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ULTRA-IRREDUCIBILITY OF INDUCED REPRESENTATIONS OF SEMIDIRECT PRODUCTS

HENRIK STETKÆR

ABSTRACT. Let the Lie group G be a semidirect product, $G = SK$, of a connected, closed, normal subgroup S and a closed subgroup K. Let Λ be a nonunitary character of S, and let K_Λ be its stability subgroup in K. Let $I^{\Lambda\mu}$, for any irreducible representation μ of K_Λ, denote the representation $I^{\Lambda\mu}$ of G induced by the representation $\Lambda\mu$ of SK_Λ. The representation spaces are subspaces of the distributions.

We show that $I^{\Lambda\mu}$ is ultra-irreducible when the corresponding Poisson transform is injective, and find a sufficient condition for this injectivity.

I. INTRODUCTION

Let G be a Lie group which is a semidirect product, $G = SK$, of a connected, closed, normal subgroup S and a compact subgroup K. Let Λ be a continuous homomorphism of S into $\mathbb{C}\backslash\{0\}$, and let $M := K_\Lambda$ be its stability subgroup in K. For any continuous irreducible representation μ of M on a complex vector space $H(\mu)$ we shall consider the representation $I^{\Lambda\mu}$ of G induced by the representation $\Lambda\mu$ of SM. $I^{\Lambda\mu}$ is a (not necessarily unitary) representation of G on a subspace of the Hilbert space $L^2(K, H(\mu))$, viz. on the subspace $L^2_\mu(K, H(\mu))$ consisting of the vectors f which satisfy the convariance condition

$$(*) \qquad f(mk) = \mu(m)[f(k)] \quad \text{for } m \in M \text{ and } k \in K.$$

The classical unitary theory, which will not be discussed here, is primarily due to Mackey. The special case of the Cartan motion group is treated in detail in [Gi]. In the nonunitary case several authors have studied the question of irreducibility of $I^{\Lambda\mu}$: [Th, Wi, Ra] are concerned with topological irreducibility for general semidirect products, and [CD] with topologically complete irreducibility in the case of the Cartan motion group. Nonunitary representations of the Cartan motion group on eigenspaces of invariant differential operators are studied in [He].

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The present paper deals with the stronger notion of ultra-irreducibility of $I^{\Lambda\mu}$ for general semidirect products with applications to the Cartan motion group, and the subgroup K need no longer be compact.

Our point of view is that there is no need for the Hilbert space frame once we accept nonunitary representations. Any $I^{\Lambda\mu}$-invariant space E of functions or distributions on K satisfying $(*)$ should be studied; the space of square integrable functions satisfying $(*)$ is just a special case. More precisely we let E be an $I^{\Lambda\mu}$-invariant subspace of the distributions $\mathscr{D}'(K, H(\mu))$ such that

$$C_\mu^\infty(K, H(\mu)) \subset E \subset \mathscr{D}_\mu'(K, H(\mu)),$$

where the subscript μ indicates that the elements should satisfy $(*)$. Particular cases are the extreme ones $C_\mu^\infty(K, H(\mu))$, $\mathscr{D}_\mu'(K, H(\mu))$ and

$$E = C_\mu(K, H(\mu)), L_\mu^p(K, H(\mu)) \quad \text{for } K \text{ compact and } 1 \le p < \infty.$$

If $I^{\Lambda\mu}$ is ultra-irreducible on one of the $I^{\Lambda\mu}$-invariant spaces between $C_\mu^\infty(K, H(\mu))$ and $\mathscr{D}_\mu'(K, H(\mu))$, it is so on all the others (Theorem II.6).

We define a Poisson transform for a semidirect product as above, and find a sufficient, algebraic condition on it to be injective (Theorem IV.2). The condition is weaker than certain of the conditions found in the literature (e.g., [Th, Theorem 4; Wi, Theorem 5.3; Ra, §2.4]) and holds for the Cartan motion group.

By help of Litvinov and Lomonosov's version of Burnside's theorem [LL1] we establish that, when the Poisson transform is injective, then the representation $I^{\Lambda\mu}$ is ultra-irreducible (Theorem V.2).

Finally we get as a corollary of the above results that those representations of the Cartan motion group which are studied by Champetier and Delorme in [CD], are ultra-irreducible (Theorem VI.1).

II. Group representations on locally convex spaces

A. Notation, terminology and basic facts. By a locally convex space we shall in this paper mean a locally convex, Hausdorff, topological vector space over the field of the complex numbers $\mathbb{C}$.

Let E be a locally convex space. Its strong topological dual will be denoted by E'. Furthermore $L(E)$, resp. $S(E)$, resp. $B(E)$, denotes the vector space of those linear maps of E into E, which are continuous, resp. are continuous with respect to the weak topology $\sigma(E, E')$, resp. map the bounded subsets of E into bounded subsets.

The ultraweak topology is the weakest locally convex topology on $B(E)$ for which the linear functionals

$$T \to \sum_{j=1}^{\infty} \lambda_j \langle Te_j, e_j' \rangle$$

on $B(E)$ are continuous. Here $\{\lambda_j\}$ ranges over $l^1(\mathbb{N})$, and $\{e_j\}$ and $\{e_j'\}$ range over the bounded sequences in E and E' respectively. A subset A of $B(E)$, equipped with the ultraweak topology, will be denoted A_{uw}.

A representation π of a group G on a locally convex space E is (here) a homomorphism of G into the group of invertible elements of $L(E)$. π is said to be ultra-irreducible if $\operatorname{span}\{\pi(G)\}$ is dense in $S(E)_{\mathrm{uw}}$. If so, then π is also topologically completely irreducible and hence topologically irreducible, too; also the commutant algebra $\pi(G)'$ reduces to the scalars.

A sufficient condition for ultra-irreducibility can be found as Corollary 8 of [LL1] or in [LL2]. It is an infinite dimensional version of Burnside's theorem.

Theorem 1 (Litvinov-Lomonosov). *Let E be a locally convex space with the property that the Fredholm operators on it possess a well-defined trace. Then a topologically irreducible representation π of a group G on E is ultra-irreducible if the closure of* $\operatorname{span}\{\pi(G)\}$ *in $L(E)_{\mathrm{uw}}$ contains a nonzero compact operator.*

If π is a strongly continuous representation of a Lie group K with right Haar measure dk on a locally convex space E, then we put

$$\pi(\phi)u := \int_K \phi(k)\pi(k)u\,dk \quad \text{for } \phi \in C_0^\infty(K),\ u \in E,$$

where the integral is defined weakly as a linear functional on E'.

Proposition 2. *Let π be a strongly continuous representation of a Lie group K on a semicomplete, locally convex space E. Then*

(α) *$\pi(\phi) \in B(E)$ for all $\phi \in C_0^\infty(K)$.*

(β) *A closed subspace of E is invariant under $\pi(K)$ iff it is invariant under $\pi(C_0^\infty(K))$.*

(γ) $\operatorname{span}\{\pi(\phi)e \mid \phi \in C_0^\infty(K),\ e \in E\}$ *is dense in E.*

(δ) $\operatorname{span}\{\pi(K)\}$ *and $\pi(C_0^\infty(K))$ have identical closures in $B(E)_{\mathrm{uw}}$.*

B. The abstract set-up and consequences. Throughout this subsection we enforce the conditions of the Abstract Set-Up below.

The Abstract Set-Up 3. C and E are semicomplete, locally convex spaces. C is a subspace of E and the inclusion map $i: C \subset E$ is continuous. G is a Lie group, K is a Lie subgroup of G and dk denotes a right Haar measure on K.

π_C is a strongly continuous representation of G on C that extends to a strongly continuous representation π of G on E in such a way that the following holds:

$\pi(\phi)E \subset C$ for each $\phi \in C_0^\infty(K)$ and the corresponding map $\pi_E^C(\phi)$ of E into C is continuous.

Easy consequences are that $\pi(\phi) \in L(E)$ and $\pi_C(\phi) \in L(C)$ for any $\phi \in C_0^\infty(K)$, and the following two lemmas:

Lemma 4 (The approximation lemma). *If W is a closed, invariant subspace of E, then $W \cap C$ is dense in W.*

Lemma 5. *Let W be an invariant subspace of C. Then W is dense in C iff it is dense in E.*

Theorem 6. *π is ultra-irreducible iff π_C is also.*

Proof. We shall only give the if part, because the other one can be treated quite similarly. So assume π_C is ultra-irreducible. To prove π is ultra-irreducible it suffices by the Hahn-Banach theorem to check the following implication for given $u \in [B(E)_{\mathrm{uw}}]'$:

$$(1) \qquad \langle u, \pi(g)\rangle = 0 \quad \text{for all } g \in G$$

implies

$$(2) \qquad \langle u, T\rangle = 0 \quad \text{for all } T \in S(E).$$

Let (1) be given. We then get

$$\langle u, \pi(g)\pi(k)\rangle = 0 \quad \text{for all } g \in G \text{ and } k \in K.$$

Composition by an element from $S(E)$, in particular by $\pi(g)$, is continuous with respect to the ultraweak topology, so from Proposition 2(δ) we get that

$$\langle u, \pi(g)\pi(\phi)\rangle = 0 \quad \text{for all } g \in G \text{ and } \phi \in C_0^\infty(K),$$

which tells us that the linear functional $T_C \to \langle u, i \circ T_C \circ \pi_E^C(\phi)\rangle$ on $B(C)$ vanishes on $\pi_C(G) \subset L(C)$. By the ultra-irreducibility of π_C it vanishes on all of $S(C)$, so

$$(3) \qquad \langle u, i \circ T_C \circ \pi_E^C(\phi)\rangle = 0 \quad \text{for all } T_C \in S(C).$$

Given $T \in S(E)$ we replace T_C in (3) by $\pi_E^C(\phi_1) \circ T \circ i \in S(C)$, where $\phi_1 \in C_0^\infty(K)$ is arbitrary, to get

$$\begin{aligned} 0 &= \langle u, i \circ \pi_E^C(\phi_1) \circ T \circ \pi_E^C(\phi)\rangle \\ &= \langle u, \pi(\phi_1) \circ T \circ \pi(\phi)\rangle. \end{aligned}$$

The continuity of left and right compositions by elements from $L(E)$ combined with Proposition 2(δ) now ensures that (2) holds. □

III. The representations I^Δ and $I^{\Delta\mu}$

Throughout this section G denotes a second countable Lie group which is the semidirect product $G = S \times_s K$ of a normal, closed, and connected subgroup S with a closed subgroup K. We let V be a finite-dimensional complex vector space.

We give $C^\infty(K, V)$ and $C_0^\infty(K, V) = \mathscr{D}(K, V)$ their usual topologies (Fréchet space and LF-space respectively) and put $\mathscr{D}'(K, V) := [\mathscr{D}(K, V')]'$. We identify $C^\infty(K, V)$ as a dense subspace of $\mathscr{D}'(K, V)$ via the natural continuous injection map $i\colon C^\infty(K, V) \subset \mathscr{D}'(K, V)$ given by

$$\langle if, \psi\rangle := \int_K \langle f(k), \psi(k)\rangle\, dk \quad \text{for } f \in C^\infty(K, V),\ \psi \in C_0^\infty(K, V').$$

We note that $C^\infty(K, V)$ and $\mathscr{D}'(K, V)$ are $C^\infty(K)$-modules.

Let Λ be a continuous homomorphism from S to $\mathbb{C}\backslash\{0\}$. By Lie group theory, $\Lambda \in C^\infty(S)$. We define a continuous representation I^Λ of G on $C^\infty(K, V)$ by (cf. [Wi, formula (2.8), p. 72])

$$(1) \qquad [I^\Lambda(sk)f](k_1) := \Lambda(k_1 s k_1^{-1}) f(k_1 k)$$

for $(s, k, k_1) \in S \times K \times K$, $f \in C^\infty(K, V)$.

Definition 1. For $u \in \mathscr{D}'(K, V)$, $\phi \in C_0^\infty(K, V')$ and $(s, k) \in S \times K$ we put

$$(2) \qquad \langle I^\Lambda(sk)u, \phi\rangle := \langle u, I^\Lambda(k^{-1}s)\phi\rangle.$$

I^Λ, defined by (2), is a continuous representation of G on $\mathscr{D}'(K, V)$, extending the representation (1) on $C^\infty(K, V)$. The stability subgroup

$$M = K_\Lambda := \{k \in K | \Lambda(ksk^{-1}) = \Lambda(s) \text{ for all } s \in S\}$$

is a closed subgroup of K. Let μ be a continuous, topologically irreducible representation of M on a complex finite dimensional vector space $V = H(\mu)$. M acts on $C^\infty(K, H(\mu))$ (or more generally on functions from K to $H(\mu)$) by

$$(3) \quad [m \cdot \phi](k) := \mu(m)[\phi(m^{-1}k)] \quad \text{for } m \in M,\ k \in K,\ \phi \in C^\infty(K, H(\mu)).$$

ϕ is a fixed point iff ϕ satisfies the covariance condition $(*)$ from the Introduction.

The action of M extends to a continuous representation of M on $\mathscr{D}'(K, H(\mu))$ which commutes with I^Λ and multiplication by functions from $C^\infty(M\backslash K)$. The vector space

$$\mathscr{D}'_\mu := \{u \in \mathscr{D}'(K, H(\mu)) | m \cdot u = u \text{ for all } m \in M\},$$

$$C_\mu^\infty := C^\infty(K, H(\mu)) \cap \mathscr{D}'_\mu \quad \text{and} \quad C_\mu := C(K, H(\mu)) \cap C_\mu^\infty$$

are therefore invariant under I^Λ and multiplication by functions from $C^\infty(M\backslash K)$. They are closed subspaces of $\mathscr{D}'(K, H(\mu))$, $C^\infty(K, H(\mu))$ and $C(K, H(\mu))$ respectively. We equip them with the topologies from

$$\mathscr{D}'(K, H(\mu)),\ C^\infty(K, H(\mu))$$

and $C(K, H(\mu))$ respectively. The restriction $I^{\Lambda\mu}$ of I^Λ from $\mathscr{D}'(K, H(\mu))$ to $\mathscr{D}'_\mu$ is then a continuous representation of G on $\mathscr{D}'_\mu$; restricting further we get that $I^{\Lambda\mu}$ defines a continuous representation (again denoted $I^{\Lambda\mu}$) of G on C_μ^∞.

Remark 2. The Abstract Set-Up holds here with $C = C_\mu^\infty$, $E = \mathscr{D}'_\mu$ and $\pi = I^{\Lambda\mu}$.

There is a continuous projection p of $C^\infty(K, H(\mu))$ onto C_μ^∞. It can be constructed as follows: Choose $\theta \in C^\infty(K)$ such that $\operatorname{supp}\{m \to \theta(mk) | k \in Q\}$ is compact for any compact subset Q of K and such that

$$\int_M \theta(mk)\, dm = 1 \quad \text{for each } k \in K$$

(cf. the proof of Lemma A.1.1 of [Wa] for the existence of such a θ).

Then

$$(Pk)(k) := \int_M \theta(mk)\mu(m)^{-1}[f(mk)]\,dm,$$

where dm is a right Haar measure on M, works.

Our final result of this section is the first place in which the irreducibility of μ is used. So far $\dim H(\mu) < \infty$ has sufficed (cf. [Wi, Lemma 3.1; Th, Lemma 3]).

Lemma 3. *If* $\mathscr{A}$ *is a dense subset of* $C^\infty(M\backslash K)$ *and* $w \in C_\mu^\infty\backslash\{0\}$, *then* $\operatorname{span}\{aI^\Lambda(k)w | a \in \mathscr{A}, k \in K\}$ *is dense in* C_μ^∞.

Proof. Since the map $a \to af$ for any fixed $f \in C_\mu^\infty$ is continuous from $C^\infty(M\backslash K)$ into C_μ^∞ we get that the closure of $\operatorname{span}\{aI^\Lambda(k)w | a \in \mathscr{A}, k \in K\}$ contains $\operatorname{span}\{\phi I^\Lambda(k)w | \phi \in C^\infty(M\backslash K), k \in K\}$, so that we may assume that $\mathscr{A} = C^\infty(M\backslash K)$. But in that case $\operatorname{span}\{aI^\Lambda(k)w | a \in \mathscr{A}, k \in K\}$ contains $P(\operatorname{span}\{\phi I^\Lambda(k)w | \phi \in C^\infty(K), k \in K\})$. Since P is surjective it suffices to prove that $\operatorname{span}\{\phi I^\Lambda(k)w | \phi \in C^\infty(K), k \in K\}$ is dense in $C^\infty(K, V)$.

Replacing w by a suitable translate we may assume that $w(e) \neq 0$.

We shall show that the annihilator in $C^\infty(K, V)'$ of

$$\operatorname{span}\{\phi I^\Lambda(k)w | \phi \in C^\infty(K), k \in K\}$$

is $\{0\}$.

By Lemma II.4 it suffices to prove that if $U \in C_0^\infty(K, V')$ is in the annihilator then $U = 0$. So assume that U is in the annihilator. Then for all $a \in C^\infty(K)$ and $k_0 \in K$ we have

$$\begin{aligned} 0 = \langle U, aI^\Lambda(k_0)w\rangle &= \int_K \langle U(k), a(k)(I^\Lambda(k_0)w)(k)\rangle\,dk \\ &= \int_K a(k)\langle U(k), (I^\Lambda(k_0)w)(k)\rangle\,dk, \end{aligned}$$

so

$$0 = \langle U(k), (I^\Lambda(k_0)w)(k)\rangle = \langle U(k), w(kk_0)\rangle \quad \text{for all } k, k_0 \in K.$$

Choosing k_0 suitably we get for any $m \in M$ that

$$0 = \langle U(k), w(m)\rangle = \langle U(k), \mu(m)[w(e)]\rangle.$$

Since $w(e) \neq 0$ and μ is irreducible we get that $U(k) = 0$. But $k \in K$ was arbitrary. □

IV. The Poisson transform

The situation and notation are as in §III.

The Lie group G is the semidirect product of a normal, closed and connected subgroup S with a closed subgroup K. $\mathfrak{k}$ will denote the Lie algebra of K

and $L(S)$ that of S. Λ is a smooth homomorphism of S into $\mathbb{C}\backslash\{0\}$, and M denotes the stability subgroup $M = K_\Lambda$.

We assume that there exists an $\mathrm{Ad}(K)$-invariant inner product $\langle\cdot,\cdot\rangle$ on $L(S)$, and extend it to $L(S)^{\mathbb{C}}$. Define $d\Lambda \in L(S)^{\mathbb{C}}$ by (cf. [Wi, p. 70])

$$\Lambda(\exp X) = e^{\langle X, d\Lambda\rangle} \quad \text{for } X \in L(S),$$

define the functions $\phi_{d\Lambda, Z} \in C^\infty(M\backslash K)$ for $Z \in L(S)^{\mathbb{C}}$ by

$$\phi_{d\Lambda, Z} := e^{\langle \mathrm{Ad}(k)Z, d\Lambda\rangle} \quad \text{for } k \in K,$$

and define the K-invariant, point separating subalgebras

$$\mathscr{A}_0(d\Lambda) := \mathrm{span}\{\phi_{d\Lambda, Z} | Z \in L(S)\}, \qquad \mathscr{A}(d\Lambda) := \mathrm{span}\{\phi_{d\Lambda, Z} | Z \in L(S)^{\mathbb{C}}\}$$

of $C^\infty(M\backslash K)$ [Wi, p. 74 and Proposition 4.4]. Note that $\mathscr{A}_0(d\Lambda)$ and $\mathscr{A}(d\Lambda)$ have the same closures in $C^\infty(M\backslash K)$ (cf. the proof of Proposition 1 below).

We will for brevity write $k \cdot Z$ for $\mathrm{Ad}(k)Z$, and $X \cdot Z$ for the action of $X \in \mathfrak{k}$ on $Z \in L(S)^{\mathbb{C}}$.

Let σ be a smooth volume element on $M\backslash K$ and let $\rho \in C^\infty(M\backslash K \times K)$ be the function satisfying

$$\int_{M\backslash K} \psi(xk)\, d\sigma(x) = \int_{M\backslash K} \psi(x)\rho(x, k)\, d\sigma(x) \quad \text{for all } \psi \in C_0^\infty(M\backslash K).$$

We define the Poisson transformation $P = P_{d\Lambda}\colon C_0^\infty(M\backslash K) \to C^\infty(L(S))$ by

$$(Pf)(X) := \int_{M\backslash K} \phi_{d\Lambda, X}(x) f(x)\, d\sigma(x) \quad \text{for } X \in L(S). \tag{1}$$

Proposition 1. *P is injective iff $\mathscr{A}(d\Lambda)$ is dense in $C^\infty(M\backslash K)$. This is in particular the case if $\mathscr{A}(d\Lambda)$ is closed under complex conjugation.*

Proof. Assume first that $\mathscr{A}(d\Lambda)$ is dense in $C^\infty(M\backslash K)$ and that

$$\int_{M\backslash K} \phi_{d\Lambda, X}(x) f(x)\, d\sigma(x) = 0 \quad \text{for some } f \in C_0^\infty(M\backslash K).$$

Since the map $Z \to \int_{M\backslash K} \phi_{d\Lambda, Z}(x) f(x)\, d\sigma(x)$ is holomorphic on $L(S)^{\mathbb{C}}$ and vanishes on the real part $L(S)$ of $L(S)^{\mathbb{C}}$, it vanishes everywhere. So

$$\int_{M\backslash K} \phi_{d\Lambda, Z}(x) f(x)\, d\sigma(x) = 0 \quad \text{for all } Z \in L(S)^{\mathbb{C}}.$$

Since $\mathscr{A}(d\Lambda)$ is dense in $C^\infty(M\backslash K)$, f has to be 0.

Assume conversely that P is injective. By the Hahn-Banach theorem we shall show the following: If $f \in \mathscr{E}'(M\backslash K)$ annihilates $\mathscr{A}(d\Lambda)$ then $f = 0$.

By the Approximation Lemma II.4 we may assume that $f \in C_0^\infty(M\backslash K)$. So we know that

$$\int_{M\backslash K} \phi_{d\Lambda, Z}(x) f(x)\, d\sigma(x) = 0 \quad \text{for all } Z \in L(S)^{\mathbb{C}}.$$

By the injectivity of P we see that $f = 0$ as desired.

The last statement follows from Stone-Weierstrass' theorem and Lemma II.5. □

The papers [Th, Wi and Ra] contain various sufficient conditions for $\mathscr{A}(d\Lambda)$ to be dense in $C^\infty(M\backslash K)$ when K is compact. (Note that $\mathscr{A}(d\Lambda)$ is dense in $C^\infty(M\backslash K)$ iff it is dense in $L^2(M\backslash K)$ (Lemma II.5).)

[He, CD and Ko] show that the Poisson transform is injective when G is the Cartan motion group. We present an extension to semidirect products, using an idea due to Koranyi [Ko]:

Theorem 2. *The Poisson transform is injective if*

$$\langle X \cdot d\Lambda, d\Lambda\rangle = 0 \quad \textit{for all } X \in \mathfrak{k}. \tag{2}$$

Remarks. (i) Condition (2) is automatically satisfied in the case of the Cartan motion group. (Combine (a) and (b), p. 295, of [Ko]. It hinges on the fact that $\operatorname{Re}\Lambda$ and $\operatorname{Im}\Lambda$ commute, both being in the (maximal) commutative subalgebra $\mathfrak{a}$.)

(ii) [Wi, Theorem 5.3, p. 78] proves that $\mathscr{A}(d\Lambda)$ is dense in $C(M\backslash K)$ if the function $k \to \langle k \cdot d\Lambda, d\Lambda\rangle$ is real-valued and K is compact. A particular case occurs if $d\Lambda$ is proportional to a vector in $L(S)$ [Th, Theorem 4]. Our condition (2) is weaker.

(iii) Another connection is to vectors of minimal length (cf. [KN]): If $d\Lambda$ is of minimal length with respect to $K^{\mathbb{C}}$ then (2) holds, (see also [Ra, §2.4]).

Proof of Theorem. Let $m := \frac{1}{2}\dim M\backslash K$ and $o := M1 \in M\backslash K$. We shall apply the method of stationary phase for complex-valued phase functions (Theorem 2.3 of [MS] or formula (X.3.5), p. 536, of [Tr]) to prove $\exp(-t\langle d\Lambda, d\Lambda\rangle)$ times the integral

$$P_{d\Lambda}(f)(td\Lambda) = \int_{M\backslash K} e^{t\langle x\cdot d\Lambda, d\Lambda\rangle} f(x)\, d\sigma(x) \quad \text{as } t \to \infty,$$

where $f \in C_0^\infty(M\backslash K)$, behaves asymptotically as $ct^{-m}f(o)$, where c is a nonzero constant that does not depend on f. To that purpose we introduce the function $a \in C^\infty(M\backslash K)$ defined by

$$a(x) := i\{\langle d\Lambda, d\Lambda\rangle - \langle x \cdot d\Lambda, d\Lambda\rangle\} \quad \text{for } x \in M\backslash K,$$

and shall then prove that

$$\int e^{ita(x)} f(x)\, d\sigma(x) \approx ct^{-m} f(o) + \cdots \quad \text{as } t \to \infty.$$

Since K acts by orthogonal transformations we see that

$$\operatorname{Im} a \geq 0 \quad \text{with equality only at } x = 0, \tag{3}$$

because $M = \{k \in K | k \cdot d\Lambda = d\Lambda\}$ [Wi, Proposition 4.3, p. 74].

For any $X \in \mathfrak{k}$ we have

$$Xa = -i\tfrac{d}{dt}\big|_{t=0}\langle \exp(tX)\cdot d\Lambda, d\Lambda\rangle = -i\langle X \cdot d\Lambda, d\Lambda\rangle = 0$$

by our assumption (2), so $x = o$ is a critical point of a. For any $X \in \mathfrak{k}$ we compute:

$$\frac{d^2}{dt^2}\Big|_{t=0} a(M \exp tX) = (-i) \frac{d^2}{dt^2}\Big|_{t=0} \langle \exp(tX) \cdot d\Lambda, d\Lambda \rangle$$
$$= (-i)\langle X^2 \cdot d\Lambda, d\Lambda \rangle = i||X \cdot d\Lambda||^2.$$

This is different from zero for any $X \in \mathfrak{k}$ off the Lie algebra of M, so $x = o$ is nondegenerate.

Theorem 2.3 of [MS] applies to functions supported in a sufficiently small neighborhood U of the critical point in question, so let us write $f = F + f_o$, where $f_o = f$ near o and $\operatorname{supp} f_o \subset U$. It is enough to show that F contributes nothing to the first term of the formula for the asymptotic behaviour.

By (3) there exists a $\delta > 0$ such that $\operatorname{Im} a \geq \delta$ on $\operatorname{supp} F$, so

$$\left| \int e^{ita(x)} F(x) \, d\sigma(x) \right| \leq \int e^{-t \operatorname{Im} a(x)} |F(x)| \, d\sigma(x) \leq e^{-t\delta} \int |F(x)| \, d\sigma(x),$$

so F does indeed contribute nothing to the asymptotic expansion.

At last we turn to the Poisson transform. We shall show that

$$\{f \in C_0^\infty(M \backslash K) | P_{d\Lambda} f(X) = 0 \text{ for all } X \in L(S)\} = \{0\}.$$

The asymptotics show us that $f(o) = 0$ for any f in the left-hand side. If f belongs to the left-hand side then so does $I^\Lambda(k)[f\rho(\cdot, k)^{-1}]$ for any $k \in K$, so

$$0 = I^\Lambda(k)[f\rho(\cdot, k)^{-1}](o) = f(Mk)\rho(Mk, k)^{-1}$$

and hence $f(Mk) = 0$. □

V. A SUFFICIENT CONDITION FOR ULTRA-IRRREDUCIBILITY

The situation is as stated in §IV. Furthermore μ is a continuous, irreducible and finite dimensional representation of M on $H(\mu) = V$.

Proposition 1. *The representation $I^{\Lambda\mu}$ of G is ultra-irreducible on C_μ^∞ if $\mathscr{A}_0(d\Lambda)$ is dense in $C^\infty(M\backslash K)$.*

Proof. Let $W \neq \{0\}$ be an invariant subspace of C_μ^∞. W is in particular invariant under $I^{\Lambda\mu}(\exp X) = \phi_{d\Lambda, X} \in \mathscr{A}_0(d\Lambda)$, so by Lemma III.3 W is dense in C_μ^∞. This proves that $I^{\Lambda\mu}$ is topologically irreducible. We continue by applying the Litvinov-Lomonosov result (Theorem II.1):

C_μ^∞ has that property that the Fredholm operators on it have well-defined traces, because there exists a continuous projection P of $C^\infty(K, V)$ onto it, and $C^\infty(K, V)$ has the property (see e.g. [JS]).

Via Proposition II.2 (δ) we see that $aI^{\Lambda\mu}(\phi) \in \operatorname{span}\{\pi(G)\}_{\text{uw}}$ for any $a \in \mathscr{A}_0(d\Lambda)$ and $\phi \in C_0^\infty(K)$, and hence by density even for all $a \in C^\infty(M\backslash K)$, so it suffices to prove that $aI^{\Lambda\mu}(\phi)$ is compact when $a \in C_0^\infty(M\backslash K)$. Now

$a = P(\psi)$ for some $\psi \in C_0^\infty(K)$, so for any $f \in C_\mu^\infty$:

$$aI^{\Lambda\mu}(\phi)f = P(\psi)I^{\Lambda\mu}(\phi)f = P(\psi I^{\Lambda}(\phi)f).$$

It now suffices to show that $\psi I^{\Lambda}(\phi): C^\infty(K, V) \to C^\infty(K, V)$ is compact. But here we may assume $V = \mathbb{C}$ because there is no twisting any longer. $\psi I^{\Lambda}(\phi)$ is compact as an integral operator with smooth, compactly supported kernel. □

The next theorem, which is our main result, deals with spaces of functions and distributions between C_μ^∞ and $\mathscr{D}_\mu'$ that satisfy the covariance condition $(*)$, e.g. $C_\mu(K, H(\mu))$ and, for K compact, $L_\mu^p(K, H(\mu))$ for $1 \le p < \infty$.

Theorem 4.11 of [Wi] and Theorem 4 of [Th] are special cases.

Theorem 2. *Let F be a semicomplete, locally convex space which is a subspace of $\mathscr{D}_\mu'$ such that $C_\mu^\infty \subset F \subset \mathscr{D}_\mu'$ with continuous inclusions.*

Assume that F is $I^{\Lambda\mu}$-invariant, and that the map $g \to I^{\Lambda\mu}(g)|F = I_F(g)$ is a strongly continuous representation of G on F.

Then I_F is an ultra-irreducible representation, if the Poisson transform is injective.

Proof. The theorem is an immediate consequence of Proposition 1 in view of Theorem II.6. □

Remark 3. Theorem 1 provides us with a proof of the "double commutant" theorem in [Wi, Theorem 1.3]: Indeed, if $\mathscr{A}(d\Lambda)$ is closed under complex conjugation then it is dense in $C^\infty(M\backslash K)$, so $I^{\Lambda\mu}$ is ultra-irreducible (Theorem 1) and thus topologically completely irreducible.

Example 4. In case of the Euclidean motion group ($S = \mathbb{R}^n$, $K = SO(n)$) our condition (2) holds iff $d\Lambda$ is proportional to a real vector. It is known that for $n > 2$ this is necessary and sufficient for $I^{\Lambda\tau}$ to be (topologically) irreducible, τ being the trivial representation [Ra, 3.5, Exemples]. According to our theorem we even get ultra-irreducibility in that case.

VI. Application to the Cartan motion group

Standard notation. Let G be a semisimple, connected, real Lie group with finite center, and let $\mathfrak{g}$ be its Lie algebra. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ and K be the corresponding maximal compact subgroup. $G_0 := \mathfrak{p} \times_s K$ is the Cartan motion group of (some) affine motions of $\mathfrak{p}$.

We let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, let $\lambda: \mathfrak{a} \to \mathbb{C}$ be real-linear, and let K^λ denote the stabilizer of λ in K.

As is usual we extend λ to all of $\mathfrak{p}$ by setting it 0 on the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{p}$ with respect to the Killing form.

We define the continuous homomorphism $\Lambda: \mathfrak{p} \to \mathbb{C}\backslash\{0\}$ by

$$\Lambda(X) := e^{i\Lambda(X)} \quad \text{for } X \in \mathfrak{p},$$

and note that $K_\Lambda = K^\lambda$.

Let μ be a continuous, irreducible representation of K^λ on $H(\mu)$. From formula (III.1) above we get a representation I^λ of G_0 on $C^\infty(K, H(\mu))$ given by

$$[I^\lambda(X, k)f](k') = e^{i\lambda(\mathrm{Ad}(k')X)} f(k'K)$$
$$\text{for } X \in \mathfrak{p},\ k, k' \in K \text{ and } f \in C^\infty(K, H(\mu)).$$

The representation extends to $\mathscr{D}'(K, H(\mu))$ as before.

By the injectiveness of the Poisson transform on the Cartan motion group we get from the previous results:

Theorem 1. *Let F be a semicomplete, locally convex space such that $C_\mu^\infty \subset F \subset \mathscr{D}'_\mu$ with continuous inclusions.*

If F is I^λ-invariant and the restriction I_F of I^λ to F is a strongly continuous representation of G_0 on F, then I_F is ultra-irreducible.

The theorem holds in particular for the representation I^λ on the space $F = L^2_\mu(K, H(\mu))$, which is the one considered in [CD, Théorème 6, p. 278]. By other methods than ours, [CD] established that I^λ is topologically completely irreducible.

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Department of Mathematics, Aarhus University, Ny Munkegade, DK-8000 Aarhus C, Denmark