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NOETHER-LEFSCHETZ LOCUS FOR SURFACES

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ABSTRACT. We generalize M. Green's Explicit Noether-Lefschetz Theorem to the family of smooth complete intersection surfaces in the higher dimensional projective spaces. Moreover, we give a new proof of the Density Theorem due to C. Ciliberto, J. Harris, and R. Miranda [5].

1. INTRODUCTION

Let $\mathbf{P}^n$ be the complex projective space of dimension n. The Noether-Lefschetz Theorem says that a general surface S of degree d in $\mathbf{P}^3$ contains only curves which are complete intersections of S with another hypersurface in $\mathbf{P}^3$ for $d \geq 4$. The word "general" is used in the following sense: A property is said to *hold at a general point of a projective variety* V, if there exists a countable union Σ of proper subvarieties of V such that the property holds at all points of $V - \Sigma$. In [21], Lefschetz proved an even more general version: A general complete intersection surface S of $n-2$ hypersurfaces in $\mathbf{P}^n$, $n \geq 3$, contains only curves that are themselves complete intersections unless S is an intersection of two quadric 3-folds in $\mathbf{P}^4$ or degree $S \leq 3$ in $\mathbf{P}^3$. We denote by Y_n the space of smooth complete intersection surfaces of type $(d_1, \dots, d_{n-2})$ in $\mathbf{P}^n$, where $2 \leq d_1 \leq d_2 \leq \cdots \leq d_{n-2}$. Let $E = \bigoplus_{i=1}^{n-2} \mathcal{O}_{\mathbf{P}^n}(d_i)$. Y_n is parametrized by an open subset, which is also denoted by Y_n, by abuse of notation, of the Grassmannian of 1-dimensional subspaces of $H^0(\mathbf{P}^n, E)$. The *Noether-Lefschetz locus* Σ_n *in* Y_n is the set of smooth surfaces in Y_n containing curves which are not complete intersections, i.e.,

$$\Sigma_n = \{S \in Y_n \mid \mathrm{Pic}(S) \text{ is not generated by the hyperplane class}\}.$$

Since the fundamental work of Noether and Lefschetz, their results have been improved in a number of interesting directions (see, e.g., [3, 7, 11, 12, 23, 26, 27]). For $n = 3$, by a mixture of Hodge-theoretic and algebraic techniques, Green [8, 10] showed that every irreducible component of Σ_3 has codimension at least $d_1 - 3$ in the family of smooth surfaces of degree d_1 in $\mathbf{P}^3$ for $d_1 \geq 3$,

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which is called the explicit Noether-Lefschetz Theorem. A generalization of this theorem to the case $n \geq 4$ is given in §2 (cf. Theorem 1). There is one new phenomenon in this case which is not present in the case of surfaces in $\mathbf{P}^3$. For example, the analog of Green's result in $\mathbf{P}^4$ holds only when a general element of a component is the intersection of two smooth 3-folds.

On the other hand, an upper bound for the codimension of the irreducible components in the case $n = 3$ is $p_g = \binom{d_1 - 1}{3}$. Ciliberto, Harris and Miranda [5] proved that over an algebraically closed field of any characteristic, for $d_1 \geq 4$, the Noether-Lefschetz locus in the family Y_3 of smooth surfaces of degree d_1 in $\mathbf{P}^3$ contains infinitely many components having maximal codimension p_g and the union of these components is Zariski dense in Y_3. Following M. Green's idea, they showed that over the complex numbers, the existence of one such component implies that the union of the components having maximal codimension p_g is dense in Y_3 in the classical topology. We will give a rather simple infinitesimal proof of this without constructing such components directly in §3.

2. A generalization of the explicit Noether-Lefschetz Theorem

Let $\Sigma_L \subset \Sigma_n$ denote the subvariety of surfaces containing lines, i.e., curves of degree 1 in $\mathbf{P}^n$. Let $G =$ Grassmannian of lines in $\mathbf{P}^n$. Then Σ_L is the image under projection on the second factor of the incidence correspondence

$$\tilde{\Sigma}_L = \{(C, S) \mid C \subset S\} \subset G \times Y_n.$$

For a line $l \subset \mathbf{P}^n$, we have an exact sequence

$$0 \to \mathscr{I}_{l|\mathbf{P}^n} \to \mathscr{O}_{\mathbf{P}^n} \xrightarrow{r} \mathscr{O}_l \to 0$$

where r is the restriction map. Tensoring with E and taking the long exact sequence of cohomology, we have

$$0 \to H^0(\mathbf{P}^n, \mathscr{I}_{l|\mathbf{P}^n} \otimes E) \to H^0(\mathbf{P}^n, E) \to H^0(\mathbf{P}^n, E \otimes \mathscr{O}_l) \to \cdots .$$

From this sequence, we can see that the fiber of the projection map $\pi_1 : \tilde{\Sigma}_L \to G$ over l is contained in $H^0(\mathbf{P}^n, \mathscr{I}_{l|\mathbf{P}^n} \otimes E)$. Since $H^0(\mathbf{P}^n, E) \to H^0(\mathbf{P}^n, E \otimes \mathscr{O}_l)$ is surjective,

$$\dim \text{ fiber of } \pi_1 = \dim H^0(\mathbf{P}^n, E) - \sum_{i=1}^{n-2} (d_i + 1) - 1.$$

So

$$\operatorname{codim}_{Y_n} \Sigma_L = \sum_{i=1}^{n-2} d_i - n.$$

We have the following explicit Noether-Lefschetz Theorem for $n \geq 4$, which generalizes the theorem of Green [10] for $n = 3$.

Theorem 1. *Let* $n \geq 4$. *An irreducible component* Σ' *of* Σ_n *has codimension at least* $\sum_{i=1}^{n-2} d_i - n$ *in* Y_n *if for a general point of* Σ', *the corresponding surface* $S = \bigcap_{i=1}^{n-2} H_i$ *has the property that* $\bigcap_{i=1}^{k} H_i$ *has no singularity for any* k *with* $d_k < d_{k+1}$.

Example. The hypothesis in Theorem 1 is necessary. For example, let $n = 4$ and $F_1 = z_0^{d_1} + z_0 z_2^{d_1 - 1} + z_1^{d_1} + z_1 z_3^{d_1 - 1}$. Then $H_1 = \{F_1 = 0\}$ has an isolated singularity at $(0, 0, 0, 0, 1)$ and has no other singularities. It is a cone over a smooth surface in $\mathbf{P}^3$ containing a line. H_1 contains the plane $z_0 = z_1 = 0$. The intersection of H_1 with any 3-fold H_2 of degree d_2 contains a plane curve of degree d_2 and therefore is in the Noether-Lefschetz locus. The component Σ' containing these complete intersection surfaces has codimension depending only on d_1, i.e., codimension of Σ' is at most $\binom{d_1+4}{4}$. If $d_2 > \binom{d_1+4}{4} + 4 - d_1$, then $\operatorname{codim}_{Y_4} \Sigma' < d_1 + d_2 - 4$.

We will give a proof of Theorem 1 using similar techniques to Green's [10]. First, we will show the following simple algebraic fact and then reduce our theorem to this.

Proposition 1. *Let* $W \subseteq H^0(\mathbf{P}^n, E)$ *be a subspace such that the evaluation map*

$$W \otimes \mathcal{O}_{\mathbf{P}^n, x} \to E_x$$

is surjective for all $x \in \mathbf{P}^n$. *Then the map*

$$W \otimes H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(k)) \to H^0(\mathbf{P}^n, E(k))$$

is surjective if $k \geq \operatorname{codim} W$.

Proof. Let $c = \operatorname{codim} W$. We can choose an increasing sequence of linear subspaces

$$W_c = W \subset W_{c-1} \subset \cdots \subset W_1 \subset W_0 = H^0(\mathbf{P}^n, E)$$

so that $\dim W_{i-1}/W_i = 1$ for $i = 1, 2, \ldots, c$. Since the evaluation map $W \otimes \mathcal{O}_{\mathbf{P}^n, x} \to E_x$ is surjective at all $x \in \mathbf{P}^n$, the kernel M_i of the map $W_i \otimes \mathcal{O}_{\mathbf{P}^n} \to E$ is a vector bundle on $\mathbf{P}^n$ sitting in the exact sequence

$$0 \to M_i \to W_i \otimes \mathcal{O}_{\mathbf{P}^n} \to E \to 0$$

for $i = 1, 2, \ldots, c$, and it is enough to show that

$$H^1(M_c \otimes \mathcal{O}_{\mathbf{P}^n}(k)) = 0 \quad \text{if } k \geq c = \operatorname{codim} W,$$

which follows from the following lemma.

Lemma. *For all* $i = 0, 1, \ldots, c$, $H^q(\mathbf{P}^n, \bigwedge^p M_i(k)) = 0$ *if* $q \geq 1$ *and* $k + q \geq p + i$.

Proof. We note that the M_i's sit in the exact sequence

$$0 \to M_i \to M_{i-1} \to \mathcal{O}_{\mathbf{P}^n} \to 0$$

and thus we have an exact sequence

$$0 \to \bigwedge^{p+1} M_i \to \bigwedge^{p+1} M_{i-1} \to \bigwedge^{p} M_i \to 0$$

for each i. Tensoring by $\mathscr{O}_{\mathbf{P}^n}(k)$ and taking the long exact sequence on cohomology, we have

$$\cdots \to H^q\left(\mathbf{P}^n, \bigwedge^{p+1} M_{i-1}(k)\right) \to H^q\left(\mathbf{P}^n, \bigwedge^{p} M_i(k)\right)$$
$$\to H^{q+1}\left(\mathbf{P}^n, \bigwedge^{p+1} M_i(k)\right) \to \cdots.$$

Let $q \geq 1$ and $k+q \geq p+i$, $i = 0, 1, \dots, c$. We will use induction on i and p to prove the lemma. First, notice that if $p \geq \operatorname{rank} M_i$, then $H^q(\mathbf{P}^n, \bigwedge^{p+1} M_i(k)) = 0$ for all $q \geq 0$ and for any $k \geq 0$.

Sublemma. *For* $i = 0$, $H^q(\mathbf{P}^n, \bigwedge^p M_0(k)) = 0$ *if* $q \geq 1$ *and* $k+q \geq p$.

To see this, we first recall (cf. [24, Lecture 14]) that a coherent sheaf F on $\mathbf{P}^n$ is said to be *m-regular,* if $H^q(\mathbf{P}^n, F(m-q)) = 0$ for $q > 0$.

From the exact sequence

$$0 \to M_0 \to H^0(\mathbf{P}^n, E) \otimes \mathscr{O}_{\mathbf{P}^n} \to E \to 0$$

tensored with $\mathscr{O}_{\mathbf{P}^n}(1-q)$, we have the long exact sequence on the cohomology

$$\cdots \to H^{q-1}(\mathbf{P}^n, E(1-q)) \to H^q(\mathbf{P}^n, M_0(1-q))$$
$$\to H^0(\mathbf{P}^n, E) \otimes H^q(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(1-q)) \to \cdots.$$

If $q = 1$, then $H^0(\mathbf{P}^n, E) \otimes H^0(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}) \to H^0(\mathbf{P}^n, E)$ is an isomorphism and $H^1(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}) = 0$, hence $H^1(\mathbf{P}^n, M_0) = 0$.

For $q > 1$, $H^{q-1}(\mathbf{P}^n, E(1-q)) = 0$ and $H^q(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(1-q)) = 0$, and hence

$$H^q(\mathbf{P}^n, M_0(1-q)) = 0.$$

Thus M_0 is 1-regular. Then $\bigwedge^p M_0$ is p-regular (see, e.g., [20, Lemma 2.7]). Since p-regularity implies $(p+1)$-regularity [24, loc.cit], the sublemma follows.

By ascending induction on i, we may assume

$$H^q\left(\mathbf{P}^n, \bigwedge^{p+1} M_{i-1}(k)\right) = 0$$

since $k+q \geq (p+1)+(i-1) = p+i$. By descending induction on p, we may assume

$$H^{q+1}\left(\mathbf{P}^n, \bigwedge^{p+1} M_i(k)\right) = 0$$

since $k+q \geq p+i$ which is equivalent to $k+(q+1) \geq (p+1)+i$. Hence

$$H^q\left(\mathbf{P}^n, \bigwedge^{p} M_i(k)\right) = 0,$$

and the lemma follows.

For a compact complex manifold M of dimension n with the associated $(1, 1)$-form ω, we recall that the primitive cohomology is

$$H_{\mathrm{pr}}^{n-k}(M) = \ker\{\omega^{k+1} : H^{n-k}(M) \to H^{n+k+2}(M)\}.$$

We denote $H_{\mathrm{pr}}^{p,q}(M) = H^{p,q}(M) \cap H_{\mathrm{pr}}^{p+q}(M)$. For a smooth hypersurface X of degree d in $\mathbf{P}^n$ with defining equation $F(z_0, \dots, z_n) = 0$, it is known (cf. [4, 15]) that there are natural Poincaré residue isomorphisms

$$H_{\mathrm{pr}}^{n-k-1,k}(X) \simeq S^{d(k+1)-n-1}/J_{F,d(k+1)-n-1} \tag{2.1}$$

where $S = \bigoplus_{k\geq 0} S^k$ is the graded ring $\mathbf{C}[z_0, \dots, z_n]$ and $J_F = \bigoplus_{k\geq d-1} J_{F,k}$ denotes the Jacobian ideal of F generated by the first partial derivatives of F. In the proof of Theorem 1, we will use this kind of algebraic representations of $H^{2,0}(S)$ and $H_{\mathrm{pr}}^{1,1}(S)$ for $S \in Y_n$.

We need the following special cases of the Bott Vanishing Theorem (cf. [2]):

Bott Vanishing Theorem. $H^p(\mathbf{P}^n, \Omega^q_{\mathbf{P}^n}(k)) = 0$ *unless*

(i) $p = q$ *and* $k = 0$,
(ii) $p = 0$ *and* $k > q$, *or*
(iii) $p = n$ *and* $k < q - n$.

We will also use the following well-known fact (see, e.g., [18, pp. 445–446]):
(2.2) Let

$$0 \to \mathscr{K}^0 \to \cdots \to \mathscr{K}^m \to 0$$

be an exact sequence of sheaves on a topological space X. Then there is a spectral sequence abutting to zero with $E_1^{p,q} = H^q(X, \mathscr{K}^p)$.

Let $B = \bigoplus_{i=1}^{n-2} \mathscr{O}_{\mathbf{P}^n}(-d_i)$. Then for $S \in Y_n$, there is a Koszul complex

$$0 \to \bigwedge^{n-2} B \to \bigwedge^{n-3} B \to \cdots \to \bigwedge^2 B \to B \to \mathscr{O}_{\mathbf{P}^n} \to \mathscr{O}_S \to 0, \tag{2.3}$$

which is exact since S is a complete intersection (see, e.g., [18, p. 688]). We denote $\mu = \sum_{i=1}^{n-2} d_i - n - 1$. For an algebraic representation of $H^{2,0}(S)$, tensoring (2.3) with $\mathscr{O}_{\mathbf{P}^n}(\mu)$ and applying (2.2), we obtain a spectral sequence abutting to zero with $E_1^{p,q} = 0$ unless $q = 0$, $q = n$, or $p = n - 1$. There is no nonzero differential other than the differentials in E_1 coming into the position $(p, 0)$ for $p = 0, 1, \dots, n - 1$. So we obtain an exact sequence

$$\cdots \to H^0\left(\mathbf{P}^n, \bigoplus_{i=1}^{n-2} \mathscr{O}_{\mathbf{P}^n}(\mu - d_i)\right) \to H^0(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(\mu)) \to H^0(\mathbf{P}^n, \mathscr{O}_S(\mu)) \to 0,$$

and hence

$$H^{2,0}(S) \simeq H^0(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(\mu))/\mathrm{im}\, H^0\left(\mathbf{P}^n, \bigoplus_{i=1}^{n-2} \mathscr{O}_{\mathbf{P}^n}(\mu - d_i)\right).$$

For an algebraic representation of $H^{1,1}_{\mathrm{pr}}(S)$ for $S \in Y_n$, we take the long exact sequence on the cohomology of the short exact sequence

$$0 \to \Theta_S \otimes K_S \to \Theta_{\mathbf{P}^n}|_S \otimes K_S \to N_{S|\mathbf{P}^n} \otimes K_S \to 0,$$

where Θ_S and $N_{S|\mathbf{P}^n}$ denote the holomorphic tangent bundle of S and the normal bundle of S in $\mathbf{P}^n$, respectively. Then we get

$$\begin{aligned} &\to H^0(S, \Theta_{\mathbf{P}^n}|_S \otimes K_S) \to H^0(S, N_{S|\mathbf{P}^n} \otimes K_S) \to H^1(S, \Theta_S \otimes K_S) \\ &\quad \to H^1(S, \Theta_{\mathbf{P}^n}|_S \otimes K_S) \to \cdots . \end{aligned}$$

So

$$\frac{H^0(S, N_{S|\mathbf{P}^n} \otimes K_S)}{\mathrm{im}\, H^0(S, \Theta_{\mathbf{P}^n}|_S \otimes K_S)} \simeq \left(\frac{H^1(S, \Omega^1_S)}{\mathrm{im}\, H^1(S, \Omega_{\mathbf{P}^n}|_S)}\right)^*$$

by Serre duality. We will show that

$$H^{1,1}_{\mathrm{pr}}(S) \simeq \frac{H^1(S, \Omega^1_S)}{\mathrm{im}\, H^1(S, \Omega^1_{\mathbf{P}^n}|_S)}. \tag{2.4}$$

Applying (2.2) to the exact sequence (2.3) tensored with $\Omega^1_{\mathbf{P}^n}$, we get a spectral sequence abutting to zero. By the Bott Vanishing Theorem, $E_1^{p,q} = 0$ unless $q = 0, n$, or $p = n-1$, or $(p, q) = (n-2, 1)$. Moreover, no nonzero differential except the differential in E_1 comes into or goes out of the position $(n-2, 1)$ or $(n-1, 1)$. So $H^1(\mathbf{P}^n, \Omega^1_{\mathbf{P}^n}) = H^1(S, \Omega^1_{\mathbf{P}^n}|_S)$. From the exact sequence (2.3) tensored with the dual E^* of E, we get a spectral sequence abutting to zero with $E_1^{p,q} = 0$ unless $q = 0$, or $q = n$, or $p = n-1$. No nonzero differential comes into the position $(n-1, 1)$. So $H^1(S, \mathscr{O}_S \otimes E^*) = 0$. We note that $N_{S|\mathbf{P}^n} = \mathscr{O}_S \otimes E$. Thus

$$\mathrm{im}\, H^1(S, \Omega^1_{\mathbf{P}^n}|_S) \simeq H^1(S, \Omega^1_{\mathbf{P}^n}|_S) \simeq H^1(\mathbf{P}^n, \Omega^1_{\mathbf{P}^n}) \simeq (\omega),$$

where ω is the associated (1,1) form of $\mathbf{P}^n$ (i.e., ω is the first Chern class $c_1(\mathscr{O}_{\mathbf{P}^n}(1))$ of $\mathscr{O}_{\mathbf{P}^n}(1)$). By Lefschetz decomposition, $H^1(\Omega^1_S) \simeq H^{1,1}_{\mathrm{pr}}(S) \oplus \omega|_S \cdot H^{0,0}(S)$. Hence we get (2.4). From the spectral sequence attached to the exact sequence (2.3) tensored with $E(\mu)$, we can see that

$$0 \to H^0(\mathbf{P}^n, E(-n-1)) \to \cdots \to H^0(\mathbf{P}^n, E(\mu)) \xrightarrow{r} H^0(S, N_{S|\mathbf{P}^n} \otimes K_S) \to 0$$

is exact. Hence

$$\frac{H^0(S, N_{S|\mathbf{P}^n} \otimes K_S)}{\mathrm{im}\, H^0(S, \Theta_{\mathbf{P}^n}|_S \otimes K_S)} \simeq \frac{H^0(\mathbf{P}^n, E(\mu))}{r^{-1}(\mathrm{im}\, H^0(\Theta_{\mathbf{P}^n}|_S \otimes K_S))}.$$

Summarizing the above computations, we obtain the following identifications:

Proposition 2.

(i) $$H^{2,0}(S) \simeq \frac{H^0(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(\mu))}{\operatorname{im} H^0(\mathbf{P}^n, \bigoplus_{i=1}^{n-2} \mathscr{O}_{\mathbf{P}^n}(\mu - d_i))},$$

(ii) $$H^{1,1}_{\mathrm{pr}}(S)^* \simeq \frac{H^0(S, N_{S|\mathbf{P}^n} \otimes K_S)}{\operatorname{im} H^0(S, \Theta_{\mathbf{P}^n}|_S \otimes K_S)} \simeq \frac{H^0(\mathbf{P}^n, E \otimes \mathscr{O}_{\mathbf{P}^n}(\mu))}{r^{-1}(\operatorname{im} H^0(S, \Theta_{\mathbf{P}^n}|_S \otimes K_S))}.$$

We also need an algebraic representation of the subspace of $H^1(S, \Theta_S)$ parametrizing the deformations of S in $\mathbf{P}^n$, that is, the image of the Zariski tangent space $T_S(Y_n)$ of Y_n at S under the Kodaira-Spencer map $\rho: T_S(Y_n) \to H^1(S, \Theta_S)$. Let $S = \bigcap_{i=1}^{n-2}\{F_i = 0\}$. $T_S(Y_n)$ is naturally isomorphic to

$$\operatorname{Hom}((S), H^0(\mathbf{P}^n, E)/(S)) \simeq H^0(\mathbf{P}^n, E)/(S),$$

where (S) denotes the 1-dimensional subspace of $H^0(\mathbf{P}^n, E)$ generated by $(F_1, \dots, F_{n-2})$. So the map $T_S(Y) \to H^0(S, N_{S|\mathbf{P}^n})$ is surjective and

$$\rho(T_S(Y_n)) = \operatorname{im}\{H^0(S, N_{S|\mathbf{P}^n}) \to H^1(S, \Theta_S)\}.$$

Tensoring the exact sequence (2.3) with $\Theta_{\mathbf{P}^n}$ and applying (2.2), we have a spectral sequence abutting to zero. By Serre duality and the Bott Vanishing Theorem, $H^0(\mathbf{P}^n, \Theta_{\mathbf{P}^n}(k))$ vanishes unless $-k-n-1 < 1-n$. Hence $E_1^{p,q} = 0$ unless $(p, q) = (n-2, 0)$, $(n-1, 0)$, $(n-2, 1)$, $(n-1, 1)$, or $q = n$. No nonzero differential except the differentials in E_1 comes into the position $(n-2, 0)$ or $(n-1, 0)$. Hence the map $\gamma_1: H^0(\mathbf{P}^n, \Theta_{\mathbf{P}^n}) \to H^0(S, \Theta_{\mathbf{P}^n}|_S)$ is an isomorphism. From the spectral sequence attached to the exact sequence (2.3) tensored with E, we can see that the map $\gamma_2: H^0(\mathbf{P}^n, E) \to H^0(S, E \otimes \mathscr{O}_S)$ is surjective. From the short exact sequence

$$0 \to \Theta_S \to \Theta_{\mathbf{P}^n}|_S \to N_{S|\mathbf{P}^n} \to 0,$$

we get the following long exact sequence which fits into a diagram:

$$\begin{array}{ccccccc} & & & & \downarrow & & \\ \longrightarrow & H^0(\mathbf{P}^n, \Theta_{\mathbf{P}^n}) & \xrightarrow{\gamma_1} & & H^0(S, \Theta_{\mathbf{P}^n}|_S) & & \\ & & & & \downarrow \alpha & & \\ \longrightarrow & H^0(\mathbf{P}^n, E) & \xrightarrow{\gamma_2} & & H^0(S, N_{S|\mathbf{P}^n}) & \longrightarrow & 0 \\ & & & & \downarrow \beta & & \\ & & & & H^1(S, \Theta_S) & & \\ & & & & \downarrow & & \end{array} \qquad (2.5)$$

Hence

$$(2.6) \qquad \rho(T_S(Y_n)) \simeq \frac{H^0(S, N_{S|\mathbf{P}^n})}{\alpha \circ \gamma_1(H^0(\mathbf{P}^n, \Theta_{\mathbf{P}^n}))} \simeq \frac{H^0(\mathbf{P}^n, E)}{\gamma_2^{-1}(\alpha(H^0(S, \Theta_{\mathbf{P}^n}|_S)))}.$$

Another preliminary fact we will use is the description of the Zariski tangent space to

$$\widetilde{Y}_n = \{(S, L) \mid S \in Y_n, \ L \in \mathrm{Pic}(S)\}.$$

The first prolongation bundle $P_1(L)$ of L is defined by an exact sequence

$$0 \to \Omega_S^1 \otimes L \to P_1(L) \to L \to 0$$

with the extension class $c_1(L) \in \mathrm{Ext}^1(L, \Omega_S^1 \otimes L) = H^1(S, \Omega_S^1)$. The computation of Zariski tangent space to the set of pairs of curves with line bundles is given in [1]. An analogous argument gives the description for the surface case: For a fixed (S, L), the Zariski tangent space $T_{(S,L)}(\widetilde{Y}_n)$ of Y_n at (S, L) maps into $H^1(S, P_1(L)^* \otimes L)$ as follows. As a complex manifold, the line bundle $L \to S$ is given by the data

$$\{U_\alpha, z_\alpha, f_{\alpha\beta}, g_{\alpha\beta}\},$$

where $\{U_\alpha\}$ is a finite open covering of S, $z_\alpha = (z_{\alpha_1}, z_{\alpha_2})$ are local coordinates in U_α, $f_{\alpha\beta}$ is the coordinate transformation on $U_\alpha \cap U_\beta$, and $g_{\alpha\beta}$ is the transition function for L. Thus two cocycle rules $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$ and $g_{\alpha\gamma} = g_{\alpha\beta} g_{\beta\gamma}$ hold in $U_\alpha \cap U_\beta \cap U_\gamma$. The first order deformation of $L \to S$ is given by

$$\{U_\alpha, z_\alpha, f_{\alpha\beta}(z_\beta, t), g_{\alpha\beta}(z_\beta, t)\}$$

satisfying

$$f_{\alpha\gamma}(z_\gamma, t) \equiv f_{\alpha\beta}(f_{\beta\gamma}(z_\gamma, t), t) \bmod t^2,$$

$$g_{\alpha\gamma}(z_\gamma, t) \equiv g_{\alpha\beta}(f_{\beta\gamma}(z_\gamma, t), t) \cdot g_{\beta\gamma}(z_\gamma, t) \bmod t^2$$

on $U_\alpha \cap U_\beta \cap U_\gamma$. Taking derivatives at $t = 0$, we can see that $\tilde{f}_{\alpha\beta} = \{\frac{\partial f_{\alpha\beta}}{\partial t} \frac{\partial}{\partial z_\alpha}\}$ is a cocycle defining a class $\tilde{f} = \{\tilde{f}_{\alpha\beta}\}$ in $H^1(S, \Theta_S)$ and that $\{\tilde{g}_{\alpha\beta} = \frac{\partial g_{\alpha\beta}}{\partial t} g_{\alpha\beta}^{-1}\}$ is a 1-cochain with coefficients in $\mathscr{O}_S$. For the coboundary map δ, $\delta(\{\tilde{g}_{\alpha\beta}\})$ is the cup product of $\tilde{f}$ with $c_1(L)$. Note that $c_1(L) = \{g_{\alpha\beta}^{-1}\, dg_{\alpha\beta}\} \in H^1(S, \Omega_S^1)$. $\sigma = \{(\tilde{f}_{\alpha\beta}, \tilde{g}_{\alpha\beta})\}$ defines a 1-cocycle with coefficients in the extension M of Θ_S by $\mathscr{O}_S$, i.e., M is defined by the exact sequence

$$0 \to \mathscr{O}_S \to M \to \Theta_S \to 0,$$

with the extension class $c_1(L)$. But $M = P_1(L)^* \otimes L$. So

$$(\sigma) \in H^1(S, P_1(L)^* \otimes L).$$

Proof of Theorem 1. Let $\widetilde{\Sigma}_n = \{(S, L) \mid S \in \Sigma_n \text{ and } L \in \mathrm{Pic}(S)\}$, and let $\pi : (S, L) \mapsto S$ be a projection. For $(S, L) \in \widetilde{Y}_n$, we have a commutative diagram

$$\begin{array}{ccc} T_{(S,L)}(\tilde{Y}_n) & \xrightarrow{\pi_*} & T_S(Y_n) \\ \downarrow & & \downarrow \rho \\ H^1(S, P_1(L)^* \otimes L) & \xrightarrow{h_1} & H^1(S, \Theta_S) \end{array}$$

where h_1 sits in the long exact sequence on cohomology

$$\to H^1(S, P_1(L)^* \otimes L) \xrightarrow{h_1} H^1(S, \Theta_S) \xrightarrow{h_2} H^2(S, \mathcal{O}_S) \to \cdots .$$

Fix $(S, L) \in \widetilde{\Sigma}_n$ with $c_1(L) \in H^{1,1}_{\mathrm{pr}}(S)$. Let Z be the union of all irreducible components of $\widetilde{\Sigma}_n$ containing (S, L). The image $T(Z)$ of the Zariski tangent space $T_S(\pi(Z))$ of $\pi(Z)$ at S under ρ is in the kernel of h_2, i.e.,

$$\rho(T_S(Y_n)) \otimes H^{1,1}_{\mathrm{pr}}(S) \xrightarrow{\cup} H^2(S, \mathcal{O}_S),$$
$$T(Z) \otimes H^0(S, K_S) \mapsto 0.$$

Equivalently,

$$\rho(T_S(Y_n)) \otimes H^0(S, K_S) \xrightarrow{\cup} H^{1,1}_{\mathrm{pr}}(S)^*,$$
$$T(Z) \otimes H^0(S, K_S) \mapsto c_1(L)^{\perp}. \tag{2.7}$$

Using the notations in the diagram (2.5), we set $T' = \gamma_2^{-1} \circ \beta^{-1}(T(Z)) \subset H^0(\mathbf{P}^n, E)$. Then $T' \supset \gamma_2^{-1}(\mathrm{im}\,\alpha) \supset \ker \gamma_2$ and the following holds:

Claim. If $S = \bigcap_{i=1}^{n-2}\{F_i = 0\}$, and F_i is a homogeneous polynomial of degree d_i such that $\bigcap_{i=1}^{k}\{F_i = 0\}$ is nonsingular for each k with $d_k < d_{k+1}$, then the evaluation map $T' \otimes \mathcal{O}_{\mathbf{P}^n, x} \to E_x$ is surjective at every $x \in \mathbf{P}^n$.

To see this, let $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbf{C}^{n-2}$ denote the ith coordinate vector, for $i = 1, \dots, n-2$. Then

$$\ker \gamma_2 \supseteq \{F_i G_k e_k \mid G_k \in H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d_k - d_i)),$$
$$d_k \geq d_i, \text{ and } i = 1, \dots, n-2\}.$$

We note that

$$\gamma_2^{-1}(\mathrm{im}\,\alpha) \supseteq \left\{ z_l \left(\frac{\partial F_1}{\partial z_j}, \dots, \frac{\partial F_{n-2}}{\partial z_j} \right) \mid l, j = 0, 1, \dots, n \right\}.$$

For a fixed $x \in \mathbf{P}^n$, let i_0 denote the smallest number such that $x \in \{F_{i_0} \neq 0\}$. Then (i) $i_0 = 1$, or (ii) $i_0 > 1$ and $d_{i_0 - 1} < d_{i_0}$, or (iii) $i_0 > 1$ and $d_{i_0-1} = d_{i_0}$. We will show that the evaluation map at x is surjective in any case.

Case (i): If $i_0 = 1$, then we can choose $G_k \in H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d_k - d_1))$, $k = 1, \dots, n-2$ so that $G_k(x) \neq 0$ for each k. So $\{F_1(x)G_k(x)e_k \mid k = 1, \dots, n-2\}$ are $n-2$ linearly independent elements in E_x.

Case (ii): If $i_0 > 1$ and $d_{i_0-1} < d_{i_0}$, then by the hypothesis of the claim $\bigcap_{i=1}^{i_0-1}\{F_i = 0\}$ has no singularity and so there is a nonvanishing $(i_0-1)\times(i_0-1)$ minor of a matrix

$$\left(\left.\frac{\partial F_i}{\partial z_j}\right|_x\right)_{\substack{i=1,\dots,i_0-1\\ j=0,1,\dots,n}},$$

say

$$\left(\left.\frac{\partial F_i}{\partial z_j}\right|_x\right)_{\substack{i=1,\dots,i_0-1\\ j=j_1,\dots,j_{i_0-1}}},$$

which has rank $i_0 - 1$. Moreover, there is some m such that $z_m(x) \neq 0$. We can choose $G_k \in H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d_k - d_{i_0}))$ for $k = i_0, i_0+1, \dots, n-2$ so that $G_k(x) \neq 0$ for each k. Then

$$\left\{z_m\left(\frac{\partial F_1}{\partial z_j}, \dots, \frac{\partial F_{n-2}}{\partial z_j}\right) \mid j = j_1, \dots, j_{i_0-1}\right\}$$
$$\cup \{F_{i_0} G_k e_k \mid k = i_0, i_0+1, \dots, n-2\}$$

provides $n-2$ linearly independent elements in E_x when evaluated at x.

Case (iii): If $i_0 > 1$ and $d_{i_0-1} = d_{i_0}$, let i_1 be the smallest number such that $d_{i_1} = \cdots = d_{i_0-1} = d_{i_0}$. Then $d_{i_1-1} < d_{i_1}$ and by the hypothesis of the claim, $\bigcap_{i=1}^{i_1-1}\{F_i = 0\}$ has no singularity. So, as in (ii) we can find $j_1, \dots, j_{i_1-1}$ such that

$$\left(\left.\frac{\partial F_i}{\partial z_j}\right|_x\right)_{\substack{i=1,\dots,i_1-1\\ j=j_1,\dots,j_{i_1-1}}}$$

has rank $i_1 - 1$. Let $G_k \in H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d_k - d_{i_0}))$ be chosen so that $G_k(x) \neq 0$ for $k = i_1, i_1+1, \dots, i_0, \dots, n-2$. Furthermore, there is some m such that $z_m(x) \neq 0$. Hence

$$\left\{z_m\left(\frac{\partial F_1}{\partial z_j}, \dots, \frac{\partial F_{n-2}}{\partial z_j}\right) \mid j = j_1, \dots, j_{i_1-1}\right\}$$
$$\cup \{F_{i_0} G_k e_k \mid k = i_1, i_1+1, \dots, n-2\}$$

defines $n-2$ linearly independent vectors in E_x when evaluated at x.

Thus, in any case, the evaluation map at x is surjective and the claim follows.

In terms of the identifications in Proposition 2 and (2.6), (2.7) implies that the evaluation map

$$T' \otimes H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(\mu)) \to H^0(\mathbf{P}^n, E(\mu))$$

is not surjective. Therefore, by Proposition 1, codim $T' \geq \sum_{i=1}^{n-2} d_i - n$. But

$$\operatorname{codim} T' \leq \operatorname{codim}_{T_S(Y_n)} T_S(\pi(Z)) \leq \operatorname{codim}_{\widetilde{Y}_n} Z.$$

Hence the theorem follows.

3. A NEW PROOF OF THE DENSITY THEOREM

In this section, we denote $d = d_1$, $Y = Y_3$, and $NL_d = \Sigma_3$. Recall (cf. [3]) that the upper bound of the codimension of irreducible components of the Noether-Lefschetz locus NL_d in the family Y of smooth surfaces of degree d in $\mathbf{P}^3$ is the geometric genus $p_g = \binom{d-1}{3}$ of any surface in Y. We will give a new proof of the following density theorem due to Ciliberto, Harris, and Miranda [5].

Theorem 2. *For $d \geq 4$, the union of all irreducible components of NL_d having codimension p_g in Y is dense in the classical topology.*

Using an infinitesimal method, we will reduce the theorem to the following proposition.

Proposition 3. *For each $d \geq 4$, there are some polynomials $G \in S^{2d-4}$ and a surface $X \in Y$ with defining equation F such that the map*

$$g\colon S^{d-4} \to S^{3d-8}/J_{F,3d-8}$$

defined by multiplication by G is injective.

Proof. Let $F = z_0^d + z_1^d + z_2^d + z_3^d$ and

$$G = \sum_{j=0}^{d-2} a_j z_0^j z_1^j z_2^{d-2-j} z_3^{d-2-j},$$

where the constant coefficients a_j's are chosen so that every possible matrix of the form

$$\begin{pmatrix} a_k & a_{k+1} & \cdots & a_{k+m} \\ a_{k+1} & a_{k+2} & \cdots & a_{k+m+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k+m} & a_{k+m+1} & \cdots & a_{k+2m} \end{pmatrix} \tag{3.1}$$

has nonzero determinant. Then we claim that g is injective:

Without loss of generality, we may assume that a nonzero element of the kernel of g is of the form

$$P = \sum_{j=m_1}^{m_2} c_j z_0^{p+j} z_1^{q+j} z_2^{r-j} z_3^{s-j},$$

where $p + q + r + s = d - 4$ and $m_1 < m_2$. This is because G belongs to the span of the set of monomials $z_0^{i_0} z_1^{i_1} z_2^{i_2} z_3^{i_3}$ satisfying the equalities

$$i_0 = i_1 = d - 2 - i_2 = d - 2 - i_3.$$

If we therefore break up S^{d-4} into the span of monomials satisfying

$$i_0 - i_1 = p - q, \quad i_0 + i_2 = p + r, \quad i_0 + i_3 = p + s,$$

where p, q, r, s vary but add up to $d-4$, and if we expand an element of $\ker g$ in terms of these subspaces, then each piece also lies in $\ker g$.

By symmetry of the role of z_0 and z_1, and of z_2 and z_3, we may assume $p \geq q$ and $r \leq s$. Then the limits of the sum above satisfy $m_1 \geq -q$ and $m_2 \leq r$. The condition that

$$P \cdot G = \sum_{j,k} a_j c_k z_0^{j+k+p} z_1^{j+k+q} z_2^{d-2-j-k+r} z_3^{d-2-j-k+s} \in J_{F,3d-8}$$

is equivalent to the system of equations

$$\sum_{j+k=l} a_j c_k = 0 \quad \text{for } s \leq l \leq d-2-p.$$

Since $m_1 \geq -p$ and $m_2 \leq s$, the two inequalities $m_1 \leq k \leq m_2$ and $s \leq k+j \leq d-2-p$ imply the inequality $0 \leq j \leq d-2$.

The coefficient matrix for the c_k's is

$$A = \begin{pmatrix} a_\alpha & a_{\alpha+1} & \cdots & a_\beta \\ a_{\alpha+1} & a_{\alpha+2} & \cdots & a_{\beta+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_\gamma & a_{\gamma+1} & \cdots & a_\delta \end{pmatrix},$$

where

$$\alpha = s - m_2, \quad \beta = s - m_1, \quad \gamma = d-2-p-m_2, \quad \delta = d-2-p-m_1.$$

The number of rows is

$$\gamma - \alpha + 1 = d-2-p-s+1 \tag{3.2}$$

and the number of columns is

$$\beta - \alpha + 1 = m_2 - m_1 + 1. \tag{3.3}$$

But (3.2)–(3.3) $= d-2-p-s-(m_2-m_1) \geq d-2-p-s-(r+q) = 2$. Therefore, g is injective provided that the appropriate minors of the matrix of the a_j's of the form (3.1) are nonvanishing, and this may be arranged by taking the ratios $|a_{j+1}/a_j|$ to increase very rapidly with j.

Proof of Theorem 2. For a smooth surface $X \in Y$, $X \in NL_d$ if and only if $H^{1,1}_{\text{pr}}(X) \cap H^2(X, \mathbf{Z}) \neq 0$. If there is a nonzero element $\gamma \in H^{1,1}_{\text{pr}}(X) \cap H^2(X, \mathbf{Q})$, then $m \cdot \gamma \in H^{1,1}_{\text{pr}}(X) \cap H^2(X, \mathbf{Z})$ for some integer m and hence $X \in NL_d$. For a given $\gamma \in H^{1,1}_{\text{pr}}(X) \cap H^2(X, \mathbf{R})$, there are some elements of $H^2_{\text{pr}}(X, \mathbf{Q})$ that are arbitrarily near to γ. We will show that one of these rational classes can be made to have type $(1, 1)$ by making a small deformation of X.

We consider the universal family $\mathscr{F}$ of smooth surfaces of degree d in $\mathbf{P}^3$:

$$\begin{array}{ccc} \mathscr{F} & \longrightarrow & Y \times \mathbf{P}^3 \\ {\scriptstyle\pi}\big\downarrow & & \\ Y & & \end{array}$$

Since π is a proper smooth map with maximal rank everywhere, Ehresmann's fibration theorem says that on a sufficiently small open neighborhood U of X, there is a fiber preserving diffeomorphism

$$\phi: \pi^{-1}(X) \times U \simeq \pi^{-1}(U) \tag{3.4}$$

so that ϕ defines a diffeomorphism $\phi_S: X \to S$ and the induced map on the cohomology $\phi_S^*: H^2(S, \mathbf{C}) \to H^2(X, \mathbf{C})$ is an isomorphism for $S \in U$.

Let $R^2\pi_*\mathbf{C}$ be the second direct image sheaf of $\pi: \mathscr{F} \to Y$, which we recall is the sheaf associated to the presheaf

$$U \to H^2(\pi^{-1}(U), \mathbf{C}),$$

where U runs through the open subsets of Y. Let R^2_{pr} be the kernel of a map

$$L: R^2\pi_*\mathbf{C} \to R^4\pi_*\mathbf{C}$$

defined as follows: For an open set $U \subset Y$ with $\pi^{-1}(U) \simeq \pi^{-1}(X) \times U$ as before,

$$H^2(\pi^{-1}(U), \mathbf{C}) \simeq H^2(X, \mathbf{C}).$$

$L_U: R^2\pi_*\mathbf{C}(U) \to R^4\pi_*\mathbf{C}(U)$ is the cup product map with the associated $(1, 1)$ form of X.

Then R^2_{pr} is a locally constant sheaf and there is a holomorphic vector bundle $\mathscr{H}$ on Y associated to it, whose fiber over $S \in Y$ is $H^2_{\mathrm{pr}}(X, \mathbf{C})$. We have a Hodge filtration $F^2 \subset F^1 \subset F^0 = \mathscr{H}$, and Hodge bundles $\mathscr{H}^{1,1} = F^1/F^2$ and $\mathscr{H}^{0,2} = F^0/F^1$, where the F^p's are holomorphic vector bundles.

For a sufficiently small open neighborhood U of X as in (3.4), we can define a smooth map $f_{\mathbf{C}}$ on the total space of $\mathscr{H}^{1,1}|_U$ as

$$f_{\mathbf{C}}: \mathscr{H}^{1,1}|_U = \{(S, \gamma) | S \in U, \gamma \in H^{1,1}_{\mathrm{pr}}(S)\} \to H^2_{\mathrm{pr}}(X, \mathbf{C}),$$
$$(S, \gamma) \longmapsto \phi_S^*(\gamma).$$

Then $f_{\mathbf{C}}$ restricts to a map

$$f: \mathscr{H}^{1,1}_{U,\mathbf{R}} = \{(S, \gamma) | S \in U, \gamma \in H^{1,1}_{\mathrm{pr}}(S) \cap H^2(S, \mathbf{R})\} \to H^2_{\mathrm{pr}}(X, \mathbf{R}).$$

We note that for the map $\pi_1: \mathscr{H}^{1,1} \to Y$, giving the bundle structure on $\mathscr{H}^{1,1}$,

$$\pi_1(f^{-1}(H^2_{\mathrm{pr}}(X, \mathbf{Q}))) = NL_d \cap U.$$

First, we will show that f has maximal rank at some $(S_0, \gamma_0) \in \mathscr{H}^{1,1}_{U,\mathbf{R}}$. Then, by the Implicit Function Theorem, this implies that

(3.5) there is an element $\gamma_V \in f(V) \cap H^2_{\mathrm{pr}}(X, \mathbf{Q})$ for each small open neighborhood V of (S_0, γ_0), and codim $\pi_1(f^{-1}(\gamma_V)) = p_g$.

In order to make the necessary computation, it is a good idea to distinguish the real tangent space $T_S(U)_{\mathbf{R}}$, the complexified tangent space $T_S(U)_{\mathbf{C}}$, and the holomorphic tangent space $T_S(U)$. There is of course a natural $\mathbf{R}$-linear isomorphism $T_S(U) \cong T_S(U)_{\mathbf{R}}$. Since df takes the tangent space of the fibers of π_1 to $H^{1,1}_{\text{pr}}(X) \cap H^2(X, \mathbf{R})$, we obtain an induced $\mathbf{R}$-linear map

$$\lambda\colon T_S(U)_{\mathbf{R}} \to \frac{H^2_{\text{pr}}(X, \mathbf{R})}{H^{1,1}_{\text{pr}}(X) \cap H^2(X, \mathbf{R})} \cong (H^{2,0}(X) \oplus H^{0,2}(X)) \cap H^2(X, \mathbf{R})$$

having maximal rank if and only if f does. Under the $\mathbf{R}$-linear identifications $T_S(U)_{\mathbf{R}} \cong T_S(U)$ and

$$(H^{2,0}(X) \oplus H^{0,2}(X)) \cap H^2(X, \mathbf{R}) \cong H^{0,2}(X),$$

the map λ is identified with the derivative of the period map

$$T_S(U) \to H^{0,2}(X).$$

By the work of Griffiths [14], the derivative of the period map is the composition of the Kodaira-Spencer map ρ with the cup product with γ, i.e.

$$T_S(Y) \xrightarrow{\rho} H^1(S, \Theta_S) \xrightarrow{\cup\gamma} H^{0,2}(S).$$

Thus λ, and hence f, has maximal rank if and only if

$$\cup\gamma\colon \rho(T_S(Y)) \to H^{0,2}(S)$$

is surjective, or equivalently,

$$H^{2,0}(S) \xrightarrow{\cup\gamma} \rho(T_S(Y))^* \quad \text{is injective.} \tag{3.6}$$

Referring to (2.6),

$$\rho(T_S(Y)) \simeq S^d / J_{F,d},$$

where F is the defining equation of S. By Macaulay's theorem (see, e.g., [9, Theorem 2.15]),

$$(\rho(T_S(Y)))^* \simeq S^{3d-8} / J_{F,3d-8}.$$

In terms of the identifications in (2.1) and above, the above map (3.6) is injective if the multiplication map

$$g\colon S^{d-4} \to S^{3d-8} / J_{F,3d-8}$$

is injective, where g is the multiplication by $G(\gamma) \in S^{2d-4}$ corresponding to γ. By Proposition 3, g is injective at (S_0, γ_0), where

$$S_0 = \{z_0^d + z_1^d + z_2^d + z_3^d = 0\}$$

and γ_0 corresponds to $G \in S^{2d-4}/J_{F,2d-4}$ with some fixed real coefficients a_i's, and hence $\gamma_0 \in H^{1,1}_{\text{pr}}(S_0) \cap H^2(S_0, \mathbf{R})$. So f has maximal rank p_g at (S_0, γ_0).

In fact, ρ composed with the cup product map $\cup\gamma$ gives rise to a holomorphic map of vector bundles on $\mathscr{H}^{1,1}$ so that we can define a map

$$\sigma : \mathscr{H}^{1,1} \to \Theta_Y^* \otimes \mathscr{H}^{0,2},$$
$$(S, \gamma) \mapsto \sigma(S, \gamma) : \Theta_Y \to \mathscr{H}^{0,2}.$$

The locus A where $\sigma(S, \gamma)$ drops rank is an analytic subvariety of $\mathscr{H}^{1,1}$. Since $\sigma(S_0, \gamma_0)$ has maximal rank p_g, A is proper. Since f has maximal rank at (S_0, γ_0), $A \cap \mathscr{H}_{\mathbf{R}}^{1,1}$ is also proper, where $\mathscr{H}_{\mathbf{R}}^{1,1} = \{(S, \gamma) | \gamma \in H_{\mathrm{pr}}^{1,1}(S) \cap H^2(S, \mathbf{R})\}$. Hence, (3.5) holds for every $(S, \gamma) \in \mathscr{H}^{1,1}$ and the theorem follows.

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