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SYMPLECTIC DOUBLE GROUPOIDS OVER POISSON $(ax + b)$-GROUPS

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Dedicated to Professor Shigeo Sasaki

ABSTRACT. First, we classify all the multiplicative Poisson structures on the $(ax + b)$-group and determine their dual Poisson Lie groups. Next, we show the existence of symplectic groupoid over the Poisson $(ax + b)$-group. Finally, by the Hamilton-Jacobi method we construct nontrivial symplectic double groupoids and conclude that for each pair of nondegenerate multiplicative Poisson structures of the $(ax + b)$-group there exists a symplectic double groupoid.

1. INTRODUCTION

Poisson Lie groups which were defined by V. G. Drinfel′d [2] are used in [7] to treat the hamiltonian structure of the Zakharov-Shabat dressing transformations of integrable systems theory. In [5], Lu and Weinstein investigate geometric properties of Poisson Lie groups and dressing transformations systematically, especially in the case of semisimple Lie groups.

Poisson structures can be understood in the framework of symplectic geometry at least locally by the notion of symplectic groupoids [1, 4, 6, 9]. We hope to understand Poisson Lie groups in this context. In [6], we stated the hope that we might construct symplectic double groupoid structures over Poisson Lie groups by the Hamilton-Jacobi method of characteristics.

In this paper, we shall try to construct symplectic double groupoid structures over the 2-dimensional Poisson $(ax + b)$-group, which is neither abelian nor semisimple. In §2, we classify the multiplicative Poisson structures on the $(ax + b)$-group and study some differences among them.

In §3, we shall show that each Poisson $(ax + b)$-group has a symplectic groupoid.

In the final section, §4, for each pair of multiplicative Poisson structures of $(ax + b)$-group, which are nondegenerate in the sense of Drinfel′d [3], we shall find two symplectic groupoids which are compatible with each other, namely

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we shall construct a symplectic double groupoid for each pair of nondegenerate Poisson $(ax+b)$-groups.

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2. Multiplicative Poisson structures of $(ax+b)$-group and their duals

Definition 2.1 (Drinfel'd [2]). A Poisson structure on a Lie group G is *multiplicative* (or *grouped*) if the multiplication map from $G \times G$ to G is a Poisson map. A Lie group with a multiplicative Poisson structure is called a *Poisson Lie group.*

Let $\mathfrak{g}$ be the Lie algebra of a Lie group G and let $\{\xi_i\}$ be a basis of $\mathfrak{g}$ so that the structure constants are given by $[\xi_i, \xi_j] = c_{ij}^k \xi_k$. Denote by ∂_i the right invariant vector field on G such that $\partial_i|_e = \xi_i$. Every exterior contravariant tensor field Π of degree 2 on G can be written as

$$\Pi(a) = \tfrac{1}{2}\pi^{ij}(a)\partial_i(a) \wedge \partial_j(a),$$

where $\pi^{ij}(a) = -\pi^{ji}(a)$.

Proposition 2.2 [2, 5]. (1) Π *is a Poisson tensor if and only if*

$$\Theta_{i,j,k}\{\pi^{il}\partial_l\pi^{jk} + \pi^{il}c_{lm}^j\pi^{mk}\} = 0,$$

where $\Theta_{i,j,k}$ *means the cyclic sum in* i, j, k. (*If the* π^{ij} *are constant, this is equivalent to the Yang-Baxter equation.*)

(2) Π *is multiplicative if and only if*

$$\pi^{ij}(ab) = \pi^{ij}(a) + \mathrm{Ad}(a)_k^i \pi^{kl}(b)\,\mathrm{Ad}(a)_l^j,$$

where $\mathrm{Ad}(a)\xi_k = \mathrm{Ad}(a)_k^l \xi_l$.

(3) *Let* $d_e\Pi\colon \mathfrak{g} \to \mathfrak{g}\wedge\mathfrak{g}$ *be the intrinsic derivative of* Π *at* e. *Then this is a* 1*-cocycle of adjoint representation of* $\mathfrak{g}$ *and its dual map* $(d_e\Pi)^*\colon \mathfrak{g}^*\wedge\mathfrak{g}^* \to \mathfrak{g}^*$ *defines a Lie algebra structure on the dual space* $\mathfrak{g}^*$. *Let* G^* *be the simply connected Lie group with Lie algebra* $\mathfrak{g}^*$. *Then, there exists uniquely a multiplicative Poisson structure, say* Π' *on* G^* *which induces the Lie algebra structure of* $\mathfrak{g}$. *Thus,* $(\mathfrak{g}, \mathfrak{g}^*)$ *has a structure of Lie bialgebra. We call* (G^*, Π') *the dual Poisson Lie group of* (G, Π).

Example 2.3. (1) Any Lie group has a Poisson Lie group structure with the zero Poisson structure.

(2) If G is abelian, its multiplicative Poisson structure must satisfy

$$\pi^{ij}(a+b) = \pi^{ij}(a) + \pi^{ij}(b) \quad \text{and} \quad \Theta_{i,j,k}\pi^{il}\partial_l\pi^{jk} = 0.$$

(3) The dual Poisson Lie group of (1) is the abelian group $\mathfrak{g}^*$ which is equipped with the Lie-Poisson bracket structure.

It is natural to classify Poisson Lie group structures by the following equivalence relation.

Definition 2.4. Two Poisson Lie groups (G_1, Π_1) and (G_2, Π_2) are isomorphic if there is a group-isomorphism $\varphi: G_1 \to G_2$ which is also a Poisson map.

If a manifold is 2-dimensional, any 2-vector field is a Poisson tensor and so finding multiplicative Poisson tensors is equivalent to finding multiplicative 2-vector fields.

Example 2.5. (1) The multiplicative Poisson structures on the abelian group $\mathbf{R}^2$ are $\{x_1, x_2\} = a_1 x_1 + a_2 x_2$, where a_1, a_2 are any constants. We have two isomorphism classes of Poisson Lie groups $\mathbf{R}^2$ with $\{x_1, x_2\} = a_1x_1 + a_2x_2$ classified by $(a_1, a_2) \neq 0$ and $(a_1, a_2) = 0$.

(2) The only multiplicative Poisson structure on the standard 2-torus $\mathbf{T}^2$ is the zero Poisson structure.

(3) The multiplicative Poisson structures on $\mathbf{S}^1 \times \mathbf{R}$ are $bx_2\partial/\partial\theta \wedge \partial/\partial x_2$, where (θ, x_2) are local coordinates on $\mathbf{S}^1 \times \mathbf{R}$ and b is any constant. Poisson Lie group isomorphism classes are parametrized by $|b|$.

The simplest nonabelian Lie group with nontrivial Poisson structures is the following $(ax+b)$-group. We consider the group of transformations which map x to $ax+b$ $(a \neq 0)$. We call it the full $(ax+b)$-group and its identity connected component the $(ax+b)$-group. The $(ax+b)$-group as a matrix group is $\{\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) | \ a > 0\}$. Under the identification of the matrix $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$ with $((\log a)/c, b/\sqrt{a})$ for some fixed nonzero constant c, we will deal with the $(ax+b)$-group as $\mathbf{R}^2_c$ with the group structure

$$(x_1, x_2) \cdot (y_1, y_2) := (x_1 + y_1, y_2 \exp(cx_1/2) + x_2 \exp(-cy_1/2)).$$

Note that these are isomorphic for different values of c and that $\mathbf{R}^2_0$ is the abelian group $\mathbf{R}^2$.

Our main result in this section is the following:

Proposition 2.6. *The multiplicative Poisson structures on the $(ax+b)$-group $\mathbf{R}^2_c$ $(c \neq 0)$ are given by*

$$\{x_1, x_2\} = 2k_1 \sinh(cx_1/2)/c + k_2x_2,$$

where k_1, k_2 are any constants.

Denote the $(ax+b)$-group with these Poisson structures by $(\mathbf{R}^2_c, (k_1, k_2))$. The isomorphism classes of multiplicative Poisson structures are given by

$$\{(0, 0)\}, \quad \{(k_1, 0) | \ k_1 \neq 0\}, \quad \textit{and} \quad \{(k_1, k_2) | \ k_1 \in \mathbf{R}\}_{k_2 \neq 0},$$

and so can be parametrized by $\{(0, k_2) | \ k_2 \in \mathbf{R}\} \cup \{(1, 0)\}$ if $c \neq 0$.

The dual Poisson Lie groups of $(\mathbf{R}_c^2, (0, 0))$, $(\mathbf{R}_c^2, (k_1, 0))$, *and* $(\mathbf{R}_c^2, (0, k_2))$ *are* $(\mathbf{R}_0^2, (0, c))$, $(\mathbf{R}_{k_1}^2, (c, 0))$, *and* $(\mathbf{R}_{k_2}^2, (0, c))$ *respectively.*

Proof. Let ξ_1, ξ_2 be the basis of the Lie algebra of the $(ax+b)$-group $\mathbf{R}_c^2$ with $\xi_1(0) = \partial/\partial x_1|_0$ and $\xi_2(0) = \partial/\partial x_2|_0$. Then, we have $[\xi_1, \xi_2] = c\xi_2$. The adjoint representation is

$$\mathrm{Ad}(x)\xi_1 = \xi_1 - cx_2 \exp(cx_1/2)\xi_2 \qquad \text{and} \qquad \mathrm{Ad}(x)\xi_2 = (\exp cx_1)\xi_2.$$

The right invariant vector fields ∂_1, ∂_2 are given by

$$\partial_1(x) = \partial/\partial x_1 + (c/2)x_2\partial/\partial x_2 \qquad \text{and} \qquad \partial_2(x) = \exp(-cx_1/2)\partial/\partial x_2.$$

It follows from Proposition 2 (2) that $\Pi(x) = \pi(x)\partial_1(x)\wedge\partial_2(x)$ is multiplicative if and only if

$$\pi(x \cdot y) = \pi(x) + \pi(y) \exp cx_1,$$

where $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbf{R}^2$. Setting $x_1 = y_1 = 0$, $x_1 = y_2 = 0$, $x_2 = y_1 = 0$, and $x_2 = y_2 = 0$, we have

$$\begin{aligned}
&\pi(0, x_2 + y_2) = \pi(0, x_2) + \pi(0, y_2), \\
&\pi(y_1, x_2 \exp(-cy_1/2)) = \pi(0, x_2) + \pi(y_1, 0), \\
&\pi(x_1, y_2 \exp(cx_1/2)) = \pi(x_1, 0) + \pi(0, y_2) \exp cx_1,
\end{aligned}$$

and

$$\pi(x_1 + y_1, 0) = \pi(x_1, 0) + \pi(y_1, 0) \exp cx_1.$$

Since $\pi(0, 0) = 0$, the first equation implies $\pi(0, t) = k_2 t$, where k_2 is constant. Denote $\pi(s, 0)$ by $q(s)$. It follows from the second or third equation that π can be written as $\pi(s, t) = q(s) + k_2 t \exp(cs/2)$. The fourth equation implies that $q(t + s) = q(t) + q(s) \exp ct$. By differentiating this equation in s at $s = 0$, we have $\dot{q}(t) = \dot{q}(0) \exp ct$. It follows from $q(0) = 0$ that $q(t) = k_1(\exp ct - 1)/c$ for some constant k_1 and $q(t) = k_1 t$ when $c = 0$.
Thus,

$$\begin{aligned}
\pi(x_1, x_2) &= q(x_1) + k_2 x_2 \exp(cx_1/2) \\
&= k_1(\exp cx_1 - 1)/c + k_2 x_2 \exp(cx_1/2).
\end{aligned}$$

Hence, multiplicative Poisson structures are

$$\begin{aligned}
\Pi &= \{k_1(\exp cx_1 - 1)/c + k_2 x_2 \exp(cx_1/2)\}\partial_1 \wedge \partial_2 \\
&= \{2k_1 \sinh(cx_1/2)/c + k_2 x_2\}\partial/\partial x_1 \wedge \partial/\partial x_2 .
\end{aligned}$$

The inner automorphism group of the full $(ax+b)$-group gives the automorphism group of the $(ax+b)$-group $\mathbf{R}_c^2$ $(c \neq 0)$, which is larger than the inner automorphism group of the $(ax+b)$-group. Thus, the automorphism group of the $(ax+b)$-group $\mathbf{R}_c^2$ is given by

$$\varphi(x_1, x_2) = (\varphi_1(x_1, x_2), \varphi_2(x_1, x_2)),$$

where

$$\varphi_1(x_1, x_2) = x_1 \quad \text{and} \quad \varphi_2(x_1, x_2) = \alpha_{21} \sinh \frac{cx_1}{2} + \alpha_{22}x_2 \quad (\alpha_{22} \neq 0).$$

Now assume that this group automorphism φ is a Poisson map of $(\mathbf{R}_c^2, (k_1, k_2))$ to $(\mathbf{R}_c^2, (h_1, h_2))$. Then $\{\varphi_1, \varphi_2\}_{(k_1, k_2)} = \{x_1, x_2\}_{(h_1, h_2)} \circ \varphi$ holds, where $\{\ ,\ \}_{(k_1, k_2)}$ means the Poisson bracket of the Poisson manifold $(\mathbf{R}_c^2, (k_1, k_2))$. Computing the both sides of the equation above, we have

$$\begin{aligned} \text{RHS} &= \frac{2h_1}{c} \sinh\left(\frac{c\varphi_1}{2}\right) + h_2\varphi_2 \\ &= \frac{2h_1}{c} \sinh\left(\frac{cx_1}{2}\right) + h_2\left(\alpha_{21} \sinh \frac{cx_1}{2} + \alpha_{22}x_2\right) \end{aligned}$$

and

$$\text{LHS} = \alpha_{22}\left(\frac{2k_1}{c} \sinh\left(\frac{cx_1}{2}\right) + k_2x_2\right).$$

Therefore, we have

$$h_1 = \alpha_{22}k_1 - c\,h_2\alpha_{21}/2, \quad \text{and} \quad h_2 = k_2.$$

Conversely, the last two equations show how to construct a Poisson Lie group automorphism φ.

Each nonzero multiplicative Poisson structure gives again the Lie algebra structure of $(ax+b)$-group and we already have all of the multiplicative Poisson structures of $(ax+b)$-group, so it is easy to see the dual Poisson Lie group. This completes the proof of Proposition 2.6. □

It is interesting to understand some difference among Poisson structures which we get in Proposition 2.6.

Notes 2.7. (1) On the $(ax+b)$-group $\mathbf{R}_c^2$ $(c \neq 0)$, since

$$\begin{aligned} (Tl_x - Tr_x)(k_1\xi_1 \wedge \xi_2) &= k_1[\xi_1(x) \wedge \xi_2(x) - \partial_1(x) \wedge \partial_2(x)] \\ &= 2k_1 \sinh(cx_1/2)\partial/\partial x_1 \wedge \partial/\partial x_2, \end{aligned}$$

the multiplicative Poisson structures which come from r-matrices in the sense of Drinfel'd [2] are $\{x_1, x_2\} = 2k_1 \sinh(cx_1/2)/c$, i.e., $(\mathbf{R}_c^2, (k_1, 0))$.

(2) The Lie bialgebra structure defined from $(\mathbf{R}_c^2, (k_1, 0))$ is degenerate and the one defined from $(\mathbf{R}_c^2, (0, k_2))$ is nondegenerate if $k_2 \neq 0$ in the sense of Drinfel'd [3].

In general, a given Poisson Lie group G with Lie algebra $\mathfrak{g}$ defines a Lie algebra structure on the dual space $\mathfrak{g}^*$. It is known that $\mathfrak{g} \oplus \mathfrak{g}^*$ has a Lie algebra structure such that $\mathfrak{g}$ and $\mathfrak{g}^*$ are subalgebras of $\mathfrak{g} \oplus \mathfrak{g}^*$ and $[\xi, \sigma]$ is defined by

$$[\xi, \sigma] = \mathrm{Ad}_{\mathfrak{g}}(\xi)(\sigma) - \mathrm{Ad}_{\mathfrak{g}^*}(\sigma)(\xi),$$

where $\xi \in \mathfrak{g}$, $\sigma \in \mathfrak{g}^*$, and $\mathfrak{g}^{**}$ is identified with $\mathfrak{g}$ naturally. This Lie algebra is called the double Lie algebra in [5] and is an example of Manin triples (cf. [5]).

Proposition 2.8. *For the multiplicative Poisson structures on $(ax+b)$-group $\mathbf{R}^2_c$ $(c \neq 0)$,*

(1) *the double Lie algebra structure defined by $\{x_1, x_2\} = k_2x_2$ $(k_2 \neq 0)$ is isomorphic with $\mathfrak{gl}(2, \mathbf{R})$, and*

(2) *the double Lie algebra structure defined by $\{x_1, x_2\} = 2k_1 \sinh(x_1/2)$ $(k_1 \neq 0)$ is isomorphic with the semidirect product of $\mathbf{R}$ and a 3-dimensional Heisenberg Lie algebra.*

Proof. Let ξ_1 and ξ_2 be the standard basis of $\mathfrak{g}$ = Lie algebra of $\mathbf{R}^2_1$ as in the proof of Proposition 2.6. Let σ^1, σ^2 be the dual basis of ξ_1 and ξ_2.

(1) The Poisson structure $\{x_1, x_2\} = k_2x_2$ implies the Lie algebra structure on $\mathfrak{g}^*$ as $[\sigma^1, \sigma^2] = k_2\sigma^2$. The double Lie algebra structure is defined by

$$\begin{aligned} &[\xi_1, \xi_2] = \xi_2, \quad &&[\xi_1, \sigma^1] = 0, \quad &&[\xi_1, \sigma^2] = -\sigma^2, \\ &[\xi_2, \sigma^1] = k_2\xi_2, \quad &&[\xi_2, \sigma^2] = -k_2\xi_1 + \sigma^1, \quad &&[\sigma^1, \sigma^2] = k_2\sigma^2. \end{aligned}$$

If we put

$$\begin{aligned} e_0 &= (\xi_1 + \sigma^1/k_2)/2, \quad & e_1 &= (\xi_2 + \sigma^2/k_2)/2, \\ e_2 &= (\xi_2 - \sigma^2/k_2)/2, \quad & e_3 &= (\xi_1 - \sigma^1/k_2)/2, \end{aligned}$$

the Lie algebra structure has e_0 as a central element and

$$[e_1, e_2] = e_3, \qquad [e_2, e_3] = -e_1, \qquad [e_3, e_1] = e_2,$$

i.e., the linear span of $\{e_1, e_2, e_3\}$ is the Lie algebra $\mathfrak{sl}(2, \mathbf{R})$ and the double Lie algebra is isomorphic with $\mathfrak{g}l(2, \mathbf{R})$.

(2) Similarly, the Poisson structure $\{x_1, x_2\} = 2\sinh x_1/2$ implies $[\sigma^1, \sigma^2] = k_1\sigma^1$ and the double Lie algebra structure is defined by

$$\begin{aligned} &[\xi_1, \xi_2] = \xi_2, \quad &&[\xi_1, \sigma^1] = k_1\xi_2, \quad &&[\xi_1, \sigma^2] = -k_1\xi_1 - \sigma^2, \\ &[\xi_2, \sigma^1] = 0, \quad &&[\xi_2, \sigma^2] = \sigma^1, \quad &&[\sigma^1, \sigma^2] = k_1\sigma^1. \end{aligned}$$

Putting

$$\begin{aligned} e_0 &= (\xi_1 - \sigma^2/k_1)/2, \quad & e_1 &= (\xi_2 - \sigma^1/k_1)/2, \\ e_2 &= (\xi_2 + \sigma^1/k_1)/2, \quad & e_3 &= (\xi_1 + \sigma^2/k_1)/2, \end{aligned}$$

we rewrite the double Lie algebra structure as

$$\begin{aligned} &[e_0, e_1] = 0, \quad &&[e_0, e_2] = e_2, \quad &&[e_0, e_3] = -e_3, \\ &[e_1, e_2] = 0, \quad &&[e_1, e_3] = 0, \quad &&[e_2, e_3] = -e_1. \end{aligned}$$

It turns out that e_1, e_2, e_3 generate a 3-dimensional Heisenberg Lie algebra, with e_0 acting on it.

Remark 2.9. It is also an interesting problem whether, for a given Poisson Lie group G, there exist Lie groups G^* and D whose Lie algebras are $\mathfrak{g}^*$ and the double Lie algebra $\mathfrak{g} \oplus \mathfrak{g}^*$ respectively such that G and G^* are subgroups of D, and $G \times G^* \to D : (g_1, g_2) \mapsto g_1 g_2$ is a diffeomorphism, i.e., D is globally factorizable into GG^* (cf. [5, 7]). When G is the $(ax+b)$-group with the Poisson structure $\{x_1, x_2\} = 2k_1 \sinh(x_1/2)$ or $\{x_1, x_2\} = k_2 x_2$ ($k_1, k_2 \neq 0$), there exists no unique factorizable Lie group D.

3. Symplectic groupoids over the Poisson $(ax+b)$-groups

In this section, we shall prove the following theorem. From here on we assume that $c = 1$.

Theorem 3.1. *There always exists a symplectic groupoid for each multiplicative Poisson structure on the $(ax+b)$-group $\mathbf{R}^2$.*

Proof. It is known that the cotangent bundle T^*G of a Lie group G with the canonical symplectic structure is a trivial symplectic groupoid over G with the zero Poisson structure. If G is our $(ax+b)$-group $\mathbf{R}^2$, we have the trivial symplectic groupoid $\mathbf{R}^4$ over $\mathbf{R}^2$ with the zero Poisson structure.

The cotangent bundle T^*G with the canonical symplectic structure is also a symplectic groupoid over the dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ with the Lie-Poisson structure (see [6] for detail). Let G be our $(ax+b)$-group $\mathbf{R}^2$ with the coordinates (x_3, x_4). Under the identification by right translations, the symplectic groupoid structure of $\mathbf{R}^4 \cong T^*G$ with equivariant momentum mappings induced from right and left action as the source and target map is as follows:

$$\begin{aligned}
&\alpha_0(x) = (x_1 + x_2x_4/2,\ x_2 \exp(-x_3/2)) && \text{(source map)},\\
&\beta_0(x) = (x_1 - x_2x_4/2,\ x_2 \exp(x_3/2)) && \text{(target map)},\\
&\varepsilon((x_1, x_2)) = (x_1, x_2, 0, 0) && \text{(identities)},\\
&(x \cdot y)_1 = x_1 - y_2y_4/2\ (= y_1 + x_2x_4/2),\\
&(x \cdot y)_2 = x_2 \exp(y_3/2)\ (= y_2 \exp(-x_3/2)),\\
&(x \cdot y)_3 = x_3 + y_3,\\
&(x \cdot y)_4 = \exp(x_3/2)y_4 + x_4 \exp(-y_3/2),
\end{aligned}$$

and the cotangent symplectic structure is

$$\Omega_0 = -dx_1 \wedge dx_3 - dx_2 \wedge dx_4,$$

where x_j, y_j and $(x \cdot y)_j$ mean the jth component of x, y and $x \cdot y$ respectively. The induced Poisson bracket of $\mathfrak{g}^*$ is given by $\{x_1, x_2\} = x_2$. Although this Poisson structure is multiplicative for the additive group structure of $\mathfrak{g}^*$, it is also multiplicative for the $(ax+b)$-group structure by Proposition 2.6 (with

$k_1 = 0$, $k_2 = 1$). Thus we have proved the existence of symplectic groupoid over $\mathbf{R}^2$ with $\{x_1, x_2\} = x_2$.

Using the following obvious Proposition 3.2 and Lemma 3.3, we have completed the proof of Theorem 3.1. □

Proposition 3.2. *Let* $(\Gamma, \alpha, \beta, \varepsilon, \Gamma_0)$ *be a symplectic groupoid and let* P *be a Poisson manifold. If* P *is Poisson diffeomorphic to* Γ_0 *by some Poisson diffeomorphism* φ *of* Γ_0 *onto* P, *then the symplectic manifold* Γ *is also a symplectic groupoid on* P *with* $\alpha' = \varphi \circ \alpha$, $\beta' = \varphi \circ \beta$, $\varepsilon' = \varepsilon \circ \varphi^{-1}$, *and the same groupoid multiplication.*

Lemma 3.3. *Nonzero multiplicative Poisson structures on* $\mathbf{R}^2$ *are Poisson isomorphic to each other.*

Proof. The Poisson structures $\{x_1, x_2\} = x_2$ and $\{x_1, x_2\} = k_2 x_2$ are Poisson isomorphic by a diffeomorphism $\varphi(x_1, x_2) = (k_2 x_1, k_2 x_2)$ of $\mathbf{R}^2$. The Poisson structures $\{x_1, x_2\} = x_2$ and $\{x_1, x_2\} = 2k_1 \sinh x_1/2$ are also Poisson isomorphic by a diffeomorphism $\varphi(x_1, x_2) = (-x_2/(k_1\theta(x_1)), x_1)$ of $\mathbf{R}^2$, where θ is a function defined by $\theta(t) = \sinh(t/2)/(t/2)$. Combining these facts and Proposition 2.6, we see that every nonzero multiplicative Poisson structure is Poisson isomorphic with $\{x_1, x_2\} = x_2$. □

Note 3.4. In [1, 4, 10], there is a general theory for constructing a local symplectic groupoid structure over a given Poisson manifold. Although we already have the cotangent bundle of the $(ax + b)$-group as a symplectic groupoid over $\mathbf{R}^2$ with $\{x_1, x_2\} = x_2$, we can also reconstruct a symplectic groupoid by the general theory described above. As expected, it turns out that this symplectic groupoid is isomorphic with the cotangent bundle. We will show briefly how to carry out this procedure and find a symplectic groupoid isomorphism from the result to the cotangent bundle.

Since the source map α is a submersion, we may assume $\alpha(x) = (x_1, x_2)$. By the symplectic realization procedure in [8], the symplectic structure we get on $\mathbf{R}^4$ is

$$\Omega = dx_1 \wedge dx_3 + x_4 \frac{\exp(-x_3) - 1 + x_3}{{x_3}^2} dx_2 \wedge dx_3 + \frac{1 - \exp(-x_3)}{x_3} dx_2 \wedge dx_4 - x_2 \frac{1 - \exp(-x_3)}{x_3} dx_3 \wedge dx_4 .$$

It turns out that the target map β is

$$\beta(x) = \left(x_1 + x_2 x_4 \frac{1 - \exp(-x_3)}{x_3}, \, x_2 \exp(-x_3) \right) .$$

The groupoid structure in $\mathbf{R}^4$ with α and β as source and target map respectively is given by

$$(x \cdot y)_1 = x_1 \left(= y_1 - x_2 x_4 \frac{1-\exp(-x_3)}{x_3}\right),$$
$$(x \cdot y)_2 = x_2 \ (= y_2 \exp x_3),$$
$$(x \cdot y)_3 = x_3 + y_3,$$

and

$$(x \cdot y)_4 = \frac{x_3 + y_3}{\exp(x_3 + y_3)} \left(x_4 \exp y_3 \frac{\exp x_3 - 1}{x_3} + y_4 \frac{\exp y_3 - 1}{y_3}\right).$$

There is a symplectic groupoid isomorphism from $(\mathbf{R}^4, \Omega)$ to $(\mathbf{R}^4, \Omega_0)$ defined by

$$u_1 = x_1 + \frac{x_2 x_4}{2} \frac{1-\exp(-x_3)}{x_3},$$
$$u_2 = x_2 \exp(-x_3), \quad u_3 = -x_3,$$

and

$$u_4 = -x_4 \frac{\exp(x_3/2) - \exp(-x_3/2)}{x_3}.$$

4. Symplectic double groupoid

The cotangent bundle T^*G of a Lie group G is one of the typical examples of symplectic groupoids. As we mentioned in §3, T^*G with the canonical symplectic structure is a symplectic groupoid over G and $\mathfrak{g}^*$ in two compatible ways. From the fact above and Theorem 4.4.4 in [11], we expect that a symplectic groupoid over any Poisson Lie group has another symplectic groupoid structure, compatible with the original one, over the dual Poisson Lie group.

In this section, we shall show that the symplectic groupoid $\mathbf{R}^4$ over a Poisson $(ax+b)$-group $\mathbf{R}^2$ of nondegenerate type, i.e., the $(ax+b)$-group with the Poisson structure $\{x_1, x_2\} = k_2 x_2$, has a "dual" symplectic groupoid structure compatible with the original one.

If we consider the $(ax+b)$-group $\mathbf{R}^2$ with the zero Poisson structure, the situation is just the cotangent bundle case and there is a "dual" symplectic groupoid over the additive group $\mathfrak{g}^* \cong \mathbf{R}^2$ with its Lie-Poisson structure.

Next we consider the symplectic groupoid $(\mathbf{R}^4; \alpha_0, \beta_0; \mathbf{R}^2)$ in §3 over the Poisson $(ax+b)$-group $\mathbf{R}^2$ with $\{x_1, x_2\} = x_2$ according to Propositions 2.6, 3.2 and 3.4.

The following results are already known.

Proposition 4.1 [11]. *The graph of the multiplication of a Poisson Lie group is a coisotropic submanifold.*

Proposition 4.2 [11]. *The inverse image of a coisotropic submanifold of the base Poisson manifold by the source map is a coisotropic submanifold of the symplectic groupoid.*

We shall apply these propositions to our symplectic groupoid, as given in Theorem 3.1. Let $\mathscr{C}_0$ be the graph of the multiplication of the $(ax+b)$-group, namely,

$$\mathscr{C}_0 := \{(\bar{u}, \bar{v}, \bar{w}) \in \mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R}^2 \mid \bar{u} \cdot \bar{v} = \bar{w}\}.$$

And let $\mathscr{C}$ be the inverse image of $\mathscr{C}_0$ by the source map, i.e.,

$$\mathscr{C} := \{(x, y, z) \in \mathbf{R}^4 \times \mathbf{R}^4 \times \mathbf{R}^4 \mid (\alpha_0(x), \alpha_0(y), \alpha_0(z)) \in \mathscr{C}_0\}.$$

Then $\mathscr{C} = h_1^{-1}(0) \cap h_2^{-1}(0)$, where

$$h_1(x, y, z) = \left(x_1 + \frac{x_2x_4}{2}\right) + \left(y_1 + \frac{y_2y_4}{2}\right) - \left(z_1 + \frac{z_2z_4}{2}\right)$$

and

$$h_2(x, y, z) = y_2 \exp\left(\frac{x_1}{2} + \frac{x_2x_4}{4} - \frac{y_3}{2}\right) + x_2 \exp\left(-\frac{y_1}{2} - \frac{y_2y_4}{4} - \frac{x_3}{2}\right) - z_2 \exp\left(-\frac{z_3}{2}\right).$$

Flowing out $\mathscr{C}_0 = \mathscr{C} \cap (\mathbf{R}^2 \oplus \mathbf{R}^2 \oplus \mathbf{R}^2)$ by $(\mathrm{T}\mathscr{C})^\perp$, we have a submanifold $\mathscr{L}$. We suspect that $\mathscr{L}$ is the graph of some groupoid structure which is compatible with the original one, but it seems hard to verify this directly, since in general [10, 11] such a flowout is not a graph nor even an embedded submanifold. Instead, we will pick up some information which we can get from $\mathscr{L}$.

Proposition 4.3. *Let $\mathscr{F}$ be the foliation of $\mathscr{C}_0 \subset \mathbf{R}^{12}$ whose leaves are defined from $(\mathrm{T}\mathscr{C})^\perp$. Let (x, y, z) be a (general) point in the leaf $\mathscr{L}$, which is flowed out from $\mathscr{C}_0$ by the hamiltonian vector fields X_{h_2} and X_{h_1}. Then, we have the following*:

(1) *If $z = y$, then*

$$x = (0, 0, y_3 + y_2y_4/2, y_4 \exp(-y_1/2)\theta(y_2y_4)).$$

(2) *If $z = x$, then*

$$y = (0, 0, x_3 - x_2x_4/2, x_4 \exp(x_1/2)\theta(x_2x_4)).$$

The function θ above is the same as the one in Proposition 3.4, *defined by $\theta(t) = (\exp(t/2) - \exp(-t/2))/t$.*

Proof. (x, y, z) is given as $\exp sX_{h_1} \circ \exp tX_{h_2}(\bar{u}, \bar{v}, \bar{w})$ for some $(\bar{u}, \bar{v}, \bar{w}) \in \mathscr{C}_0$ and suitable times s, t, where $\exp tX_{h_j}$ means the flow of the hamiltonian vector field X_{h_j} $(j = 1, 2)$. Let $\exp tX_{h_2}(\bar{u}, \bar{v}, \bar{w}) = (u, v, w)$.

The flow of $\exp sX_{h_1}$ is integrated as $(\exp sX_{h_1})(u, v, w) = (x, y, z)$, where

$$x_1 = u_1, \quad x_2 = u_2 \exp(-s/2), \quad x_3 = u_3 + s, \quad x_4 = u_4 \exp(s/2),$$

$$y_1 = v_1, \quad y_2 = v_2 \exp(-s/2), \quad y_3 = v_3 + s, \quad y_4 = v_4 \exp(s/2),$$
$$z_1 = w_1, \quad z_2 = w_2 \exp(-s/2), \quad z_3 = w_3 + s, \quad z_4 = w_4 \exp(s/2).$$

From this explicit formula, the condition $y = z$ in (1) implies $v = w$. Concerning Hamilton's equations $\frac{d}{dt}(x, y, z) = X_{h_2}(x, y, z)$, i.e.,

$$\begin{aligned}
\dot{x}_1 &= \frac{x_2}{2}\exp A, & \dot{x}_2 &= -\frac{x_2 y_2}{4}\exp B,\\
\dot{x}_3 &= \frac{y_2}{2}\exp B, & \dot{x}_4 &= \exp A + \frac{x_4 y_2}{4}\exp B,\\
\dot{y}_1 &= \frac{y_2}{2}\exp B, & \dot{y}_2 &= \frac{x_2 y_2}{4}\exp A,\\
\dot{y}_3 &= -\frac{x_2}{2}\exp A, & \dot{y}_4 &= -\frac{x_2 y_4}{4}\exp A + \exp B,\\
\dot{z}_1 &= \frac{z_2}{2}\exp\left(-\frac{z_3}{2}\right), & \dot{z}_2 &= 0,\\
\dot{z}_3 &= 0, \quad \text{and} & \dot{z}_4 &= \exp\left(-\frac{z_3}{2}\right),
\end{aligned}$$

where

$$A = \left(-x_3 - y_1 - \frac{y_2 y_4}{2}\right)/2 \quad \text{and} \quad B = \left(x_1 - y_3 + \frac{x_2 x_4}{2}\right)/2,$$

it turns out that

$$x_1 - \frac{x_2 x_4}{2}, \quad y_1 - \frac{y_2 y_4}{2}, \quad z_1 - \frac{z_2 z_4}{2},$$
$$x_2 \exp\left(\frac{x_3}{2}\right), \quad y_2 \exp\left(\frac{y_3}{2}\right), \quad z_2 \exp\left(\frac{z_3}{2}\right),$$

are the first integrals of our differential system. Under the initial condition $(\bar{u}, \bar{v}, \bar{w}) \in \mathscr{C}_0$, if $y = z$, i.e., $v = w$, then we have $\bar{v} = \bar{w}$ and so $\bar{u} = 0$. Thus, we have $x_1 = x_2 = 0$ identically. Now our Hamilton's equations become simple and we have $y_2 = \bar{v}_2$, $y_3 = 0$, and $B = 0$. We solve

$$u_3 = \frac{\bar{v}_2}{2}t, \quad v_1 = \frac{\bar{v}_2}{2}t + \bar{v}_1, \quad v_4 = t.$$

Since $A = -\frac{3}{4}\bar{v}_2 t - \frac{1}{2}\bar{v}_1$, we have the equation

$$\dot{x}_4 = \frac{\bar{v}_2}{4}x_4 + \exp\left(-\frac{\bar{v}_1}{2} - \frac{3}{4}\bar{v}_2 t\right),$$

and we have the solution

$$u_4 = \exp\left(\frac{\bar{v}_2}{4}t - \frac{\bar{v}_1}{2}\right)(1 - \exp(-\bar{v}_2 t))/\bar{v}_2.$$

Since $s = y_3$, $t = y_4 \exp(-y_3/2)$, $\bar{v}_2 = y_2 \exp(y_3/2)$, and $\bar{v}_1 = y_1 - y_2 y_4/2$, we have

$$x = \left(0, 0, y_3 + \frac{y_2 y_4}{2}, y_4 \exp\left(-\frac{y_1}{2}\right)\theta(y_2 y_4)\right).$$

Thus, we have completed the proof of (1). The proof of (2) is the same as in (1), so we omit it. □

If the submanifold $\mathscr{L}$ which is obtained by flowing out the submanifold $\mathscr{C}_0$ along $(T\mathscr{C})^{\perp}$ is a graph of some groupoid structure, then the points in Proposition 4.3 are the image of the source and target map respectively. In the next theorem, we shall show that they are actually the source and target maps of a symplectic groupoid structure and that $\mathscr{L}$ is an open submanifold of the graph of multiplication of that groupoid.

Theorem 4.4. (1) *Define maps* α *and* β *as*

$$\alpha(x) = \left(0, 0, x_3 + \frac{x_2x_4}{2}, x_4 \exp\left(-\frac{x_1}{2}\right)\theta(x_2x_4)\right)$$

and

$$\beta(x) = \left(0, 0, x_3 - \frac{x_2x_4}{2}, x_4 \exp\left(\frac{x_1}{2}\right)\theta(x_2x_4)\right),$$

where θ *is the function in Proposition* 4.3. *Then* α *and* β *are projections of* $\mathbf{R}^4$ *onto* $\mathbf{R}^2 \cong (0, 0) \times \mathbf{R}^2$ *and satisfy*

$$\begin{aligned} &\{\alpha_1, \alpha_2\} = -\alpha_2, \qquad &&\{\alpha_1, \beta_1\} = 0, \qquad &&\{\alpha_1, \beta_2\} = 0, \\ &\{\alpha_2, \beta_1\} = 0, &&\{\alpha_2, \beta_2\} = 0, &&\{\beta_1, \beta_2\} = \beta_2, \end{aligned}$$

where the Poisson bracket is induced from the canonical symplectic structure $\Omega_0 = -dx_1 \wedge dx_3 - dx_2 \wedge dx_4$.

(2) *There is a symplectic groupoid structure in* $(\mathbf{R}^4, \Omega_0)$ *such that* α *and* β *are the source and target map respectively, and the graph of multiplication contains* $\mathscr{L}$.

(3) *The two groupoid structures defined from* (α_0, β_0) *and* (α, β) *are compatible in the following sense:*

$$(x \cdot y) * (u \cdot v) = (x * u) \cdot (y * v)$$

where " $\cdot$ *" and "* $*$ *" denote the groupoid multiplications of those groupoids defined by* (α_0, β_0) *and* (α, β) *respectively.*

Proof. (1) Since (1) comes from a direct computation, we omit its proof.

(2) Suppose first that there exists a groupoid multiplication $*$ with the source and target map α and β. Let $z = x * y$. Then, we have $\alpha(z) = \alpha(x)$ and $\beta(z) = \beta(y)$ under the compatibility condition $\beta(x) = \alpha(y)$. These imply

$$\begin{aligned} &\left(z_3 + \frac{z_2z_4}{2}\right) - \left(x_3 + \frac{x_2x_4}{2}\right) = 0, \\ &z_4\theta(z_2z_4)\exp\left(-\frac{z_1}{2}\right) - x_4\theta(x_2x_4)\exp\left(-\frac{x_1}{2}\right) = 0, \\ &\left(z_3 - \frac{z_2z_4}{2}\right) - \left(y_3 - \frac{y_2y_4}{2}\right) = 0, \\ &z_4\theta(z_2z_4)\exp\left(\frac{z_1}{2}\right) - y_4\theta(y_2y_4)\exp\left(\frac{y_1}{2}\right) = 0, \\ &\left(x_3 - \frac{x_2x_4}{2}\right) - \left(y_3 + \frac{y_2y_4}{2}\right) = 0, \\ &x_4\theta(x_2x_4)\exp\left(\frac{x_1}{2}\right) - y_4\theta(y_2y_4)\exp\left(-\frac{y_1}{2}\right) = 0. \end{aligned}$$

From the first and third equations above, we have

$$z_3 = x_3 - \frac{y_2y_4}{2} \quad \left(= y_3 + \frac{x_2x_4}{2}\right).$$

The second, fourth, and sixth equations yield

$$z_4(\exp(z_1 - x_1 - y_1) - 1) = 0.$$

Thus, z_1 should be x_1+y_1. $z_1 = x_1+y_1$ and the second equation above imply that

$$z_4\theta(z_2z_4) = x_4\theta(x_2x_4)\exp\left(\frac{y_1}{2}\right) \quad \left(= y_4\theta(y_2y_4)\exp\left(-\frac{x_1}{2}\right)\right).$$

We have

$$z_2z_4 = x_2x_4 + y_2y_4$$

from the first, third and fifth equation above. If $z_4 \neq 0$, from $z_2z_4 = x_2x_4 + y_2y_4$, the second and fourth equation we have

$$\begin{aligned} z_2 &= x_2\frac{x_4}{z_4} + y_2\frac{y_4}{z_4} \\ &= x_2\frac{\theta(z_2z_4)}{\theta(x_2x_4)}\exp\left(-\frac{z_1}{2} + \frac{x_1}{2}\right) + y_2\frac{\theta(z_2z_4)}{\theta(y_2y_4)}\exp\left(\frac{z_1}{2} - \frac{y_1}{2}\right), \end{aligned}$$

namely, we have

$$\frac{z_2}{\theta(z_2z_4)} = \frac{x_2}{\theta(x_2x_4)}\exp\left(-\frac{y_1}{2}\right) + \frac{y_2}{\theta(y_2y_4)}\exp\left(\frac{x_1}{2}\right).$$

This equation makes sense even if $z_4 = 0$. We can verify that the set $\mathcal{M}$ of common zeros of the functions below defines a groupoid structure with the source and target maps α and β respectively.

$$\begin{aligned} f_1 &= z_1 - (x_1 + y_1), \\ f_2 &= \frac{z_2}{\theta(z_2z_4)} - \left(\frac{x_2}{\theta(x_2x_4)}\exp\left(-\frac{y_1}{2}\right) + \frac{y_2}{\theta(y_2y_4)}\exp\left(\frac{x_1}{2}\right)\right), \\ f_3 &= z_3 - \left(x_3 - \frac{y_2y_4}{2}\right), \\ f_4 &= z_4\theta(z_2z_4) - x_4\theta(x_2x_4)\exp\left(\frac{y_1}{2}\right), \\ f_5 &= \left(x_3 - \frac{x_2x_4}{2}\right) - \left(y_3 + \frac{y_2y_4}{2}\right), \\ f_6 &= x_4\theta(x_2x_4)\exp\left(\frac{x_1}{2}\right) - y_4\theta(y_2y_4)\exp\left(-\frac{y_1}{2}\right). \end{aligned}$$

Since df_j $(j = 1, \dots, 6)$ are linearly independent, $\mathcal{M}$ which is the graph of groupoid multiplication $*$, is a 6-dimensional manifold; it is a lagrangian submanifold because

$$\{f_3, f_6\} = -f_6/2, \qquad \{f_5, f_6\} = f_6$$

and the other Poisson brackets of f_j $(j = 1, \dots, 6)$ are all zero. Thus, we have another symplectic groupoid structure $(\mathbf{R}^4, \Omega_0)$ over the Poisson manifold $\mathbf{R}^2$ with $\{x_3, x_4\} = -x_4$.

To show that $\mathscr{L} \subset \mathscr{M}$, it is enough to see X_{h_1} and X_{h_2} are tangent to $\mathscr{M}$, i.e., $\{h_j, f_k\} = 0$ on $\mathscr{M}$ for $j = 1, 2$, $k = 1, \dots, 6$ because $\mathscr{C}_0 \subset \mathscr{M}$. To see $\{h_j, f_k\} = 0$ on $\mathscr{M}$, we use the following algebraic lemma.

Lemma 4.5. *For any functions P, Q on $\mathbf{R}^{12}$, we have*
(1) $\{P + Qf_j, f_k\} = \{P, f_k\}$ *on* $\mathscr{M}$, *and*
(2) $\{P\exp(Qf_j), f_k\} = \{P, f_k\}$ *on* $\mathscr{M}$.

We continue the proof of Theorem 4.4. Since $h_1 \equiv 0 \bmod (f_1, \dots, f_6)$, we have $\{h_1, f_k\} = 0$ on $\mathscr{M}$. We also see $h_2 z_4 \equiv 0 \bmod (f_1, \dots, f_6)$, and have $\{h_2 z_4, f_k\} = 0$ on $\mathscr{M}$, i.e.,

$$h_2\{z_4, f_k\} + \{h_2, f_k\} z_4 = 0 \quad \text{on } \mathscr{M}.$$

This implies that $\{h_2, f_k\} = 0$ on $\mathscr{M} \cap \{z_4 \neq 0\}$ and so $\{h_2, f_k\} = 0$ on $\mathscr{M}$.
(3) We first recall that

$$(x \cdot y)_2 (x \cdot y)_4 = (x * y)_2 (x * y)_4 = x_2 x_4 + y_2 y_4.$$

Assume that (x, y), (u, v) are composable pairs with respect to "$\cdot$", (x, u), (y, v) are composable pairs with respect to "$*$", $(x \cdot y, u \cdot v)$ is a composable pair with respect to "$*$" and $(x * u, y * v)$ is a composable pair with respect to "$\cdot$". Then the compatibility conditions are

$$x_1 - \frac{x_2 x_4}{2} = y_1 + \frac{y_2 y_4}{2},$$

$$x_2 \exp\left(\frac{x_3}{2}\right) = y_2 \exp\left(-\frac{y_3}{2}\right),$$

$$u_1 - \frac{u_2 u_4}{2} = v_1 + \frac{v_2 v_4}{2},$$

$$u_2 \exp\left(\frac{u_3}{2}\right) = v_2 \exp\left(-\frac{v_3}{2}\right),$$

$$x_3 - \frac{x_2 x_4}{2} = u_3 + \frac{u_2 u_4}{2},$$

$$x_4 \exp\left(\frac{x_1}{2}\right) \theta(x_2 x_4) = u_4 \exp\left(-\frac{u_1}{2}\right) \theta(u_2 u_4),$$

$$y_3 - \frac{y_2 y_4}{2} = v_3 + \frac{v_2 v_4}{2},$$

$$y_4 \exp\left(\frac{y_1}{2}\right) \theta(y_2 y_4) = v_4 \exp\left(-\frac{v_1}{2}\right) \theta(v_2 v_4),$$

$$(x \cdot y)_3 - \frac{(x \cdot y)_2 (x \cdot y)_4}{2} = (u \cdot v)_3 + \frac{(u \cdot v)_2 (u \cdot v)_4}{2},$$

$$(x \cdot y)_4 \exp\left(\frac{(x \cdot y)_1}{2}\right) \theta((x \cdot y)_2 (x \cdot y)_4)$$
$$= (u \cdot v)_4 \exp\left(-\frac{(u \cdot v)_1}{2}\right) \theta((u \cdot v)_2 (u \cdot v)_4),$$

$$(x * u)_1 - \frac{(x * u)_2 (x * u)_4}{2} = (y * v)_1 + \frac{(y * v)_2 (y * v)_4}{2},$$

and

$$(x*u)_2 \exp\left(\frac{(x*u)_3}{2}\right) = (y*v)_2 \exp\left(-\frac{(y*v)_3}{2}\right).$$

Concerning the first and third components, we have directly that

$$\begin{aligned}
&\{(x\cdot y)*(u\cdot v)\}_1 - \{(x*u)\cdot(y*v)\}_1 \\
&\quad = \{(x\cdot y)_1 + (u\cdot v)_1\} - \left\{(x*u)_1 - \frac{(y*v)_2(y*v)_4}{2}\right\} \\
&\quad = x_1 - \frac{y_2y_4}{2} + u_1 - \frac{v_2v_4}{2} - \left\{x_1 + u_1 - \frac{y_2y_4 + v_2v4}{2}\right\} \\
&\quad = 0
\end{aligned}$$

and

$$\begin{aligned}
&\{(x\cdot y)*(u\cdot v)\}_3 - \{(x*u)\cdot(y*v)\}_3 \\
&\quad = \{(x\cdot y)*(u\cdot v)\}_3 - (x*u)_3 - (y*v)_3 \\
&\quad = (x\cdot y)_3 - \frac{(u\cdot v)_2(u\cdot v)_4}{2} - (x*u)_3 - (y*v)_3 \\
&\quad = x_3 + y_3 - \frac{u_2u4 + v_2v_4}{2} - \left(x_3 - \frac{u_2u_4}{2}\right) - \left(y_3 - \frac{v_2v_4}{2}\right) \\
&\quad = 0.
\end{aligned}$$

Using the composability conditions and the property

$$(s+t)\theta(s+t) = s\theta(s)\exp(-t/2) + t\theta(t)\exp(s/2),$$

we can rewrite both sides of the second components as follows:

$$\begin{aligned}
\{(x\cdot y)*(u\cdot v)\}_2 &= \theta(x_2x_4 + y_2y_4 + u_2u_4 + v_2v_4) \\
&\quad \cdot\left[\frac{(x\cdot y)_2}{\theta(x_2x_4 + y_2y_4)}\exp\left(-\frac{(u\cdot v)_1}{2}\right) + \frac{(u\cdot v)_2}{\theta(u_2u_4 + v_2v_4)}\exp\left(\frac{(x\cdot y)_1}{2}\right)\right] \\
&= \left[\exp\left\{-\frac{(u\cdot v)_1}{2} - \frac{u_2u_4 + v_2v_4}{2}\right\}\left\{\frac{(u\cdot v)_2(u\cdot v)_4}{(x\cdot y)_4} + (x\cdot y)_2\right\}\right. \\
&\qquad \cdot (x_2x_4 + y_2y_4) + \exp\left\{\frac{(x\cdot y)_1}{2} + \frac{x_2x_4 + y_2y_4}{2}\right\} \\
&\qquad\qquad \left.\cdot\left\{(u\cdot v)_2 + \frac{(x\cdot y)_2(x\cdot y)_4}{(u\cdot v)_4}\right\}(u_2u_4 + v_2v_4)\right] \\
&\quad \div (x_2x_4 + y_2y_4 + u_2u_4 + v_2v_4) \\
&= \exp\left\{-\frac{(u\cdot v)_1}{2} - \frac{u_2u_4 + v_2v_4}{2}\right\}\frac{x_2x_4 + y_2y_4}{(x\cdot y)_4} \\
&\quad + \exp\left\{\frac{(x\cdot y)_1}{2} + \frac{x_2x_4 + y_2y_4}{2}\right\}\frac{u_2u_4 + v_2v_4}{(u\cdot v)_4} \\
&= x_2\exp\left\{\frac{y_3}{2} - \frac{(u\cdot v)_1}{2} - \frac{u_2u_4 + v_2v_4}{2}\right\} \\
&\quad + u_2\exp\left\{\frac{v_3}{2} + \frac{(x\cdot y)_1}{2} + \frac{x_2x_4 + y_2y_4}{2}\right\}.
\end{aligned}$$

Also, we have

$$\begin{aligned}\{(x*u)\cdot(y*v)\}_2 &= (x*u)_2 \exp\left(\frac{(y*v)_3}{2}\right)\\ &= \theta(x_2x_4+u_2u_4)\left\{\frac{x_2}{\theta(x_2x_4)}\exp\left(-\frac{u_1}{2}\right)+\frac{u_2}{\theta(u_2u_4)}\exp\left(\frac{x_1}{2}\right)\right\}\\ &\quad\cdot\exp\left(\frac{(y*v)_3}{2}\right)\\ &= \left[\exp\left(-\frac{u_1}{2}-\frac{u_2u_4}{2}\right)(u_2u_4x_2+x_2{}^2x_4)\right.\\ &\qquad\qquad\left.+\exp\left(\frac{x_1}{2}+\frac{x_2x_4}{2}\right)(u_2{}^2u_4+u_2x_2x_4)\right]\\ &\quad\cdot\exp\left(\frac{(y*v)_3}{2}\right)\Big/(x_2x_4+u_2u_4)\\ &= \left[x_2\exp\left(-\frac{u_1}{2}-\frac{u_2u_4}{2}\right)+u_2\exp\left(\frac{x_1}{2}+\frac{x_2x_4}{2}\right)\right]\exp\left(\frac{(y*v)_3}{2}\right).\end{aligned}$$

Since we have

$$\begin{aligned}&\left[\frac{y_3}{2}-\frac{(u\cdot v)_1}{2}-\frac{u_2u_4+v_2v_4}{2}\right]-\left[-\frac{u_1}{2}-\frac{u_2u_4}{2}+\frac{(y*v)_3}{2}\right]\\ &\quad=\frac{y_3}{2}-\left(u_1-\frac{v_2v_4}{2}\right)/2-\frac{u_2u_4}{2}-\frac{v_2v_4}{2}+\frac{u_1}{2}+\frac{u_2u_4}{2}-\left(y_3-\frac{v_2v_4}{2}\right)/2\\ &\quad=0\end{aligned}$$

and

$$\begin{aligned}&\left[\frac{v_3}{2}+\frac{(x\cdot y)_1}{2}+\frac{x_2x_4+y_2y_4}{2}\right]-\left[\frac{x_1}{2}+\frac{x_2x_4}{2}+\frac{(y*v)_3}{2}\right]\\ &\quad=\frac{v_3}{2}+\left(x_1-\frac{y_2y_4}{2}\right)/2+\frac{x_2x_4+y_2y_4}{2}-\frac{x_1}{2}-\frac{x_2x_4}{2}-\left(v_3+\frac{y_2y_4}{2}\right)/2\\ &\quad=0,\end{aligned}$$

we finally have

$$\{(x\cdot y)*(u\cdot v)\}_2=\{(x*u)\cdot(y*v)\}_2.$$

Using the same argument on the fourth component, or from the relations

$$\{(x\cdot y)*(u\cdot v)\}_4=\frac{x_2x_4+y_2y_4+u_2u_4+v_2v_4}{\{(x\cdot y)*(u\cdot v)\}_2}$$

and

$$\{(x*u)\cdot(y*v)\}_4=\frac{x_2x_4+y_2y_4+u_2u_4+v_2v_4}{\{(x*u)\cdot(y*v)\}_2},$$

we also have

$$\{(x\cdot y)*(u\cdot v)\}_4=\{(x*u)\cdot(y*v)\}_4. \quad\square$$

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