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THE FROBENIUS-PERRON OPERATOR ON SPACES OF CURVES

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ABSTRACT. Let $\tau: R^2 \to R^2$ be a diffeomorphism which leaves a compact set A invariant. Let $B \subset A$ be such that τ can map out of B. Assume that τ has a hyperbolic fixed point p in B. Let $\mathscr{C}$ be a space of smooth curves in B. We define a normalized Frobenius-Perron operator on the vector bundle of Lipschitz continuous functions labelled by the curves in $\mathscr{C}$, and use it to prove the existence of a unique, smooth conditionally invariant measure μ on a segment V^u of the unstable manifold of p. A formula for the computation of f^*, the density of μ, is derived, and $\mu(\tau^{-1}V^u)$ is shown to be equal to the reciprocal of the maximal modulus eigenvalue of the Jacobian of τ at p.

1. INTRODUCTION

An intensively developing area of modern ergodic theory concerns the understanding of deterministic dynamical systems $\tau: R^n \to R^n$, whose behavior is asymptotically very complex. The first approach to this problem was purely geometric, the crucial assumption being the existence of an invariant set Λ which is uniformly hyperbolic. Although Axiom A diffeomorphisms satisfy this condition, it is in general difficult to establish uniform hyperbolicity in examples and for some dynamical systems, such as the Henon map and the Duffing system, it has been shown that the condition fails to exist. The main geometric consequence of uniform hyperbolicity is the foliation of the stable manifold in a neighborhood of the hyperbolic attractor. In this setting, the existence of strange attractors can be proved. Plykin's attractor is a typical example.

The second approach in the study of strange attractors is probabilistic; it deemphasizes the geometry and is concerned with invariant measures on the hyperbolic attractor Λ. In [7], the existence of a unique measure on Λ is proved. This measure 'displays' the time averages of points in sets of positive Lebesgue measure near the attractor, which can have Lebesgue measure 0 itself. Although this approach requires less than the purely geometric approach, it still needs the existence of a uniformly hyperbolic attractor. In this note, we shall use a probabilistic approach to the problem of strange attractors, but we shall not insist on the existence of a uniformly hyperbolic attractor.

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We will work on spaces of curves in a neighborhood of a segment of the unstable manifold of a fixed point. Our intention is to study the dynamics of probability density functions on these curves. Whereas in the uniformly hyperbolic case, a neighborhood of Λ is the union of pieces of stable manifolds of points in Λ, the present approach has the potential to bestow 0 measure to points or arcs where the desired foliation fails to exist.

Let $\tau: R^n \to R^n$ be piecewise smooth, and let $A \subset R^n$ be a compact set which is invariant under τ, i.e., $\tau(A) \subset A$. Let $\mathscr{C}$ denote a space of smooth curves in A. (A precise definition is given in §2.) Let p be a fixed point of τ, which is assumed to be hyperbolic. Let $W^u(p)$ be the unstable manifold of p. Let Γ be a curve in a neighborhood of $W^u(p)$, which is transversal to $W^s(p)$. Under τ, Γ is transformed to the curve $\tau(\Gamma)$, which according to the λ-lemma is uniformly close to $W^u(p)$. Let f be a probability density function on Γ. Then τ induces a probability density function f_* on $\tau(\Gamma)$. We denote the operator transforming f into f_* by $\overline{P}_\tau$. The operator which takes (Γ, f) to $(\tau(\Gamma), f_*)$ is called the Frobenius-Perron operator and is denoted by $\overline{\mathscr{P}}_\tau$. It is an operator on a vector bundle of C^0 spaces and will be discussed in detail in §3.

We shall restrict our attention to a segment of $W^u(p)$. For example, in the Henon map, we restrict our attention to the first quadrant; there, the connected segment of $W^u(p)$ passing through the fixed point is a strictly decreasing function which is concave [1]. We denote this curve by $V^u(p)$.

With the foregoing example in mind, we let B be a subset of A, and we let $\mathscr{C}$ denote a space of curves in B which intersect $W^s(p)$ only once in a neighborhood of $V^u(p)$. At this point, a difficulty arises: since τ is expanding in the direction of $V^u(p)$, points in the neighborhood of $V^u(p)$ will eventually leave the set B. Therefore, if we start with a density function f on a curve Γ in B transversal to $W^s(p)$ in a neighborhood of $V^u(p)$, some of the mass of the measure associated with f will be dispersed outside the set B. To retain a probability density function on the image curve $\tau(\Gamma)$ remaining in B, we must normalize $\overline{P}_\tau f$ on $\tau(\Gamma)$. We, therefore, define a new operator, $P_\tau: L_1(\Gamma) \to L_1(\tau(\Gamma))$ by

$$P_\tau f = \frac{\overline{P}_\tau f}{\|\overline{P}_\tau f\|_{|L_1(\tau(\Gamma))}}.$$

We remark that P_τ is not a Frobenius-Perron operator associated with a map. We shall refer to P_τ as the conditional Frobenius-Perron operator. It is a nonlinear Markov operator, i.e., $P_\tau f \geq 0$ if $f \geq 0$, and $\|P_\tau\|_{L_1(\tau(\Gamma))} = \|f\|_{L_1(\Gamma)}$.

A measure μ is called conditionally invariant if there exists a constant α, $0 < \alpha < 1$, such that $\mu(\tau^{-1}E) = \alpha\mu(E)$ for all Borel sets E. It can be easily shown that $P_\tau f^* = f^*$ if and only if the measure $f^*\, ds$ on $V^u(p)$ is conditionally invariant with respect to τ, where ds denotes the arclength differential on $V^u(p)$. The notion of conditionally invariant measures was used in [3, 4].

A conditionally invariant measure μ on a set S can be interpreted as follows: we start with a very large number of points in S and iterate those points under τ. After some time, the distribution of points remaining in S is given by μ.

In §2, we fix the notation for Frobenius-Perron operators on spaces of curves. In §3, we use the notion of regularity of a function to prove the existence of a unique, smooth conditionally invariant measure μ on a segment of the unstable manifold of a fixed point for the homeomorphism τ. Furthermore, we show that $\alpha = 1/|\lambda_1|$, where λ_1 is the eigenvalue of maximum modulus of the Jacobian matrix of τ at the fixed point p. We also derive a formula for f^*, the density of μ, in terms of λ_1.

In §3, we start with Lebesgue measure on a neighborhood of V^u and show that the sequence of measures obtained by iterating it under P_τ converges weakly to μ.

Remark. The idea of considering transformations on spaces of curves has applications. For example, in [6], the motion of rotary drills is modelled by a transformation on a circle: everytime a tip of the circular drill strikes rock, it spins deterministically to a different point on the drill. The shape of the DNA molecule is the well-known double helix. In the course of division, each strand undergoes deformations, which can be viewed as a transformation from one curve to another. A third example is the human spine, which may be modelled by a curve. Each step is a dynamical process through a family of curves which are deformations of the original curve.

2. Notation

In this section, we fix our notation, present definitions, and auxiliary results. Without loss of generality, we shall work in R^2. We consider a transformation $\tau: R^2 \to R^2$ which has a hyperbolic fixed point p. The Henon map is an example. Let B be a bounded neighborhood of p. (For the Henon map, we consider the part of the trapping region that lies in the first quadrant.) Let $V^u = W^u(p) \cap B$ be a connected segment of the unstable manifold of p which lies in B and contains p. For the Henon map, V^u is described in [1]; it is a strictly decreasing function which is concave and whose graph can be approximated by solutions of functional equations. Let $V^s = W^s(p) \cap B$ be a connected piece of the stable manifold of p which lies in B and contains p. Let Γ be a smooth curve in B which intersects V^s transversely and only once at the point x_0. See Figure 1.

With the motivation of the preceding discussion, we present the following notation.

Let $\mathscr{C}'$ be a family of smooth curves in R^2 of finite length. For each $\Gamma \in \mathscr{C}'$, let us distinguish a point $a_\Gamma \in \operatorname{int} \Gamma$ such that the arclengths from a_Γ to both ends of Γ are greater than the fixed positive numbers q and r. Let each $\Gamma \in \mathscr{C}'$ be parametrized by its arclength, i.e., by $\gamma : [-q_\Gamma, r_\Gamma]$, where $q_\Gamma + r_\Gamma$ is the total length of Γ and $\gamma(0) = a_\Gamma$. Let $\mathscr{C}$ be the set of curves belonging

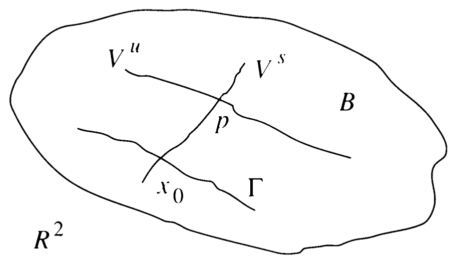

FIGURE 1

to $\mathscr{C}'$, which are parametrized by arclength on the fixed interval $I = [-q, r]$. We define a metric on $\mathscr{C}$ by

$$\delta(\Gamma_1, \Gamma_2) = \|\gamma_1(t) - \gamma_2(t)\| + \|\gamma_1'(t) - \gamma_2'(t)\| + \|\gamma_1''(t) - \gamma_2''(t)\|,$$

where γ_1, γ_2 are the arclength parametrizations of Γ_1 and Γ_2, respectively, and $\|\gamma\| = \sup_{t \in I} \|\gamma(t)\|$.

Let $\Gamma \in \mathscr{C}$ and let $L_1(\Gamma) = L_1(\Gamma, \mathscr{B}, ds_\Gamma)$ be the L_1 space on Γ, where ds_Γ is the arclength measure on Γ and $\mathscr{B}$ is the Borel σ-algebra generated by open arcs of Γ. We now want to define a space of functions on the curves in $\mathscr{C}$: let

$$\mathscr{L} = \{f_{|\Gamma} : \Gamma \in \mathscr{C}, \ f \in C^0(\Gamma)\}.$$

Let $\Gamma_1, \Gamma_2 \in \mathscr{C}$ and let $f_2 \in C^0(\Gamma_1)$, $f_2 \in C^0(\Gamma_2)$. Then define

(1) $$\eta(f_1, f_2) = \delta(\Gamma_1, \Gamma_2) + \|f_1(\gamma_1(t)) - f_2(\gamma_2(t))\|.$$

Clearly, η is a metric. The functions f_1 and f_2 are equal in the η-metric if and only if $\Gamma_1 = \Gamma_2 = \Gamma$ and $f_1 = f_2$ on Γ.

In this setting, we invoke a version of the λ-lemma [2] suited to our needs. Referring to Figure 1, we state:

Proposition 1. *For any $\varepsilon > 0$, there exists an integer n such that $\tau^n(\Gamma) \cap B$ is ε-close to V^u in the C^2-topology, i.e., $\delta(\tau^n(\Gamma) \cap B, V^u) < \varepsilon$.*

3. THE FROBENIUS-PERRON OPERATOR ON CURVES

Let us consider two smooth curves Γ_1 and Γ_2 in R^2, both parametrized on the fixed interval I: for $i = 1, 2$ by $\gamma_i : I \to R^2$, respectively, where $\gamma_i \in C^2(I, R^2)$. Let $\tau \in C^2(R^2, R^2)$, i.e., τ is a C^2 transformation of the plane. We assume that $\tau(\Gamma_1) \supset \Gamma_2$. We consider the spaces $L_1(\Gamma_i) = L_1(\Gamma_i, \mathscr{B}, ds_{\Gamma_i})$, $i = 1, 2$, and define the Frobenius-Perron operator $\overline{P}_\tau$: $L_1(\Gamma_1) \to L_1(\Gamma_2)$ induced by τ. First, let us consider the following diagram:

$$\begin{array}{ccc} \Gamma_1 & \xleftarrow{\ \tau^{-1}\ } & \Gamma_2 \\ {\scriptstyle \gamma_1}\uparrow & & \uparrow{\scriptstyle \gamma_2} \\ I & \xleftarrow[\ T^{-1}\ ]{} & I \end{array}$$

Then $T^{-1}: I \to I$ is conjugate to $\tau^{-1}_{|\Gamma_2}$ and is given by $T^{-1} = \gamma_1^{-1}\tau^{-1}\gamma_2$. The basic property that we impose on $\overline{P}_\tau$ is that for any $E \subset \Gamma_2$ and any $f \in L_1(\Gamma_1)$,

$$\int_{\tau^{-1}(E)} f\, ds_{\Gamma_1} = \int_E \overline{P}_\tau f\, ds_{\Gamma_2}.$$

This says that given a density function f on Γ_1, $\overline{P}_\tau f$ is the density function on Γ_2 induced by τ. We can write this expression in the following equivalent form:

$$\int_{\gamma_1^{-1}\tau^{-1}(E)} f(\gamma_1)|\gamma_1'|\, dm = \int_{\gamma_2^{-1}(E)} \overline{P}_\tau f(\gamma_2)|\gamma_2'|\, dm,$$

where m is Lebesgue measure on I. After changing the variable, the right-hand side of the above equality becomes

$$\int_{T^{-1}\gamma_2^{-1}(E)} \overline{P}_\tau f(\gamma_2 T)|\gamma_2'(T)|\,|T'|\, dm,$$

and since $\gamma_1^{-1}\tau^{-1}(E) = T^{-1}\gamma_2(E)$, we have

$$f(\gamma_1)|\gamma_1'| = \overline{P}_\tau f(\gamma_2 T)|\gamma_2'|\,|T'|.$$

This formula justifies the following definition.

Definition 1. The Frobenius-Perron operator $\overline{P}_\tau: L_1(\Gamma_1) \to L_1(\Gamma_2)$ is defined by the formula

$$\overline{P}_\tau f(x) = \frac{f(\tau^{-1}(x))|\gamma_1'(\gamma_1^{-1}\tau^{-1}(x))|}{|\gamma_2'(\gamma_2^{-1}(x))|\,|T'(\gamma_1^{-1}\tau^{-1}(x))|} \tag{2}$$

for any $f \in L_1(\Gamma_1)$, any $x \in \Gamma_2$, and where γ_1 and γ_2 are parametrizations of Γ_1 and Γ_2, respectively.

Remarks. (1) Since this definition is equivalent to the equality

$$\int_{\tau^{-1}(E)} f\, ds_{\Gamma_1} = \int_E \overline{P}_\tau f\, ds_{\Gamma_2},$$

$E \in \mathscr{B}(\Gamma_2)$, it does not depend on the parametrizations γ_1 and γ_2.

(2) If all the curves are parametrized by arclength, formula (2) simplifies to

$$\overline{P}_\tau f(x) = \frac{f(\tau^{-1}(x))}{|T'(\gamma_1^{-1}\tau^{-1}(x))|}, \tag{3}$$

where $x \in \Gamma_2$ and $f \in L_1(\Gamma_1)$. In the sequel, we shall use this formula with a slightly different notation. Let $D_\gamma\tau(x) = T'(\gamma^{-1}x)$, where $T = \gamma^{-1} \circ \tau \circ \gamma$. Then

$$\overline{P}_\tau f(x) = \frac{f(\tau^{-1}(x))}{|D_\gamma\tau(\tau^{-1}(x))|}.$$

Note that since τ is not a real valued function, $D_\gamma\tau$ is not a directional derivative in the ordinary sense.

We shall now define the regularity functional on a curve. Let Γ be a smooth curve in R^2: Γ is given by $\gamma: I \to R^2$, where $\gamma \in C^2(I, R^2)$. Let $\mathrm{Lip}^+(\Gamma)$ be the space of positive, real-valued Lipschitz continuous functions. Let $f \in \mathrm{Lip}^+(\Gamma)$. Motivated by the definition in [3], we define the regularity functional of f on Γ by

$$\mathrm{Reg}_\Gamma f = \sup_{x\in\Gamma} \frac{|D_\Gamma f(x)|}{|f(x)|},$$

where $D_\Gamma f(x) = (f \circ \gamma)'(\gamma^{-1}(x))/|\gamma'(\gamma^{-1}(x))|$ is the derivative of f in the direction of the curve Γ at x. (For the arclength parametrization γ of Γ, we obtain $D_\Gamma f(x) = (f \circ \gamma)'(\gamma^{-1}(x))$.)

It is easy to check that if $f \in \mathrm{Lip}^+(\Gamma)$ and $\tau \in C^2(R^2, R^2)$, then $\overline{P}_\tau f \in \mathrm{Lip}^+(\Gamma)$. We can now prove

Lemma 1. *We have*

$$\mathrm{Reg}_{\Gamma_2}(\overline{P}_\tau f) \le \mathrm{Reg}_{\Gamma_1}(f) \left(\sup_{x\in\Gamma_2} \frac{|\gamma_1'(\gamma_1^{-1}\tau^{-1}(x))|}{|\gamma_2'(\gamma_2^{-1}(x))|\,|T'(T^{-1}\gamma_2^{-1}(x))|} \right) + M,$$

where $T^{-1} = \gamma_1^{-1}\tau^{-1}\gamma_2$, and $M > 0$ is a constant independent of f.

Proof. Let

$$A(x) = \frac{|\gamma_1'(\gamma_1^{-1}\tau^{-1}(x))|}{|\gamma_2'(\gamma_2^{-1}(x))|\,|T'(T^{-1}\gamma_2^{-1}(x))|}.$$

Then

$$\begin{aligned} D_{\Gamma_2}(\overline{P}_\tau f) &= D_{\Gamma_2}(f(\tau^{-1}x)A(x)) \\ &= D_{\Gamma_2}(f(\tau^{-1}x))A(x) + f(\tau^{-1}x)D_{\Gamma_2}A(x) \\ &= \frac{(f\circ\tau^{-1}\circ\gamma_2)'(t)}{|\gamma_2'(t)|}A(x) + f(\tau^{-1}x)D_{\Gamma_2}A(x), \end{aligned}$$

where $t = \gamma_2^{-1}(x)$. Since $\tau^{-1}\gamma_2 = \gamma_1 T^{-1}$, we have

$$\begin{aligned} (f\circ\tau^{-1}\circ\gamma_2)'(t) &= (f\circ\gamma_1\circ T^{-1})'(t) = (f\circ\gamma_1)'(T^{-1}(t))(T^{-1})'(t) \\ &= D_{\Gamma_1} f(\tau^{-1}x)|\gamma_1'(\gamma_1^{-1}\tau^{-1}(x))|(T^{-1})'(\gamma_2^{-1}(x)). \end{aligned}$$

On the other hand, we have

$$M(x) \equiv D_{\Gamma_2}A(x) = \frac{\frac{d}{dt}|\gamma_1'(T^{-1}t)/\gamma_2'(t)T'(T^{-1}(t)|}{|\gamma_2'(t)|},$$

which is a bounded function on Γ_2 and does not depend on f. Letting $M = \sup_{x\in\Gamma_2} M(x)$, we obtain the desired inequality. □

Let $d_\Gamma(x, y)$ denote the arclength distance between the two points x and y on Γ.

Lemma 2. *If* $d_{\Gamma_1}(\tau^{-n}x, \tau^{-n}y) \le C\alpha^n d_{\Gamma_2}(x, y)$ *for* $n = 1, 2, \dots$, *then*

$$\sup_{x\in\Gamma_2} \frac{|\gamma_1'(\gamma_1^{-1}\tau^{-n}(x))|\,|(T^{-n})'(\gamma_2^{-1}(x))|}{|\gamma_2'(\gamma_2^{-1}(x))|} \le C\alpha^n$$

for $n = 1, 2, \dots$.

Proof. Let

$$A(x) = \frac{|\gamma_1'(\gamma_1^{-1}\tau^{-n}(x))|\,|(T^{-n})'(\gamma_2^{-1}(x))|}{|\gamma_2'(\gamma_2^{-1}(x))|}.$$

Then we have

$$\begin{aligned}
A(x) &= \frac{|\gamma_1'(\gamma_1^{-1}\tau^{-n}(x))|}{|\gamma_2'(\gamma_2^{-1}(x))|} \lim_{y\to x} \frac{|T^{-n}\gamma_2^{-1}y - T^{-n}\gamma_2^{-1}x|}{|\gamma_2^{-1}y - \gamma_2^{-1}x|} \\
&= \frac{|\gamma_1'(\gamma_1^{-1}\tau^{-n}(x))|}{|\gamma_2'(\gamma_2^{-1}(x))|} \lim_{y\to x} \frac{|\gamma_1^{-1}\tau^{-n}y - \gamma_1^{-1}\tau^{-n}x|}{|\gamma_2^{-1}y - \gamma_2^{-1}x|} \\
&= \frac{|\gamma_1'(\gamma_1^{-1}\tau^{-n}(x))|}{|\gamma_2'(\gamma_2^{-1}(x))|} \\
&\quad \times \lim_{y\to x} \frac{|\gamma_1^{-1}\tau^{-n}y - \gamma_1^{-1}\tau^{-n}x|/d_{\Gamma_1}(\tau^{-n}y, \tau^{-n}x)}{|\gamma_2^{-1}y - \gamma_2^{-1}x|/d_{\Gamma_2}(y, x)} \frac{d_{\Gamma_1}(\tau^{-n}y, \tau^{-n}x)}{d_{\Gamma_2}(y, x)} \\
&\le C\alpha^n \frac{|\gamma_1'(\gamma_1^{-1}\tau^{-n}(x))|}{|\gamma_2'(\gamma_2^{-1}(x))|} \frac{|\gamma_2'(\gamma_2^{-1}(x))|}{|\gamma_1'(\gamma_1^{-1}\tau^{-n}(x))|} = C\alpha^n. \quad \square
\end{aligned}$$

Lemma 3. *If* Γ_1 *is close to* V^u *in the* C^2*-topology and* $\Gamma_2 = \tau(\Gamma_1) \cap B$, *then the hypothesis of Lemma* 2 *holds with* $\alpha < 1$.

Proof. The proof follows from the smoothness of τ and the contractive property of τ^{-1} on V^u. □

Recall that $\mathscr{C}$ is the space of smooth curves on B and that $\mathscr{L} = \{f_{|\Gamma} : \Gamma \in \mathscr{C}, f \in \mathrm{Lip}^+(\Gamma)\}$. As in the introduction, we consider the operator $\mathscr{P} : \mathscr{L} \to \mathscr{L}$ defined by

$$\mathscr{P}(\Gamma, f) = (\tau(\Gamma), P_\tau f),$$

where $f \in \mathrm{Lip}^+(\Gamma)$ and P_τ is the conditional Frobenius-Perron operator

$$P_\tau f = \frac{\overline{P}_\tau f_{|\Gamma_2}}{\|\overline{P}_\tau f_{|\Gamma_2}\|_{|L_1(\Gamma_2)}},$$

where $\Gamma_2 = \tau(\Gamma) \cap B$.

We are now ready to state the main result, which will be proved in the subsequent lemmas.

Theorem 1. *Let $\Gamma \in \mathscr{C}$ intersect V^s only once, and let $f \in \mathrm{Lip}(\Gamma)$. Then*

(1) $\lim_{n\to\infty} \mathscr{P}^n(\Gamma, f) = (\Gamma^*, f^*)$ *exists,* $\Gamma^* = V^u$, *and* f^* *is the density function of a conditionally invariant measure.*

(2) $\mathscr{P}(\Gamma^*, f^*) = (\Gamma^*, f^*)$.

(3) f^* *is unique, Lipschitz continuous (if* τ *is* C^∞, *then* f^* *is* C^∞*), and bounded away from* 0.

Lemma 4. *Let* $(\Gamma, f) \in \mathscr{L}$ *and* $(\Gamma_n, f_n) = \mathscr{P}^n(\Gamma, f)$, $n = 1, 2, \ldots$. *Then there exists a constant* K *such that* $\mathrm{Reg}_{\Gamma_n}(f_n) \le K$ *for all* n.

Proof. We know that $\Gamma_n \to V^u$ in the C^2-topology as $n \to \infty$. By Lemma 3, there exists an integer N such that for all $n > N$, the hypothesis of Lemma 2 holds for some n_0 with $C\alpha^{n_0} < 1$. Let $n = N + n_0 \cdot k + r$, $0 \le r < n_0$. Since $\Gamma_n \to V^u$ in the C^2-topology, the constant M of Lemma 1 can be chosen for all Γ_n's at once. Let M_1 be the analogous constant for the transformation τ^{n_0} on the curve Γ_n with $n > N$. We can also find constants C_1 and C_2, where

$$\sup_{n \le N} \left(\sup_{x \in \Gamma_n} |(T^{-1})'(\gamma_n^{-1} x)| \right) = C_1$$

and

$$\sup_{n > N} \left(\sup_{x \in \Gamma_n} |(T^{-1})'(\gamma_n^{-1} x)| \right) = C_2 .$$

By Lemma 1, we have

$$\begin{aligned}
\mathrm{Reg}_{\Gamma_n} P_\tau^n f &\le \mathrm{Reg}_{\Gamma_n} P_\tau^{n_0 k} P_\tau^r P_\tau^N f \\
&\le (C\alpha^{n_0})^k \mathrm{Reg}_{\Gamma_{N+r}} P_\tau^r P_\tau^N f + ((C\alpha^{n_0})^{k-1} + \cdots + C\alpha^{n_0} + 1) M_1 \\
&\le (C\alpha^{n_0})^k C_2 \mathrm{Reg}_{\Gamma_n} P_\tau^N f + (C_2^{r-1} + \cdots + C_2 + 1) M \\
&\quad + ((C\alpha^{n_0})^{k-1} + \cdots + C\alpha^{n_0} + 1) M_1 \\
&\le (C\alpha^{n_0})^k C_2^r C_1^N \mathrm{Reg}_\Gamma f + (C_1^{N-1} + \cdots + C_1 + 1) M \\
&\quad + (C_2^{r-1} + \cdots + C_1 + 1) M + ((C\alpha^{n_0})^{k-1} + \cdots + C\alpha^{n_0} + 1) M_1 .
\end{aligned}$$

It is easy to see that the last expression is bounded by some constant K independent of n, although it may depend on Γ and f. □

Lemma 5. *The functions* $\{f_n\}_{n=1}^\infty$ *are uniformly bounded.*

Proof. For any $n = 1, 2, \ldots$, we have

$$\mathrm{Reg}_{\Gamma_n}(f_n) = \sup_{x \in \Gamma_n} \frac{|(f_n \circ \gamma_n)'(\gamma_n^{-1} x)|}{|f_n(x)|} \le K .$$

Thus we can write

$$\sup_{t \in I} \frac{|(f_n \circ \gamma_n)'(t)|}{|f_n \circ \gamma_n(t)|} \le K .$$

Hence, for any $t_1, t_2 \in I$, we have

$$|\ln(f_n \circ \gamma_n)(t_1) - \ln(f_n \circ \gamma_n)(t_2)| \le \left| \int_{t_1}^{t_2} \mathrm{Reg}_{\Gamma_n}(f_n)\, dt \right| \le K|I|$$

or

$$\frac{f_n(\gamma_n(t_1))}{f_n(\gamma_n(t_2))} \le \exp(K|I|).$$

Since $\int_I (f_n \circ \gamma_n)(t)\, dt = 1$, there exists a point $t_0 \in I$ such that $(f_n \circ \gamma_n)(t_0) = 1/|I|$. Thus there exists a constant K_1 such that $\|f_n\| \le K_1$ for $n = 1, 2, \ldots$. □

Corollary 1. *The proof of Lemma* 5 *also shows that*

$$f_n(x) \ge \exp(-K|I|) \quad \textit{for all } x \in \Gamma_n.$$

Lemma 6. *The functions* $\{f_n\}_{n=1}^{\infty}$ *are uniformly equicontinuous.*

Proof. We have $\mathrm{Reg}_{\Gamma_n}(f_n) \le K$ and $\|f_n\| \le K_1$ for $n = 1, 2, \ldots$. Hence,

$$\|(f_n \circ \gamma_n)'\| \le KK_1 \quad \text{for } n = 1, 2, \ldots,$$

which implies that $f_n \circ \gamma_n$ and f_n satisfy the same Lipschitz condition for $n = 1, 2, \ldots$. □

Lemma 7. *The sequence* $\{f_n\}_{n=1}^{\infty}$ *is precompact in the* $\mathscr{L}$*-topology.*

Proof. The proof follows directly from Lemmas 5 and 6 and the Arzella-Ascoli Theorem. □

Lemma 8. *Let* $(\Gamma_1, f), (\Gamma_2, g) \in \mathscr{L}$, *where* f *and* g *are bounded functions which are also bounded away from* 0. *Then there exists a constant* $W > 0$ *such that*

$$\eta(\mathscr{P}^n(\Gamma_1, f), \mathscr{P}^n(\Gamma_2, g)) \le W\eta((\Gamma_1, f), (\Gamma_2, g))$$

for all $n = 1, 2, \ldots$.

Proof.

$$\eta(\mathscr{P}^n(\Gamma_1, f), \mathscr{P}^n(\Gamma_2, g)) = \delta(\tau^n(\Gamma_1) \cap B, \tau^n(\Gamma_2) \cap B) + \|P_\tau^n f \circ \gamma_{\tau^n(\Gamma_1)\cap B}, P_\tau^n g \circ \gamma_{\tau^n(\Gamma_2)\cap B}\|.$$

The first summand is smaller than $W_1\delta(\Gamma_1, \Gamma_2)$ for some $W_1 > 0$, by the λ-lemma. Mimicking Proposition 1 of [3], we estimate the second summand as follows.

Let χ_{Γ_1} and χ_{Γ_2} be functions identically equal to 1 on Γ_1 and Γ_2, respectively. Let $\Gamma_1^{(n)} = \tau^n(\Gamma_1) \cap B$, $\Gamma_2^{(n)} = \tau_2^n(\Gamma) \cap B$, $n = 1, 2, \ldots$. Let $\beta_n(f) = \|\overline{P}_\tau^n(f)\|_1$, $\beta_n(g) = \|\overline{P}_\tau^n(g)\|_1$, $\beta_n(\chi_{\Gamma_1}) = \|\overline{P}_\tau^n(\chi_{\Gamma_1})\|_1$, $\beta_n(\chi_{\Gamma_2}) = \|\overline{P}_\tau^n(\chi_{\Gamma_2})\|_1$, $n = 1, 2, \ldots$. These are all positive numbers since $f, g, \chi_{\Gamma_1}, \chi_{\Gamma_2}$ are bounded away from 0.

We have

$$\begin{aligned}
&\|P_\tau^n f\circ\gamma_{\Gamma_1^{(n)}} - P_\tau^n g\circ\gamma_{\Gamma_2^{(n)}}\| \\
&= \left\| \frac{\overline{P}_\tau^n f\circ\gamma_{\Gamma_1^{(n)}}}{\beta_n(f)} - \frac{\overline{P}_\tau^n g\circ\gamma_{\Gamma_2^{(n)}}}{\beta_n(g)} \right\| \\
&\le \frac{1}{\beta_n(f)}\|\overline{P}_\tau^n f\circ\gamma_{\Gamma_1^{(n)}} - \overline{P}_\tau^n g\circ\gamma_{\Gamma_2^{(n)}}\| + \left\| \frac{\overline{P}_\tau^n g\circ\gamma_{\Gamma_2^{(n)}}}{\beta_n(g)} \frac{\beta_n(f)-\beta_n(g)}{\beta_n(f)} \right\| \\
&\le \frac{1}{\beta_n(f)}\|\overline{P}_\tau^n f\circ\gamma_{\Gamma_1^{(n)}} - \overline{P}_\tau^n g\circ\gamma_{\Gamma_2^{(n)}}\| \\
&\quad + \|P_\tau^n g\||I|\frac{1}{\beta_n(f)}\|\overline{P}_\tau^n f\circ\gamma_{\Gamma_1^{(n)}} - \overline{P}_\tau^n g\circ\gamma_{\Gamma_2^{(n)}}\| \\
&= (1+\|P_\tau^n g\||I|)\frac{1}{\beta_n(f)} \left\| \frac{f(\tau^{-n}(\gamma_{\Gamma_1^{(n)}}))}{|D_{\Gamma_1}\tau^n(\tau^{-n}(\gamma_{\Gamma_1^{(n)}}))|} - \frac{f(\tau^{-n}(\gamma_{\Gamma_1^{(n)}}))}{|D_{\Gamma_1}\tau^n(\tau^{-n}(\gamma_{\Gamma_1^{(n)}}))|} \right\| \\
&\le (1+\|P_\tau^n g\||I|)\left(\frac{1}{\beta_n(f)}\|\overline{P}_\tau^n\chi_{\Gamma_1}\|\,\|f(\tau^{-n}(\gamma_{\Gamma_1^{(n)}})) - g(\tau^{-n}(\gamma_{\Gamma_2^{(n)}}))\| \right.\\
&\qquad\qquad \left. + \frac{1}{\beta_n(f)}\|g\|\,\|\overline{P}_\tau^n\chi_{\Gamma_1} - \overline{P}_\tau^n\chi_{\Gamma_2}\|\right) \\
&\le (1+\|P_\tau^n g\||I|)\left(\frac{\beta_n(\chi_{\Gamma_1})}{\beta_n(f)}\|P_\tau^n\chi_{\Gamma_1}\|\|f-g\| \right.\\
&\qquad\qquad \left. + \|g\|\,\frac{\beta_n(\chi_{\Gamma_1})}{\beta_n(f)}\left\|P_\tau^n\chi_{\Gamma_1} - P_\tau^n\chi_{\Gamma_2}\frac{\beta_n(\chi_{\Gamma_2})}{\beta_n(\chi_{\Gamma_1})}\right\|\right).
\end{aligned}$$

Moreover, we have

$$(\beta_n(\chi_{\Gamma_1})/\beta_n(f)) \le 1/\inf f,$$

and, by Lemma 5, $\|P_\tau^n g\|$, $\|P_\tau^n\chi_{\Gamma_1}\|$, $\|P_\tau^n\chi_{\Gamma_2}\|$ are uniformly bounded. By the λ-lemma, $(\beta_n(\chi_\Gamma)/\beta_n(\chi_\Gamma))$ is uniformly bounded and the bound depends only on the angle at which Γ_1 and Γ_2 intersect V^s.

Summing up we obtain the conclusion of the lemma with W depending only on the angle at which Γ_1 and Γ_2 intersect V^s and upper and lower bounds on f and g. □

Lemma 9. *The operator* $\mathscr{P}$ *has a unique fixed point* (Γ^*, f^*) *with* $\Gamma^* = V^u$ *and* $f^* \in \mathrm{Lip}^+(V^u)$.

Proof. Let us consider $\tau = \tau_{|V^u}$ and $P_\tau = P_{\tau|V^u}$. Let

$$\mathscr{K} = \left\{ f\in \mathrm{Lip}^+(V^u),\ f>0,\ \int_0^1 f\,ds_{V^u} = 1 \right\}.$$

It is easy to see that $P_\tau(\mathscr{K}) \subset \mathscr{K}$. Using arguments analogous to those in the proof of Lemma 1, we can show that for $f\in\mathscr{K}$,

$$\mathrm{Reg}_{V^u} P_\tau f \le \beta\,\mathrm{Reg}_{V^u} f + M,$$

where $0 < \beta < 1$ and $M > 0$ is independent of β. Let $\rho = M/(1-\beta)$ and $H_\rho = \{f \in \mathscr{K} : \mathrm{Reg}_{V^u} f \le \rho\}$. Then $P_\tau H_\rho \subset H_\rho$. Since regularity is a convex function, H_ρ is convex. Also, H_ρ is compact in $C^0(V^u)$. The proof of this follows from arguments analogous to those in Lemmas 5, 6, and 7.

Since P_τ is a normalization of a continuous operator, it is itself a continuous operator on H_ρ. By the Schauder-Tychonoff Theorem, P_τ has a fixed function f^* in H_ρ.

We will now prove the uniqueness of f^*. Let us assume that there exist two fixed functions f_1 and f_2 for P_τ in H_ρ. Let α_i be the constant such that

$$\overline{P}_\tau f_i = \alpha_i P_\tau f_i,$$

i.e., $\alpha_i = \int_{\tau^{-1}(V^u)} f_i \, ds_{V^u}$ for $i = 1, 2$.

First we will prove that $\alpha_1 = \alpha_2$. Assume $\alpha_1 \ne \alpha_2$ and let $\alpha_1 > \alpha_2$. We have

$$\overline{P}^n_\tau(f_1) = \alpha_1^n P^n_\tau(f_1) = \alpha^n f_1,$$
$$\overline{P}^n_\tau(f_2) = \alpha_2^n P^n_\tau(f_2) = \alpha^n f_2.$$

By Lemma 5 and Corollary 1 there exists $\beta > 0$ such that $\beta f_1 \ge f_2$. Thus we have

$$\beta \alpha_2^n f_2 = \beta \overline{P}^n_\tau(f_2) = \overline{P}^n_\tau(\beta f_2) \ge \overline{P}^n_\tau(f_1) = \alpha_1^n f_1.$$

Therefore,

$$f_2 \ge (\alpha_1/\alpha_2)^n f_1/\beta,$$

which is impossible since $(\alpha_1/\alpha_2)^n \to \infty$, as $n \to \infty$.

Let $\alpha = \alpha_1 = \alpha_2$. We shall now prove that $f_1 = f_2$. Since $\alpha_1 = \alpha_2$, we have that

$$\int_{\tau^{-n}(V^u)} f_1 \, ds_{V^u} = \int_{\tau^{-n}(V^u)} f_2 \, ds_{V^u}$$

for $n = 1, 2, \ldots$. By continuity of f_1 and f_2, it follows that $f_1(p) = f_2(p) = c$, where p is the fixed point of τ. Since f_i is a fixed point of P_τ, $i = 1, 2$,

$$f_i(x) = P_\tau f_i(x), \qquad x \in V^u.$$

Therefore,

$$f_i(x) = \frac{f_i(\tau^{-1}x)}{\alpha |D_\gamma \tau(\tau^{-1}x)|}$$

and

$$f_i(\tau^{-1}x) = \alpha f_i(x) |D_\gamma \tau(\tau^{-1}x)|,$$

where γ is the arclength parametrization of V^u. It is easy to show inductively that we have

$$f_i(\tau^{-n}x) = \alpha^n f_i(x) |D_\gamma \tau(\tau^{-n}x)|$$

for $x \in V^u$, $n = 1, 2, \ldots$, $i = 1, 2$. Since f_1 and f_2 are continuous and $\tau^{-n}x \to p$ as $n \to \infty$, we obtain:

$$f_i(p) = f_i(x) \lim_{n\to\infty} \alpha^n |D_\gamma \tau^n(\tau^{-n}x)|.$$

Since $f_1(p) = f_2(p) = c$, we have

$$f_1(x) = f_2(x) = \frac{c}{\lim_{n\to\infty} \alpha^n |D_\gamma \tau^n(\tau^{-n}x)|}.$$

Therefore, $f_1 = f_2$. □

Corollary 2. *The unique Lipschitz continuous, positive fixed function f^* of $P_{\tau|V^u}$ is given by*

$$f^*(x) = \frac{f^*(p)}{\lim_{n\to\infty} \alpha^n |D_\gamma \tau^n(\tau^{-n}x)|}$$

for $x \in V^u$, and the pair (V^u, f^*) is the unique fixed point of $\mathscr{P}$.

Corollary 3. *The number α for which $\mu(\tau^{-1}V^u) = \alpha\mu(V^u)$ or*

$$\int_{\tau^{-1}(V^u)} f^*\, ds_{V^u} = \alpha \int_{V^u} f^*\, ds_{V^u}$$

is equal to $1/|D_\gamma\tau(p)| = 1/|\lambda_1|$, where λ_1 is the maximal eigenvalue of the Jacobian matrix of τ at the fixed point p.

Proof. Since $D_\gamma\tau(x) = T'(\gamma^{-1}x)$ and $T^n = \gamma^{-1} \circ \tau^n \circ \gamma$, it can be shown that

$$D_\gamma \tau^n(\tau^{-n}x) = D_\gamma\tau(x) D_\gamma\tau(\tau^{-1}x) \cdots D_\gamma\tau(\tau^{-n+1}x) D_\gamma\tau(\tau^{-n}x).$$

Since $\tau^{-n}x \to p$ as $n \to \infty$ and $D_\gamma\tau^n$ is continuous, the only possible way that $\lim_{n\to\infty} \alpha^n |D_\gamma\tau^n(\tau^{-n}x)|$ can exist is that $\alpha = 1/D_\gamma\tau(p)|$.

Recall that $D_\gamma\tau(p) = T'(\gamma^{-1}p)$. Since $T(t) = \gamma^{-1} \circ \tau \circ \gamma(t)$, $T'(t) = J_{\gamma^{-1}}(\tau\gamma(t)) J_\tau(\tau\gamma(t)) J_\gamma(t)$, where J_ρ is the Jacobian of the transformation ρ. Let t_0 be the point in I such that $\gamma(t_0) = p$, the fixed point of τ. Then

$$T'(\gamma^{-1}p) = J_{\gamma^{-1}}(p) J_\tau(p) J_\gamma(\gamma^{-1}p).$$

Notice that $J_\gamma(t_0)$ is a vector in the direction of the tangent line to V^u at p. Hence

$$J_\tau(p) J_\gamma(\gamma^{-1}p) = \lambda_1 J_\gamma(\gamma^{-1}p),$$

where λ_1 is the maximal eigenvalue of $J_\tau(p)$. Thus

$$|T'(\gamma^{-1}p)| = |\lambda_1 J_{\gamma^{-1}}(p) J_\gamma(\gamma^{-1}p)| = |\lambda_1|. \quad \square$$

Lemma 10. *Let $(\Gamma, f) \in \mathscr{L}$ and $(\Gamma_n, f_n) = \mathscr{P}^n(\Gamma, f)$, $n = 1, 2, \ldots$. Then the sequence $\{(\Gamma_n, f_n)\}_{n=1}^\infty$ converges in $\mathscr{L}$ to the unique fixed point of $\mathscr{P}$, (V^u, f^*).*

Proof. We proved in Lemma 7 that the sequence $\{(\Gamma_n, f_n)\}_{n=1}^{\infty}$ is precompact in $\mathscr{L}$. Let (V^u, g) be a limit point of the sequence, i.e., there exists a subsequence $\{(\Gamma_{n_k}, f_{n_k})\}$ such that

$$(\Gamma_{n_k}, f_{n_k}) \overset{\mathscr{L}}{\to} (V^u, g).$$

We will prove that $(P_{\tau|V^u})^{n_{k+1}-n_k} g \to g$ in $\mathrm{Lip}^+(V^u)$. We have

$$\begin{aligned} \|(P_{\tau|V^u})^{n_{k+1}-n_k} g - g\| &= \eta(\mathscr{P}^{n_{k+1}-n_k}(V^u, g), (V^u, g)) \\ &\le \eta(\mathscr{P}^{n_{k+1}-n_k}(V^u, g), \mathscr{P}^{n_{k+1}-n_k}(\mathscr{P}^{n_k}(\Gamma, f))) \\ &\quad + \eta(\mathscr{P}^{n_{k+1}}(\Gamma, f), (V^u, g)) \\ &\le W\eta((V^u, g), \mathscr{P}^{n_k}(\Gamma, f)) + \eta(\mathscr{P}^{n_{k+1}}(\Gamma, f), (V^u, g)) \\ &\to 0 \quad \text{as } k \to \infty. \end{aligned}$$

Thus, $(P_{\tau^u|V^u})^{n_{k+1}-n_k} g \to g$ in $\mathrm{Lip}^+(V^u)$.

There are two possibilities:

(a) The sequence $\{(n_{k+1} - n_k)\}_{k=1}^{\infty}$ is bounded. Then g is a periodic point of $P_{\tau|V^u}$. Since the reasoning of Lemma 9 applies to τ^n as well as to τ, $g = f^*$.

(b) The sequence $\{(n_{k+1} - n_k)\}_{k=1}^{\infty}$ is unbounded. Then g is a limit of some sequence $\{(P_{\tau}^{n_i} g)\}_{i=1}^{\infty}$. Using the results of Proposition 6 and Theorem 3 of [3] applied to the space V^u, the transformation $\tau_{|V^u}$, and the operator $P_{\tau|V^u}$, we again obtain $g = f^*$. □

Remarks. (1) Lemma 10 does not necessarily hold if we start the sequence $\{(\Gamma_n, f_n)\}_{n=1}^{\infty}$ from a density function which is not positive or Lipschitz continuous. For example, in the case of the Henon mapping, if we start with a density f on V^u, which is 0 on one side of the fixed point and positive on the other side, then there exist at least two limit densities for $\{f_n\}_{n=1}^{\infty}$.

(2) Using the methods in the appendix to [3], we can prove that if τ is C^{∞}, then the conditionally invariant density f^* is also C^{∞} and that the convergence

$$\mathscr{P}^n(\Gamma, f) \to (V^u, f^*)$$

takes place in any C^r-topology for both the curves and the densities on them.

(3) Recall that $V^u = W^u(p) \cap B$. Hence V^u and the conditionally invariant measure depend on B. We show this dependence explicitly by writing $V_B^u = W^u(p) \cap B$ and $\mu_B = \mu$. Corollary 3 proves that $\mu_B(\tau^{-1} V_B^u) = 1/|\lambda_1|$, where λ_1 is the maximal eigenvalue of the Jacobian of τ at p; that is, $\mu_B(\tau^{-1} V_B^u)$ does not depend on B. This phenomenon exists in a simpler setting. Consider the map $\tau\colon R \to R$, defined by $\tau(x) = sx$, where $s > 1$. Let $B = [-\beta_1, \beta_2]$ be any interval containing the origin. It is easy to show that μ_B is Lebesgue measure on B. Hence $\mu_B(\tau^{-1} B) = 1/s$, independent of B.

(4) The conditional Frobenius-Perron operator P_τ is defined by normalizing the Frobenius-Perron operator $\overline{P}_\tau$. Another normalized Frobenius-Perron operator can be obtained by defining $P'_\tau = \overline{P}_\tau + (1 - \alpha)$, where $\alpha = \mu(\tau^{-1} V^u)$

and μ is the conditional invariant measure obtained by using P_τ. P'_τ can be interpreted as follows: $\overline{P}_\tau f$ causes $1-\alpha$ of the mass of the density f to be removed from the set B, and the number $1-\alpha$ reflects the insertion of this mass back into B. It is easy to see that $P'_\tau f^* = f^*$, where $f^* = P_\tau f^*$ is the density of the conditional invariant measure.

4. Convergence to the conditionally invariant measure

In this section, we consider the following problem. Let $\mathscr{U}$ be any neighborhood of the fixed point with $\mathscr{U} \subset B$. Let l be the normalized Lebesgue measure on $\mathscr{U}$, or any absolutely continuous measure on $\mathscr{U}$ having a Lipschitz continuous density function. We consider the sequence of probability measure:

$$\psi_n = \frac{\tau_*^n l_{|B}}{\tau_*^n l(B)}.$$

We will show that $\psi_n \to \mu$ in the weak topology, where μ is the conditionally invariant measure on V^u, the existence of which was proved in the previous section.

Let $\mathscr{W}$ be any C^2 vector field on $\mathscr{U}$ such that V^u is an integral curve of $\mathscr{W}$ and $\mathscr{W}$ is transversal to V^s. Let the family $\{\Gamma_x\}_{x\in V^s\cap\mathscr{U}}$ be the foliation of $\mathscr{U}$ into integral curves of $\mathscr{W}$. We will need the following lemma in the sequel.

Lemma 11. *The convergence of $P_\tau^n f_{\Gamma_x}$ to f^* as $n\to\infty$ proved in Theorem* 1 *is uniform in $x \in V^s$.*

Proof. By Theorem 1, we know that $P_\tau^n f_{\Gamma_x} \to f^*$ in the $\mathscr{L}$-topology, for any $x \in V^s$. From the λ-lemma for families of curves transversal to V^s and depending continuously on $x \in V^s$, we know that $\tau^n(\Gamma_x)\cap B \to V^u$ uniformly in the δ-metric. The speed of convergence depends only on the angle of intersection of Γ_x and V^s, and this angle can be made uniformly bounded away from zero by a proper choice of the vector field $\mathscr{W}$. By definition

$$\overline{P}_\tau^n f_{\Gamma_x}(u) = \frac{f_{\Gamma_x}(\overline{\tau}^n(u))}{|D_{\Gamma_x}\tau^n(\tau^{-n}(u))|}.$$

Since $\{f_{\Gamma_x}\}_{x\in V^s}$ is a uniformly equicontinuous, uniformly bounded family of functions, τ is at least a C^2-transformation, and $\tau^n(\Gamma_x)\cap B \to V^u$ uniformly, the functions $\overline{P}_\tau^n f_{\Gamma_{x_1}}$ and $\overline{P}_\tau^n f_{\Gamma_{x_2}}$ are uniformly close for all x_1 and $x_2 \in V^s$. This implies that the numbers $\rho(\tau^n\Gamma_{x_1})$ and $\rho(\tau^n\Gamma_{x_2})$ are uniformly close for $x_1, x_2 \in V^s$, where

$$\rho(\tau^n\Gamma_x) = \int_{\tau^n(\Gamma_x)\cap B} \overline{P}_\tau^n f_{\Gamma_x}(\omega)\, ds_{\tau^n(\Gamma_x)}(\omega).$$

Hence we obtain the desired uniform convergence with respect to $x \in V^s$ in $\mathscr{L}$. □

Theorem 2. *Let $\mathscr{U} \subset B$ be a neighborhood of p. Let υ be an absolutely continuous (with respect to Lebesgue measure) probability measure on $\mathscr{U}$ having a Lipschitz continuous density function. Then the sequence*

$$\psi_n = \frac{\tau_*^n \upsilon_{|B}}{\tau_*^n \upsilon(B)}$$

converges in the weak topology to the conditionally invariant measure μ of Theorem 1.

Proof. The measure υ can be represented as an integral of conditional measures,

$$\upsilon(h) = \int_{V^s} \left(\int_{\Gamma_x} h(y)\, d\upsilon_{\Gamma_x}(y) \right) d\bar{\upsilon}(x)$$

for any continuous function h on $\mathscr{U}$. The measure υ_{Γ_x} is absolutely continuous (with respect to arclength measures on Γ_x) and has a Lipschitz continuous density. The measure $\bar{\upsilon}$ is absolutely continuous with respect to the arclength measure on V^s. Therefore, we have

$$\upsilon(h) = \int_{V^s} \left(\int_{\Gamma_x} h(y) f_{\Gamma_x}(y)\, ds_{\Gamma_x}(y) \right) f(x)\, ds_{V^s}(x),$$

where f_{Γ_x}, $x \in V^s$, and f are positive, Lipschitz continuous functions.

Now, for any positive integer n, we have

$$\begin{aligned} \tau_*^n \upsilon(h) = \upsilon(h \circ \tau^n) &= \int_{V^s} \left(\int_{\Gamma_x} h(\tau^n y) f_{\Gamma_x}(y)\, ds_{\Gamma_x}(y) \right) f(x)\, ds_{V^s}(x) \\ &= \int_{\tau^n(V^s)} \left(\int_{\tau^n(\Gamma_x)} h(\omega) \overline{P}_{\tau^n} f_{\Gamma_x}(\omega)\, ds_{\tau^n(\Gamma_x)} \right) \frac{f(\tau^{-n}u)}{|D_{V^s}\tau^n(\tau^{-n}u)|}\, ds_{V^s}(u). \end{aligned}$$

This implies

$$\frac{\tau_*^n \upsilon(h)}{\tau_*^n \upsilon(B)} = \int_{\tau^n(V^s)} \frac{\rho(\tau^n \Gamma_x)}{\tau_*^n \upsilon(B)} \left(\int_{\tau^n(\Gamma_x)} h(\omega)((P_\tau)^n f_{\Gamma_x})(\omega)\, ds_{\tau^n(\Gamma_x)} \right) f_n(u)\, ds_{V^s}(u),$$

where

$$f_n(u) = \frac{f(\tau^{-n}u)}{|D_{V^s}\tau^n(\tau^{-n}u)|}, \qquad \rho(\tau^n \Gamma_x) = \int_{\tau^n(\Gamma_x)\cap B} \overline{P}_\tau^n f_{\Gamma_x}(\omega)\, ds_{\tau^n(\Gamma_x)},$$

and $u = \tau^n(x)$.

By Lemma 11, we know that $P_\tau^n f_{\Gamma_x} \to f^*$ in $\mathscr{L}$ uniformly in $x \in V^s$. Also, by Lemma 11, the numbers $\rho(\tau^n \Gamma_x)$ depend continuously on $x \in V^s$. It is also easy to see that

$$\int_{\tau^n(V^s)} \frac{\rho(\tau^n \Gamma_x)}{\tau_*^n \upsilon(B)} f_n(u)\, ds_{V^s}(u) = 1.$$

Using all these facts and the continuity of h, we obtain

$$\frac{\tau_*^n \upsilon(h)}{\tau_*^n \upsilon(B)} \to \int_{V^u} h(\omega) f^*(\omega)\, ds_{V^u}$$

as $n \to \infty$. $\square$

5. Application to the Henon map

The application of the foregoing theory to the Henon map is straightforward. The fixed point $p = (.631, .189)$ is hyperbolic; this allows us to use the λ-lemma in a neighborhood of p. B can be any open set in the trapping region of the Henon map which does not intersect the vertical line $x = -.325$ where the eigenvalues of the Jacobian of τ are equal to 1 in modulus. The intersection of the trapping region and the first quadrant is the set B used in [1].

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