial D)$ to $\Lambda^p_{1/2m}(\partial D)$, which implies T and S are compact operators from $L^p_{0,1}(\partial D)$ to $L^p(\partial D)$.

(2.13) **Theorem.** *Let f and D be the same as in Theorem 1 and let T and S be defined as in (2.11) and (2.12). Then*

$$(2.14)\qquad f = \overline{\partial}_b(Tf - Sf)$$

in the distribution sense.

Since f satisfies the compatibility conditions and $L^*(\zeta, z)$ is $\overline{\partial}_\zeta$-closed for $\zeta\in D$, we have $L^-_{\partial D}f = 0$ for $z\in\mathbf{C}^2\setminus D$ in (2.9).

From Lemmas (2.5) and (2.7), for any smooth $(n, n-2)$ form ψ on ∂D we have

$$\begin{aligned}
&\int_{\partial D} f(z) \wedge \psi(z)\\
&= \lim_{\varepsilon\to 0^+} \int_{z\in\partial D} [B^+_{\partial D} f(z-\varepsilon v(z)) - B^-_{\partial D} f(z+\varepsilon v(z))] \wedge \psi(z)\\
&= \lim_{\varepsilon\to 0^+} \int_{z\in\partial D} [\bar\partial_z R^+_{\partial D} f(z-\varepsilon v(z)) - \bar\partial_z R^-_{\partial D} f(z+\varepsilon v(z))] \wedge \psi(z)\\
&= \lim_{\varepsilon\to 0^+} \int_{z\in\partial D} [R^+_{\partial D} f(z-\varepsilon v(z)) - R^-_{\partial D} f(z+\varepsilon v(z))] \wedge \bar\partial_z \psi(z)\\
&= \lim_{\varepsilon\to 0^+} \int_{z\in\partial D} [Tf(z) - Sf(z)] \wedge \bar\partial_b \psi(z)
\end{aligned}$$

which implies (2.15) in the weak sense as defined in (1.2) in §1.

3. Proof of Theorems 1 and 2

In this section we shall estimate the solution derived in (2.14) and justify Lemma 2.10 at the same time. By integrating over the λ variable in (2.6) and (2.6′), we have

$$R^+_{\partial D} f(z) = \int_{\zeta\in\partial D} f(\zeta) \wedge K(\zeta, z) \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n \quad \text{for } z \in D, \tag{3.1}$$

$$R^-_{\partial D} f(z) = \int_{\zeta\in\partial D} f(\zeta) \wedge K^*(\zeta, z) \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n \quad \text{for } z \in \mathbb{C}^n\setminus\overline{D}, \tag{3.1′}$$

where $K(\zeta, z) = \sum_{q=0}^{n-2} K_q(\zeta, z)$ and

$$K_q(\zeta, z) = \frac{1}{(2\pi i)^n} \frac{\det(\bar\zeta - \bar z, P(\zeta, z), \overbrace{d\bar\zeta, \dots, d\bar\zeta}^{(n-2-q)\text{-times}}, \overbrace{\bar\partial_\zeta P(\zeta, z), \dots, \bar\partial_\zeta P(\zeta, z)}^{q\text{-times}})}{\Phi(\zeta, z)^{q+1} |\zeta - z|^{2(n-2-q)+1}}. \tag{3.2}$$

We define $K^*(\zeta, z) = K(z, \zeta)$.

We note that $K(z, \zeta)$ and $K^*(\zeta, z)$ have only singularities at $\zeta = z$. We shall first prove that the kernels K and K^* are integrable. In fact, we prove

(3.3) **Lemma.** *There exist constants M_1 and M_2 such that*

$$\int_{\partial D} |K(\zeta, z)|^a d\sigma_\zeta < M_1 \quad \textit{uniformly for } z \in D, \tag{3.4}$$

$$\int_{\partial D} |K^*(\zeta, z)|^a d\sigma_\zeta < M_2 \quad \textit{uniformly for } z \in D^\delta\setminus\overline{D}, \tag{3.4′}$$

where a is any number such that

$$1 \le a < \gamma = (2(n-1)(m-1) + 2n)/(2(n-1)(m-1) + 2n - 1).$$

Proof. Let K_q be defined as in (3.2). We shall prove

$$\int_{\partial D} |K_q(\zeta, z)|^a d\sigma_\zeta < M \quad \text{uniformly for } z \in D, \tag{3.5}$$

where a is any number such that $0 < a < a_q$ and

$$a_q = \frac{2(q+1)(m-1)+2n}{2(q+1)(m-1)+2n-1}.$$

Then (3.4) will be proved since $a_0 < a_1 < \cdots < a_{n-2} = \gamma$.

Since $K(\zeta, z)$ is smooth except at $\zeta = z$, it is clear we only need to estimate (3.5) when $|\zeta - z|$ is small. We also note that only the complex tangential part of $K(\zeta, z)$ will be integrated. Lete $B_x(r)$ be a ball in $\mathbf{C}^n$ centered at x with radius r. We cover the boundary ∂D by a finite collection of balls $\{B_{x_i}(\eta_0)\}_{i=1}^N$ such that $x_i \in \partial D$ and η_0 is small. We also require that $B_{x_i}(\eta_0/2)$ covers a tubular neighborhood of ∂D. We shall estimate (3.5) for $z \in B_{x_i}(\eta_0/2)$ for some fixed i, and denote $B_{x_i}(\eta_0/2) = B$ and $B_{x_i}(\eta_0) = \widetilde{B}$. Thus for $|\zeta - z| < \eta_0/2$ we have $\zeta \in \widetilde{B}$. If η_0 is small enough, one of the coordinates of $\zeta_1, \zeta_2, \ldots, \zeta_n$ must be bounded away from zero on $\widetilde{B}$. Assuming that coordinate is ζ_n, i.e., $|\operatorname{Re}\zeta_n| \geq \varepsilon > 0$, we can choose a basis for the $(0, 1)$ complex tangential vector fields $\overline{L}_1, \ldots, \overline{L}_{n-1}$ such that

$$\overline{L}_j = \rho_{\bar{n}}\frac{\partial}{\partial\overline{\zeta}_j} - \rho_{\bar{j}}\frac{\partial}{\partial\overline{\zeta}_n}, \qquad j = 1, \ldots, n-1. \tag{3.6}$$

Using (ii) of Lemma 2.1, $\zeta_1, \ldots, \zeta_{n-1}, \overline{\zeta}_1, \ldots, \overline{\zeta}_{n-1}$ and $\operatorname{Im}\Phi(\zeta, z)$ are linearly independent and thus form a coordinate system on $\widetilde{B} \cap \partial D$.

We introduce new real coordinates $t \equiv (t_1, \tau_1, \ldots, t_{n-1}, \tau_{n-1}, t_{zn-1}) \equiv (t', t_{2n-1})$ on $\widetilde{B} \cap \partial D$ such that

$$\begin{cases} t'_k = t_k + i\tau_k = z_k - \zeta_k, \\ t_{2n-1} = \operatorname{Im}\Phi(\zeta, z). \end{cases} \tag{3.7}$$

Let $y = |\rho(z)|$; then under these coordinates we have from Lemma 2.1,

(3.8)

$$|d\sigma_\zeta| \leq c\,dt_1\,d\tau_1 \cdots dt_{n-1}\,d\tau_{n-1}\,dt_{2n-1} \equiv c\,dt \equiv c\,dt'\,dt_{2n-1},$$

$$|\zeta - z| \geq c(|t'| + y),$$

$$|\Phi(\zeta, z)|$$

$$\geq c\left\{y + |t_{2n-1}| + \sum_{k=1}^{n-1}[(x_k - t_k)^{2n_k-2} + (y_k - \tau_k)^{2m_k-2}]|t'_k|^2 + |t'_k|^{2m_k}\right\}.$$

From now on, we shall always assume $n_k, m_k > 1$ (the other cases are simpler), since from (3.6) and the definition of the p_i's, we have

$$|\overline{L}_i p_k(\zeta, z)| \leq \delta_{ik}[|\rho_{k\bar{k}}(\zeta)| + (|\xi_k|^{2n_k-3} + |\eta_k|^{2m_k-3})|\zeta_k - z_k|] \tag{3.9}$$

for $i, k = 1, 2, \ldots, n-1$ and

$$|\overline{L}_i p_n(\zeta, z)| \leq c(|\xi_i|^{2n_i-1} + |\eta_i|^{2m_i-1}), \qquad i = 1, \ldots, n-1. \tag{3.10}$$

In t coordinates, (3.9) and (3.10) become

(3.11)

$$|\overline{L}_i p_k| \leq \delta_{ik}[(|x_k - t_k|^{2n_k-2} + |y_k - \tau_k|^{2m_k-2}) + (|x_k - t_k|^{2n_k-3} + |y_k - \tau_k|^{2n_k-3})|t'_k|],$$

$$|\overline{L}_i p_n| \leq c(|x_k - t_k|^{2n_x-1} + |y_k - \tau_k|^{2m_k-1}). \tag{3.11'}$$

To prove (3.5), we note that

$$|K_q(\zeta, z)| \leq \sum{}' \frac{\prod_{j=1}^q |\overline{L}_{k_j} p_{k_j}(\zeta, z)|}{|\Phi|^{q+1}|\zeta - z|^{2s+1}}, \tag{3.12}$$

where $\sum'$ is summed over all q-tuples $1 \leq k_1 < k_2 < \cdots < k_q \leq n$, $s = n - 2 - q$.

To show that each term on the right of (3.12) is in $L^a(\partial D)$ uniformly for all $z \in D$, we need the following lemma.

(3.13) **Lemma.** *For $\alpha > 1$, $j \geq 1$, $A > 0$, and $I = \{(t, \tau)|t^2 + \tau^2 \leq 1|\}$, there exists a constant c satisfying the following assertions.*

$$\int_I \frac{|x - t|^j \, dt \, d\tau}{(|x - t|^j(t^2 + \tau^2) + A)^{\alpha+1}} \leq cA^{-\alpha}, \tag{i}$$

$$\int_I \frac{|x - t|^{j-1}|t| \, dt \, d\tau}{(|x - t|^j(t^2 + \tau^2) + A)^{\alpha+1}} \leq cA^{-\alpha}, \tag{ii}$$

$$\int_I \frac{|x - t|^{j-1}|t| \, dt \, d\tau}{(|x - t|^j(\tau^2 + t^2) + (\sqrt{t^2 + \tau^2})^{j+2} + A)^{\alpha+1}} \leq cA^{-\alpha}, \tag{iii}$$

$$\int_I \frac{|\xi|^{j-1}|t| \, dt \, d\tau}{(|\xi|^j(\tau^2 + t^2) + (\sqrt{t^2 + \tau^2})^{j+2} + A)^{\alpha+1}} \leq cA^{-\alpha}, \tag{iv}$$

$$\int_I \frac{|\xi|^{j-1}|\tau| \, dt \, d\tau}{(|\xi|^j(\tau^2 + t^2) + (\sqrt{t^2 + \tau^2})^{j+2} + A)^{\alpha+1}} \leq cA^{-\alpha}, \tag{v}$$

uniformly for $|x|$, $|\xi| \leq 1$.

Proof. Parts (i), (ii), and (iii) of this lemma are proved in [4]. Since this lemma is crucial in the estimates that follow we shall include a proof for (i)–(iii).

To prove (i), we divide the domain of integration into three parts:

(a) If $|t| \leq \frac{1}{2}|x|$,

$$\int_I \frac{|x - t|^j \, dt \, d\tau}{(|x - t|^j(t^2 + \tau^2) + A)^{\alpha+1}} \leq c \int_I \frac{|x|^j \, dt \, d\tau}{(|x|^j(t^2 + \tau^2) + A)^{\alpha+1}}$$
$$\leq c \int_0^1 \frac{|x|^j r \, dr}{(|x|^j r^2 + A)^{\alpha+1}} \leq cA^{-\alpha}.$$

(b) If $|x| \le \frac{1}{2}|t|$,

$$\int_I \frac{|x-t|^j\,dt\,d\tau}{(|x-t|^j(t^2+\tau^2)+A)^{\alpha+1}} \le \int_I \frac{|t|^j\,dt\,d\tau}{(|t|^j(t^2+\tau^2)+A)^{\alpha+1}}$$
$$= c\int_0^{2\pi}\int_0^1 \frac{|r\cos\theta|^j r\,dr\,d\theta}{(r^{j+2}|\cos\theta|^j+A)^{\alpha+1}} \le cA^{-\alpha}.$$

(c) If $\frac{1}{2}|x| \le |t| \le 2|x|$, we have $3|t| \ge |t|+|x| \ge |x-t|$. Making a change of variable $x-t=\tilde{t}$, we have

$$\int_I \frac{|x-t|^j\,dt\,d\tau}{(|x-t|^j(t^2+\tau^2)+A)^{\alpha+1}} \le \int_{|(\tilde{t},\tau)|\le 2} \frac{|\tilde{t}|^j\,d\tilde{t}\,d\tau}{(|\tilde{t}|^j(t^2+\tau^2)+A)^{\alpha+1}}$$
$$\le c\int_0^{2\pi}\int_0^2 \frac{r^{j+1}|\cos\theta|^j\,dr\,d\theta}{(r^{j+2}|\cos\theta|^j+A)^{\alpha+1}} \le cA^{-\alpha}.$$

Part (i) is proved. (ii) can be proved in exactly the same way. To prove (iii), we divide the domain of integration into two parts:

(a) If $|\tau| \le |x-t|$

$$\int_I \frac{|x-t|^{j-1}|\tau|\,dt\,d\tau}{(|x-t|^j(t^2+\tau^2)+(\sqrt{t^2+\tau^2})^{j+2}+A)^{\alpha+1}}$$
$$\le \int_I \frac{|x-t|^j\,dt\,d\tau}{(|x-t|^j(t^2+\tau^2)+A)^{\alpha+1}} \le cA^{-\alpha} \quad \text{(by (i))}.$$

(b) If $|\tau| \ge |x-t|$

$$\int_I \frac{|x-t|^{j-1}|\tau|\,dt\,d\tau}{(|x-t|^{j-1}(t^2+\tau^2)+(\sqrt{t^2+\tau^2})^{j+2}+A)^{\alpha+1}}$$
$$\le \int_I \frac{|t|^j\,dt\,d\tau}{((\sqrt{t^2+\tau^2})^{j+2}+A)^{\alpha+1}}$$
$$\le c\int_0^{2\pi}\int_0^1 \frac{r^{j+1}|\sin\theta|^j\,dr\,d\theta}{(r^{j+2}+A)^{\alpha+1}} \le cA^{-\alpha}.$$

To prove (iv), we decompose I into two parts:

(a) If $|t| \le |\xi|$, then

$$\int_I \frac{|\xi|^{j-1}|t|\,dt\,d\tau}{(|\xi|^j(t^2+\tau^2)+(\sqrt{t^2+\tau^2})^{j+2}+A)^{\alpha+1}}$$
$$\le \int_I \frac{|\xi|^j\,dt\,d\tau}{(|\xi|^j(t^2+\tau^2)+A)^{\alpha+1}} \le c\int_0^1 \frac{|\xi|^j r\,dr}{(|\xi|^j r^2+A)^{\alpha+1}} \le cA^{-\alpha}.$$

(b) if $|\xi| \le |t|$, then

$$\int_I \frac{|\xi|^{j-1}|t|\,dt\,d\tau}{(|\xi|^j(t^2+\tau^2)+(\sqrt{t^2+\tau^2})^{j+2}+A)^{\alpha+1}} \le c\int_0^1 \frac{r^{j+1}\,dr}{(r^{j+2}+A)^{\alpha+1}} \le cA^{-\alpha}.$$

Part (v) can be proved exactly the same way and the lemma is proved.

To finish the proof of Lemma 3.3, using (3.7), (3.8), and (3.12), it suffices to show that $(k_1, \dots, k_q) = (1, 2, \dots, q)$ and the other terms in (3.12) can be estimated similarly.

Let

$$I_q^a = \int_{|t|\le 1} \frac{\prod_{j=1}^q [|x_j - t_j|^{2n_j-2} + |y_j - \tau_j|^{2m_j-2} + (|x_j - t_j|^{2n_j-3} + |y_j - \tau_j|^{2m_j-3})|t_j'|]^a \, dt}{\{|t_{2n-1}| + \sum_{k=1}^{n-1}[((x_k - t_k)^{2n_k-2} + (y_k - \tau_k)^{2m_k-2}]|t_k'|^2 + |t_k'|^{2m_k})\}^{q+1}|t'|^{2s+1}}.$$

We shall prove that there exists a constant M such that

$$I_q^a < M \tag{3.14}$$

uniformly for $|z| \le 1$, for all $a < a_q$.

To prove (3.14), if $q = 0$, we have

$$\begin{aligned} I_0^a &\le \int_{|t|\le 1} \frac{1}{(|t_{2n-1}| + |t'|^{2m})^a |t'|^{(2n-3)a}}\, dt \\ &\le \int_{|t'|\le 1} \frac{1}{|t'|^{2m(a-1)}|t'|^{(2n-3)a}}\, dt' \\ &\le c\int_0^1 \frac{r^{2n-3}\, dr}{r^{2m(a-1)+(2n-3)a}} < \infty \end{aligned}$$

since $a < (2m+2n-2)/(2m+2n-3) = a_0$ and $2m(a-1)+(2n-3)(a-1) < 1$.

When $q \ge 1$, we let

$$t = (t_1, \tau_1, \dots, t_q, \tau_q, t'', t_{2n-1}),$$

where $t'' = (t_{q+1}, \tau_{q+1}, \dots, t_{n-1}, \tau_{n-1})$ and $dt'' = dt_{q+1}d\tau_{q+1} \cdots dt_{n-1}\, d\tau_{n-1}$.

Let

$$A_j = |x_j - t_j|^{2n_j-2} + |y_j - \tau_j|^{2m_j-2} + (|x_j - t_j|^{2n_j-3} + |y_j - \tau_j|^{2m_j-3})|t_j'|$$

and

$$B_j = [(x_j - t_j)^{2n_j-2} + (y_j - \tau_j)^{2m_j-2}]|t_j'|^2 + |t'|^{2m_j}.$$

Then (3.14) becomes

$$I_q^a \le \int_{|t|\le 1} \frac{(\prod_{j=1}^q A_j)}{(|t_{2n-1}| + \sum_{k=1}^{n-1} B_k)^{(q+1)a}|t'|^{(2s+1)a}}\, dt$$

since $a \ge 1$. We integrate the right-hand side with respect to $t_1, \tau_1, t_2, \tau_2, \dots$ and t_q, τ_q respectively using Lemma 3.13 q-times. Then

$$\begin{aligned} I_q^a &\le c\int_{|(t'', t_{2n-1})|\le 1} \frac{1}{(|t_{2n-1}| + |t''|^{2m})^{(q+1)a-q}|t''|^{(2s+1)a}}\, dt_{2n-1}\, dt'' \\ &\le c\int_{|t''|\le 1} \frac{1}{|t''|^{2m(q+1)(a-1)}|t|^{(2s+1)a}}\, dt'' \\ &= c\int_0^1 \frac{r^{2s+1}\, dr}{r^{2m(q+1)(a-1)+(2s+1)a}} < \infty \end{aligned}$$

since $a < a_q = (2(q+1)(m-1)+2n)/(2(q+1)(m-1)+2n-1)$ and claim (3.5) is proved. (3.4′) can be similarly proved. This proves Lemma 3.3.

(3.15) **Lemma.** *There exists a constant* c, *depending only on* m *and* p, *such that*

$$\sup_{z\in\partial D_\delta} |\operatorname{grad} R^+_{\partial D} f(z)| \le c\delta^{-1+1/2m-\beta_0/2mp}\|f\|_p \tag{3.16}$$

and

$$\sup_{z\in\partial D^\delta} |\operatorname{grad} R^-_{\partial D} f(z)| \le c\delta^{-1+1/2m-\beta_0/2mp}\|f\|_p, \tag{3.16'}$$

where $\beta_0 = 2(n-1)(m-1)+2n$, $\beta_0 \le p \le \infty$, *and* $\|f\|_p = \|f\|_{L^p_{0,1}(\partial D)}$.

Proof. We shall first prove (3.16) when $p=\infty$ and $p=\beta_0$. Let K_q be defined as in (3.2). Then

$$|\operatorname{grad} K_q(\zeta, z)| \le \sum{}' \frac{\prod_{j=1}^q |\overline{L}_{k_j} p_{k_j}|}{|\Phi|^{q+1}|\zeta - z|^{2s+2}} + \sum{}' \frac{\prod_{j=1}^a |\overline{L}_{k_j} p_{k_j}|}{|\Phi|^{q+2}|\zeta - z|^{2s+1}}, \tag{3.17}$$

where $\sum'$ is summed over all increasing q-tuples $(k_1, k_2, \dots, k_q)$, $1 \le k_1 \le k_2 \le \cdots \le k_q = n$, and $q+s = n-2$.

Thus (3.16) will be proved for $p=\infty$ if we can show

$$I_1 = \int_{\partial D} \frac{\prod_{j=1}^q |\overline{L}_{k_j} p_{k_j}|}{|\Phi|^{q+1}|\zeta - z|^{2s+2}}\, d\sigma_\zeta \le c(\log|\rho(z)|)^2, \tag{3.18}$$

$$I_2 = \int_{\partial D} \frac{\prod_{j=1}^q |\overline{L}_{k_j} p_{k_j}|}{|\Phi|^{q+2}|\zeta - z|^{2s+2}}\, d\sigma_\zeta \le c|\rho(z)|^{-1+1/2m}; \tag{3.19}$$

and (3.17) will be proved for $p=\beta_0$ if one can show

$$I_3 = \int_{\partial D} \frac{\prod_{j=1}^q |\overline{L}_{k_j} p_{k_j}||f(\zeta)|}{|\Phi|^{q+1}|\zeta - z|^{2s+2}}\, d\sigma_\zeta \le c|\rho(z)|^{-1}\|f\|_{\beta_0}, \tag{3.20}$$

$$I_4 = \int_{\partial D} \frac{\prod_{j=1}^q |\overline{L}_{k_j} p_{k_j}||f(\zeta)|}{|\Phi|^{q+1}|\zeta - z|^{2s+2}}\, d\sigma_\zeta \le c|\rho(z)|^{-1}\|f\|_{\beta_0}. \tag{3.21}$$

We shall estimate (3.18)–(3.21) assuming $k_1 = 1, \dots, k_q = q$. All the other terms can be similarly estimated.

Using the local coordinates t introduced in (3.7) and the estimates (3.11), (3.11′), we have (using the same notation as before)

$$I_1 \le c\int_{|t|\le 1} \frac{\prod_{j=1}^q A_j}{(|t_{2n-1}| + y + \sum_{k=1}^{n-1} B_k)^{q+1}(|t'| + y)^{2s+2}}\, dt. \tag{3.18'}$$

Integrating with respect to dt, $d\tau_1$, $dt_2\,d\tau_2, \dots, dt_q\,d\tau_q$ respectively and applying Lemma 3.13 q times, we have

$$\begin{aligned} I_1 &\le c\int_{|(t'',t_{2n-1})|\le 1} \frac{dt''\,dt_{2n-1}}{(|t_{2n-1}|+y+|t''|^{2m})(|t'|+y)^{2s+2}} \\ &\le c\int_{|t''|\le 1} \frac{\log y}{(y+|t''|)^{2s+2}}\,dt'' \le c\int_0^1 \frac{(\log y)}{r+y}\,dr \le c(\log y)^2. \end{aligned}$$

To prove (3.19), we repeat the same arguments. Then we have

$$\begin{aligned} I_2 &\le c\int_{|(t'',t_{2n-1})|\le 1} \frac{1}{(|t_{2n-1}|+y+|t''|^{2m})^2(|t''|+y)^{2s+1}}\,dt_{2n-1}\,dt'' \\ &\le c\int_{|t''|\le 1} \frac{1}{(y+|t''|^{2m})(|t''|+y)^{2s+1}}\,dt'' + C_1 \\ &\le c\int_0^1 \frac{dr}{y+r^{2m}} \le cy^{-1+1/2m}. \end{aligned}$$

Thus (3.19) is proved. To prove (3.11), by Hölder's inequality we have

$$\begin{aligned} I_3 &\le \left[\int_{\partial D}\left(\frac{\prod_{j=1}^q \overline{L}_j p_j|}{|\Phi|^{q+1}|\zeta-z|^{(2s+2)}}\right)^{a_q} d\sigma_\zeta\right]^{1/a_q} \|f\|_{\beta_0} \\ &\le \left[\int_{\partial D} \frac{\prod_{j=1}^q |\overline{L}_j p_j|}{|\Phi|^{(a+1)a_q}|\zeta-z|^{(2s+2)a_q}}\,d\sigma_\zeta\right]^{1/a_q} \|f\|_{\beta_0} \end{aligned}$$

since $1/a_q + 1/\beta_0 = 1$. Integrating $dt_1\,d\tau_1, \dots, dt_q\,d\tau_q$ and applying Lemma 3.13 q times, we have

$$\begin{aligned} I_3 &\le c\left[\int_{|(t'',t_{2n-1})|\le 1} \frac{dt_{2n-1}dt''}{(|t_{2n-1}|+y+|t''|^{2m})^{(q+1)a_q-q}(|t''|+y)^{(2s+2)a_q}}\right]^{1/a_q} \|f\|_{\beta_0} \\ &\le c\left[\int_{|t''|\le 1} \frac{dt''}{(y+|t''|^{2m})^{(q+1)(a_q-1)}(|t''|+y)^{(2s+2)a_q}}\right]^{1/a_q} \|f\|_{\beta_0} \\ &\le c\left[\int_0^1 \frac{dr}{(y+r^{2m})^{(q+1)(a_q-1)}(r+y)^{(2s+1)(a_q-1)}(r+y)^{a_q}}\right]^{1/a_q} \|f\|_{\beta_0} \\ &\le c\left[y^{-a_q+\varepsilon}\int_0^1 \frac{dr}{(y+r^{2m})^{(q+1)(a_q-1)}r^{(2s+1)(a_q-1)+\varepsilon}}\right]^{1/a_q} \|f\|_{\beta_0} \end{aligned}$$

(3.20′)

$$v = r/y^{1/2m} = c\left[y^{-a_q+\varepsilon-\varepsilon/2m}\int_0^{1/y^{1/2m}} \frac{dv}{(1+v^{2m})^{(q+1)(a_q-1)}v^{(2s+1)(a_q-1)}+\varepsilon}\right]^{1/a_q},$$

where $\varepsilon > 0$ and we have used

$$(q+1)(a_q-1) + \frac{1}{2m}[(2s+1)(a_q-1)+\varepsilon] - \frac{1}{2m} = \frac{\varepsilon}{2m}.$$

Since $2m(q+1)(a_q-1)+(2s+1)(a_q-1)+\varepsilon>1$ and $(2s+1)(a_q-1)+\varepsilon<1$ if ε is sufficiently small, it follows that the integrand in (3.20′) is integrable,

$$I_3 \le c(y^{-a_q+\varepsilon-\varepsilon/2m})^{1/a_q}\|f\|_{\beta_0} \le cy^{-1}\|f\|_{\beta_0},$$

and (3.20) is proved.

Finally, we proceed as before to prove (3.21). We have

$$\begin{aligned} I_4 &\le c\left[\int_{|(t'',t_{2n-1})|\le 1} \frac{dt_{2n-1}\,dt''}{(|t_{2n-1}|+y+|t''|^{2m})^{(q+2)a_q-q}(y+|t''|)^{(2s+1)a_q}}\right]^{1/a_q}\|f\|_{\beta_0} \\ &\le c\left[\int_{|t''|\le 1} \frac{1}{(y+|t''|^{2m})^{(q+2)a_q-(q+1)}(y+|t''|)^{(2s+1)a_q}}\,dt''\right]^{1/a_q}\|f\|_{\beta_0} \\ &\le c\left[\int_0^1 \frac{1}{(y+r^{2m})^{(q+2)a_q-(q+1)}r^{(2s+1)(a_q-1)}}\right]^{1/a_q}\|f\|_{\beta_0} \\ &= c\left[y^{-a_q}\int_0^{1/y^{1/2m}} \frac{dv}{(1+v^{2m})^{(q+2)a_q-(q+1)}v^{(2s+1)(a_q-1)}}\right]^{1/a_q}\|f\|_{\beta_0} \\ &\le cy^{-1}\|f\|_{\beta_0} \end{aligned}$$

since

$$(q+2)a_q-(q+1)+\frac{1}{2m}(2s+1)(a_q-1)-\frac{1}{2m}=a_q,$$
$$2m[(q+2)a_q-(q+1)]+(2s+1)(a_q-1)>1,$$

and $(2s+1)(a_q-1)<1$. (3.32) is proved. Thus (3.17) is proved for $p=\beta_0$ and ∞ since y is equivalent to δ.

To prove (3.16) for $\beta_0<p<\infty$, we use an interpolation argument. By the linearity property in (3.16), we may assume $\|f\|_P=1$. Let $f=f_0+f_1$, where $f_1(z)=f(z)$ if $|f|\le h$ and $f_1=0$ otherwise (h is a positive constant to be determined later). Then $f_0\in L^{\beta_0}_{0,1}(\partial D)$ and $f_1\in L^{\infty}_{0,1}(\partial D)$. Using (3.17) for $p=\beta_0$ and $p=\infty$ we have

$$\begin{aligned} \text{(3.22)}\quad \sup_{z\in\partial D_\delta}|\operatorname{grad} R^+_{\partial D}f(z)| &\le \sup_{z\in\partial D_\delta}[|\operatorname{grad} R^+_{\partial D}f_0(z)|+|\operatorname{grad} R^+_{\partial D}f_1(z)|] \\ &\le c(\delta^{-1}\|f_0\|_{\beta_0}+\delta^{-1+1/2m}\|f_1\|_\infty). \end{aligned}$$

Since $\|f_1\|_\infty\le h$ and $\|f_0\|_\infty\ge h$,

$$\begin{aligned} \|f_0\|_{\beta_0} &= \left[\int_{|f|>h}|f|^{\beta_0}\,d\sigma_\zeta\right]^{1/\beta_0} \le \left[h^{\beta_0-h}\int_{|f|>h}|f|^P\,d\sigma_\zeta\right]^{1/\beta_0} \\ &= h^{1-P/\beta_0}\|f\|_P^{P/\beta_0}=h^{1-P/\beta_0}. \end{aligned}$$

Thus if we choose $h=\delta^{-\beta_0/2mp}$, then (3.22) becomes

$$\sup_{z\in\partial D_\delta}|\operatorname{grad} R^+_{\partial D}f(z)|\le c\delta^{-1+1/2m-\beta_0/2mp}\|f\|_p$$

and (3.17) is proved. (3.16′) can be proved similarly and Lemma 3.15 is proved.

(3.23) **Lemma.** *For every* $1 \le p \le \infty$, *there exists a constant* c *such that*

$$\|\operatorname{grad} R^+_{\partial D} f(z)\|_{L^p(\partial D_\delta)} \le c\delta^{-1+1/2m}\|f\|_{L^p(\partial D)} \tag{3.24}$$

and

$$\|\operatorname{grad} R^-_{\partial D} f(z)\|_{L^p(\partial D^\delta)} \le c\delta^{-1+1/2m}\|f\|_{L^p(\partial D)}. \tag{3.24′}$$

Proof. When $p = \infty$, this is proved in Lemma 3.16. To prove (3.24) when $p = 1$, using Fubini's theorem and letting σ_z denote the surface area on ∂D_δ, we have

$$\begin{aligned}
\int_{\partial D_\delta} & |\operatorname{grad}_\zeta R^+_{\partial D} f(z)|\, d\sigma_z \\
&\le \int_{\partial D_\delta}\int_{\partial D} |\operatorname{grad}_\zeta K(\zeta, z)|\,|f(\zeta)|\, d\sigma_\zeta\, d\sigma_z \\
&= \int_{\partial D}\left[\int_{\partial D_\delta} |\operatorname{grad} K(\zeta, z)|\, d\sigma_z\right] |f(\zeta)|\, d\sigma_\zeta \\
&= \sum_{q=0}^{n-2}\int_{\partial D_\delta} |\operatorname{grad} K_q(\zeta, z)\, d\sigma_z |f(\zeta)|\, d\sigma_\zeta.
\end{aligned}$$

To prove (3.24), following (3.17) it suffices to show

$$\int_{\partial D_\delta} \frac{\sum' \prod_{j=1}^q |\bar{L}_{k_j} p_{k_j}|}{|\Phi|^{q+1}|\zeta - z|^{2s+2}}\, d\sigma_z \le c(\log\delta)^2 \tag{3.25}$$

and

$$\int_{\partial D_\delta} \frac{\sum' \prod_{j=1}^a |\bar{L}_{k_j} p_{k_j}|}{|\Phi|^{q+2}|\zeta - z|^{2s+2}}\, d\sigma_z \le c\delta^{-1+1/2m}. \tag{3.26}$$

Since in (3.25) and (3.26) the kernels have singularities only at $z = \zeta$, we shall localize in a small neighborhood $|\zeta - z| < \varepsilon$ and use a change of variables similar to (3.7) (viewing ζ as a parameter and the z_k's as variables). Then (3.25) and (3.26) can be proved exactly in the same way as (3.18) and (3.19) except that we invoke (iv) and (v) of Lemma 3.6. We omit the details here and (3.24) is proved for $p = 1$. For $1 < p < \infty$, it is an easy consequence of the Marcinkiewicz interpolation theorem (see e.g. Stein [26]) and the lemma is proved.

Proof of Theorems 1 *and* 2. Now we can easily prove Theorems 1 and 2. Using Lemma 3.3, the kernels are in $L^a(\partial D)$ for $a < \gamma$, thus by the application of a lemma in Folland-Stein (see Lemma 15.3 in [7]), (i), (ii), and (iii) in Theorem 1 follow easily.

To prove (iv) of Theorem 1 and Theorem 2, we use the following version of the Hardy-Littlewood lemma: For $0 < \alpha < 1$, if a function $v \in C^1(D)$

satisfies $\|\operatorname{grad} v(z)\|_{L^p(\partial D_\delta)} \le C\delta^{-1+\alpha}$ uniformly in δ, then $v \in \Lambda_\alpha^p(\partial D)$. For a proof of this lemma, see Krantz [15] or Ramanov [17]. Theorems 1 and 2 are proved using Lemma 3.23. It remains to prove Lemma 2.10. From Theorem 2, Tf and Sf are bounded operators from $L_{0,1}^p(\partial D)$ to $\Lambda_{1/2m}^p(\partial D)$. Since $\Lambda_{1/2m}^p(\partial D)$ is compact in $L^p(D)$, the lemma is proved.

4. The Poincaré-Lelong equation and the proof of Theorem 3

In this section we shall prove Theorem 3 using the estimates obtained in §3. By the Poincaré-Lelong equation, for any holomorphic function h in D, we have

$$\frac{1}{2\pi i}\partial\overline{\partial} h = M_h, \tag{4.1}$$

where M_h is the zero divisor of h. One can reduce this problem to solve

$$\partial\overline{\partial} u = \alpha, \tag{4.2}$$

where α is a positive d-closed current with finite measure coefficients. We are looking for solutions u in (4.2) with L^1 boundary values. To do this we proceed as in Henkin [10] and Skoda [24] (see also Rudin [19] and Shaw [23]).

(4.3) **Lemma.** *Let α be a positive d-closed $(1, 1)$ form on $\overline{D}$, i.e.,*

$$\alpha = \sum_{i,j=1}^{n} \alpha_{i\bar{j}} dz_i \wedge d\overline{z}_j, \qquad \alpha_{i\bar{j}} \in C^\infty(\overline{D}),$$

and $\alpha_{i\bar{j}}$ is positive definite. Then there exists a $(0, 1)$ form f on $\overline{D}$ such that

(i) $\overline{\partial} f = 0$,
(ii) $\partial f - \overline{\partial f} = \alpha$,
(iii) *there exists a constant c depending only on D such that*

$$\|f\|_{L_{0,1}^1(\partial D)} \le c\|\alpha\|_{L_{1,1}^1(D))}. \tag{4.4}$$

Proof. Since D is star-shaped, it follows from the classical Poincaré lemma that if we define

$$f(z) = \sum_{k=1}^{n} f_k d\overline{z}_k, \quad \text{where } f_k = \sum_{j=1}^{n} z_j \int_0^1 t\alpha_{j\bar{k}}(tz)\, dt, \tag{4.5}$$

then f is $\overline{\partial}$-closed and $\partial f - \overline{\partial f} = \alpha$. (For details, see e.g. Rudin [19, Theorem 17.2.7].) Since α is positive, we have $2|\alpha_{i\bar{j}}| \le \alpha_{i\bar{i}} + \alpha_{j\bar{j}}$ for all

$i, j = 1, 2, \ldots, n$. Thus from (4.5)

$$\begin{aligned}\int_{\partial D} |f(z)|\, d\sigma_z &\le \sum_{j,k} \int_{\partial D} |z_j| \int_0^1 |t\alpha_{j\bar{k}}(tz)|\, dt\, d\sigma_z \\ &\le c \sum_{j,k} \int_{\partial D} \int_0^1 |\alpha_{j\bar{k}}(tz)|\, dt\, d\sigma_z \\ &\le c \sum_{j} \int_{\partial D} \int_0^1 \alpha_{j\bar{j}}(tz)\, dt\, d\sigma_z \\ &\le c \sum_j \|\alpha_{j\bar{j}}\|_{L^1(D)} \le c\|\alpha\|_{L^1_{1,1}(D)}\end{aligned}$$

and the lemma is proved.

(4.6) **Lemma.** *Let f be a smooth $(0, 1)$ form on D such that $\bar{\partial} f = 0$ on D. Then there exists a smooth function u such that*

(4.7)
(i) $\bar{\partial} u = f$ *on* D,
(ii) $\|u\|_{L^1(\partial D)} \le c\|f\|_{L^1(\partial D)}$, *where c is independent of f.*

Proof. Let $u = B_D f + R^+_{\partial D} f$, where $R^+_{\partial D} f(z)$ is defined as in (2.6) and $B_D f$ is defined by

$$B_D f = \frac{1}{(2\pi i)^n} \int_D f(\zeta) \wedge B(\zeta, z) \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n,$$

where $B(\zeta, z) = \det(\bar{\zeta} - \bar{z}, d\bar{\zeta}, \ldots, d\bar{\zeta})_{n\times n}/|\zeta - z|^{2n}$, i.e., u is the Grauert-Lieb Henkin solution of $\bar{\partial}$ on D. Then $\bar{\partial} u = f$ and it follows from [4] that $u \in \Lambda_{1/2m}(\partial D)$. One can also modify the argument of Siu to show uniform bounds for the derivatives of u and obtain $u \in C^\infty(\bar{D})$. We shall omit the details since our main application is (4.7).

To prove (4.7), we note that when $z \in \partial D$, it follows from Romanov [17] (see also Shaw [23]) that $u(z) = R^+_{\partial D} f(z) - R^-_{\partial D} f(z)$, i.e., $\bar{\partial}_b u = f$ and u is exactly the solution constructed in §2. Using estimate (i) in Theorem 1, we have proved $\|u\|_{L^{\gamma-\varepsilon}}(\partial D) \le c\|f\|_{L^1_{0,1}}(\partial D)$ and the lemma is proved.

(4.8) **Lemma.** *Let α be a positive d-closed $(1, 1)$ form on D with smooth coefficients in $\bar{D}$. Then there exists a function u such that $u \in C^\infty(\bar{D})$ and*

(i) $\partial\bar{\partial} u = \alpha$,
(ii) $\|u\|_{L^1(\partial D)} \le c\|\alpha\|_{L^1_{1,1}(D)}$.

Proof. This is an easy consequence of Lemmas 4.3 and 4.6.

Using Lemma 4.8 and the Poincaré-Lelong equation, one can easily prove Theorem 3 by an approximation argument. We shall omit the details and refer the readers to [19] and [23] for details.

5. A SPECIAL DOMAIN

In this section we consider an example which shows that the Hölder exponent $1/2m - \beta_0/2mp$ obtained in Theorem 1 is the best possible. This example is a modification of an example of Stein (see Kerzman [11] and Krantz [15]). Let D be the complex ellipsoid in $\mathbf{C}^n$ defined by $D = \{z \in \mathbb{C}^n \mid |z_1|^2 + \cdots + |z_{n-1}|^2 + |z_n|^{2m} < 1\}$. We shall assume $\beta_0 < p \le \infty$. First we consider $\beta_0 < p < \infty$.

5.1. Let

$$f(z) = \tau \frac{d\overline{z}_n}{(z_1 - 1)^b}, \quad \text{where } b = \frac{2(n-1)(m-1) + 2n}{2mp} = \frac{\beta_0}{2mp}$$

and define $(z_1 - 1)^b$ to be the branch with $0 < \operatorname{Arg}(z_1 - 1)^b < 2\pi$ for all $z_1 \notin [1, \infty)$.

(5.2) **Lemma.** *$f(z)$ is $\overline{\partial}_b$-closed and $f \in L^s_{0,1}(\partial D)$ for all $s < p$.*

Proof. Since $f = \tau(\overline{\partial}\overline{z}_n/(z_1 - 1)^b)$, f is $\overline{\partial}_b$-closed. Since

$$(5.3) \qquad |z_j|^2 \le 1 - |z_1|^2 \le 2(1 - |z_1|), \qquad j = 1, \dots, n-1,$$

and

$$(5.4) \qquad |z_n|^{2m} \le 1 - |z_1|^2 \le 2(1 - |z_1|)$$

on ∂D, if we parametrize ∂D near a neighborhood U of $(1, 0, \dots, 0)$ by $(y_1, y_2, y_3, \dots, z_n)$, then from (5.3) and (5.4)

$$\begin{aligned} \int_U |f|^s \, d\sigma_z &\le \int_U \frac{1}{|1 - z_1|^{sb}} \, d\sigma_z \\ &= \int_U \frac{1}{|1 - z_1|^{s/p}(1 - |z_1|)^{(n-2)s/p}} \, d\sigma_z \\ &\le \int_U \frac{1}{|1 - z_1|^{s/p}|z_2|^{2s/p} \cdots |z_{n-1}|^{2s/p}|z_n|^{2s/p}} \, d\sigma_z \\ &\le c\int_0^1 \frac{dy_1}{y_1^{s/p}} \left(\int_0^1 \frac{dx_2\, dy_2}{|z_2|^{2s/p}}\right) \cdots \left(\int_0^1 \frac{dx_n\, dy_n}{|z_n|^{2s/p}}\right) < \infty \quad \text{since } \frac{s}{p} < 1. \end{aligned}$$

Next we shall assume $p > \beta_0$.

(5.5) **Theorem.** *If u is a solution satisfying $\overline{\partial}_b u = f$ in the weak sense, then $u \notin \Lambda_\alpha(\partial D)$ for any $\alpha > 1/2m - \beta_0/2mp = \alpha_0$. However, there exists a solution $v \in \Lambda_{\alpha_0}(\partial D)$.*

Proof. Let $v(z) = \overline{z}_n/(z_1 - 1)^b$. Then $v \in \Lambda_{\alpha_0}(\partial D)$. To see this, we note that $v \in C^\infty(\overline{D} \backslash x_0)$, where $x_0 = (1, 0, \dots, 0)$. We choose a small neighborhood U around x_0 such that $|\rho(z)| \le c|1 - x_1|$. Then

$$\begin{aligned} (5.6) \qquad \sup_{z \in D \cap U} \frac{1}{|z_1 - 1|^b} &\le c\frac{1}{|1 - x_1|^b} \le c|\rho(z)|^{-b} \\ &\le c|\rho(z)|^{-1+\alpha_0} \quad \text{since } 1 - b \ge \alpha_0. \end{aligned}$$

Using (5.4) we have

$$(5.7)\quad \sup_{z\in D\cap U} \frac{|\bar{z}_n|}{|z_1-1|^{b+1}} \le c\frac{(1-|z_1|)^{1/2m}}{|1-z_1|^{b+1}} \le c\frac{(1-x_1)^{1/2m}}{(1-x_1)^{b+1}} \le c|\rho(z)|^{-1+1/2m-b}.$$

It follows from (5.6), (5.7), and the Hardy-Littlewood lemma that $v \in \Lambda_{\alpha_0}(\partial D)$.

For any solution $u \in \Lambda_\alpha(\partial D)$, if $\alpha > \alpha_0$, then $u - v \in \Lambda_{\alpha_0}(\partial D)$ and $\bar{\partial}_b(u-v) = 0$. It follows from a theorem of Bochner's that $u - v$ is the boundary value of a holomorphic function h in D. Let $u = v + h$; then $u \in \Lambda_\alpha(D)$. Let $\Gamma_1 = \{z \in \partial D | z_1 = 1-\varepsilon,\ z_2 = 0, \dots, z_{n-1} = 0,\ |z_n|^{2m} = \varepsilon\}$, $\Gamma_2 = \{z \in \partial D | z_1 = 1-2\varepsilon,\ z_2 = 0, \dots, z_{n-1} = 0,\ |z_n|^{2m} = \varepsilon\}$, and $\delta = \varepsilon^{1/2m}$. Then

$$(5.8)\quad \left|\int_{\Gamma_1} \tilde{u}\, dz_n - \int_{\Gamma_2} \tilde{u}\, dz_n\right|$$
$$= \left|\int_{\Gamma_1} \tilde{u}(1-\varepsilon, 0, \dots, 0, z_n)\, dz_n - \int_{\Gamma_2} \tilde{u}(1-2\varepsilon, 0, \dots, 0, z_n)\, dz_n\right|$$
$$= \delta\left|\int_0^{2\pi} [u(1-\varepsilon, 0, \dots, 0, \delta e^{i\theta}) - u(1-2\varepsilon, 0, \dots, 0, \delta e^{i\theta})]\, d\theta\right| \le c\varepsilon^{\alpha+1/2m}.$$

On the other hand, let Ω_1 and Ω_2 be the plane region in D bounded by Γ_1 and Γ_2 respectively. Then by Stokes' theorem we have

$$(5.9)\quad \left|\int_{\Gamma_1} \tilde{u}\, dz_n - \int_{\Gamma_2} \tilde{u}\, dz_n\right| = \left|\int_{\Omega_1} \bar{\partial}_{z_n}\tilde{u}\, dz_n - \int_{\Omega_2} \bar{\partial}_{z_n}\tilde{u}\, dz_n\right|$$
$$= \left|\int_{\Omega_1} \frac{dz_n\, d\bar{z}_n}{(-\varepsilon)^b} - \int_{\Omega_2} \frac{dz_n\, d\bar{z}_n}{(-2\varepsilon)^b}\right| = 2\pi\left|\frac{\delta^2}{(-\varepsilon)^b} - \frac{\delta^2}{(-2\varepsilon)^b}\right| = 2\pi\frac{\delta^2}{\varepsilon^b}\left|1 - \frac{1}{2^b}\right|.$$

Since $0 < b < 1/2m$, we have $|1 - 1/2^b| \ge C_0 > 0$. Combining this with (5.8) and (5.9), we have $|1 - 1/2^b|\delta^2/\varepsilon^b \le c\varepsilon^{\alpha+1/2m}$, which is impossible if $\alpha > 1/2m - b$. Thus $u \notin \Lambda_\alpha(\partial D)$ for any $\alpha > 1/2m - b$. When $p = \infty$, let $f(z) = d\bar{z}_n/\log(z_1 - 1)$. Then $f \in C_{0,1}(\partial D)$ and f is $\bar{\partial}_b$-closed. The same arguments as before will show that there exists no solution $u \in \Lambda_\alpha(\partial D)$ for any $\alpha > 1/2m$ and the theorem is proved.

Note added in proof. Recently Fefferman, Kohn, and Machedon have proved Hölder estimates on CR manifolds with a diagonalizable Levi form (see [27]).

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